A QUADRATICALLY CONVERGENT ALGORITHM FOR SOLVING INFINITE DIMENSIONAL INEQUALITIES

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A Quadratically Convergent Algorithm for Solving Infinite Dimensional Inequalities*

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Abstract

Many design problems can be formulated as determining a parameter to satisfy conventional and infinite dimensional constraints. An algorithm, with quadratic rate of convergence, for solving such inequalities, is presented.

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1. Introduction

Many design problems, such as the design of circuits, control systems and structures [1,2,3], can be formulated as determining a parameter vector $z \in \mathbb{R}^n$ which satisfies:

$$g^j(z) \leq 0 \ , \quad j = 1,\ldots,p, \tag{1}$$

and

$$\phi^j(z,\alpha) \leq 0 \ , \quad j = 1,\ldots,m, \tag{2}$$

for all $\alpha \in A_j$. The set A_j is a compact interval of the real line. The infinite dimensional constraint $\phi^j(z,\alpha) \leq 0$, for all $\alpha \in A_j$, $j = 1,\ldots,m$, may be expressed as:

$$f^j(z) \leq 0 \ , \quad j = 1,\ldots,m, \tag{3}$$

where

$$f^j(z) \triangleq \max\{\phi^j(z,\alpha),\ \alpha\in A_j\} \tag{4}$$

Without loss of generality, so as to simplify exposition, we shall assume that $A_1 = A_2 = \ldots = A_m = A = [\alpha_0,\alpha_c]$. Polak and Mayne [4] describe an algorithm for minimization subject to constraints (1) and (2); an improved algorithm, due to Gonzaga, Polak and Trahan, is presented in [5]. Both of these first order algorithms are easily modified to solve inequalities (1) and (2).

In this paper we present an algorithm which solves (1) and (2) and has a quadratic rate of convergence. It is, perhaps, worth mentioning that it is easy to extend conventional algorithms, in a formal manner, to cope with infinite dimensional constraints. However, the resultant algorithms are conceptual, i.e., at each iteration of the main algorithm an infinite process is required (e.g. to solve $\min\{\phi(z,\alpha)\,|\,\alpha\in A\}$). The achievement in [4] and [5] is the development of implementable algorithms (i.e., requiring a finite process at each iteration). Similarly it is easy to extend Newton's method to solve infinite dimensional constraints; the resultant algorithm, while it has a quadratic rate of convergence, is conceptual. However, under certain assumptions which are satisfied by a reasonably wide class of problems, it is possible to obtain an implementable, quadratically convergent algorithm.

We present in the next section a locally convergent algorithm with a quadratic rate of convergence and in §3 we show how the algorithm may be stabilized to ensure global convergence. Finally, in §4, we present an implementable algorithm.

2. A Quadratically Convergent Algorithm

To state our assumptions it is helpful to introduce a few definitions. Let $f(\cdot): \mathbb{R}^n \to \mathbb{R}^m$ denote $(f^1(\cdot), f^2(\cdot), \ldots, f^m(\cdot))^T$, $g(\cdot): \mathbb{R}^n \to \mathbb{R}^p$ denote $(g^1(\cdot), g^2(\cdot), \ldots, g^p(\cdot))^T$ and let $\phi(\cdot,\cdot): \mathbb{R}^n \times A \longrightarrow \mathbb{R}^m$ denote $(\phi^1(\cdot,\cdot), \ldots, \phi^m(\cdot,\cdot))^T$. Let $\psi(\cdot): \mathbb{R}^n \to \mathbb{R}$ be defined by:

$$\psi(z) \triangleq \max\{f^j(z),\ j \in \underline{m};\ g^j(z),\ j \in \underline{p}\} \tag{5}$$

and let $\psi(\cdot)_+: \mathbb{R}^n \to \mathbb{R}$ be defined by:

$$\psi(z)_+ \triangleq \max\{0, \psi(z)\}. \tag{6}$$

The feasible set F clearly satisfies:

$$F \triangleq \{z \mid f(z) \leqq 0,\ g(z) \leqq 0\} = \{z \mid \psi(z) \leq 0\}, \tag{7}$$

so that $z \in F$ if and only if $\psi(z)_+ = 0$. The "ε-most-active" constraints are specified by the sets $J_\varepsilon^f(z)$, $J_\varepsilon^g(z)$ defined by:

$$J^f_\varepsilon(z) \triangleq \{j \in \underline{m} \mid f^j(z) \geq \psi(z)_+ - \varepsilon\}, \tag{8}$$

$$J^g_\varepsilon(z) \triangleq \{j \in \underline{p} \mid g^j(z) \geq \psi(z)_+ - \varepsilon\}, \tag{9}$$

where $\underline{m} \triangleq \{1,2,\ldots,m\}$, etc. The set of points in A at which $\phi^j(z,\alpha)$ is "ε-most active" is defined by:

$$A^j_\varepsilon(z) \triangleq \{\alpha \in A \mid \phi^j(z,\alpha) \geq \psi(z)_+ - \varepsilon\}. \tag{10}$$

The algorithm makes use of local maximizers of $\phi^j(z,\cdot)$, $j = 1,\ldots,m$; the set of local maximizers in $A^j_\varepsilon(z)$ is defined to be:

$$\tilde{A}^j_\varepsilon(z) \triangleq \{\alpha \in A^j_\varepsilon(z) \mid \alpha \text{ is a local maximizer of } \phi^j(z,\cdot)\}. \tag{11}$$

Let $\varepsilon_0 > 0$ be given. For all $z \in \mathbb{R}^n$, all $j \in \underline{m}$, let $k^j(z)$ denote the cardinality of $\tilde{A}^j_{\varepsilon_0}(z)$. Now suppose that $z \in \mathbb{R}^n$ is such that

(i) $k^j(z)$ is finite for all $j \in \underline{m}$.

(ii) for any $\alpha^j_k \in \tilde{A}_{\varepsilon_0}(z) \cap (\alpha_0,\alpha_c)$, $\phi_{\alpha\alpha}(z,\alpha^j) > 0$, $j \in \underline{m}$, $k \in \underline{k}^j(z)$,

(iii) for any $\alpha^j_k \in \tilde{A}_{\varepsilon_0}(z)$ such that $\alpha^i_k = \alpha_0(\alpha_c)$, $\phi^j(z,\alpha^j_k) < 0$ (>0), $j \in \underline{m}$, $k \in \underline{k}^j(z)$.

Then

$$\tilde{A}_{\varepsilon_0}(z) = \{\alpha^j_1, ,\alpha^j_2,\ldots,\alpha^j_{k^j(z)}\} \tag{12}$$

and there exists a ball $B_z = \{z' \mid \|z'-z''\| < \rho\}$ (of radius ρ) together with continuously differentiable functions $\alpha^j_k(\cdot) : B_z \to A$, $j \in \underline{m}$, $k \in \underline{k}^j(z)$, such that $\alpha^j_k(z) = \alpha^j_k$ for all $j \in \underline{m}$, $k \in \underline{k}^j(z)$ and $\alpha^j_k(z')$ is a local maximizer of $\phi^j(z',\cdot)$. For all $\alpha^j_k \in (\alpha_0,\alpha_c)$, (i.e. unconstrained local maximizers) the existence of such a ball and functions follows from the fact that $\phi_\alpha^j(z,\alpha_k^j) = 0$ via the Implicit Function Theorem and (ii) above. For all $\alpha_k^j = \alpha_0(\alpha_c)$, because of the continuity of $\phi^j(\cdot,\cdot)$ and (iii), there must exist a ball B_z such that $\alpha_k^j(z') \equiv \alpha_k^j$ is a local maximizer of $\phi^j(z',\cdot)$ for all $z' \in B_z$.

Next, for all $j \in \underline{m}$, $k \in k^j(z)$, let $\eta_k^j(\cdot) : B_z \to \mathbb{R}$ be defined by

$$\eta_k^j(z') \triangleq \phi^j(z',\alpha_k^j(z')). \tag{13}$$

If $\alpha_k^j(z') \equiv \alpha_k^j(z)$ for $z' \in B_z$, then $\eta_k^j(\cdot)$ is obviously differentiable and

$$\frac{\partial \eta_k^j(z)}{\partial z} = \phi_z^j(z',\alpha_k^j(z)) \text{ for all } z' \in B_z. \tag{14}$$

If $\alpha_k^j(z) \in (\alpha_0,\alpha_c)$ is an unconstrained local maximizer, then it must satisfy

$$\phi_{\alpha\alpha}^j(z,\alpha_k^j(z))[(\frac{\partial}{\partial z}\ \alpha_k^j(z)] + \phi_{\alpha z}^j\ (z,\alpha_k^j(z)) = 0 \tag{15}$$

Hence, because of (ii),

$$(\frac{\partial}{\partial z})\alpha_k^j(z) = -\phi_{\alpha\alpha}^j\ (z,\alpha_k^j(z))^{-1}\phi_{\alpha z}^j\ (z,\alpha_k^j(z)). \tag{15a}$$

Consequently $\eta_k^j(\cdot)$ is again differentiable and $(\frac{\partial}{\partial z})\eta_k^j(z)$ is given again by

$$\begin{aligned}(\partial/\partial z)\eta_k^j(z) &= \phi_z^j(z,\alpha_k^j(z)) + \phi_\alpha^j(z,\alpha_k^j(z))[(\partial/\partial z)\alpha_k^j(z)] \\ &= \phi_z^j(z,\alpha_k^j(z))\end{aligned} \tag{16}$$

Thus, (16) holds because either $\frac{\partial}{\partial z}\ \alpha_k^i(z) =$ or $\phi_\alpha(z,\alpha_k^j(z)) = 0$. Since the formulas (13), (14) and (16) are not always valid, we shall use the formal definition $\eta_k^j(z) = \phi(z,\alpha_k^j)$, $\alpha_k^j \in A_{\varepsilon_0}(z)$ and its "gradient" $\bar{\nabla}\eta_k^j(z)$, defined by:

$$\bar{\nabla}\eta_k^j(z) \triangleq \phi_z^j(z,\alpha_k^j(z))^T \tag{17}$$

in place of $\nabla\eta_k^j(z)$.

