DESIGN OF MULTIVARIABLE FEEDBACK SYSTEMS WITH STABLE PLANT

by

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ABSTRACT

This paper considers, in a general algebraic framework, the design of a unity-feedback multivariable system with a stable plant. The method is based on a simple parametrization of the four closed-loop transfer functions in terms of P, the plant transfer function, and $Q = H_{y_1u_1}$. In particular the I/O transfer function $H_{y_2u_1} = PQ$. Using the framework of rational transfer functions, we show that the closed-loop system will be exp. stable if and only if Q is exp. stable. Furthermore if both P and Q are strictly proper then the controller is also strictly proper.

Algorithms are given for obtaining strictly proper controllers such that the resulting I/O map is decoupled, all its poles can be chosen by the designer, and the same holds for zeros except, of course, for the $\mathbb{C}_+$-zero prescribed by the $\mathbb{C}_+$-zero of the plant. A discussion is included to temper these results by the constraints imposed by noise and plant saturation.

I. Introduction

In the past several decades, the problem of multi-variable design, e.g., closed-loop stability, pole-zero assignment, disturbance rejection, etc., has been investigated intensively by many authors (see e.g. the paper collections [MacF. 1, Part IV], [Sai. 1]). In general, the design methods can be classified into two categories: 1) the time-domain approach, and 2) the frequency-domain approach. In the time-domain approach, the system is described by a state-space model and the design specifications are expressed in terms of an appropriate (in particular, quadratic) performance index; the feedback design is obtained by solving the corresponding optimal control problem (see e.g. [IEEE 1], [Kwa. 1], [And. 1] Based on transfer-function matrices, the frequency-domain approach tackles the design problem in various ways (see e.g. [MacF. 1, pp. 303-317], [MacF. 2]): we mention more specifically the algebraic approach (see e.g. [Ros. 1], [Pec. 1], [Des. 1]), the geometric approach (see e.g. [Won. 1]) and the complex-variable approach (see e.g. [MacF. 3]). Sain has recently emphasized the importance of algebraic thinking in system engineering [Sai. 2].

In this paper, we study the design of <u>linear, unity-feedback systems</u> with a <u>stable</u> plant. Using the work of several authors, we formulate a refined version of a stability theorem in a general algebraic set up by using the parametrization proposed by Zames in [Zam. 1]. Considering rational transfer-function matrices, we propose a design method for obtaining a decoupled I/O map with pole-zero assignment in each (decoupled) I/O channel. This design is based on the fact that, in the unity-feedback configuration, the I/O map of the closed-loop system is simply related to that of the plant. The design algorithm displays clearly the constraint(s) imposed by the $\mathbb{C}_+$-zeros of the plant on the decoupled I/O map. The proposed design method, although described for the rational transfer function case, applies to any algebra of transfer functions within which algorithms are available for obtaining coprime factorizations of transfer functions.

The paper ends by a discussion of the limitations inherent in the algebraic approach resulting from saturation and system noise.

In Appendix A, we perform a number of useful calculations in an arbitrary noncommutative ring: thus by following these calculations the reader has performed several of them for the price of one (see Table I)!

Notations

$a := b$ means "a denotes b." $\mathbb{R}$:= field of real numbers; $\mathbb{C}$:= field of complex numbers; $\mathbb{R}_+$:= set of nonnegative real numbers; $\mathbb{C}_+$, $(\mathbb{C}_-)$:= set of complex numbers such that Re $z \geq 0$, (Re $z \leq 0$, resp.). For any set A, $A^{n\times n}$ denotes the class of all nxn arrays with elements in A, and $\mathring{A}$ denotes the interior of A. Thus $\mathring{\mathbb{C}}_-$ denotes the open left half-plane. $\mathbb{C}_p(s)$, $(\mathbb{C}_{p,o}(s))$ denotes the class of all proper, (strictly proper, resp.), rational functions with coefficients in $\mathbb{C}$. $R(0)$, $(R_o(0))$ denotes the class of all proper, (strictly proper, resp.) elements of $\mathbb{C}_p(s)$ that are analytic in $\mathbb{C}_+$. If $d(s)$ is a polynomial, ∂d := degree of d, $Z[d]$:= set of zeros of d. If $P \in \mathbb{R}(s)^{n\times n}$, $Z[P]$:= set of zeros of transmission of P, $P[P]$:= set of poles of P.

II. Algebras of I/O Maps

We consider inputs, errors and outputs to be functions defined on T (typically, $T = \mathbb{R}_+$ for the continuous-time case, $T = \mathbb{N}$ for the discrete-time case), into some normed space V (typically, $V = \mathbb{R}^{n_i}, \mathbb{R}^{n_e}, \mathbb{R}^{n_o}$ or $\mathbb{C}^{n_i}, \mathbb{C}^{n_e}, \mathbb{C}^{n_o}$). The function space $F := \{f : T \to V\}$ is a linear space over $\mathbb{R}$ or $\mathbb{C}$. Let $(L, \|\cdot\|)$ denote a Banach subspace of F, i.e.,

$$f \in L := f : T \to V \quad \text{and} \quad \|f\| < \infty$$

(typically, $L = L^n_\infty, L^n_2, L^n_1, \ell^n_\infty \ldots$). Call L_e the corresponding extended space of L [Des. 2] [Vid. 1].

Let θ_L denote the zero element in L (L_e). Let $\mathbb{A}$ denote the non-commutative algebra with identity I, consisting of <u>linear causal</u> maps from L_e <u>into</u> L_e. Let $\tilde{\mathbb{A}}$ be an algebra of <u>linear causal</u> maps defined on L_e (but not necessarily <u>into</u> L_e) and such that $\mathbb{A}$ is a subring of $\tilde{\mathbb{A}}$ (Equivalently, $\tilde{\mathbb{A}}$ is a super-ring of $\mathbb{A}$; for example, in the continuous-time case, $\tilde{\mathbb{A}}$ includes the time differentiation operators, hence its range is larger than L_e since it includes distributions). Let $\mathbb{A}_s$ denote the <u>radical</u> (see Appendix) of $\mathbb{A}$. Let $(\mathbb{B}, \|\cdot\|)$ denote[†] the Banach algebra of <u>continuous</u> (linear and causal) elements of $\mathbb{A}$, i.e.,

$$\mathbb{B} := \{P \in \mathbb{A} : \|P\| := \sup_{\substack{u \in L \\ u \neq \theta_L}} \frac{\|Pu\|}{\|u\|} < \infty\} \quad .$$

We refer to the elements of $\mathbb{B}$ as the <u>$\mathbb{B}$-stable</u> maps: indeed, we have, for $P \in \mathbb{B}$

$$\|Pu\| \leq \|P\| \cdot \|u\| \qquad \forall u \in L$$

and[§]

$$\|Pu\|_T \leq \|P\| \cdot \|u\|_T \qquad \forall T \in \mathcal{T},\ \forall u \in L_e$$

Let Θ denote the zero element in $\mathbb{A}$, ($\mathbb{B}$). Let $\mathbb{B}_s := \mathbb{A}_s \cap \mathbb{B}$. Table I shows some typical examples of $\mathbb{A}$, $\tilde{\mathbb{A}}$, $\mathbb{A}_s$, $\mathbb{B}$ and $\mathbb{B}_s$.

III. <u>Closed-loop $\mathbb{B}$-stability</u>

We consider the unity-feedback system shown in Fig. 1. The inputs are u_1 and u_2, d_o is the output disturbance; they are all elements in L_e.

