IDENTIFICATION OF GROUPS OF ε-COHERENT GENERATORS

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ABSTRACT

It is observed that for a power system after a disturbance certain groups of generators have similar waveforms for their rotor-angle response curves. We define generators to be coherent if the waveforms of the rotor-angle curves are identical. In practice, however, they may be very close but not identical. We say that generators are ε-coherent if the maximum of the difference of their angle curves is less than ε. We present some necessary and sufficient conditions for a group of generators to be ε-coherent under a set of disturbances. The condition is expressed in terms of the controllability Grammian of the model. Singular-value decomposition of the Grammian is used to provide insights to the results. We also derive an approximate expression for the Grammian and suggest a practical algorithm for identifying groups of ε-coherent generators.

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I. INTRODUCTION

It has been observed that for a power system after a disturbance certain groups of generators have similar waveforms for their rotor-angle response curves. This phenomenon, called coherency of generators, has been utilized effectively to construct a reduced-order model of the external system, known as dynamic equivalents, for power system transient stability study [1]. For generators to form a coherent group it depends on the type and location of the disturbance. The conventional approach to identify coherent groups of generators is examining the response curves obtained from numerical integration of a set of simplified linearized differential equations of the system [1]. Engineering judgement is required to select a disturbance so that coherency will prevail for other disturbances.

In a previous paper [2] we have studied the phenomenon of coherency analytically via the mathematical model of the system. For analytical studies we define coherency to mean the angle curves of the generators are identical. We have characterized coherency in terms of the controllability subspace of the system model. We have also proposed an algorithm to identify coherency directly from the parameters of the model.

In practice the generator rotor-angle curves may be very close but not identical. We call such cases ε-coherency. To be more precise, we say two generators are ε-coherent if the maximum of the difference of their rotor-angle waveforms (ignoring the linear shift of the whole curve due to different initial rotor-angles) is less than ε.

Avramovic et. al. [4] have suggested a "slow coherency" to mean that the slow modes in the difference of the angle curves are small.

Our definition of ε-coherency is direct. In this paper we present some necessary and sufficient conditions for a group of generators to be ε-coherent under a set of disturbances. The conditions are expressed in terms of the controllability Grammian of the model. Singular value decomposition of the Grammian is used to gain insights to the results. The conditions may be used to identify which sets of generators will form ε-coherent groups. Our definition of ε-coherency is the same as Sastry and Varaiya [3]. Our approach is more constructive.

The Grammian can be obtained by the solution of a linear differential equation. We show that because of the properties of our model, for ε-coherency identification we can use the steady-state solution of the differential equation as an approximate expression for the Grammian. The steady-state solution involves a Liapunov equation for which efficient solution algorithm exists. Based on these results we suggest a practical algorithm for identifying groups of ε-coherent generators by (1) solving a set of special linear algebraic equations and (2) identifying ε-coherency from the elements of the solution.

II. MODELING

1. Power System Model

The same models for generators and network as in [2] are used here. Three types of disturbances are considered, namely, load shedding, generator dropping, and line switching. The modeling of load shedding and generator dropping remains the same as before. However a new procedure of modeling line switching is introduced, which simplifies the analysis considerably, as we shall see shortly.

The linearized swing equations for the generators are used:

$$M\,\Delta\dot{\underline{\omega}} = \Delta\underline{PM} - \Delta\underline{PG} - D\,\Delta\underline{\omega}$$
$$\Delta\dot{\underline{\delta}} = \Delta\underline{\omega} \qquad (1)$$

where Δ = deviation

$\underline{\omega} = (\omega_1,\ldots,\omega_n)$ of rotor speeds

$\underline{\delta} = (\delta_1,\ldots,\delta_n)$ of rotor angles

$M = \mathrm{diag}(M_1,\ldots,M_n)$ of machine inertia constants

$\underline{PM} = (PM_1,\ldots,PM_n)$ of mechanical power inputs

$\underline{PG} = (PG,\ldots,PG_n)$ of electrical power outputs

$D = \mathrm{diag}(D_1,\ldots,D_n)$ of machine damping constants

We assume that during the time of interest $\Delta\,\underline{PM} = 0$

The linearized decoupled real power flow equations are used for the network:

$$\begin{bmatrix} \Delta PG \\ \Delta PL \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} H_{gg} & H_{g\ell} \\ H_{\ell g} & H_{\ell\ell} \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\theta \end{bmatrix} \qquad (2)$$

where

PG: vector of real power injections at generator internal buses

PL: vector of real power injections at load buses

δ: vector of phase angles at generator internal buses

= generator rotor angles

θ: vector of phase angles at lead buses

H: matrix of partial derivatives

For load shedding at bus i, we model it as changes in the real power injection,

$$\Delta PL = (0,\ldots 1,\ldots 0)^T\, P_i \qquad (3)$$

where 1 occurs in the $\underline{i}$th position and P_i is the amount of power disconnected. For generator dropping, we model it as load shedding at the load bus to which the generator is connected.

We are going to model line switching simply as load changes. Suppose that for the system after removing the line connecting bus i and bus j (Fig. 1b), $\Delta\underline{\theta}^*(t)$ is the vector of bus angles deviation at time t. Let

$$\Delta P_i(\Delta\underline{\theta}^*(t)) := (H_{\ell\ell})_{ij}(\Delta\theta_i^*(t)-\Delta\theta_j^*(t)) \tag{4}$$

Now consider the system with line ij connected and the power injections at bus i and bus j changed by $\Delta P_i(\Delta\underline{\theta}^*(t))$ and $-\Delta P_i(\Delta\underline{\theta}^*(t))$, respectively (Fig. 1c). Clearly we get exactly the same set of equations for this system as the one before without line ij. So we model line switching as changes in load at buses i and j.

$$\Delta PL = (0,\ldots 1,\ldots -1,\ldots 0)^T \; \Delta P_i(\Delta\underline{\theta}^*(t)) \tag{5}$$

where 1 and -1 occur at position i and j, respectively. Note that P_i, being a function of $\Delta\underline{\theta}^*(t)$, is not known a priori. However, as we have shown previously [2] and will see again shortly, the exact waveform of $\Delta P_i(\cdot)$ does not enter into our analytic characterization of coherency.

In general, a set of k disturbances can be modeled as

$$\Delta PL = E\,\underline{u}(t) \tag{6}$$

where $E = [e_1 \vdots e_2 \vdots \cdots e_k]$, $e_j = (0,\ldots 1,\ldots 0)^T$ and $u_j(t) = P_j$ if jth disturbance is a load shedding or generator dropping,
$e_j = (0,\ldots 1,\ldots -1,\ldots 0)^T$ and $u_j(t) = \Delta P_i(\Delta\underline{\theta}^*(t))$ if jth disturbance is a line switching. Note that for any time interval τ of interest, we have

$$\int_0^{\tau} \|\underline{u}(t)\|^2 dt \leq K^2 \tag{7}$$

for some $K \geq 0$.

Combining (1)(2) and (6), we obtain a standard linear system model

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}, \quad \underline{x}(0) = \underline{0} \tag{8}$$

$$\text{where } \dot{\underline{x}} = \begin{bmatrix} \Delta\underline{\omega} \\ \Delta\underline{\delta} \end{bmatrix} \in \mathbb{R}^{2n} \tag{9}$$

$$A \triangleq \begin{bmatrix} -M^{-1}D & -M^{-1}(H_{gg}-H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g}) \\ I & 0 \end{bmatrix} \tag{10}$$

$$B \triangleq \begin{bmatrix} -M^{-1}H_{g\ell}H_{\ell\ell}^{-1}E \\ 0 \end{bmatrix} \tag{11}$$

2. Coherency and ε-coherency

The idea of coherent generators arises from the observed phenomenon that after a disturbance the rotor-angle curves of some generators have almost identical waveform. We therefore define coherency as follows. Two generators 'i' and 'j' are said to be coherent if $\delta_i(t) - \delta_j(t)$ = a constant for $t \geq 0$, or equivalently, $\Delta\delta_i(t) - \Delta\delta_j(t) = 0$ for $t \geq 0$. If the waveforms of two generator rotor-angle curves are "close" but not identical in a time interval $[0,\tau]$, we would like to call it "almost" coherent.‡ To be more precise, we say generators 'i' and 'j' are ε-coherent in the interval $[0,\tau]$ if

‡Avramovic et. al. [4] define slow-coherency as follows: generator i and j are said to be slow-coherent iff slow modes of $\Delta\delta_i(t) - \Delta\delta_j(t)$ are small. Our definition in (12) is the same as Sastry and Varaiya [3].

$$\max_{t\in[0,\tau]} |\Delta\delta_i(t) - \Delta\delta_j(t)| \leq \varepsilon \tag{12}$$

III. GRAMMIAN

Our criterion for testing ε-coherency will be given (Theorem 1) in terms of the singular value decomposition of the reachability Grammian. In this section we introduce the relevant concepts.