Since $\tilde{A}_{\varepsilon_0}^j$ is a set of local maximizers which always includes the global maximizer of $\phi^j(z,\cdot)$ it follows that $\phi^j(z,\alpha) \leq 0$ for all $\alpha \in A$ if an only if $\eta_k^j(z) \leq 0$ for all $j \in \underline{m}$, all $k \in \underline{k}^j(z)$; (i.e., $\psi(z) = \max\{\eta_k^j(z) \mid k \in \underline{k}^j(z),\ j \in \underline{m}\}$). The latter, finite set of inequalities will be employed in the algorithm which is, in essence, Newton's method applied to these inequalities.

Algorithm 1.

Data: $z_0 \in \mathbb{R}^n$, $\varepsilon_0 > 0$.

Step 0: Set $i = 0$.

Step 1: If $g(z_i) \leq 0$ and $\eta_k^j(z_i) \leq 0$ for all $j \in J_{\varepsilon_0}^f(z_i)$, all $k \in \underline{k}^j(z_i)$, stop.

Step 2: Compute v_i to solve: $\min\{\|v\|^2 \mid g^j(z_i) + \langle \nabla g^j(z_i), v\rangle \leq 0$ for all $j \in J_{\varepsilon_0}^g(z_i)$; $\eta_k^j(z_i) + \langle \bar{\nabla}\eta_k^j(z), v\rangle \leq 0$, for all $j \in J_{\varepsilon_0}^f(z_i)$, all $k \in \underline{k}^j(z_i)\}$.

Step 3: Set $z_{i+1} = z_i + v_i$. Set $i = i+1$. Go to Step 1. □

A straightforward extension of Newton's method would require the replacement of the finite set $\underline{k}^j(z_i)$ (corresponding to $\tilde{A}_{\varepsilon_0}^j(z_i)$) by an infinite set corresponding to $A_\varepsilon(z_i)$). Of course Algorithm 1 is still conceptual since the determination of the local maximizers requires an infinite process. We will show later how an implementable version of the algorithm may be obtained. Firstly, however, we establish the (local) convergence properties of Algorithm 1. Our assumptions are:

Assumption 1. $g(\cdot): \mathbb{R}^n \to \mathbb{R}^m$ and $\phi(\cdot,\cdot): \mathbb{R}^n \times A \longrightarrow \mathbb{R}$ are three times continuously differentiable.

<u>Assumption 2</u>. For all $z \in \mathbb{R}^n$, all $\varepsilon \in [0,\varepsilon_0]$, all $j \in J_\varepsilon^f(z)$, $\hat{A}_\varepsilon^j(z)$ is a finite set.

<u>Assumption 3</u>. For all $z \in \delta F \triangleq \{z' | \psi(z')=0\}$, for all $\alpha \in \tilde{A}_{\varepsilon_0}^j(z)$, for all $j \in J_{\varepsilon_0}^f(z)$,

(i) if $\alpha \in (\alpha_0, \alpha_c)$, then $\phi_{\alpha\alpha}(z,\alpha) > 0$;

(ii) if $\alpha = \alpha_0$, then $\phi_\alpha(z,\alpha) < 0$;

(iii) if $\alpha = \alpha_c$, then $\phi_\alpha(z,\alpha) > 0$.

<u>Assumption 4</u>. For all $z \in \bar{F}^c$, $\{\nabla g^j(z),\ j \in J_{\varepsilon_0}^g(z);\ \bar{\nabla}\eta_k^j(z),\ k \in \underline{k}^j(z),\ j \in J_{\varepsilon_0}^f(z)\}$ is a set of linearly independent vectors. □

We require the following preliminary result:

<u>Lemma 2.1</u>. Let $z^* \in \delta F$. Then there exists an $\varepsilon_z > 0$ such that for all $z \in B(z^*,\varepsilon_z)$, for all $k \in \underline{k}^j(z^*)$, for all $j \in J_{\varepsilon_0}^f(z^*)$, there exist $\alpha_k^j(\cdot)\colon B(z^*,\varepsilon_z) \longrightarrow \mathbb{R}$ and corresponding $\eta_k^j(\cdot)\colon B(z^*,\varepsilon_z) \longrightarrow \mathbb{R}$, unique, well defined and continuously differentiable, such that $\alpha_k^j(z) \in A_{\varepsilon_0}(z)$ and $\eta_k^i(z) = \phi^j(z,\alpha_k^j(z))$ for all $z \in B(z^*, \varepsilon_z)$.

<u>Proof</u>. For all $k \in \underline{k}^j(z)$, for all $j \in J_{\varepsilon_0}^f(z)$, $\alpha_k^j(z)$ is a solution of:

$$\phi_\alpha(z,\alpha) = 0 \tag{18}$$

The desired result follows from Assumptions 1-3 and the implicit function theorem. □

Our main result, which establishes the (local) convergence properties of Algorithm 1, follows:

Theorem 2.1. Suppose Assumptions 1-4 are satisfied and let $\{z_i\}$ denote an infinite sequence generated by Algorithm 1. For any $z^* \in \delta F$, for any $\gamma \in (0,1)$, there exists an $\varepsilon^* \in (0,\infty)$, an $\bar{\varepsilon} \in (0,\varepsilon^*]$ and a $M \in (0,\infty)$ such that:

(i) $\|v(z)\| \leq \gamma\|z-z^*\|$ and $M\|v(z)\| \leq \gamma$ for all $z \in B(z^*,\varepsilon^*)$.

(ii) $\|v(A_1(z))\| \leq M\|v(z)\|^2$ where $A_1(z) \triangleq z+v(z)$, for all $z \in B(z^*,\varepsilon^*)$.

(iii) If $z_0 \in B(z^*,\bar{\varepsilon})$ then $z_i \in B(z^*,\varepsilon^*)$ for all $i = 0,1,2,\ldots$.

(iv) If $z_0 \in B(z^*,\bar{\varepsilon})$ then $z_i \to \tilde{z} \in \delta F \cap B(z^*,\varepsilon^*)$, with a quadratic rate of convergence, as $i \to \infty$.

Proof. To simplify the notation in the proof, we restrict our attention to the case where $p = 0$ and $m = 1$, i.e., $F = \{z|\phi(z,\alpha) \leq 0,\ \alpha \in A\}$. Hence we discard $g(\cdot)$ and replace $\phi^j(\cdot)$, $j = 1,\ldots,m$, by $\phi(\cdot)$, $\alpha_k^j(\cdot)$ by $\alpha_k(\cdot)$, $A_\varepsilon^j(\cdot)$ by $A_\varepsilon(\cdot)$, $\tilde{A}_\varepsilon^j(\cdot)$ by $\tilde{A}_\varepsilon(\cdot)$, $k^j(\cdot)$ by $k(\cdot)$, etc. The extension of the proof to the general case is simple but tedious.

For all $z \in \mathbb{R}^n$ let $v(z)$ denote the solution (if it exists) of:

$$\min\{\|v\|^2 | \eta_k(z) + \langle\bar{\nabla}\eta_k(z),v\rangle \leq 0,\ k \in \underline{k}(z)\} \tag{19}$$

Let $z^* \in \delta F$ and let $K \subset \underline{k}(z^*)$ be defined by:

$$K \triangleq \{k \in \underline{k}(z^*) | \eta_k(z^*) \geq -\varepsilon_0/2\} \tag{20}$$

There exists a neighborhood $B(z^*,\varepsilon_1)$ of z^* in which $K \subset \underline{k}(z)$ for all z in this neighborhood. (Note that an arbitrarily small perturbation in z can convert a stationary point of $\phi(z,\cdot)$ into a local maximum, thus increasing the set $\tilde{A}_{\varepsilon_0}(z)$. However, if $\eta_k(z) = -\varepsilon_0$, an arbitrarily small perturbation in z can cause $\alpha_k(z)$ to leave $\tilde{A}_\varepsilon(z)$.) From Implicit Function Theorem and Lemma 2.1, there exists an $\varepsilon_2 \in (0,\varepsilon_1]$ and an $\varepsilon_\alpha > 0$ such that:

(a) the maps $\alpha_k(\cdot)$ map $B(z^*,\varepsilon_2)$ into $B(\alpha_k(z^*),\varepsilon_\alpha)$, $k \in K$, and are continuously differentiable (so that $\bar{\nabla}\eta_k(z) = \nabla\eta_k(z)$ for all $z \in B(z^*,\varepsilon_1)$, all $k \in K$);

(b) the sets $B(\alpha_k(z^*),\varepsilon_\alpha)$, $k \in K$, are disjoint;

(c) $K \subset \underline{k}(z)$ for all $z \in B(z^*,\varepsilon_2)$.