[†]We shall use $\|\cdot\|$ to denote both the norm in L and the induced norm in $\mathbb{B}$.

[§]$\|f\|_T$, with $T \in \mathcal{T}$ and $f \in L_e$, denotes the norm of the function f truncated at T.

Formally (i.e. assuming that all the expressions involved are well-defined), the maps $H_{yu}:\begin{bmatrix}u_1\\u_2\end{bmatrix}\to\begin{bmatrix}y_1\\y_2\end{bmatrix}$ and $H_{eu}:\begin{bmatrix}u_1\\u_2\end{bmatrix}\to\begin{bmatrix}e_1\\e_2\end{bmatrix}$, representing the closed-loop system, are given by

$$H_{yu}=\begin{bmatrix}C(I+PC)^{-1} & -CP(I+CP)^{-1}\\ PC(I+PC)^{-1} & P(I+CP)^{-1}\end{bmatrix}\tag{3.1}$$

and

$$H_{eu}=\begin{bmatrix}(I+PC)^{-1} & -P(I+CP)^{-1}\\ C(I+PC)^{-1} & (I+CP)^{-1}\end{bmatrix}\tag{3.2}$$

respectively. Note the relations:

$$H_{yu}=J\,H_{eu}-J\tag{3.3}$$

and

$$H_{eu}=I_2-JH_{yu}\tag{3.4}$$

where

$$J:=\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix},\quad I_2:=\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}$$

An important consequence of (3.3) and (3.4) is that: $H_{yu}\in\mathbb{A}^{2x2}$ $\Leftrightarrow H_{eu}\in\mathbb{A}^{2x2}$, and similarly with $\mathbb{A}$ replaced by $\mathbb{B}$ (but not true for $\mathbb{A}_s$ or $\mathbb{B}_s$).

Definition 3.1.

The unity-feedback system of Fig. 1, with $P\in\mathbb{A}$, $C\in\tilde{\mathbb{A}}$ and

$$H_{yu}\in\mathbb{A}^{2x2}\tag{3.5}$$

is called the system (P,Q), where

$$Q:=C(I+PC)^{-1}\tag{3.6}$$

is an element in $\mathbb{A}$ (the latter follows from (3.1) and (3.5)). ¤

Remarks 3.1.

(a) Assumption (3.5) does not imply that the controller C is an element of $\mathbb{A}$. For example let $\mathbb{A}=\mathbb{C}_p(s)$ and let $p(s),\ q(s)\in\mathbb{C}_p(s)$, We may have $c(s) \notin \mathbb{C}_p(s)$: take $p(s) = 1$, $q(s) = s/(s+1)$, hence $c(s) = s$, which is <u>not proper</u>.

(b) Assumption (3.5) excludes singular cases by requiring that <u>all</u> the <u>closed-loop transfer functions</u> be in $\mathbb{A}$.

As a consequence of the assumptions made in defining the system (P,Q), it turns out (see (A.34)) that

$$H_{yu} = \begin{bmatrix} Q & -QP \\ PQ & P(I-QP) \end{bmatrix} \in \mathbb{A}^{2x2} \tag{3.11}$$

and similarly,

$$H_{eu} = \begin{bmatrix} I-PQ & -P(I-QP) \\ Q & I-QP \end{bmatrix} \in \mathbb{A}^{2x2} \tag{3.12}$$

The importance of equ. (3.11) and (3.12) is that the <u>closed-loop</u> behavior of the unity-feedback system shown in Fig. 1 is completely described in terms of sums and products of P and Q; no inverses are required! (Compare with (3.1) and (3.2) above.)

<u>Remarks 3.2.</u>

a) For design purposes, the crucial observation is that the I/O map of the system shown in Fig. 1 has the simple form

$$\boxed{H_{y_2u_1} = PQ} \tag{3.13}$$

and, since $H_{y_2d_o} = H_{e_1u_1}$, the map from the <u>output disturbance</u> d_o to the <u>output</u> y_2 is:

$$\boxed{H_{y_2d_o} = I-PQ} \tag{3.14}$$

b) Note that equ. (3.11) to (3.14) are valid irrespective of whether P and/or Q are stable.

Definition 3.3.

The unity-feedback system (P,Q) is said to be $\mathbb{B}$-stable iff

$$H_{yu} \in \mathbb{B}^{2x2}$$

or equivalently,

$$H_{eu} \in \mathbb{B}^{2x2} .$$

Remark 3.3.

Note that, as a consequence of the unity feedback, $H_{e_1 d_o} = -H_{e_1 u_1}$ and $H_{e_2 d_o} = -H_{e_2 u_1}$, hence, for design purposes, any output disturbance d_o can be replaced by an equivalent input $u_1 = -d_o$.

With $\mathbb{B}$-stability defined, we can state and prove the following stabilization theorem:

Theorem 3.4. (Closed-loop $\mathbb{B}$-stability).

Consider the unity-feedback system (P,Q) of Fig. 1 (hence, by definition, $P,Q \in \mathbb{A}$, $C \in \tilde{\mathbb{A}}$ and $H_{yu} \in \mathbb{A}^{2x2}$);

(i) u.t.c., if

$$P \in \mathbb{B} \tag{3.20}$$

then†

$$Q \in \mathbb{B} \Leftrightarrow H_{yu} \in \mathbb{B}^{2x2} \tag{3.21}$$

and

$$Q \in \mathbb{B}_s \Leftrightarrow H_{yu} \in \mathbb{B}^{2x2} \text{ and } C \in \mathbb{A}_s; \tag{3.22}$$

(ii) u.t.c., if

$$P \in \mathbb{B}_s \tag{3.23}$$

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†Equation (3.21) is a slight generalization of [Des. 3, Theorem III]. Equation (3.22) is a result obtained by Zames in [Zam. 1].

then

$$Q \in \mathbb{B} \Leftrightarrow H_{yu} \in \mathbb{B}^{2x2} \text{ and } C \in \mathbb{A} \tag{3.24}$$

and

$$Q \in \mathbb{B}_s \Leftrightarrow H_{yu} \in \mathbb{B}_s^{2x2} \text{ and } C \in \mathbb{A}_s. \tag{3.25}$$

¤

Comments 3.4.

(a) To appreciate the scope of this theorem refer to Table I.

(b) With $P \in \mathbb{B}$, (3.21) shows that as Q ranges over all of $\mathbb{B}$, the $\mathbb{B}$-stability of the closed-loop system (P,Q) is guaranteed; $Q \in \mathbb{B}$ parametrizes all $\mathbb{B}$-stable closed-loop systems (P,Q). Note, however, C is not guaranteed to be $\mathbb{B}$-stable.

(c) Concerning (3.21), $P \in \mathbb{B}$ and $Q \in \mathbb{B}$ does not imply

$$(I-PQ)^{-1} \in \mathbb{A},$$

hence the resulting controller $C = Q(I-PQ)^{-1}$ may not be in $\mathbb{A}$; by assumption, it is in $\tilde{\mathbb{A}}$ (see Def. 3.1, above).

(d) The most realistic case is $P \in \mathbb{B}_s$; then (3.25) shows that restricting Q to be in $\mathbb{B}_s$ delivers C in $\mathbb{A}_s$. In the rational case, a strictly proper plant P and a strictly proper parameter Q give a strictly proper controller C. Hence, with $P \in \mathbb{B}_s$, the problem is to choose $Q \in \mathbb{B}_s$ so that the other design objectives are satisfied.