1. Grammian and Reachability Set.

Consider the linear time-invariant system

$$\underline{\dot{x}} = A\underline{x} + B\underline{u} \qquad \underline{x}(0) = \underline{0} \tag{13}$$

where the admissible input $\underline{u}$ satisfying the constraint

$$\int_0^\tau \|\underline{u}(t)\|^2 dt \leq K^2 \tag{14}$$

The <u>reachability Grammian</u> at τ is defined to be the matrix W_τ^2,

$$W_\tau^2 := \int_0^\tau e^{At}BB^Te^{A^Tt}\, dt \tag{15}$$

Note that W_τ^2 is real, symmetric and positive semidefinite. Therefore we have

$$W_\tau^2 = U\Sigma^2U^T \tag{16}$$

where $\Sigma^2 = \mathrm{diag}(\sigma_1^2,\sigma_2^2,\ldots\sigma_r^2,0,\ldots0)$, $\sigma_1^2 \geq \sigma_2^2 \ldots \geq \sigma_r^2 > 0$, are the real eignevalues of W_τ^2 and columns of U are the corresponding eigenvectors. We define

$$W_\tau := U\Sigma U^T \tag{17}$$

where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots \sigma_r, 0, \ldots 0)$, $\sigma_1 \geq \sigma_2 \ldots \geq \sigma_r > 0$.

The set of reachable states at τ is given by

$$S_\tau = \{\underline{x} | \underline{x} = \int_0^\tau e^{A(\tau-t)} B\underline{u}(t)\, dt,\ \int_0^\tau \|\underline{u}(t)\|^2 dt \leq K^2\} \tag{18}$$

and the set of reachable states in $[0,\tau]$ is given by

$$S_{[0,\tau]} = \{\underline{x} | \exists t \in [0,\tau] \text{ s.t. } \underline{x} = \int_0^t e^{A(t-t')} B\underline{u}(t')dt' \text{ and } \int_0^\tau \|\underline{u}(t)\|^2 dt \leq K^2\} \tag{19}$$

It turns out that these two sets are identical, i.e. $S_\tau = S_{[0,\tau]}$. Moreover, they are identical to the image S under the map W_τ of the ball with radius K,

$$S := \{\underline{x} | \underline{x} = W_\tau \underline{p},\ \|\underline{p}\| \leq K\} \tag{20}$$

This provides a very nice characterization of the reachability set. We state the foregoing as a fact.

<u>Fact 1</u> $\quad S_{[0,\tau]} = S_\tau = S.$

2. Singular Value Decomposition of the Grammian.

Consider the set

$$S = \{\underline{x} | \underline{x} = W_\tau \underline{p},\ \|\underline{p}\| \leq K\} \tag{21}$$

where $W_\tau = U\Sigma U^T$ and $\Sigma = \mathrm{diag}(\sigma_1, \ldots \sigma_r, 0 \ldots 0)$ (22)

If we change coordinates to a basis formed by the columns of U, i.e.,

$\underline{x} = U\underline{x}'$, $\underline{p} = U\underline{p}'$, then the set S can be described as follows:

$$S = \{\underline{x}' \mid (\frac{x_1'}{\sigma_1})^2 + (\frac{x_2'}{\sigma_2})^2 + \ldots + (\frac{x_r'}{\sigma_r}) \leq K^2,\ x_{r+1}' = x_{2n}' = 0\} \tag{23}$$

Thus S is an r-dimensional elliposoid in $\mathbb{R}^{2n}$ whose axes are the eigenvectors corresponding to the nonzero eigenvalues of W_τ and the length of the ith axis is σ_i.