Let $\hat{A}$ denote $\bigcup_{k\in K} B(\alpha_k(z^*),\varepsilon_\alpha)$. For all $z \in B(z^*,\varepsilon_2)$, $\max\{\phi(z,\alpha) \mid \alpha\in\hat{A}\}$ $= \max\{\eta_k(z) \mid k\in K\}$ and $\max\{\phi(z,\alpha) \mid \alpha\in\hat{A}^c\} = \max\{\eta_k(z) \mid k\in\underline{k}(z)\backslash\underline{k}(z^*)\}$. Since $z^* \in \delta F$, $\psi(z^*) = 0$. Let $\delta \triangleq \max\{\phi(z^*,\alpha) \mid \alpha \in \hat{A}^c\} < 0$. There exists an $\varepsilon_3 \in (0,\varepsilon_2]$ such that, for all $z \in B(z^*,\varepsilon_3)$:

$$\eta_k(z) = \phi(z,\alpha_k(z)) \leq \delta/2 \,, \tag{21}$$

for all $k \in \underline{k}(z)\backslash K$, since, for such z and k the uniqueness property of the implicit function theorem implies $\alpha_k(z) \in \hat{A}^c$.

Now let $v_1(z)$ denote the solution of:

$$\min\{\|v\|^2 \mid \eta_k(z) + \langle\bar{\nabla}\eta_k(z),v\rangle \leq 0,\ k\in K\} \,. \tag{22}$$

It follows from Proposition A.1 of the Appendix and the linear independence of $\nabla\bar{\eta}_k(z^*)$, $k \in K$, that $\|v_1(z)\| \to 0$ as $z \to z^*$. Hence there exists an $\varepsilon_4 \in (0,\varepsilon_3]$ such that, for all $z \in B(z^*,\varepsilon_4)$:

$$\eta_k(z) + \langle\bar{\nabla}\eta_k(z),v_1(z)\rangle \leq 0 \,, \quad k \in K \,,$$

and (using (21)):

$$\eta_k(z) + \langle\bar{\nabla}\eta_k(z),v_1(z)\rangle \leq \delta/4 < 0, \quad k \in \underline{k}(z)\backslash K \,,$$

since $\|\bar{\nabla}\eta_k(z)\|$ is bounded in $B(z^*,\varepsilon_3)$. Hence, for all $z \in B(z^*,\varepsilon_4)$, $\|v(z)\| \le \|v_1(z)\|$ (so that $\|v(z)\| \longrightarrow 0$ as $z \to z^*$). Consequently, for all $z \in B(z^*,\varepsilon_4)$

$$\eta_k(z) + \langle \bar{\nabla}\eta_k(z), v(z)\rangle \le \delta/4 < 0 \ , \quad \text{for all } k \in \underline{k}(z)/K.$$

Hence, for all $z \in B(z^*,\varepsilon_4)$, the constraints in (19), with $k \in \underline{k}(z)\backslash K$, remain inactive and may be neglected. Thus, for all $z \in B(z^*,\varepsilon_4)$, $v(z) = v_1(z)$ so that, for such z, $v(z)$ is a Newton step for the problem of determining a $z \in \mathbb{R}^n$ to satisfy the <u>finite</u> set of inequalities $\eta_k(z) \le 0$, $k \in K$. It follows from Proposition A1 of the Appendix that there exists an $\varepsilon^* \in [0,\varepsilon_4]$ such that (i) and (ii) hold.

To prove (iii) and (iv) suppose that $\bar{\varepsilon} \in [0,\gamma\varepsilon^*]$ satisfies $\|v(z_0)\| \le (1-\gamma)^2\varepsilon^*$ for all $z_0 \in B(z^*,\bar{\varepsilon})$; such an $\bar{\varepsilon}$ exists by virtue of (i). Also from (i) and (ii), $\|v(A_1(z))\| \le M\|v(z)\|\|v(z)\| \le \gamma\|v(z)\|$ for all $z \in B(z^*)$. Hence, if $z_0 \in B(z^*,\bar{\varepsilon})$, $z_1 \triangleq z_0 + v(z_1) \in B(z^*,\bar{\varepsilon}+\|v(z_1)\|)$ where $\|v(z_1)\| \le \gamma\|v(z_0)\| \le \gamma(1-\gamma)^2\varepsilon^* \le (1-\gamma)\varepsilon^*$. Hence $\bar{\varepsilon}+\|v(z_1)\| \le \gamma\varepsilon^* + (1-\gamma)\varepsilon^* = \varepsilon^*$ so that $z_1 \in B(z_0,\varepsilon^*)$. Suppose now there exists a finite sequence $\{z_0,z_1,\ldots,z_j\}$ such that $z_0 \in B(z^*,\bar{\varepsilon})$, $z_i \in B(z^*,\varepsilon^*)$, and $z_i = A_1(z_{i-1}) = z_{i-1} + v(z_{i-1})$ for $i = 1,\ldots,j$. Then $z_{j+1} \triangleq A_1(z_j)$ exists and:

$$\|z_{j+1}-z_0\| \le \sum_{i=0}^{j} \|v(z_j)\| \le [1+\gamma+\gamma^2+\cdots+\gamma^j]\|v(z_0)\| \le \|v(z_0)\|/(1-\gamma)$$

Hence $\|z_{j+1}-z^*\| \le \|z_0-z^*\| + \|v(z_0)\|/(1-\gamma) \le \gamma\varepsilon^* + (1-\gamma)\varepsilon^* = \varepsilon^*$ so that $\{z_0,z_1,\ldots,z_{j+1}\}$ satisfies $z_0 \in B(z^*,\bar{\varepsilon})$, $z_i \in B(z^*,\varepsilon^*)$ and $z_i = A_1(z_{i-1})$ for $i = 1,\ldots,j+1$. By induction there exists an infinite sequence $\{z_i\}$ satisfying $z_0 \in B(z^*,\bar{\varepsilon})$, $z_i \in B(z^*,\varepsilon^*)$ and $z_i = A_1(z_{i-1})$ for

$i = 1,2,3,\ldots$. Also $z_i \to \tilde{z} \in B(z^*,\varepsilon^*)$. Since $\{z_i\}$ is an infinite sequence, so that the stopping condition in Step 1 is never satisfied, it follows that $\tilde{z} \in \delta F$. □

3. A Stabilized Algorithm

The quadratically convergent algorithm described above is merely locally convergent, i.e., it will generate a sequence converging to $z^* \in \delta F$ if the initial point of the sequence lies in $G_1 \triangleq B(z^*,\varepsilon^*)$. To stabilize the algorithm (i.e. to make it globally convergent) we make use of an algorithm model specially designed for this purpose [6, Algorithm model 3]. Let $A_1(\cdot): G_1 \to \mathbb{R}^n$ be defined by Steps 1 and 2 of Algorithm 1, i.e.

$$A_1(z) \triangleq z + v(z) \tag{23}$$

where $v(\cdot)$ is defined in (19). An infinite sequence $\{z_i\}$ generated by Algorithm 1 satisfies $z_{i+1} = A_1(z_i)$, $i = 0,1,2,\ldots$.

Proposition 3.1. Suppose $\{z_i\}$ is an infinite sequence such that $z_i \to \hat{z}$ and $\|v(z_i)\| \to 0$ as $i \to \infty$. Then $\hat{z} \in F$.

Proof. Since for each z, $\psi(z) = \eta_k(z)$ for some $k \in k(z)$, it follows from Step 2 that for all i

$$\psi(z_i) + \langle \bar{\nabla}\eta_{k_{z_i}}(z_i), v(z_i)\rangle \leq 0 \text{ for some } k_{z_i} \in k(z_i) \tag{23a}$$

Now, $\psi(z_i) \to \psi(\hat{z})$ by continuity, $v(z_i) \to 0$ by assumption, and $\bar{\nabla}\eta_{k_{z_i}}(z_i)$ is bounded for all i. Consequently, (23a) implies that $\psi(\hat{z}) \leq \overline{\lim}\, (\psi(z_i) + \langle \bar{\nabla}\eta_{k_{z_i}}(z_i), v(z_i)\rangle) \leq 0$ which completes our proof. □

To stabilize the algorithm we test the Newton step $v(z)$; if $v(z)$ is unsuitable we employ a (first order) algorithm specified by a map $A_2(\cdot)$: $\mathbb{R}^n \to \mathbb{R}^n$, i.e., a sequence $\{z_i\}$ generated by the first order algorithm satisfies $z_{i+1} = A_2(z_i)$, $i = 0,1,2,\ldots$. The first order algorithm employs, for all $z \in \mathbb{R}^n$, all $\varepsilon > 0$, a search direction $w_\varepsilon(z)$ defined to be the solution of the quadratic program (see [8])

$$\tilde{\theta}_\varepsilon(z) \triangleq \min\{\tfrac{1}{2}\|w\|^2 + \max\{\langle \nabla g^j(z), w\rangle,\ j \in J^g_\varepsilon(z);\ \langle \nabla_z \phi^j(z,\alpha), w\rangle,\ \alpha \in A^j_\varepsilon(z),\ j \in J^f_\varepsilon(z)\}$$

$$= \min\{\tfrac{1}{2}\|w\|^2 \,|\, w \in \mathrm{co}\{\nabla g^j(z),\ j \in J^g_\varepsilon(z);\ \nabla_z \phi^j(z,\alpha),\ \alpha \in A^j_\varepsilon(z),\ j \in J^f_\varepsilon(z)\} \tag{24}$$

The map $A_2(\cdot)$ can now be defined:

$$A_2(z) = z + \lambda(z) w_{\varepsilon(z)}(z) \tag{25}$$

where $\varepsilon(z)$ is the largest $\varepsilon \in \{1, 1/2, 1/4, \ldots\} \cup \{0\}$ such that for a <u>given</u> $\delta \in (0,1]$:

$$\tilde{\theta}_\varepsilon(z) \leq -\delta\varepsilon, \tag{26}$$

and $\lambda(z)$ is the largest $\lambda \in \{1, \beta, \beta^2, \ldots\}$, such that for a <u>given</u> $\beta \in (0,1)$

$$\psi(z + \lambda w_{\varepsilon(z)}(z)) - w(z) \leq -\lambda\delta\varepsilon/2\,. \tag{27}$$

It follows from [8] and Lemmas 1 and 3 of [6] that for all $z \in \bar{F}^c$ there exists a $\mu > 0$ and a $\rho > 0$ such that

$$\psi(A_2(z')) - \psi(z') \leq -\mu\,, \tag{28}$$

for all $z' \in B(z,\rho)$. The stabilized version of Algorithm 1, based on Algorithm Model 3 of [6] can now be presented.