Proof.

First, we recall (3.11):

$$H_{yu} = \begin{bmatrix} Q & -QP \\ PQ & P(I-QP) \end{bmatrix} \in \mathbb{A}^{2x2} \tag{3.11}$$

$\Leftarrow$ The implications (3.21), (3.22), (3.24) and (3.25), in this direction, are immediate by (3.11) since $Q = H_{y_1u_1}$.

$\Rightarrow$ By (3.11), and the closure properties of the rings $\mathbb{B}$ and $\mathbb{B}_s$, respectively,

$$P \in \mathbb{B},\ Q \in \mathbb{B} \Rightarrow H_{yu} \in \mathbb{B}$$
$$P \in \mathbb{B}_s,\ Q \in \mathbb{B}_s \Rightarrow H_{yu} \in \mathbb{B}_s$$

Hence the first part of the implications (3.21), (3.22), (3.24) and (3.25) are proven.

Now, calculating in the ring $\tilde{\mathbb{A}}$ ($\supset \mathbb{A} \supset \mathbb{B} \supset \mathbb{B}_s$), we obtain

$$\begin{aligned}(I+PC)^{-1} &= I - PC(I+PC)^{-1} && \text{(by (A.4))}\\ &= I - PQ \in \mathbb{A} \subset \tilde{\mathbb{A}} && \text{(by (3.11));}\end{aligned}$$

then, by (A.27),

$$C = Q(I-PQ)^{-1} \in \tilde{\mathbb{A}}\,.$$

Hence,

$$P \in \mathbb{B} \subset \mathbb{A} \text{ and } Q \in \mathbb{B}_s \subset \mathbb{A}_s$$
$$\Rightarrow (I-PQ)^{-1} \in \mathbb{A} \qquad (\mathbb{A}_s \text{ is the radical of } \mathbb{A})$$
$$\Rightarrow C = Q(I-PQ)^{-1} \in \mathbb{A}_s \quad (\mathbb{A}_s \text{ is the radical of } \mathbb{A}).$$

A fortiori,

$$P \in \mathbb{B}_s \subset \mathbb{B} \quad \text{and} \quad Q \in \mathbb{B}_s \Rightarrow C \in \mathbb{A}_s.$$

Finally,

$$P \in \mathbb{B}_s \subset \mathbb{A}_s \quad \text{and} \quad Q \in \mathbb{B} \subset \mathbb{A}$$
$$\Rightarrow (I-PQ)^{-1} \in \mathbb{A}$$
$$\Rightarrow C = Q(I-PQ)^{-1} \in \mathbb{A}\,.$$

This completes the proof of equivalences (3.21)-(3.25). ◻

Remarks 3.4.

(a) Since $H_{y_2u_1} = PQ$ and $H_{y_2d_o} = I - PQ$, the parametrization by Q gives us direct control (through Q) of both the input-output map $H_{y_2u_1}$ and the output-disturbance sensitivity operator $H_{y_2d_o}$. Note that this does not imply that we can achieve simultaneously the design goals with respect to $H_{y_2u_1}$ and $H_{y_2d_o}$. However, suppose that the supports of the spectrum of the reference input u_1 and the output disturbance d_o are essentially the same, then by (3.13) and (3.14),

$$H_{y_2u_1} \approx I \Leftrightarrow H_{y_2d_o} \approx \theta$$

with the same degree of approximation over the frequency band of interest.

Note that the output-disturbance desensitization requirement is more restrictive than the requirement that the I/O map $H_{y_2u_1}$ be "nice." For example, for the I/O map

$$H_{y_2u_1}(s) = \begin{bmatrix} d_1(s) & x_{12}(s) & \cdots & x_{1m}(s) \\ x_{21}(s) & d_2(s) & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ x_{m1}(s) & \cdots & \cdots & d_m(s) \end{bmatrix} \in R(0)^{m \times m} \tag{3.35}$$

to have good control capability, we would like to have, over the band of interest, all the off-diagonal elements, $|x_{ij}(j\omega)|$'s small and

$$|d_i(j\omega)| \approx 1 \qquad i = 1,\ldots,m, \quad \forall\omega \in \text{band}, \tag{3.36}$$

(together with $\measuredangle\, d_i(j\omega)$'s that yield acceptable step responses). Since

$$H_{y_2d_o}(s) = I - H_{y_2u_1}(s) = \begin{bmatrix} 1-d_1(s) & -x_{12}(x) & \cdots & -x_{1m}(s) \\ -x_{21}(x) & 1-d_2(s) & \ddots & \vdots \\ -x_{m1}(s) & \cdots & \cdots & 1-d_m(s) \end{bmatrix} \tag{3.37}$$

For output-disturbance desensitization, we must have, over the band of interest, all $|x_{ij}(j\omega)|$'s small and

$$|1-d_i(j\omega)| \approx 0 \qquad i = 1,\ldots,m. \tag{3.38}$$

Note that (3.38) is equivalent to

$$d_i(j\omega) \approx 1 \qquad i = 1,\ldots,m, \ \forall\omega \in \text{band}, \tag{3.39}$$

which is more restrictive than (3.36).

(b) In many applications, e.g., in a fail-safe closed-loop system or for ease of maintenance, we would like to have the resulting controller C in $\mathbb{B}_s$ or in $\mathbb{B}$. Since with P in $\mathbb{B}_s$, the condition $Q \in \mathbb{B}_s$ does not imply that $C \in \mathbb{B}_s$, it would be desirable to have a <u>numerically convenient</u> description of the class of all Q in $\mathbb{B}_s$ with its corresponding controller C in $\mathbb{B}_s$.

(c) With the plant P <u>fixed</u> in $\mathbb{B}$ (or $\mathbb{B}_s$), when the design parameter Q ranges over all $\mathbb{B}$ (or $\mathbb{B}_s$), the corresponding input-output map is given by $H_{y_2u_1} = PQ$ (see (3.13)); hence, the I/O map is restricted by the presence of the left factor P. We will see later that this implies <u>limitations</u> on the achievable $H_{y_2u_1}$'s.

Theorem 3.4 has therefore the following design implications that we formulate as the

<u>Design Theorem 3.5</u>

Suppose that we wish to design a unity-feedback system (P,Q) as shown in Fig. 1. Then, given <u>any</u> $P \in \mathbb{B}_s$, $\forall H_{y_2u_1} \in \mathbb{B}_s$ such that $H_{y_2u_1} = PQ$ for <u>some</u> $Q \in \mathbb{B}_s$ $\exists$ $C \in \mathbb{A}_s$ for which the system (P,Q) is $\mathbb{B}$-stable and has the specified I/O map $H_{y_2u_1}$.

Remark 3.5: Since in a number of algebras of transfer functions there are procedures for stabilizing by feedback any unstable transfer function (see e.g. [You. 1], [Cal. 4], [Che. 1], [Ant. 1]), the theorem above applies to any $P \in /A_s$ provided that, whenever $P \notin \mathbb{B}$, P is first stabilized by local feedback.