Remark: In this particular case where W_τ is real, symmetric and positive semidefinite, the eigenvalues and the eigenvectors of W_τ coincide with the singular values and the singular vectors of W_τ, and $W_\tau = U\Sigma U^T$ is also the singular value decomposition (SVD) of W_τ. The readers are reminded of the many desirable numerical properties enjoyed by the SVD [5-8].

3. Differential Equation of the Grammian

The reachability Grammian W_τ^2 can be obtained from the solution of a linear matrix differential equation as stated in the following fact, whose proof is immediate by the definition of W_τ^2.

Fact 2 [9, pp. 84] The reachability Grammina W_t^2 satisfies the following linear matrix differential equation

$$\dot{X} = AX + XA^T + BB^T,\ X(0) = 0 \tag{24}$$

IV ε-COHERENCY IDENTIFICATION

1. Criterion for ε-coherency identification.

Theorem 1. Consider the power system model (8) generators 'i' and 'j' are ε-coherent in $[0,\tau]$ for a set of disturbances satisfying

$\int_0^\tau \|\underline{u}(t)\|^2\, dt \leq K^2$, if and only if

$$(C1) \sum_{m=1}^{r} \sigma_m^2 (U_{im} - U_{jm})^2 \leq \frac{\varepsilon^2}{K^2} \tag{25}$$

or equivalently

$$(C2) \overline{W}_{ii} + \overline{W}_{jj} - 2\,\overline{W}_{ij} \leq \frac{\varepsilon^2}{K^2} \tag{26}$$

where $i = \overline{i} + n$, $j = \overline{j} + n$, $W_\tau = U\Sigma U^T$, $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots \sigma_r, 0 \ldots 0)$, and U_{im}, $\overline{W}_{im}$ are the $\underline{im}$th elements of U, W_τ^2 respectively.

Consider the column vectors $\underline{u}_{\cdot m}$ of U,

$$U = [\underline{u}_{\cdot 1} \vdots \underline{u}_{\cdot 2} \vdots \cdots \underline{u}_{\cdot 2n}]$$

Let us weight each column vector $\underline{u}_{\cdot m}$ by its corresponding singular value σ_m,

$$U\Sigma = [\sigma_1 \underline{u}_{\cdot 1} \vdots \cdots \sigma_r \underline{u}_{\cdot r} \vdots \underline{0} \ldots \underline{0}] \tag{27}$$

Recall that the orthonormal vectors $\underline{u}_{\cdot 1}, \ldots, \underline{u}_{\cdot r}$ are the directions of the axes of the elliposid S, which is the set of reachable states, and the singular values $\sigma_1, \ldots \sigma_r$ are the lengths of the corresponding axes. Our condition (25) asserts that if there are identical rows in the matrix

$$[\sigma_1 \underline{u}_{\cdot 1} \vdots \ldots \sigma_r \underline{u}_{\cdot r}] \tag{28}$$

then the corresponding generators are coherent; and if there are rows in the matrix (27) that are almost identical (in the sense of the Euclidean distance between the row vectors), then the corresponding generators are ε-coherent.

2. Procedure for ε-coherency Identification

The following procedure may be used for identifying ε-coherent generators. However Step 2 involves the solution of a set of linear differential equations.

1. Determine ε, τ. Estimate K.

2. Solve

$$\dot{X} = AX + XA^T + BB^T,\ X(0) = 0$$

and set

$$W_\tau^2 = X(\tau)$$

3. Perform singular value decomposition of W_τ^2 and obtain

$$W_\tau = U\Sigma U^T$$

4. Identify groups of ε-coherent generators by the sets of "almost identical" rows (condition (C1)) of the matrix $[\sigma_1 \underline{u}_{\cdot 1} \vdots \ldots \sigma_r \underline{u}_{\cdot r}]$

We may also identify ε-coherent generators directly from W_τ^2 using (C2). Thus, Steps 3 and 4 above may be replaced by the following Step 3'.

3'. Identify groups of ε-coherent generators by checking condition (C2), i.e., the set of generators H will be ε-coherent if for any i,j in H, we have

$$\bar{W}_{ii} + \bar{W}_{jj} - 2\bar{W}_{ij} \leq \frac{\varepsilon^2}{K^2}.$$

Remark: Condition (C1) provides more insights to the result and it is similar to our previous results [2]. Condition (C2) may be more computationally advantageous.