Algorithm 2.

Data: $\gamma, \beta \in (0,1)$, $\delta \in (0,1]$, $k \in [1,\infty)$, $z_0 \in \mathbb{R}^n$.

Step 0: Set $i = 0$, $j = 0$.

Step 1: If $\psi(z_i)_+ = 0$, stop.

Step 2: If a solution $v(z_i)$ of (19) exists and $\|v(z_i)\| \le k\gamma^j$, set $z_{i+1} = A_1(z_i)$, set $j = j+1$, set $i = i+1$ and go to Step 1. Else proceed.

Step 3: Set $z_{i+1} = A_2(z_i)$, set $i = i+1$ and go to Step 1. □

Theorem 3.1. Let $\{z_i\}$ be a sequence generated by Algorithm 2:

(i) If $\{z_i\}$ is finite, its last point is feasible.

(ii) If $\{z_i\}$ is infinite and bounded, any accumulation point z^* of $\{z_i\}$ is feasible and there exists an integer N such that $z_{i+1} = A_1(z_i)$ for all $i \ge N$ (so that $z_i \to z^*$ quadratically).

Proof. (i) This follows from Step 1.

(ii) Suppose $\{z_i\}$ is infinite and bounded.

(a) If there exists an integer N such that $z_{i+1} = A_2(z_i)$ for all $i \ge N$ it follows from [8] that any accumulation point z^* of $\{z_i\}$ lies in F.

(b) If (a) does not occur then there exists an infinite subset S of $\{0,1,2,\ldots\}$ such that $z_{i+1} = A_1(z_i)$ for all $i \in S$; thus there exists an accumulation point z^* and a subset S_1 of S such that $z_i \to z^*$ as $i \to \infty$, $i \in S_1$. Hence $j \to \infty$; since $\|v(z_i)\| \le k\gamma^j$, $\|v(z_i)\| \to 0$ as $i \to \infty$, $i \in S_1$. From Proposition 3.1 $z^* \in F$. Clearly, referring to Theorem 2.1, for some finite $i \in S_1$, $z_i \in B(z^*,\bar{\varepsilon})$ is such that $\|v(z_i)\| \le \gamma$. Because $\|v(z_i)\| \le k\gamma^j$, $z_{i+1} = A_1(z_i)$. From Theorem 2.1, $\|v(z_{i+1})\| \le \gamma\|v(z_i)\| \le k\gamma^{j+1}$ so that $z_{i+2} = A_1(z_{i+1})$, i.e. the sequence $\{z_i, z_{i+1}, z_{i+2}, \ldots\}$ is generated by A_1 and so converges quadratically to a $\tilde{z} \in \delta F \cap B(z^*,\varepsilon^*)$ as $i \to \infty$. Hence $\tilde{z} = z^*$.

(c) We now show that (a) does not occur so that the conclusions of (b) always hold (i.e. $z_i \to z^* \in \delta F$ quadratically) if $\{z_i\}$ is a bounded, infinite sequence. If (a) occurs, j remains constant at J, say, for $i \geq N$. Since $\{z_i\}$ is compact it has an accumulation point z^* in F. If z^* is in the interior of F, then, by virtue of the test in Step 1, $\{z_i\}$ is a finite sequence; hence $z^* \in \delta F$. From Theorem 2.1 there exists a finite $I \geq N$ such that $z_I \in B(z^*,\varepsilon^*)$ and $\|v(z_I)\| \leq \gamma^J$. Since $j = J$ when $i = I \geq N$ it follows that $z_{I+1} = A_1(z_I)$ which contradicts the assumption that $z_{i+1} = A_2(z_i)$ for all $i \geq N$. Hence (a) does not occur. □

4. An Implementable Algorithm

Algorithm 2, although it replaces the infinite set A by the finite sets $\tilde{A}^j_\varepsilon(z)$, $j \in \underline{m}$, is conceptual since Step 1 requires the computation of $\psi(z)_+$ and Steps 2 and 3 the determination of the set of local maximizers of $\phi^j(z,\cdot)$, $j \in \underline{m}$.

To make the algorithm implementable we approximate $A \triangleq [\alpha_0,\alpha_c]$ by the finite set:

$$A_q \triangleq \{\alpha \in A \,|\, \alpha = \alpha_0 + k\Delta q,\ k = 0,1,2,\ldots,q\}, \tag{29}$$

where $\Delta q = (\alpha_c - \alpha_0)/q$. The points in A_q will be referred to as <u>mesh points</u>. Similarly $f^j_q(\cdot)$, an approximation to $f^j(\cdot)$, $j \in \underline{m}$, is defined by:

$$f^j_q(z) \triangleq \max\{\phi^j(z,\alpha) \,|\, \alpha \in A_q\}. \tag{30}$$

The functions $\psi_q(\cdot)$ and $\psi_q(\cdot)_+ : \mathbb{R}^n \to \mathbb{R}$ are defined by:

$$\psi_q(z) \triangleq \max\{g^j(z),\ j \in \underline{p};\ f^j_q(z),\ j \in \underline{m}\}, \tag{31}$$

$$\psi_q(z)_+ \triangleq \max\{\psi_q(z),0\}. \tag{32}$$

The approximate ε-most-active constraint sets are defined as follows:

$$A^j_{q,\varepsilon}(z) \triangleq \{\alpha \in A_q \,|\, \phi^j(z,\alpha_q) \geq \psi_q(z)_+ - \varepsilon\}, \tag{33}$$

$$\tilde{A}^j_{q,\varepsilon}(z) \triangleq \{\alpha \in A^j_{q,\varepsilon}(z) \,|\, \alpha \text{ is a left local maximizer of } \phi^j(z,\cdot)\}, \tag{34}$$

$$J^f_{q,\varepsilon}(z) \triangleq \{j \in \underline{m} \mid f^j_q(z) \geq \psi_q(z)_+ - \varepsilon\}, \tag{35}$$

$$J^g_{q,\varepsilon}(z) \triangleq \{j \in \underline{p} \mid g^j(z) \geq \psi_q(z)_+ - \varepsilon\}. \tag{36}$$

In (34), a left local maximizer is defined as follows: if $A^j_{q,\varepsilon}(z) = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_s\}$ then $\bar{\alpha}_i$ is a left local maximizer of $\phi^j(z,\cdot)$ in $A^j_{q,\varepsilon}(z)$ if $\phi^j(z,\bar{\alpha}_i) > \phi^j(z,\bar{\alpha}_{i-1})$ and $\phi^j(z,\bar{\alpha}_i) \geq \phi^j(z,\bar{\alpha}_{i+1})$.

We first discuss an implementable version of $A_1(\cdot)$. We recall that $A_1(z) \triangleq z + v(z)$ where $v(z)$ is defined as the solution of:

$$\min\{\|v\|^2 \mid g^j(z) + \langle \nabla g^j(z), v\rangle \leq 0,\ j \in J^g_{\varepsilon_0}(z);\ \eta^j_k(z) + \langle \bar{\nabla}\eta^j_k(z), v\rangle \leq 0,$$
$$j \in J^f_{\varepsilon_0}(z),\ k \in \underline{k}^j(z)\}. \tag{37}$$

It follows from (17) that the second set of constraints in (37) is equivalent to:

$$\phi^j(z,\alpha) + \phi^j_z(z,\alpha)v \leq 0, \tag{38}$$

for all $\alpha \in A^j_{\varepsilon_0}(z)$, all $j \in J^f_{\varepsilon_0}(z)$.