If the plant P in Fig. 1 is nonlinear, then from (3.1), it is easy to show the following result:

Corollary 3.6. (Nonlinear System $\mathbb{B}$-stability)

Let $\mathbb{B}_{NL}$ be the class of nonlinear $\mathbb{B}$-stable maps from L_e into L_e (more precisely,

$$P \in \mathbb{B}_{NL} \Leftrightarrow \exists \gamma(P) < \infty \ni \|Pe_2\| \leq \gamma(P)\cdot\|e_2\|, \quad \forall e \in L).$$

Consider the unity-feedback system in Fig. 1. If $P \in \mathbb{B}_{NL}$, then

$$H_{yu} \in \mathbb{B}_{NL}^{2x2} \Leftrightarrow C(I+PC)^{-1} \in \mathbb{B}_{NL} \text{ and } (I+PC)^{-1} \in \mathbb{B}_{NL}$$ ¤

Remark 3.6.

Note that the class $\mathbb{B}_{NL}$ is no longer a ring: the right-distributive law fails!

IV. Design Procedures

Youla et al. have shown that any stabilizing proper controller can be parametrized in terms of given matrices [You. 1]; however, the resulting formula for the controller and the closed-loop I/O transfer-function matrix are rather complicated. Zames discovered that, if the plant is stable, a simple parametrization exists for both the closed-loop stability and the output-disturbance sensitivty [Zam. 1]. In the case of unity-feedback systems with a stable plant, we observed that the relationship between the closed-loop I/O map $H_{y_2u_1}$ and the plant transfer-function P is particularly simple if one uses the parametrization proposed by Zames: $H_{y_2u_1} = PQ$ (see (3.13)). The design procedures described below are based on this observation.

Although the examples are developed for the rational transfer function case, it is clear that the procedure will immediately apply to those cases where the matrix transfer functions can be factorized appropriately and where the notion of zero makes sense (in this connection for the algebra $\hat{\mathcal{B}}(0)$ see [Cal. 1-4].

In this section, we consider a linear, time-invariant plant $P(s) \in \mathcal{R}_o(0)^{m\times m}$ (equivalently, $P(s)$ is strictly proper and exp. stable) and we introduce procedures for obtaining a strictly proper controller such that

(i) the closed-loop unity-feedback system in Fig. 1 is exp. stable,

(ii) the I/O map $H_{y_2u_1}$ is decoupled and strictly proper; and

(iii) in each diagonal element of $H_{y_2u_1}(s)$, the poles and the zeros (in addition to the $\mathbb{C}_+$-zeros of $P(s)$) can be specified by the designer.

From the previous results, the design starts by choosing an appropriate diagonal $H_{y_2u_1}(s) \in \mathcal{R}_o(0)^{m\times m}$ such that

(i) equation (3.13), $H_{y_2u_1}(s) = P(s)\,Q(s)$, has a solution $Q(s)$ in $\mathcal{R}_o(0)^{m\times m}$ (Hence, by Theorem 3.4, the closed-loop system is exp. stable and the resulting controller is strictly proper);

(ii) the decoupled I/O map $H_{y_2u_1}(s)$ satisfies design specifications such as zero asymptotic error for step responses, appropriate bandwidth, etc.

We consider two design situations depending on whether the plant P(s) has $\mathbb{C}_+$-zeros or not.

4.1. Design Procedure for P(s) with no $\mathbb{C}_+$-zeros

Algorithm 4.1. (P(s) has no $\mathbb{C}_+$-zeros)

Data $P(s) \in R_o(0)^{m \times m}$, $Z[P] \subset \mathring{\mathbb{C}}_-$.

Step 1. Calculate $P(s)^{-1}$

Step 2. Choose the polynomials $n_1(s),\ldots n_m(s)$ and $d_1(s),\ldots, d_m(s)$ in

$$H_{y_2u_1}(s) := \mathrm{diag}[\frac{n_1(s)}{d_1(s)}, \ldots, \frac{n_m(s)}{d_m(s)}] \tag{4.1}$$

such that

$$Q(s) := P(s)^{-1}H_{y_2u_1}(s) \tag{4.3}$$

is exp. stable and strictly proper. More precisely, we choose the polynomials $n_1(s),\ldots,n_m(s)$ and $d_1(s),\ldots,d_m(s)$, such that, for $j = 1,\ldots,m$,

$$\text{(i)} \quad Z[d_j] \subset \mathring{\mathbb{C}}_- \tag{4.4}$$

$$\text{(ii)}^\dagger \quad \partial d_j > \partial n_j + \partial\gamma_j[P^{-1}] \tag{4.5}$$

Comments: Since $P[P^{-1}] = Z[P] \subset \mathring{\mathbb{C}}_-$, condition (4.4) guarantees that Q is exp. stable. Condition (4.5) guarantees that Q(s) is strictly proper.

Step 3. Calculate the required controller transfer function:

$$C(s) = P(s)^{-1}\mathrm{diag}[\frac{n_1(s)}{d_1(s)-n_1(s)}, \ldots, \frac{n_m(s)}{d_m(s)-n_m(s)}] \tag{4.6}$$

End of Algo. 4.1 □

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† For $M(s) \in \mathbb{R}(s)^{m \times m}$, $\gamma_j[M]$ denotes the jth column of M(s) and $\partial\gamma_j[M]$ denotes the largest degree difference between the numerator and the denominator among the m rational functions in $\gamma_j[M]$.

Remark 4.1: Eqn. (4.6) shows that, in principle, given any list of polynomials $(n_i(s))_1^m$, a stable controller is always possible by approximate choice of the polynomials $d_i(s)$, $1 \le i \le m$.

Example 4.1.

Consider

$$P(s) = \frac{1}{(s+2)^2(s+3)} \begin{bmatrix} s^2+8s+10 & 3s^2+7s+4 \\ 2s+2 & 3s^2+9s+8 \end{bmatrix} \in R_o(0)^{2x2}$$

with a right-coprime factorization given by

$$P(s) = N_{pr}(s)\, D_{pr}(s)^{-1} = \begin{bmatrix} s+4 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} s^2+3s+4 & 2 \\ 2 & s+4 \end{bmatrix}^{-1}$$

Since $Z[P] = Z[N_{pr}] = Z[\det\ N_{pr}] = \{-2\} \subset \mathring{\mathbb{C}}_-$, we can apply Algo. 4.1.

Now, from (4.3), calculate

$$Q(s) := P(s)^{-1} H_{y_2u_1}(s)$$

$$= \begin{bmatrix} \dfrac{(3s^2+9s+8)n_1(s)}{3(s+2)d_1(s)} & \dfrac{-(3s^2+7s+4)n_2(s)}{3(s+2)d_2(s)} \\ \dfrac{-2(s+1)n_1(s)}{3(s+2)d_1(s)} & \dfrac{(s^2+8s+10)n_2(s)}{3(s+2)d_2(s)} \end{bmatrix} \tag{4.7}$$

In order that $Q(s) \in R_o(0)^{2x2}$, we must have

$$\text{(i)} \quad Z[d_1] \subset \mathring{\mathbb{C}}_- \quad \text{and } Z[d_2] \subset \mathring{\mathbb{C}}_- \tag{4.8}$$

$$\text{(ii)} \quad \partial d_1 \ge \partial n_1 + 2 \text{ and } \partial d_2 \ge \partial n_2 + 2 \tag{4.9}$$