V. APPROXIMATE METHOD FOR ε-COHERENCY IDENTIFICATION

In this section we will make use of the special properties of our power system model (8-11) and develop an approximate method for identifying ε-coherent generators.

1. Some Analytic Properties of the Model

We make the following reasonable assumptions about the system.

(A1) The network is connected.

(A2) The lines are lossless with positive reactance.

(A3) The initial angles across the lines satisfy

$$|\theta_i(0) - \theta_j(0)| < \frac{\pi}{2} \quad \forall_{i,j}.$$

<u>Fact 3</u> Under the assumptions (A1) - (A3). The eigenvalues of the power system matrix A (eq. 10) have the following properties.

(i) All the eigenvalues of A lie in the closed left half plane.

(ii) There is no eigenvalues of A on the imaginery axis except at the origin.

(iii) A has exactly one eigenvalue at the origin.

Remark: The results of Fact 3 are what one would expect intuitively. Since we have assumed $\Delta \underline{PM} = 0$, and the generators have positive damping $D > 0$, without any disturbance the system should settle down, i.e., back to synchronism. It makes no difference if we shift all generator angles by the same amount, since it is the angle difference and its derivative that count. Therefore the system may settle down to the same frequencies $\Delta \underline{W} = 0$ but to some other angles $\Delta \underline{\delta}$ different by a constant term. Thus we expect the eigenvalues of A to lie on the open left half plane plus one at the origin (for the constant term in $\Delta \underline{\delta}$).

2. An Approximate Expression for the Grammian

By making use of the properties (Fact 3) of the power system model we have the following asymptotic expression for the reachability Grammian W_t^2.

Theorem 2. Consider the power system model (8). Suppose that assumptions (A1) - (A3) hold. Let W^2 be the solution of

$$AX + XA^T = -BB^T \tag{29}$$

with minimum Frobenius norm.†

Then

$$W_\tau^2 \rightarrow W^2 + \beta\tau \, \underline{\xi} \, \underline{\xi}^T$$

where $\underline{\xi}^T = (0,\ldots 0,1,\ldots 1)$ is a vector whose first n components are zeros and the last n components are ones, and β is a constant.

Remark: If all the eigenvalues of A except the one at the origin have real part less than $-|a|$, then for $\tau >> \frac{1}{|a|}$,

$$W^2 + \beta\tau \, \underline{\xi} \, \underline{\xi}^T$$

is a good approximation of W_τ^2.

Fact 1 asserts that any reachable state $\underline{x}$ in $[0,\tau]$ can be expressed as $\underline{x} = W_\tau \underline{p}$ for some $\|\underline{p}\| \leq K$. This implies that [7,8]

$$\underline{x} = W_\tau^2 \underline{z} \quad \text{for some } \underline{z} \tag{30}$$

If we use the approximation

$$W_\tau^2 \approx W^2 + \beta\tau \, \underline{\xi} \, \underline{\xi}^T \tag{31}$$

then we have

$$\underline{x} \approx (W^2 + \beta\tau \, \underline{\xi} \, \underline{\xi}^T) \, \underline{z} \tag{32}$$

†The Frobenius norm $\|\cdot\|_F$ of a matrix X is defined by $\|X\|_F := \left(\sum_{i,j} x_{ij}^2\right)^{1/2}$ [5, pp. 173].

In coherency identification, we are interested in $(\Delta\delta_i - \Delta\delta_j)$. Recall that the first n components of $\underline{x}$ are the Δw_k's and the last n components of $\underline{x}$ are the $\Delta\delta_k$'s. Therefore as far as coherency identification is concerned we can just focus on W^2 and ignore the term $\beta\tau\, \underline{\xi}\, \underline{\xi}^T$. But W^2 is the solution of an algebraic equation (29) whereas W_τ^2 is the solution of a differential equation (24). Thus we expect the computational effort using W^2 for ε-coherency identification will be much less.

3. Algorithm for the Approximate Method of ε-coherency Identification

The following algorithm can be used as an approximate method for identifying ε-coherent generators.

Step 1. Determine ε,τ. Estimate K.