In the implementable version of the algorithm, the set of local maximizers $\tilde{A}^j_{\varepsilon_0}(z)$, $j \in J^f_{\varepsilon_0}(z)$ is not available. The set $\tilde{A}^j_{q,\varepsilon_0}(z)$ of approximate local maximizers (whose cardinality is $k^j_q(z) \leq k^j(z)$), $j \in J^f_{q,\varepsilon_0}(z)$ is available. However, for $\alpha \notin \{\alpha_0, \alpha_c\}$, we can employ our knowledge of ϕ and its partial derivatives at $(z, \alpha^j_{q,k}(z))$ to obtain a better estimate of the left hand side of (38) than that obtained by replacing $\tilde{A}^j_{\varepsilon_0}(z)$, $j \in J^f_{\varepsilon_0}(z)$, by $\tilde{A}^j_{q,\varepsilon_0}(z)$, $j \in J^f_{q,\varepsilon_0}(z)$. Expanding $\phi^j_\alpha(z,\cdot)$ to first order yields:

$$\phi^j_\alpha(z,\alpha+\delta\alpha) = \phi^j_\alpha(z,\alpha) + \phi^j_{\alpha\alpha}(z,\alpha)\delta\alpha + e \tag{39}$$

Setting the left hand side of (39) equal to zero, and ignoring e, yields the $\delta\alpha$, $\delta\alpha^j(z,\alpha)$ say, which, for given z, approximately reduces ϕ^j_α to zero. Clearly

$$\delta\alpha^j(z,\alpha) = -\phi^j_{\alpha\alpha}(z,\alpha)^{-1}\ \phi^j_\alpha(z,\alpha). \tag{39a}$$

Substituting this $\delta\alpha$ into the linearized constraint inequality:

$$\phi^j(z,\alpha) + \phi^j_\alpha(z,\alpha)\delta\alpha + [\phi^j_z(z,\alpha) + \phi^j_{\alpha z}(z,\alpha)\ \delta\alpha]\ \delta z \leq 0 \tag{40}$$

yields (with $v \triangleq \delta z$) a possible replacement for (38):

$$\begin{aligned}&[\phi^j(z,\alpha) - \phi^j_{\alpha\alpha}(z,\alpha)^{-1}\phi^j_\alpha(z,\alpha)^2] \\ &\quad + [\phi^j_z(z,\alpha) - \phi^j_{\alpha z}(z,\alpha)\phi^j_{\alpha\alpha}(z,\alpha)^{-1}\phi^j_\alpha(z,\alpha)]v \leq 0,\end{aligned} \tag{41}$$

for all $\alpha \in \tilde{A}^j_{q,\varepsilon_0}(z)$, $j \in J^f_{q,\varepsilon_0}(z)$. However, as discussed in §2, $\phi^j_{\alpha\alpha}(z,\alpha)^{-1}$ is not necessarily bounded. For all $q > 0$, all $j \in \underline{m}$, let $\delta\alpha^j_q(\cdot,\cdot)$ be defined by:

$$\begin{aligned}\delta\alpha^j_q(z,\alpha) &\triangleq \delta\alpha^j(z,\alpha) \quad \text{if} \quad |\delta\alpha^j(z,\alpha)| \leq 2\Delta q \ \text{and}\ (\alpha+\delta\alpha) \in A \\ &\triangleq 2\Delta q \quad \text{if} \quad \delta\alpha^j(z,\alpha) > 2\Delta q \ \text{and}\ (\alpha+2\Delta q) \in A \\ &\triangleq \alpha_c - \alpha \ \text{if}\ \alpha + \delta\alpha^j(z,\alpha) > \alpha_c \ \text{and}\ \alpha + 2\Delta q > \alpha_c \\ &\triangleq -2\Delta q \quad \text{if} \quad \delta\alpha^j(z,\alpha) < -2\Delta q \ \text{and}\ \alpha-2\Delta q \in A \\ &\triangleq \alpha_0 - \alpha \ \text{if}\ \alpha + \alpha\delta < 0 \ \text{and}\ \alpha - 2\Delta q < \alpha_0 \\ &\left.\begin{aligned}&\triangleq 0 \ \text{if}\ \alpha = \alpha_c \ \text{and}\ \phi^j_\alpha(z,\alpha) > 0 \\ &\triangleq 0 \ \text{if}\ \alpha = \alpha_0 \ \text{and}\ \phi^j_\alpha(z,\alpha) < 0\end{aligned}\right\} \text{overrides the above}.\end{aligned} \tag{42}$$

Our replacement for (38) is:

$$\begin{aligned}&[\phi^j(z,\alpha) + \phi^j_\alpha(z,\alpha)\delta\alpha^j_q(z,\alpha)] \\ &\quad + [\phi^j_z(z,\alpha) + \phi^j_{\alpha z}(z,\alpha)\delta\alpha^j_q(z,\alpha)]v \leq 0\end{aligned} \tag{43}$$

for all $\alpha \in \tilde{A}^j_{q,\varepsilon_0}(z)$, $j \in J^f_{q,\varepsilon_0}(z)$.

Our approximation $A_{q,1}$ to A_1 can now be specified:

$$A_{q,1}(z) \triangleq z + v_q(z) \tag{44}$$

where $v_q(z)$ is defined as the solution of:

$$\min\{\|v\|^2 \,|\, g^j(z) + g^j_z(z)v \le 0;\ j \in J^g_{\varepsilon_0}(z);\ [\phi^j(z,\alpha) + \phi^j_\alpha(z,\alpha)\delta\alpha^j_q(z,\alpha)] + [\phi^j_z(z,\alpha) + \phi^j_{\alpha z}(z,\alpha)\delta\alpha^j_q(z,\alpha)]v \le 0,\ \alpha \in \tilde{A}^j_{q,\varepsilon_0},\ j \in J^f_{q,\varepsilon_0}(z)]\} . \tag{45}$$

Before proceeding to analyze $A_{q,1}$ we examine the closeness of the approximation (43) to (38). Let $z^* \in \delta F$. For all $j \in J^f_{\varepsilon_0/2}(z^*)$ let $K^j \subset \underline{k}^j(z^*)$ be defined by:

$$K^j \triangleq \{k \in \underline{k}^j(z^*) \,|\, \eta^j_k(z^*) \ge -\varepsilon_0/2\}. \tag{46}$$

We require the following preliminary result:

<u>Lemma 4.1</u>. There exists an $\varepsilon^* > 0$, an $\varepsilon_\alpha > 0$ and an integer q^* such that:

(a) for each $k \in K^j$, $j \in J^f_{\varepsilon_0/2}(z^*)$, $\alpha^j_k(\cdot)$ is continuously differentiable from $B(z^*,\varepsilon^*)$ into $B(\alpha^j_k(z^*),\varepsilon_\alpha)$;

(b) for each $j \in J^f_{\varepsilon_0/2}(z^*)$, the sets $B(\alpha^j_k(z^*),\varepsilon_\alpha)$, $k \in K^j$ are disjoint;

(c) $K^j \subset \underline{k}^j(z)$ for all $j \in J^f_{\varepsilon_0/2}(z^*)$, for all $z \in B(z^*,\varepsilon^*)$;

(d) $\phi^j(z,\cdot)$ has a unique maximizer, $\alpha^j_{q,k}(z)$ say, in $\tilde{A}^j_{q,\varepsilon_0}(z) \cap B(\alpha^j_k(z^*),\varepsilon_\alpha)$ and $|\alpha^j_k(z)-\alpha^j_{q,k}(z)| \le \Delta q$ for all $k \in K^j$, all $j \in J^f_{\varepsilon_0/2}(z^*)$, all $q \ge q^*$.

<u>Proof</u>. (a), (b) and (c) are slight generalizations of Lemma 2.1 and statements (a), (b) and (c) occurring in the proof of Theorem 2.1. It is evident that ε^* can be chosen so that $J^f_{\varepsilon_0/2}(z^*) \subset J^f_{\varepsilon_0}(z)$ for all $z \in B(z^*,\varepsilon^*)$. (d) follows from the strict positivity of $\phi_{\alpha\alpha}(z^*,\alpha)$ for all $\alpha \in K^j/\{\alpha_0,\alpha_c\}$, all $j \in J^f_{\varepsilon_0/2}(z^*)$, and Assumption 1 so that ε^* can be chosen so that for all $j \in J^f_{\varepsilon_0/2}(z^*)$, $\phi^j(z,\cdot)$ is strictly convex, in each of the disjoint intervals $B(\alpha^j_k(z^*,\varepsilon_\alpha))$, $k \in K^j$. □

Lemma 4.2. Let $z^* \in \delta F$. Then there exists an $\varepsilon^* \in (0,\infty)$, an $\varepsilon_\alpha \in (0,\infty)$, a $c \in (0,\infty)$ such that for all $j \in J^f_{\varepsilon_0/2}(z^*)$, all $k \in K^j$, all $z \in B(z^*,\varepsilon^*)$, all $\alpha^j_k(z^*) \notin \{\alpha_0,\alpha_c\}$, all $\alpha \in B(\alpha^j_k(z^*),\varepsilon_\alpha)$:

(i) $|(\alpha^j_k(z)-\alpha) - \delta\alpha^j(z,\alpha)| \leq c|\alpha^j_k(z)-\alpha|^2$

(ii) $|\phi^j(z,\alpha^j_k(z)) - [\phi^j(z,\alpha) + \phi^j(z,\alpha)\delta\alpha^j(z,\alpha)]| \leq c|\alpha^j_k(z)-\alpha|^2$

(iii) $|\phi^j_z(z,\alpha^j_k(z)) - [\phi^j_z(z,\alpha) + \phi^j_{\alpha z}(z,\alpha)\delta\alpha^j(z,\alpha)]| \leq c|\alpha^j_k(z)-\alpha|^2$

where $\delta\alpha^j(z,\alpha)$ is defined as in (39a).