By choosing $n_1(s) = n_2(s) = 1$ and $d_1(s) = d_2(s) = (\frac{s}{\omega_n})^2 + \sqrt{2}\,(\frac{s}{\omega_n}) + 1$, with $\omega_n > 0$, the <u>decoupled</u> I/O map $H_{y_2u_1}(s)$ is given by

$$H_{y_2u_1}(s) = \frac{1}{(\frac{s}{\omega_n})^2 + \sqrt{2}\,(\frac{s}{\omega_n}) + 1}\, I \in R_o(0)^{2x2} \tag{4.10}$$

which has (i) zero asymptotic error for any step input, since $H_{y_2u_1}(s)\Big|_{s=0} = I$ (ii) corresponding bandwidth of ω_n rad/s and 5% settling time of $\frac{3\sqrt{2}}{\omega_n}$ s for each decoupled channel. From (4.5), the required controller is given by

$$C(s) = P(s)^{-1}\,\mathrm{diag}\left[\frac{1}{d_1(s)-1}\,,\,\frac{1}{d_2(s)-1}\right]$$

$$= \frac{\left(\frac{\omega_n}{6}\right)}{s\left(\frac{s}{2}+1\right)\left(\frac{s}{\sqrt{2}\,\omega_n}+1\right)} \begin{bmatrix} 3s^2+9s+8 & -(3s^2+7s+4) \\ -2(s+1) & s^2+8s+10 \end{bmatrix}$$

Note that the pole at $s = 0$ of $C(s)$ is a direct consequence of $H_{y_2u_1}(0) = I$. ◻

4.2. Design procedure for P(s) with $\mathbb{C}_+$-zeros

If the plant $P(s)$ has $\mathbb{C}_+$-zeros, then $P(s)^{-1}$ has a $\mathbb{C}_+$-pole at each $\mathbb{C}_+$-zero of $P(s)$. Then, the use of equ. (4.3) would lead to an unstable $Q(s)$ and hence unstable closed-loop system by Theorem 3.4. Hence, the only way to have $Q(s)$ exp. stable is to have $H_{y_2u_1}(s)$ cancell all the $\mathbb{C}_+$-poles of $P(s)^{-1}$.

Algorithm 4.2 ($P(s)$ has $\mathbb{C}_+$-zeros)

Data. $P(s) \in R_o(0)^{m\times m}$, $Z[P] \cap \mathbb{C}_+ \neq \phi$

Step 1. Obtain a right-coprime factorization of $P(s)$:

$$P(s) = N_{pr}(s)\,D_{pr}(s)^{-1}$$

where $N_{pr}(s),\, D_{pr}(s) \in \mathbb{R}[s]^{m\times m}$.

Step 2. Calculate $[\gamma_{ij}]_{m\times m} := N_{pr}(s)^{-1}$ (4.11)

Step 3. Choose the polynomials $n_{1+}(s),\ldots,n_{m+}(s)$, where, for each j, $n_{j+}(s) \in \mathbb{R}[s]$ is of least-degree such that for $i = 1,\ldots,m$,

$$\gamma_{ij}(s)\; n_{j+}(s) \text{ is analytic in } \mathbb{C}_+ \tag{4.13}$$

Comment: For each j, the polynomial $n_{j+}(s)$ must cancel all the $\mathbb{C}_+$- poles of all the m rational functions $\gamma_{1j}(s),\, \gamma_{2j}(s),\, \ldots,\, \gamma_{mj}(s)$ in the jth column of $N_{pr}(s)^{-1}$.

Step 4. Choose the polynomials $\tilde{n}_1(s),\ldots,\tilde{n}_m(s)$ and $d_1(s),\ldots,d_m(s)$ in

$$H_{y_2u_1}(s) := \text{diag}\,[\frac{n_{1+}(s)\tilde{n}_1(s)}{d_1(s)}, \ldots, \frac{n_{m+}(s)\tilde{n}_m(s)}{d_m(s)}] \tag{4.15}$$

such that for $j = 1,\ldots,m$,

(i) $Z[d_j] \subset \mathring{\mathbb{C}}_-$, (4.16)

(ii) the polynomial $\tilde{n}_j(s)$ is chosen freely,

(iii) $\partial d_j > \partial n_{j+} + \partial\tilde{n}_j + \partial\gamma_j[P^{-1}]$. (4.18)

Comments: From (4.3), conditions (4.13) and (4.16) guarantee that Q has no $\mathbb{C}_+$-poles; condition (4.18) guarantees that Q is strictly proper.

Step 5. Calculate the required controller transfer function: let

$$n_j(s) := n_{j+}\tilde{n}_j(s) \qquad j = 1,\ldots,m$$

then

$$C(s) = P(s)^{-1}\,\text{diag}[\frac{n_1(s)}{d_1(s)-n_1(s)},\ldots,\frac{n_m(s)}{d(s)-n_m(s)}] \tag{4.23}$$

End of Algo. 4.2 ¤

Remarks 4.2 (a) In Appendix C we show that, for $j = 1,\ldots,m$, the polynomial $n_{j+}(s)$ (in step 3) cancels all the $\mathbb{C}_+$-poles of the jth column of $N_{pr}(s)^{-1}$ if and only if $n_{j+}(s)$ cancels all the $\mathbb{C}_+$-poles of the jth column of $P(s)^{-1}$. Since calculating the inverse of a polynomial matrix is much easier than calculating that of a rational matrix, it is computationally more attractive to consider $N_{pr}(s)^{-1}$ rather than $P(s)^{-1}$.

(b) Equation (4.23) shows that a stable controller is always possible: indeed, after the polynomials $n_{j+}(s)$ and $\tilde{n}_j(s)$, $1 \leq j \leq m$, have been chosen polynomials $d_j(s)$'s can always be found so that, for $j = 1,\ldots,m$, the polynomial $d_j(s) - \tilde{n}_{j+}(s)\, n_j(s)$ is strictly Hurwitz. This conclusion agrees with Theorem 2 of [You. 2].

Example 4.2.

Consider

$$P(s) = \frac{1}{(s+2)^2(s+3)} \begin{bmatrix} 3s+8 & 2s^2+6s+2 \\ s^2+6s+2 & 3s^2+7s+8 \end{bmatrix} \in \mathcal{R}_o(s)^{2x2}$$

which has a right-coprime factorization:

$$P(s) = N_{pr}(s)\, D_{pr}(s)^{-1} = \begin{bmatrix} 3 & 2 \\ s+2 & 3 \end{bmatrix} \begin{bmatrix} s^2+3s+4 & 2 \\ 2 & s+4 \end{bmatrix}^{-1}$$

Since $Z[P] = Z[N_{pr}] = Z[\det N_{pr}] = \{2.5\} \subset \mathbb{C}_+$ we use Algo. 4.2. Now,

$$N_{pr}(s)^{-1} = \begin{bmatrix} \frac{-1.5}{s-2.5} & \frac{1}{s-2.5} \\ \frac{0.5(s+2)}{s-2.5} & \frac{-1.5}{s-2.5} \end{bmatrix}$$

We choose $n_{1+}(s) = n_{2+}(s) = s - 2.5$. Then, from (4.3).

$$Q(s) = -0.5 \begin{bmatrix} \frac{(3s^2+7s+8)\tilde{n}_1(s)}{d_1(s)} & \frac{-(2s^2+6s+2)\tilde{n}_2(s)}{d_2(s)} \\ \frac{-(s^2+6s+2)\tilde{n}_1(s)}{d_1(s)} & \frac{(3s+8)\tilde{n}_2(s)}{d_2(s)} \end{bmatrix}$$

To guarantee $Q(s) \in \mathcal{R}_o(0)^{2x2}$, we choose $\tilde{n}_1(s) = \tilde{n}_2(s) = 1$ and $d_1(s), d_2(s) \in \mathbb{R}[s]$ such that

(i) $\partial d_1 \geq 3$ and $\partial d_2 \geq 3$

(ii) $Z[d_1] \subset \Lambda_1 \subset \mathring{\mathbb{C}}_-$ and $Z[d_2] \subset \Lambda_2 \subset \mathring{\mathbb{C}}_-$, where Λ_1 and Λ_2 are regions in $\mathring{\mathbb{C}}_-$ of desirable closed-loop poles for channel 1 and channel 2, respectively.