Step 2. Find the solution W^2 of

$$AX + XA^T = -BB^T$$

with minimum Frobenius norm. There is an efficient algorithm for the solution of this problem [10].

Step 3. Identify ε-coherency condition by checking

$$\overline{W}_{ii} + \overline{W}_{jj} - 2\overline{W}_{ij} \leq \frac{\varepsilon^2}{K^2}$$

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APPENDIX

1. Proof of Fact 1

(i) $S_\tau = S$: See [7,8]

(ii) $S_{[0,\tau]} = S_\tau$.

It is obvious that $S_\tau \subset S_{[0,\tau]}$. We are going to show $S_{[0,\tau]} \subset S_\tau$ by contradiction. Suppose $S_{[0,\tau]} \not\subset S_\tau$ then

$$\exists t',\ 0 \leq t' \leq \tau, \text{ and } u(\cdot),\ \int_0^\tau \|u(t)\|^2 dt \leq K^2 \text{ such that}$$

$$\underline{x}(t') = \int_0^{t'} e^{A(t'-t)} B\underline{u}(t)dt \notin S_\tau$$

$$\text{Let } \hat{\underline{u}}(t) = \begin{array}{ll} \underline{u}[t-(\tau-t')] & \text{for } \tau - t' \leq t \leq \tau \\ 0 & 0 \leq t < \tau - t' \end{array}$$

then $\int_0^\tau \|\hat{u}(t)\|^2 dt \leq K^2$ and $\hat{\underline{x}}(\tau) = \int_0^\tau e^{A(\tau-t)} B\hat{\underline{u}}(t)dt = \underline{x}(t')$. But by definition of S_τ, $\hat{\underline{x}}(\tau) \in S_\tau$ this implies $\hat{\underline{x}}(\tau) = \underline{x}(t') \in S_\tau$ *

2. Proof of Theorem 1

By Fact 1 for any $x(t)$, $\exists \underline{p}(t)$, $\|\underline{p}(t)\| \leq K$, $\forall t \in [0,\tau]$ and $\underline{x}(t) = W_\tau \underline{p}(t)$ for $t \in [0,\tau]$.

Consider phase angles of generator $\bar{i}$, $\bar{j}$:

$$x_i(t) - x_j(t) = \sum_{m=1}^{2n} (W_{im} - W_{jm}) P_m(t)$$

For a fixed t', and let $h_m := W_{im} - W_{jm}$

$$\max_{\int \|u\|^2 dt \leq k^2} |x_i(t') - x_j(t')| \leq \max\{|\langle h, p\rangle| \mid \|p\| \leq K\} = \|h\| \cdot K$$

Since $\|h\|\cdot K$ is independent of t, we have

$$\max_{t\in[0,\tau]} |x_i(t) - x_j(t)| \le K[\sum_{m=1}^{2n} (W_{im}-W_{jm})^2]^{1/2}$$

$$= K\|\alpha^T W_\tau\| \qquad \alpha \triangleq (0\ldots \underset{\substack{\uparrow\\ i}}{1} \ldots \underset{\substack{\uparrow\\ j}}{-1},\ldots 0)$$

$$= K(\alpha^T W_\tau W_\tau^T \alpha)^{1/2}$$

$$= K(\alpha^T W_\tau^2 \alpha)^{1/2}$$

Similarly using the fact that $\|p\| \le K$ implies $\|U^T p\| \le K$, we have

$$\max_{t\in[0,\tau]} |x_i(t) - x_j(t)| \le K[\sum_{m=1}^{r} \sigma_m^2 (u_{im}-u_{jm})^2]^{1/2}$$

3. Proof of Fact 3

We first establish two lemmas for the proof.

Lemma 1: $H_{\ell\ell}^{-1}$ is positive definite.

Proof: $H_{\ell\ell}$ is the node admittance matrix of the network obtained by connecting all generator internal buses to ground. Under assumptions (A1) - (A3) this is equivalent to a positive resistive network, hence $H_{\ell\ell}$ is positive definite this implies that $H_{\ell\ell}^{-1}$ is positive definite.

Lemma 2: $H_{gg} - H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g}$ is nonnegative definite and has one and only one zero eigenvalue.