Proof. Let $c \in (0,\infty)$ be such that $\max\{|\phi^j_{\alpha\alpha}(z,\alpha)|, |\phi^j_{\alpha\alpha}(z,\alpha)^{-1}|\} \leq c, |\phi^j_{\alpha\alpha z}(z,\alpha)| \leq c$ and $|\phi^j_{\alpha\alpha\alpha}(z,\alpha)| \leq c$ for all $j \in J^f_{\varepsilon_0/2}(z)$, all $k \in K^j$, all $\alpha \in B(\alpha^j_k(z),\varepsilon_\alpha)$, all $z \in B(z^*,\varepsilon^*)$. From the definition of $\alpha^j_k(z)$:

$$0 = \phi^j_\alpha(z,\alpha^j_k(z)) = \phi^j_\alpha(z,\alpha) + \phi^j_{\alpha\alpha}(z,\alpha)(\alpha^j_k(z)-\alpha) + e^j(z,\alpha) \tag{47}$$

where, from the mean value theorem, $|e^j(z,\alpha)| \leq c|\alpha^j_k(z)-\alpha|^2$ for all j, k, α and z satisfying the above conditions. Comparing (47) with the definition of $\delta\alpha^j(\cdot)$ (see (39a) yields (i). To prove (ii) we note that:

$$\begin{aligned}\phi^j(z,\alpha^j_k(z)) &= \phi^j(z,\alpha) + \phi^j_\alpha(z,\alpha)\delta\alpha^j(z,\alpha) \\ &\quad + \phi^j_\alpha(z,\alpha)[(\alpha^j_k(z)-\alpha) - \delta\alpha^j(z,\alpha)] + e^j_2(z,\alpha)\end{aligned} \tag{48}$$

where $|e^j_2(z,\alpha)| \leq c|\alpha^j_k(z)-\alpha|^2$ for all relevant j, k, α and z. Part (iii) is similarly proven. □

We can now establish a major property of $A_{q,1}(\cdot)$.

Theorem 4.1. Let $z^* \in \delta F$, $\bar{c} \in (0,\infty)$, $\gamma \in (0,1)$ be given, and let $\{z_i\}$, $\{q_i\}$ be infinite sequences satisfying $z_{i+1} = z_i + v_{q_i}(z_i)$, $q_i \geq \bar{c}/\|v_{q_i}(z_i)\|$, $i = 0,1,2,\ldots$. Then there exists an $e^* \in (0,\infty)$, an $\bar{\varepsilon} \in (0,\varepsilon^*]$, an $M \in (0,\infty)$ and an integer q^* such that:

(i) $\|v_q(z)\| \leq M\|z-z^*\|$ and $M\|v_q(z)\| \leq \gamma$ for all $z \in B(z^*,\varepsilon^*)$, all $q \geq \max\{\bar{c}/\|v_q(z)\|,q^*\}$

(ii) $\|v_{q'}(z')\| \leq M\|v_q(z)\|^2$ for all $z' = z+v_q(z)$, all $z \in B(z^*,\varepsilon^*)$, all $q \geq \max\{\bar{c}/\|v_q(z)\|, q^*\}$, all $q' \geq \max\{\bar{c}/\|v_{q'}(z')\|, q^*\}$

(iii) if $z_0 \in B(z^*,\bar{\varepsilon})$ then $z_i \in B(z^*,\varepsilon^*)$ for all $i = 0,1,2,\ldots$

(iv) if $z_0 \in B(z^*,\bar{\varepsilon})$ then $z_i \to \tilde{z} \in \delta F \cap B(z^*,\varepsilon^*)$, with a quadratic rate of convergence, as $i \to \infty$.

Proof. Again, for simplicity, we consider the case when $p = 0$ (no conventional constraints) and $m = 1$ (one infinite dimensional constraint). Hence we discard the superscript "j" in the sequel.

Let $z^* \in \delta F$, $\gamma \in (0,1)$ and ε^*, ε_α, c and q^* be as in Lemmas 4.1 and 4.2. From (45), $v_q(z)$ is the solution of:

$$\min\{\|v\|^2 \,|\, [\phi(z,\alpha) + \phi_\alpha(z,\alpha)\delta\alpha_q(z,\alpha)] + [\phi_z(z,\alpha) + \phi_{z\alpha}(z,\alpha)\delta\alpha_q(z,\alpha)]v \leq 0,\ \alpha \in \tilde{A}_{q,\varepsilon_0}(z)\} \tag{49}$$

From (22) $v_1(z)$ is the solution of:

$$\min\{\|v\|^2 \,|\, \phi(z,\alpha) + \phi_z(z,\alpha)v \leq 0,\ \alpha \in \tilde{A}_{\varepsilon_0}(z) \cap \hat{A}\} \tag{50}$$

where, as above:

$$\hat{A} \triangleq \bigcup_{k\in K} B(\alpha_k(z^*),\varepsilon_\alpha)$$

We define $v_{q,1}(\cdot)$, our approximation to $v_1(\cdot)$, as follows:

$$\min\{\|v\|^2 \,|\, [\phi(z,\alpha) + \phi_\alpha(z,\alpha)\delta\alpha_q(z,\alpha)] + [\phi_z(z,\alpha) + \phi_{z\alpha}(z,\alpha)\delta\alpha_q(z,\alpha)]v \leq 0,\ \alpha \in \tilde{A}_{q,\varepsilon_0}(z) \cap \hat{A}\} \tag{51}$$

From Lemma 4.1(a), for all $z \in B(z^*,\varepsilon^*)$, all $q \geq q^*$, $\tilde{A}_{\varepsilon_0}(z) \cap \hat{A} = \{\alpha_k(z) \,|\, k \in K\}$, $\tilde{A}_{q,\varepsilon_0}(z) \cap \hat{A} = \{\alpha_{q,k}(z) \,|\, k \in K\}$ and $|\alpha_k(z) - \alpha_{q,k}(z)| \leq \Delta q$ for all $k \in K$. From Lemma 4.2(i), $|\delta\alpha(z,\alpha_{q,k}(z))| \leq |\alpha_k(z) - \alpha_{q,k}(z)| + c|\alpha_k(z) - \alpha_{q,k}(z)|^2$, for all $(z,k) \in Q \triangleq$ the set of (z,k) satisfying

$z \in B(z^*,\varepsilon^*)$, $\alpha_k(z) \notin \{\alpha_0,\alpha_c\}$, $k \in K$. From Lemma 4.1(d), $|\delta\alpha(z,\alpha_{q,k}(z))| \leq \Delta q + c\Delta q^2$ so that q^* can be chosen so that if $\alpha_k(z) \in \{\alpha_0,\alpha_c\}$, so does $\alpha_{q,k}(z)$ and $|\delta\alpha(z,\alpha_{q,k}(z))| < 2\Delta q$ (and $\delta\alpha_q(z,\alpha_{q,k}(z)) = \delta\alpha(z,\alpha_k(z))$) for all $(z,k) \in Q$, with $q \geq q^*$. Hence, from Lemmas 4.1 and 4.2, $v_1(z)$ may be expressed as the solution of:

$$\min\{\|v\|^2 \mid B(z)v + b(z) \leq 0\} \tag{52}$$

and $v_{q,1}(z)$ as the solution of:

$$\min\{\|v\|^2 \mid B_q(z)v + b_q(z) \leq 0\}, \tag{53}$$

where:

$$\|B(z) - B_q(z)\| \leq c\Delta q^2, \tag{54}$$

and:

$$\|b(z) - b_q(z)\| \leq c\Delta q^2, \tag{55}$$

for all $z \in B(z^*,\varepsilon^*)$, all $q \geq q^*$ (from (50), the elements of $b(z)$ are $\phi(z,\alpha)$, $\alpha \in \tilde{A}_{\varepsilon_0}(z) \cap \hat{A}$ and the rows of $B(z)$ are $\phi_z(z,\alpha)$, $\alpha \in \tilde{A}_{\varepsilon_0}(z) \cap \hat{A}$; $b_q(z)$ and $B_q(z)$ are similarly constructed using (51) instead of (50)).

It follows from Proposition A2 in the Appendix that ε^* can be chosen so that:

$$\|v_{q,1}(z)\| \leq M\|z - z^*\| \tag{56}$$

$$M\|v_{q,1}(z)\| \leq \gamma \tag{57}$$

$$\|v_{q',1}(z')\| \leq M\|v_{q,1}(z)\| \tag{58}$$

for all $z \in B(z^*,\varepsilon^*)$, all $z' = z + v_q(z)$, all $q \geq \max\{\bar{c}/\|v_{q,1}(z)\|, q^*\}$, all $q' \geq \max\{\bar{c}/\|v_{q',1}(z')\|, q^*\}$. Since $\phi(z^*,\alpha) \leq \delta < 0$ for all $\alpha \in \hat{A}^c$, since $\phi_\alpha(\cdot,\cdot)$, $\phi_z(\cdot,\cdot)$ and $\phi_{z\alpha}(\cdot,\cdot)$ are continuous, since

$|\delta\alpha_q(z,\alpha)| \leq 2\Delta q$ for all (z,q) and since $v_{1,q}(\cdot)$ satisfies (56), it is clear that ε^* and q^* can be chosen so that:

$$[\phi(z,\alpha)+\phi_\alpha(z,\alpha)\delta\alpha_q(z,\alpha)] + [\phi_z(z,\alpha)+\phi_{z\alpha}(z,\alpha)\delta\alpha_q(z,\alpha)]v_{1,q}(z) < 0 \tag{59}$$

for all $z \in B(z^*,\varepsilon^*)$, all $q \geq \max\{\bar{c}/\|v_{1,q}(z)\|,q^*\}$, all $\alpha \in \tilde{A}_{q,\varepsilon_0}(z) \cap \hat{A}^c$; it follows from (49) and (51) that, for (z,q,α) satisfying these conditions, $v_{1,q}(z)$ satisfies the constraints in (49), so that $\|v_q(z)\| \leq \|v_{1,q}(z)\|$. Hence:

$$[\phi(z,\alpha)+\phi_\alpha(z,\alpha)\delta\alpha_q(z,\alpha)] + [\phi_z(z,\alpha)+\phi_{z\alpha}(z,\alpha)\delta\alpha_q(z,\alpha)]v_q(z) < 0 \tag{60}$$

for all $z \in B(z^*,\varepsilon^*)$, all $q \geq \max\{\bar{c}/\|v_q(z)\|,q^*\}$, all $\alpha \in \tilde{A}_{q,\varepsilon_0}(z) \cap \hat{A}^c$; hence $v_q(z) = v_{1,q}(z)$ for such (z,q). Hence:

$$\|v_q(z)\| \leq \gamma\psi(z) \tag{61}$$

$$M\|v_q(z)\| \leq \gamma \tag{62}$$

$$\|v_{q'}(z')\| \leq M\|v_q(z)\| \tag{63}$$

for all $z \in B(z^*,\varepsilon^*)$ such that $z' = z+v_q(z) \in B(z^*,\varepsilon^*)$, all $q \geq \max\{\bar{c}/\|v_q(z)\|,q^*\}$, all $q' \geq \max\{\bar{c}/\|v_{q'}(z')\|,q^*\}$. This proves (i) and (ii); (iii) and (iv) follow directly from (i) and (ii) as shown in the proof of Theorem 2.1. □