(iii) $d_1(0) = d_2(0) = -2.5$

Then, the resulting decoupled I/O map is given by

$$H_{y_2u_1}(s) = \mathrm{diag}\left[\frac{s-2.5}{d_1(s)}, \frac{s-2.5}{d_2(s)}\right]$$

with $H_{y_2u_1}(s)\Big|_{s=0} = I$ (i.e,, with zero asymptotic error for any step input). From (4.23), the corresponding controller is given by

$$C(s) = \begin{bmatrix} \dfrac{-0.5(3s^2+7s+8)}{d_1(s)-(s-2.5)} & \dfrac{(s^2+3s+1)}{d_2(s)-(s-2.5)} \\[2ex] \dfrac{0.5(s^2+6s+2)}{d_1(s)-(s-2.5)} & \dfrac{-0.5(3s+8)}{d_2(s)-(s-2.5)} \end{bmatrix}$$

V. Basic Design Limitations on the Design

The algorithms above suggest that, using unity-feedback around a stable plant, we can always obtain a decoupled I/O map that satisfies any given design specifications (within the constraints imposed by the $\mathbb{C}_+$-zeros of the plant)! As everyone knows, practical considerations impose limitations on the "achievable benefits" of feedback, see e.g. [Hor. 1]. We emphasize here four sources of limitations:† Plant dynamics, Saturation, Noise and Uncertainty.

†These four limitations are the results of many discussions with many people, J. C. Lozier and G. Zames, in particular.

5.1. Plant Dynamics

(i) Any plant has some intrinsic dominant time constants; these may be in the milliseconds, seconds, minutes or hours range. This, together with saturation, imposes a time scale on the achievable I/O maps of the closed-loop system.

(ii) A plant sometimes exhibits a zero of transmission on the $j\omega$-axis or in the open right half-plane. Again, because of saturation, a zero close to the $j\omega$-axis imposes limitations on the achievable loop gain about that frequency. A right half-plane zero attracts, under increasing loop gains, closed-loop eigenvalues; hence, stability requirements impose a limitation on the achievable loop gain, hence desensitization.

5.2. Saturation

The linear model of the plant remains valid provided that the plant input-signals remain below the saturation level; otherwise, the linear model is no longer applicable. As an illustration of how saturation could occur in a linearly designed feedback system, we consider a s.i.s.o. system with plant given by $p(s) = \frac{1}{s(s/2+1)}$. A compensator $c(s) = \frac{12(1+s/4.7)}{1+s/85.3}$ is used to achieve required performance specifications of a) velocity error constant $> 5\ \text{sec.}^{-1}$, b) phase margin $\geq 45°$, c) $1.2 \leq$ maximum magnitude of closed-loop frequency response ≤ 1.5, and d) 25 rad/sec $<$ bandwidth $<$ 60 rad/sec. [Sau. 1, P. 492]. Then, even though the closed-loop system may behave reasonably well with an error signal of certain magnitude at $\omega = 0.5$ rad/s, any error signal at $\omega = 30$ rad/s with the same magnitude will very likely saturate the plant because $|c(j30)| \approx 6.1\ |c(j0.5)|$. Note also that, in Example 4.1 with bandwidth

$\omega_n = 15$ rad/s, then $|H_{y_1u_1}(j15)| = |Q_{11}(j15)| \approx 11|Q_{11}(j0)| = |H_{y_1u_1}(j0)|$; hence, inputs of frequency in the neighborhood of 15 rad/s must be kept small to avoid plant saturation. Note that the saturation level always specifies the <u>largest</u> signal possible at the <u>output</u> of the plant.

5.3. <u>Noise</u>

To actually control the plant, we need <u>actuators</u> to drive the input and <u>sensors</u> to measure the output; hence, in order to effectively control the plant, we must make sure that the noise sources associated with actuators and sensors do not swamp the input signals and the measurements, respectively.

5.4. <u>Uncertainty</u>†

For design purposes, it is convenient to distinguish two types of uncertainty which deteriorate the system performance when the feedback system is designed on the basis of a nominal plant:

(i) the <u>modeling uncertainty</u> caused by approximations in the modeling of the dynamics: e.g., by linearizing nonlinear dynamics, by neglecting high-frequency modes, delays, small interactions, etc.

(ii) the <u>parameter uncertainty</u> (i.e., variation of the plant parameters) caused by manufacturing tolerances, loading, aging, etc.

5.5. <u>A more realistic design problem formulation</u>

With the above considerations in mind, we suggest that one way to bring some realism in the design process is as follows:

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†For a general discussion of this point see [Zam. 1].

Let us consider two noise inputs: one associated with the system input u_1 and one associated with the measurement of y_2. These two noises enter the system at the summing point. Since $Q(j\omega) = H_{y_1 u_1}(j\omega)$ with $u_2 \equiv 0$), if, over the band of interest,† $\|Q(j\omega)\|_2 = \sigma_{max}[(j\omega)]$ is too large, then these noises will saturate the plant. Thus the consideration of noises and the plant saturation forces us to put a bound on

$$\sigma_{max}[Q(j\omega)], \quad \text{over the band of interest} \tag{5.1}$$

More precisely, suppose that the designer chooses a bound L constant over the band, then the choice of Q in algorithms 4.1 and 4.2 is subject to the additional constraint:

$$\sigma_{max}[Q(j\omega)] \leq L \qquad \forall\omega \in [0,\omega_b] \tag{5.2}$$

where ω_b is the highest frequency in the band of interest.

As an illustration of this approach, in Example 4.1, the chosen Q(s) (implicit in (4.10)) is parametrized by ω_n and hence the problem is to choose an "optimal" ω_n. This can be done by nonlinear programming using Q(s) specified by (4.7) and subject to inequality (5.2) with, say, $\omega_b \geq 1.5\ \omega_n$. In view of the smooth nature of Q(s) in (4.7), the infinite set of inequalities (5.2) can, for design purposes, be replaced, say, by a dozen inequalities by choosing frequencies equally spaced over $[0,\omega_b]$. This problem is being investigated at present.

It is our conviction that algebraic methods of design as described in Sec. 4 make sense only when they are incorporated with inequality constraints such as (5.2).

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† $\sigma_{max}[M]$, for $M \in \mathbb{C}^{m\times m}$, denotes the largest singular value of M. (See e.g. [Ste. 1]).

VI. Conclusions

The thrusts of this paper are the following:

I) Within a general algebraic framework, the closed-loop transfer functions H_{eu} and H_{yu} can be given a simple parametric form in terms of P and Q (see (3.11) and (3.12)). II) Based on [Des. 3], [Zam. 1] and [Des. 1], we proved that: if $P \in \mathbb{B}_s$, then $Q \in \mathbb{B}_s \Leftrightarrow H_{yu} \in \mathbb{B}_s$ and $C \in \mathbb{A}_s$. Hence the problem of guaranteeing the closed-loop stability and the strict properness of the controller C is automatically solved by taking $Q \in \mathbb{B}_s$; furthermore, nothing is lost by doing so! III) For design purposes, the important observation is that $H_{y_2u_1} = PQ$, where $P \in \mathbb{B}_s$ is given and if Q is chosen in $\mathbb{B}_s$, stability is guaranteed! IV) Using coprime factorizations, for any given exp. stable P, our algorithms construct a strictly proper controller that results in a decoupled I/O map $H_{y_2u_1}$ in each channel of which we can prescribe the poles and also the zeros (of course in addition to the $\mathbb{C}_+$-zeros required by those of P.) Note that in any case, we can always choose the poles so that the resulting controller is exp. stable. This contribution should be viewed as an extension of [Pec. 1]. V) The results above must be tempered by the realization that these algebraic results must face the limitations imposed by system noise and by saturation.