Proof: By construction, we have

$$H_{gg}\mathbb{I} + H_{g\ell}\mathbb{I}' = \underline{0} \tag{33}$$

$$H_{\ell g}\mathbb{I} + H_{\ell\ell}\mathbb{I}' = \underline{0} \tag{34}$$

where $\mathbb{I}$, $\mathbb{I}'$ are vectors having all elements 1, with appropriate dimensions.

We rewrite (34) as

$$\mathbb{I}' = -H_{\ell\ell}^{-1}H_{\ell g}\mathbb{I} \tag{35}$$

Substituting (35) into (33) we obtain

$$(H_{gg}-H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g})\mathbb{I} = \underline{0}.$$

This implies that $\mathbb{I}$ is an eigenvector of $H_{gg} - H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g}$ with eigenvalue $\lambda = 0$.

Now since the network is assumed to be connected,

$$\left[\begin{array}{c|c} H_{gg} & H_{g\ell} \\ \hline H_{\ell g} & H_{\ell\ell} \end{array}\right] \in R^{NxN} \quad \text{is nonnegative definite and}$$

with rank N - 1, i.e.,

$$(\underline{x}^T|\underline{y}^T)\left[\begin{array}{c|c} H_{gg} & H_{g\ell} \\ \hline H_{\ell g} & H_{\ell\ell} \end{array}\right](\frac{\underline{x}}{\underline{y}}) \geq 0 \tag{36}$$

The equality holds if and only if

$$(\frac{\underline{x}}{\underline{y}}) = k(\frac{\mathbb{I}}{\mathbb{I}'}) \text{ for some k.}$$

Let $\underline{y} = -H_{\ell\ell}^{-1}H_{\ell g}\underline{x}$ then we can write (36) as

$$\underline{x}^T(H_{gg}-H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g})\underline{x} \geq 0.$$

The equality holds iff

$$\underline{x} = k\mathbb{I}.$$

This implies that

$$\dim N(H_{gg}-H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g}) = 1$$

Hence $H_{gg}-H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g}$ is nonnegative definite, and has exactly one eigenvalue $\lambda = 0$.

Proof of Fact 3.

Assume that $\begin{bmatrix} \underline{x}_1 \\ \hline \underline{x}_2 \end{bmatrix}$ is an eigenvalue of A with eigenvalue λ.

$$\begin{bmatrix} -M^{-1}D & -M^{-1}(H_{gg}-H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g}) \\ I & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \hline \underline{x}_2 \end{bmatrix} = \lambda \begin{bmatrix} \underline{x}_1 \\ \hline \underline{x}_2 \end{bmatrix}$$

$$\Rightarrow -M^{-1}D\underline{x}_1 - M^{-1}(H_{gg}-H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g})\underline{x}_2 = \lambda\underline{x}_1$$

$$\underline{x}_1 = \lambda\underline{x}_2$$

$$\Rightarrow -M^{-1}D\lambda\underline{x}_2 - M^{-1}(H_{gg}-H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g})\underline{x}_2 = \lambda^2\underline{x}_2$$

$$\Rightarrow -\lambda\underline{x}_2^T D\underline{x}_2 - \underline{x}_2^T(H_{gg}-H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g})\underline{x}_2 = \lambda^2\underline{x}_2^T M\underline{x}_2$$

But $\underline{x}_2 \neq \underline{0}$, otherwise $\underline{x}_1 = \lambda\underline{x}_2 = \underline{0}$

Let $d \triangleq \underline{x}_2^T D\underline{x}_2$, $e \triangleq \underline{x}_2^T(H_{gg}-H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g})\underline{x}_2$, $f \triangleq \underline{x}_2^T M\underline{x}_2$

then $f\lambda^2 + d\lambda + e = 0$

and $d > 0$, $e \geq 0$, $f > 0$.

Hence (i) $\mathrm{Re}\lambda(A) \leq 0$

(ii) $\mathrm{Re}\lambda(A) = 0 \Rightarrow \lambda(A) = 0$

Besides, rank $A = n + \text{rank } (H_{gg}-H_{g\ell}H_{\ell\ell}^{-1}H_{\ell g})$

$= 2n - 1$ and the fact that A can not have generalized eigenvector at origin implies that A has exactly one eigenvalue at the origin.