To stabilize the algorithm we require a globally convergent first order algorithm. We employ the obvious modification (essentially removing f^0) of Algorithm II (with $\pi = 2$) given in [5]. For all $\varepsilon > 0$, all $q > 0$ this algorithm employs a search direction $w_{q,\varepsilon}(z)$ defined as the solution of:

$$\theta_{q,\varepsilon}(z) = \min\{\tfrac{1}{2}\|v\|^2 + \max\{g_z^j(z)v,\ j\in J_\varepsilon^g(z);\ \phi_z^j(z,\alpha)v,\ \alpha\in\tilde{A}_\varepsilon^j(z),\ j\in J_\varepsilon^f(z)\}\}. \tag{64}$$

Our final (implementable) algorithm is:

<u>Algorithm 3</u>.

<u>Data</u>: γ, $\beta \in (0,1)$, $\delta \in (0,1]$, $k \in [1,\infty)$, $\bar{c}$, M, ε_0, μ, $\nu \in (0,\infty)$, q_{-1}, a positive integer, $z_0 \in \mathbb{R}^n$.

<u>Step 0</u>: Set $i = 0$, $j = 0$.

<u>Step 1</u>: Set $\varepsilon = \varepsilon_0$.

<u>Step 2</u>: Determine q_i, the minimum $q \in \{q_{i-1}+1, q_{i-1}+2, \ldots\}$ such that $q_i \geq \bar{c}/\|v_{q_i}(z_i)\|$. If $v_q(z_i)$ exists for $q \in \{q_{i-1},\ldots,q_i\}$ and if $\|v_{q_i}(z_i)\| \leq k\gamma^j$ then: set $z_{i+1} = z_i + v_{q_i}(z_i)$, set $j = j+1$, set $i = i+1$ and go to Step 1. Else proceed.

<u>Step 3</u>: Determine q_i, the minimum $q \in \{q_{i-1}, q_{i-1}+1, \ldots\}$ such that $\tilde{A}^j_{q_i,0}(z_i)$ does not contain two adjacent mesh points for all $j \in \underline{m}$.

<u>Step 4</u>: If $\tilde{\theta}_{q_i,\varepsilon}(z_i) > -\delta\varepsilon$, $\varepsilon \leq \mu/2q_i$ and $\psi_{q_i}(z_i)_+ \leq \nu/2q_i$, set $q_i = q_i + 1$ and go to Step 1. If $\tilde{\theta}_{q_i,\varepsilon}(z_i) > -\delta\varepsilon$ but either $\varepsilon > \mu/2q_i$ or $\psi_{q_i}(z_i)_+ > \nu/2q_i$, set $\varepsilon = \varepsilon/2$ and go to Step 2. If $\tilde{\theta}_{q_i,\varepsilon}(z_i) \leq -\delta\varepsilon$ proceed.

<u>Step 5</u>: Compute λ_i, the largest $\lambda \in \{M, \beta M, \beta^2 M, \ldots\}$ such that:

$\psi_{q_i}(z_i + \lambda w_{q_{i,\varepsilon}}(z_i)) - \psi_{q_i}(z_i) \leq -\alpha\delta\varepsilon/2$. Set $z_{i+1} = z_i + \lambda_i w_{q_i,\varepsilon}(z_i)$. Set $i = i+1$ and go to Step 1. □

<u>Theorem 4.2</u>. Let $\{z_i\}$ be an infinite bounded sequence generated by Algorithm 3. Then, any accumulation point z^* of $\{z_i\}$ is feasible and there exists an integer N such that $z_{i+1} = z_i + v_{q_i}(z_i)$ for all $i \geq N$ so that $z_i \to z^*$ quadratically.

Proof. Again we assume $p = 0$ and $m = 1$.

(a) Suppose there exists an integer N such that Step 3 is entered for all $i \geq N$. It follows from Theorem 3 in [5] that any accumulation point z^* of $\{z_i\}$ is feasible.

(b) If (a) does not occur, then there exists an infinite subset S of $\{0,1,2,\ldots\}$ such that Step 2 is entered for all $i \in S$; hence $z_{i+1} = z_i + v_{q_i}(z_i)$ for all $i \in S$. There exists an accumulation point z^* of $\{z_i\}_{i \in S}$ and a subset S_1 of S such that $z_i \to z^*$ as $i \to \infty$, $i \in S_1$. Since $j \to \infty$ as $i \to \infty$, $i \in S_1$ and since $\|v_{q_i}(z_i)\| \leq k\gamma^j$ for all $i \in S$, it follows that $\|v_{q_i}(z_i)\| \to 0$ as $i \to \infty$, $i \in S_1$. Since q_i is increased by at least unity and time Step 2 is entered it follows that $q_i \to \infty$ $(\Delta q_i \to 0)$ as $i \to \infty$, $i \in S_1$. It follows from (42) that, for all $\alpha_i \in \tilde{A}_{q,\varepsilon_0}(z_i)$, $|\delta\alpha_{q_i}(z_i,\alpha_i)| \to 0$ as $i \to \infty$, $i \in S_1$. It then follows that $z^* \in F$ (for, if not, $\psi(z^*) > 0$, and, since $z_i \to z^*$ and $q_i \to \infty$ as $i \to \infty$, $i \in S_1$, it follows from (49) that there exists an integer i_1 and a $\mu > 0$ such that $\phi(z_i,\alpha_i) > 0$ for some $\alpha_i \in A_{q_i,\varepsilon_0}(z_i)$ for all $i \geq i_1$, $i \in S_1$; hence, from (49), there exists a $\nu > 0$ such that $\|v_{q_i}(z_i)\| > \nu$, for all $i \geq i_1$, $i \in S_1$, contradicting the convergence of $\|v_{q_i}(z_i)\|$ to zero).

The remainder of the proof is identical to the corresponding portion of the proof of Theorem 3.1, with Theorem 4.1 replacing Theorem 2.1 in the proof. □

5. Conclusion

An algorithm, with a quadratic rate of convergence, for solving both finite and infinite dimensional inequalities has been presented. The key assumption is that the function specifying each infinite dimensional inequality has a finite number of local maxima (see Assumption 2). It is thus plausible that Newton's method, applied to these local maxima, will constitute a quadratically convergent algorithm. In essence, this paper presents a stabilized, implementable version of such a procedure. Stabilization is achieved by employing a first order procedure, presented in [5], whenever the Newton step is unsatisfactory. Implementability is achieved by employing a discrete mesh A_q in place of the (infinite-dimensional) set A, and refining the mesh suitably at each iteration to ensure quadratic convergence. The final algorithm (Algorithm 3) is a slight modification of the first order algorithm in [5]; an extra step (Step 2) computes the Newton step and employs it if satisfactory; a quadratic program is, in any case, required for the first order algorithm (see (64)) so here the extra programming to compute $v_{q_i}(z_i)$ (see (45)) is slight.

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Appendix

The analysis in the main text requires an analysis of Newton's method applied to a finite number of inequalities. Thus, in the proof of Theorem 2.1, it is shown that the step $v(z)$ is a Newton step for the problem of solving $\eta_k(z) \leq 0$, $k \in K$. In this appendix, therefore, we consider the problem of satisfying the inequality $b(z) \leqq 0$ where $b(\cdot): \mathbb{R}^n \to \mathbb{R}^t$ is twice continuously differentiable. For all $z \in \mathbb{R}^n$ let $B(z) \triangleq b_z(z)$. Let F_b denote the feasible set $\{z | b(z) \leqq 0\}$. Let $v_1(z)$ denote the solution, when it exists, of $\min\{\|v\|^2 | B(z)v + b(z) \leqq 0\}$.

Proposition A1. Suppose that:

(i) $z^* \in \delta F_b$, $\gamma \in (0,1)$;

(ii) $b(\cdot): \mathbb{R}^n \to \mathbb{R}^t$, $t \leq n$, is twice continuously differentiable ;

(iii) $\nabla b^i(z^*)$, $i = 1,\ldots,t$ are linearly independent.

Then there exists an $\varepsilon^* \in (0,\infty)$ and an $M \in (0,\infty)$ such that:

(i) $\|v_1(z)\| \leq M\|z-z^*\|$ and $M\|v_1(z)\| \leq \gamma$ for all $z \in B(z^*,\varepsilon^*)$;

(ii) $\|v_1(z')\| \leq M\|v_1(z)\|^2$, for all $z \in B(z^*,\varepsilon^*)$, all $z' = z + v_1(z)$.