VII. Acknowledgement

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Appendix A. Algebraic Manipulations in Rings

Let $\mathcal{R}$ denote any non-commutative ring with identity I, (see e.g. [Sig. 1], [MacL. 1]). Let $\tilde{\mathcal{R}}$ be a super-ring of $\mathcal{R}$ (i.e., $\mathcal{R}$ is a subring of $\tilde{\mathcal{R}}$). Let $\mathcal{R}_s$ denote the radical of $\mathcal{R}$ [Nai. 1], i.e., by definition, $\mathcal{R}_s \subset \mathcal{R}$ and $M \in \mathcal{R}_s$ iff $\forall N \in \mathcal{R}$

$$\text{(i)} \quad MN \in \mathcal{R}_s, \; NM \in \mathcal{R}_s \tag{A.1}$$

and

$$\text{(ii)} \quad (I+MN)^{-1} \in \mathcal{R}, \; (I+NM)^{-1} \in \mathcal{R}. \tag{A.2}$$

It is easy to see that the radical $\mathcal{R}_s$ is a subring of $\mathcal{R}$. Note that

$\tilde{\mathcal{R}} \supset \mathcal{R} \supset \mathcal{R}_s$

The reader may want to keep a specific example in mind: take $\tilde{\mathcal{R}} = \mathbb{R}(s)^{m\times m}$, $\mathcal{R} = \mathbb{R}_p(s)^{m\times m}$, $\mathcal{R}_s = \{P \in \mathbb{R}_p(s)^{m\times m}$ and strictly proper$\}$.

Lemma A.1.

Let $M \in \mathcal{R}$ and $(I+M)^{-1} \in \mathcal{R}$, then

$$(I+M)^{-1} = I - M(I+M)^{-1} \tag{A.4}$$

$$= I - (I+M)^{-1}M \tag{A.5}$$

$\square$

Proof: The right-hand sides of (A.4) and (A.5) are respectively equal to

$$[(I+M) - M](I+M)^{-1} = (I+M)^{-1} \quad \text{(by left distributivity in } \mathcal{R})^{\dagger}$$

and

$$(I+M)^{-1}[(I+M) - M] = (I+M)^{-1} \quad \text{(by right distributivity in } \mathcal{R})$$

Remark A.1. (Nonlinear case): (A.4) holds for nonlinear M (since only the left-distributive law has been used); (A.5), however, does not.

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† By "left distributivity" we mean that the multiplication is distributive to the left [Bou. 1].

Lemma A.2.

Let $P, C \in \mathcal{R}$.

(i) $(I+PC)^{-1} \in \mathcal{R} \Leftrightarrow (I+CP)^{-1} \in \mathcal{R}$ (A.7)

and

$$(I+CP)^{-1} = I - C(I+PC)^{-1}P \tag{A.8}$$

(ii) If $(I+PC)^{-1} \in \mathcal{R}$ (or equivalently, by (i), $(I+CP)^{-1} \in \mathcal{R}$),

then

$$P(I+CP)^{-1} = (I+PC)^{-1}P \in \mathcal{R} \tag{A.9}$$

$\square$

Proof:

(i) $\Rightarrow$ By assumption, $(I+PC)^{-1} \in \mathcal{R}$. We claim that $(I+CP)^{-1}$ is given by (A.8). Using repeatedly the distributive law in $\mathcal{R}$, we obtain

$$\begin{aligned}(I+CP)\,[I - C(I+PC)^{-1}P] &= I+CP - (I+CP)C\cdot(I+PC)^{-1}P \\ &= I+CP - C(I+PC)\cdot(I+PC)^{-1}P \\ &= I \\ [I - C(I+PC)^{-1}P]\,(I+CP) &= I+CP - C(I+PC)^{-1}\cdot P(I+CP) \\ &= I+CP - C(I+PC)^{-1}(I+PC)P \\ &= I\end{aligned}$$

Hence, $(I+CP)^{-1} = I - C(I+PC)^{-1}P \in \mathcal{R}$

$\Leftarrow$ Repeat the calculation with P and C interchanged.

(ii) Using left and right distributivity in $\mathcal{R}$, we obtain

$$(I+PC)P = P+PCP = P(I+CP); \tag{A.10}$$

then by pre- and post-multiplying (A.10) by $(I+PC)^{-1}$ and $(I+CP)^{-1}$ respectively, (A.9) follows. $\square$

Remark A.2: When P is nonlinear, right distributivity does not hold and hence (A.7), (A.8) and (A.9) are not true.

Lemma A.3.

Let

$$P, C \in \mathcal{R} \quad \text{and} \quad (I+PC)^{-1} \in \mathcal{R} \tag{A.21}$$

Let

$$Q := C(I+PC)^{-1} \tag{A.22}$$

(hence, $Q \in \mathcal{R}$ by (A.21)). U.t.c., the following equalities hold and all expressions are in $\mathcal{R}$:

(i) $$(I+PC)^{-1} = I - PQ; \tag{A.23}$$

equivalently,

$$I+PC = (I-PQ)^{-1}. \tag{A.24}$$

(ii) $$(I+CP)^{-1} = I - QP; \tag{A.25}$$

equivalently,

$$I+CP = (I-QP)^{-1}. \tag{A.26}$$

(iii) $$C = Q(I-PQ)^{-1} \tag{A.27}$$

¤

<u>Comment A.3</u>:

The equivalence stated in (i) and (ii) are immediate by inverting both sides and noting that P, C, Q and $I \in \mathcal{R}$ imply that $I+PC$, $I+CP$, $I-PQ$ and $I-QP \in \mathcal{R}$.

<u>Proof</u>:

(i) $(I+PC)^{-1} = I - PC(I+PC)^{-1}$ (by (A.4))

$= I - PQ$ (by (A.22))

(ii) $(I+CP)^{-1} = I - C(I+PC)^{-1}P$ (by (A.8))

$= I - QP$ (by (A.22))

(iii) $C = C(I+PC)^{-1}\,(I+PC)$

$= Q(I-PQ)^{-1}$ (by (A.22) and (A.24)).

¤

<u>Remark</u>: Note that, in Lemmas A.1, A.2 and A.3, the ring $\mathcal{R}$ is arbitrary; in applications, $\mathcal{R}$ is chosen to suit the needs. ¤

Lemma A.4.

Let $P \in \mathcal{R}$ and $C \in \tilde{\mathcal{R}}$

Let

$$H_{yu} := \begin{bmatrix} C(I+PC)^{-1} & -CP(I+CP)^{-1} \\ PC(I+PC)^{-1} & P(I+CP)^{-1} \end{bmatrix} \in \mathcal{R}^{2x2} \tag{A.32}$$

Let

$$Q := C(I+PC)^{-1} \tag{A.33}$$

(hence $Q \in \mathcal{R}$ by (A.32)). U.t.c.,

$$H_{yu} := \begin{bmatrix} Q & -QP \\ PQ & P(I-QP) \end{bmatrix} \in \mathcal{R}^{2x2} \tag{A.34}$$

¤

Proof:

By inspection, $H_{y_1u_1} = Q$ and $H_{y_2u_1} = PQ$. Since $H_{y_2u_1}$ and $H_{y_1u_2}$ are in $\mathcal{R}$ by (A.32), using (A.4), we conclude that $(I+PC)^{-1} \in \mathcal{R} \subset \tilde{\mathcal{R}}$ and $(I+CP)^{-1} \in \mathcal{R} \subset \tilde{\mathcal{R}}$; now, performing calculations in the ring $\tilde{\mathcal{R}} \supset \mathcal{R}$, we obtain

$$\begin{aligned} H_{y_2u_1} &= -CP(I+CP)^{-1} = -C(I+PC)^{-1}P && \text{(by (A.9))} \\ &= -QP \\ H_{y_2u_2} &= P(I+CP)^{-1} = P(I-QP) && \text{(by (A.25))} \end{aligned}$$

Hence, formula (A.34) is established. ¤

Appendix B: The Radical of $\hat{B}(0)$.

By definition [Cal. 1], $\hat{B}(0)$ consists of all transfer function $\hat{f}$ of the form $\hat{f} = \hat{n}/\hat{d}$, where $\hat{n} \in \hat{A}_-(0)$, $\hat{d} \in \hat{A}_-^{\infty}(0)$. By definition, $\hat{B}_o(0)$ consists of all elements $\hat{g} \in \hat{B}(0)$ such that $\hat{g}(s) \to 0$ as $|s| \to \infty$ in C_+, in the precise sense described as follows: $\forall \varepsilon > 0$, $\exists \rho > 0$ such that

$$|s| > \rho \text{ and } \mathrm{Re}\, s \geq 0 \text{ imply that } |\hat{g}(s)| < \varepsilon. \tag{B.1}$$

Lemma: $\hat{B}_o(0)$ is the radical of $\hat{B}(0)$.

Proof. a) $\forall \hat{f} \in \hat{B}(0)$ and $\forall \hat{g} \in \hat{B}_o(0)$, we have $\hat{f}(s)\hat{g}(s) \to 0$ in the sense of (B.1): indeed, for all sufficiently large ρ, there is an $M < \infty$ such that $|\hat{f}(s)| \leq M$, $\forall s \in \mathbb{C}_+$ with $|s| \geq \rho$; hence the conclusion follows since $\hat{g} \in \hat{B}_o(0)$.

b) We claim that $\forall \hat{f} \in \hat{B}(0)$ and $\forall \hat{g} \in \hat{B}_o(0)$, $[1 + (\hat{f}\hat{g})(s)]^{-1} \in \hat{B}(0)$. Using (A.1), and the boundedness at infinity of $\hat{f}$, we see that for ρ sufficiently large

$$2 > |1 + (\hat{f}\hat{g})(s)| > 1/2 \qquad \forall s \in \mathbb{C}_+ \text{ with } |s| > \rho.$$

Hence, by [Cal. 1, Thm 3.7], the inverse of $1+\hat{f}\hat{g}$ is in $\hat{B}(0)$.

c) If $\hat{g} \in \hat{B}(0)$, but $\hat{g} \notin \hat{B}_o(0)$, then $\omega \mapsto \hat{g}(j\omega)$ is asymptotically, for $|\omega| \to \infty$, almost periodic. Then it is easy to choose an $\hat{f} \in \hat{B}(0)$ such that $[1 + (fg)(s)]^{-1}$ has an infinite number of $\mathbb{C}_+$-poles, hence is not in $\hat{B}(0)$.

Appendix C.

Theorem C.1. Let $P(s) = N_{pr}(s)D_{pr}(s)^{-1}$ be a right coprime factorization, then for $j = 1,\ldots,m$, the jth column of $P(s)^{-1}$ has a pole p with highest order ℓ_j if and only if the jth column of $N_{pr}(s)^{-1}$ has a pole p with highest order ℓ_j.

Proof: This theorem is easy to prove using the Smith-McMillan form. For our purpose here, we need only consider $p \in \mathbb{C}_+$. Furthermore, we have $\det D_{pr}(s) \neq 0$ in $\mathbb{C}_+$ by stability of $P(s)$. Consider the Laurent expansion of the jth column of $N_{pr}(s)^{-1}$; call ξ_j the $\mathbb{C}^m$-vector made up of the coefficients of the term in $(s-p)^{-\ell_j}$. Since $P(s)^{-1} = D_{pr}(s)N_{pr}(s)^{-1}$ and since $D_{pr}(s)$ is analytic in $\mathbb{C}$ and nonsingular in $\mathbb{C}_+$, the corresponding $\mathbb{C}^m$-vector made up of the coefficients of the term in $(s-p)^{-\ell_j}$ in $P(s)^{-1}$ is

$$\eta_j := D_{pr}(p)\ \xi_j$$

Since $\det[D_{pr}(p)] \neq 0$, $\xi_j \neq \theta_m$ if and only if $\eta_j \neq \theta_m$. ◻

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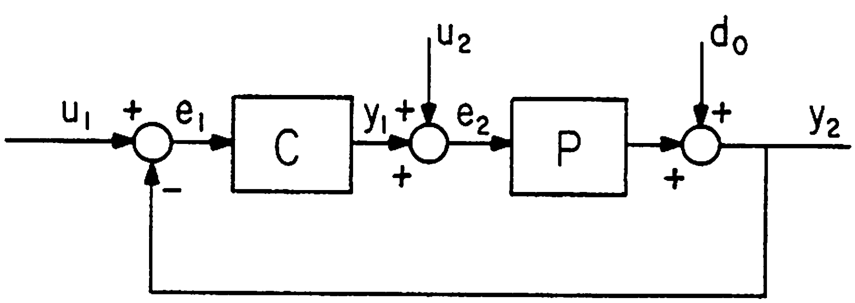
Table I: Examples of $\tilde{\mathbb{A}}$, $\mathbb{A}$, $\mathbb{A}_s$, $\mathbb{B}$ and $\mathbb{B}_s$				
$\tilde{\mathbb{A}}$	$\mathbb{C}(s)$	$\mathbb{C}(s)^{m\times m}$		
$\mathbb{A}$	$\mathbb{C}_p(s)$	$\mathbb{C}_p(s)^{m\times m}$	$\hat{\mathcal{B}}(0)$	$\hat{\mathcal{B}}(0)^{m\times m}$
$\mathbb{A}_s$	$\mathbb{C}_{p,o}(s)$	$\mathbb{C}_{p,o}(s)^{m\times m}$	$\hat{\mathcal{B}}_o(0)$	$\hat{\mathcal{B}}_o(0)^{m\times m}$
$\mathbb{B}$	$\mathcal{R}(0)$	$\mathcal{R}(0)^{m\times m}$	$\hat{\mathcal{A}}_-(0)$	$\hat{\mathcal{A}}(0)^{m\times m}$
$\mathbb{B}_s$	$\mathcal{R}_o(0)$	$\mathcal{R}_o(0)^{m\times m}$	$\hat{\mathcal{A}}_{o-}(0)$	$\hat{\mathcal{A}}_{o-}(0)^{m\times m}$

Figure Caption

Fig. 1. The unity-feedback system (P,Q).



u1
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e1
C
y1
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u2
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P
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