Proof. The proof is standard, and is repeated to provide a basis for the perturbed problem considered in Proposition A2. Since $B(z^*)$ has t linearly independent rows, then $t \leq n$ and there exists a permutation matrix P such that

$$B(z^*) = [B_1(z^*), B_2(z^*)]P \tag{A1}$$

where $B_1(z^*) \in \mathbb{R}^{t\times t}$ is invertible. Let $B_1(\cdot)$, $B_2(\cdot)$ be defined by:

$$B(z) = [B_1(z), B_2(z)]P \tag{A2}$$

Since $B_1(\cdot): \mathbb{R}^n \to \mathbb{R}^{t\times t}$ is continuous and $B_1(z^*)$ is invertible, there exists [7] an $\varepsilon_1 \in (0,\infty)$ and an $M_1 \in (0,\infty)$ such that $B_1(z)$ is invertible and $\|B_1(z)^{-1}\| \leq M_1$ for all $z \in B(z^*,\varepsilon_1)$.

It is easily checked that a solution of:

$$B(z)v + b(z) = b(z^*) \leqq 0 \tag{A3}$$

is:

$$v_2(z) = P\begin{pmatrix} B_1(z)^{-1}(b(z^*) - b(z)) \\ 0 \end{pmatrix} \tag{A4}$$

and, hence, that $\|v_1(z)\| \leq \|v_2(z)\| \leq M_1\|b(z)-b(z^*)\| \leq M_2\|z-z^*\|$ for all $z \in B(z^*,\varepsilon_1)$ and some $M_2 \in (0,\infty)$.

Choose $\varepsilon_2 = \varepsilon_1/(1+M_2)$. Then, since $\|z_1+v_1(z)-z^*\| \leq \|z-z^*\| + M_2\|z-z^*\|$ it follows that $z+v_1(z) \in B(z^*,\varepsilon_1)$ for all $z \in B(z^*,\varepsilon_2)$. For all z let z' denote $z+v_1(z)$. A solution of:

$$B(z')v + b(z') = B(z)v_1(z) + b(z) \leqq 0 \tag{A5}$$

is:

$$v_3(z') = P\begin{pmatrix} -B_1(z')^{-1}(b(z')-b(z)-B(z)v_1(z)) \\ 0 \end{pmatrix} \tag{A6}$$

Since $B(z) \triangleq b_z(z)$ and since $b(\cdot)$ is twice continuously differentiable, it follows that $\|b(z')-b(z)-B(z)v_1(z)\| \leq M_3\|v_1(z)\|$ for all $z \in B(z^*,\varepsilon_2)$ and some $M_3 \in (0,\infty)$. Hence $\|v_1(z')\| \leq \|v_3(z')\| \leq M_1M_3\|v_1(z)\|^2$ for all $z \in B(z^*,\varepsilon_2)$.

Setting $M = \max\{M_2,M_1M_3\}$ and $\varepsilon^* = \min\{\varepsilon_2,\gamma/M^2\}$ yields $\|v_1(z)\| \leq M\|z-z^*\|$, $M\|v_1(z)\| \leq M^2\|z-z^*\| \leq M^2\varepsilon^* \leq \gamma$ and $\|v_1(z')\| \leq M\|v_1(z)\|^2$ for all $z \in B(z^*,\varepsilon^*)$, thus proving (i) and (ii). □

The above theorem suffices for Theorem 2.1. Theorem 4.1 requires the analysis of an algorithm (for solving the inequality $b(z) \leqq 0$) which employs an approximate Newton step $v_{q,1}(z)$ obtained by solving $\min\{\|v\|^2 | B_q(z) + b_q(z) \leq 0\}$ where $\|B(z)-B_q(z)\| \leq c\Delta q$ and $\|b(z)-b_q(z)\| \leq c\Delta q^2$ for all $z \in B(z^*,\varepsilon_1)$, all $q \geq q_1$ where $z^* \in \delta F_b$ and $c \in (0,\infty)$ (see (53)-(55), ε_1 replacing ε^*, q_1 replacing q^*). For any $\bar{c} \in (0,\infty)$, $q^* \geq 0$ an integer, let $Q(z) \triangleq \max\{\bar{c}/\|v_{q,1}(z)\|, q^*\}$.

Proposition A2. Let z^*, γ and $b(\cdot)$ satisfy hypotheses (i)-(iii) of Proposition A1. Let $v_{q,1}(\cdot)$ be defined as above and let $B_q(\cdot)$ and $b_q(\cdot)$ satisfy the above inequalities. Let $\bar{c} \in (0,\infty)$. Then there exists an $\varepsilon^* \notin (0,\infty)$, an $M \in (0,\infty)$ and a positive integer q^* such that:

(i) $\|v_{q,1}(z)\| \leq M\|z-z^*\|$ and $M\|v_{q,1}(z)\| \leq \gamma$ for all $z \in B(z^*,\varepsilon^*)$, all $q \geq Q(z)$;

(ii) $\|v_{q',1}(z')\| \leq M\|v_{q,1}(z)\|^2$ for all $z \in B(z^*,\varepsilon^*)$, all $q \geq Q(z)$, all $z' = z + v_{q,1}(z)$, all $q' \geq Q(z')$.

As before

Proof. As before,

$$B(z^*) = [B_1(z^*), B_2(z^*)]P \tag{A7}$$

where $B_1(z^*)$ is invertible and P is a permutation matrix. Let $B_{q,1}(\cdot)$, $B_{q,2}(\cdot)$ be defined by:

$$B_q(z) = [B_{q,1}(z), B_{q,2}(z)]P. \tag{A8}$$

It follows from our assumptions that $B_{q,1}(z) \to B_1(z^*)$ as $(z,q) \to (z^*,\infty)$ and, hence, [7], that there exists an $\varepsilon_1 \in (0,\infty)$, a positive integer q_1 and an $M_1 \in (0,\infty)$ such that $B_{q,1}(z)$ is invertible and $\|B_{q,1}(z)^{-1}\| \leq M_1$ for all $z \in B(z^*,\varepsilon_1)$. It then follows that a solution of:

$$B_q(z)v + b_q(z) = b(z^*) \leq 0 \tag{A9}$$

is:

$$v_{q,2}(z) = P\begin{pmatrix} B_q(z)^{-1}(b(z^*)-b_q(z)) \\ 0 \end{pmatrix} \tag{A10}$$

and

$$\|v_{q,1}(z)\| \leq \|v_{q,2}(z)\| \leq M_1\|b_q(z)-b(z^*)\| \leq M_2[\|z-z^*\| + 1/q^2] \tag{A11}$$

for all $z \in B(z^*,\varepsilon_1)$, all $q \geq q_1$ and some $M_2 \in (0,\infty)$.

Choose $\varepsilon_2 = \varepsilon_1/[2(1+M_2)]$ and $q_2 = \max\{q_1, (2M_2/\varepsilon_1)^{1/2}\}$. Then $\|z + v_{q,1}(z) - z^*\| \leq \|z-z^*\| + M_2[\|z-z^*\| + 1/q^2] \leq (\varepsilon_1/2)/(1+M_2) + M_2(\varepsilon_1/2)/(1+M_2) + \varepsilon_1/2 = \varepsilon_1$, i.e. $z + v_{q,1}(z) \in B(z^*,\varepsilon_1)$ for all $z \in B(z^*,\varepsilon_2)$, all $q \geq q_2$.

A solution of:

$$B_{q'}(z')v + b_{q'}(z') = B_q(z)v_{q,1}(z) + b_q(z) \leq 0 \tag{A12}$$

where $z' \triangleq z + v_{q,1}(z)$ is:

$$v_{q,3}(z') = P\begin{pmatrix} B_{q',1}(z')^{-1}(b_{q'}(z')-b_q(z)-B_q(z)v_{q,1}(z)) \\ 0 \end{pmatrix} \tag{A13}$$

Hence:

$$\|v_{q',1}(z')\| \leq M_1\|b(z')-b(z)+B(z)v_{q,1}(z)\| + M_1c/(q')^2 + M_1c/q^2 + M_1(c/q)\|v_{q,1}(z)\| \tag{A14}$$

for all $z \in B(z^*,\varepsilon_2)$, all $q \geq q_2$. If we choose q (and q') such that $q \geq \bar{c}/\|v_{q,1}(z)\|$ (and $q' \geq \bar{c}/\|v_{q',1}(z')\|$) then, from (A11): $\|v_{q,1}(z)\| \leq M_2\|z-z^*\| + (M_2/\bar{c}^2)\|v_{q,1}(z)\|$ for all $z \in B(z^*,\varepsilon_2)$, all $q \geq \max\{\bar{c}/\|v_{q,1}(z)\|, q_2\}$. Hence there exists an $\varepsilon_3 \in (0,\varepsilon_2]$ and an $M_3 > M_2$ such that:

$$\|v_{q,1}(z)\| \leq M_3\|z-z^*\| \tag{A15}$$

for all $z \in B(z^*,\varepsilon_3)$, all $q \geq \max\{\bar{c}/\|v_{q,1}(z)\|,q_2\}$. Similarly, from (A14) there exists an $\varepsilon_4 \in (0,\varepsilon_3]$ and an $M_4 \in (0,\infty)$ such that:

$$\|v_{q',1}(z')\| \leq M_4\|v_{q,1}(z)\|^2 \tag{A16}$$

for all $z \in B(z^*,\varepsilon_4)$, all $q \geq \max\{\bar{c}/\|v_{q,1}(z)\|,q_2\}$, all $z' = z+v_{q,1}(z)$, all $q' \geq \max\{\bar{c}/\|v_{q',1}(z')\|,q_2\}$. The desired result follows from (A15) and (A16). □