4. Proof of Theorem 2

Consider the linear map from R^{2nx2n} to R^{2nx2n}

$$L : X \rightarrow AX + XA^T$$

Then the eigenvalue of L [11, pp. 235-239] are $\lambda = \lambda_i + \lambda_j$, i, j = 1,...,2n, λ_i, λ_j are eigenvalues of A. But from Fact 3, $Re\lambda(A) \leq 0$, and A has only one eigenvalue at the origin. Hence $Re\lambda(L) \leq 0$, and L has exactly one eigenvalue $\lambda = 0$. This implies that

$$\text{rank}(L) = 2n \times 2n - 1$$

Let $\{\underline{x}_i\}_{i=1}^{2n}$ be a set of generalized eigenvectors of A, then $\{\underline{x}_i\}$spans R^{2n}. [12,13]. $B = \sum_{i=1}^{2n} \underline{x}_i\underline{\alpha}_i^T$, $B \in R^{2nxr}$, $\underline{\alpha}_i \in R^{rx1}$. Thus

$$W_\tau^2 = \int_0^\tau e^{At}BB^Te^{A^Tt}dt$$

$$= \int_0^\tau \Big(\sum_{k=1}^{\sigma}\sum_{\ell=0}^{m_k-1} t^\ell e^{\lambda_k t}P_{j\ell}(A)\Big)\Big(\sum_{i=1}^{2n}\underline{x}_i\underline{\alpha}_i^T\Big)\Big(\sum_{i=1}^{2n}\underline{x}_i\underline{\alpha}_i^T\Big)^T\Big(\sum_{k=1}^{\sigma}\sum_{\ell=0}^{m_k-1} t^\ell e^{\lambda_k t}P_{j\ell}(A)\Big)^T dt$$

$$= \int_0^\tau \sum_{\substack{i,j \\ \lambda_i+\lambda_j\neq 0}} \underline{\alpha}_i^T\underline{\alpha}_j f_{ij}(t)e^{(\lambda_i+\lambda_j)t}\underline{x}_i\underline{x}_j^T dt + \int_0^\tau \beta\,\underline{\xi}\,\underline{\xi}^T dt$$

$$=: H(\tau) + \beta\tau\,\underline{\xi}\,\underline{\xi}^T$$

where m_k is the multiplicity of eigenvalue λ_k of minimal polynomial of A.

Since $Re(\lambda_i+\lambda_j) < -|a|$ for $\lambda_i + \lambda_j \neq 0$

Hence as $\tau \to \infty, \dot{H}(\tau) \to 0$ and $\dot{W}_\tau^2 = AW_\tau^2 + W_\tau^2 A^T + BB^T$. Thus

$$AH(\tau) + H(\tau)A^T + BB^T \to \beta \, \underline{\xi} \, \underline{\xi}^T \quad (\because A\underline{\xi} = \underline{0})$$

$$\Rightarrow AH(\tau) + H(\tau)A^T \approx - BB^T + \beta \, \underline{\xi} \, \underline{\xi}^T \text{ for large } \tau$$

Now if W^2 is the pseudo solution of $AX + XA^T = -BB^T$ with minimum Frobenius norm. i.e.

$$W^2 = \arg \min_X \| AX + XA^T + BB^T \|_F$$

then $H(\tau) = W^2 + r \, \underline{\xi} \, \underline{\xi}^T$ for some constant r. (since $\mathcal{N}(L) = \text{span } \underline{\xi} \, \underline{\xi}^T$)

But $H(\tau) := \int_0^\tau \sum_{\substack{i,j \\ \lambda_i+\lambda_j \neq 0}} \underline{\alpha}_i^T \underline{\alpha}_j f_{ij}(t) e^{(\lambda_i+\lambda_j)t} \underline{x}_i \underline{x}_j^T \, dt \in \mathcal{R}(L)$

and $W^2 \in \mathcal{R}(L^*) = \mathcal{R}(L) = \text{sp}\{\underline{x}_i \underline{x}_j^T\}_{\underline{x}_i \underline{x}_j^T \neq \underline{\xi} \, \underline{\xi}^T}$

$\Rightarrow \quad r = 0$

$\therefore W_\tau^2 \to W^2 + \beta\tau \, \underline{\xi} \, \underline{\xi}^T.$

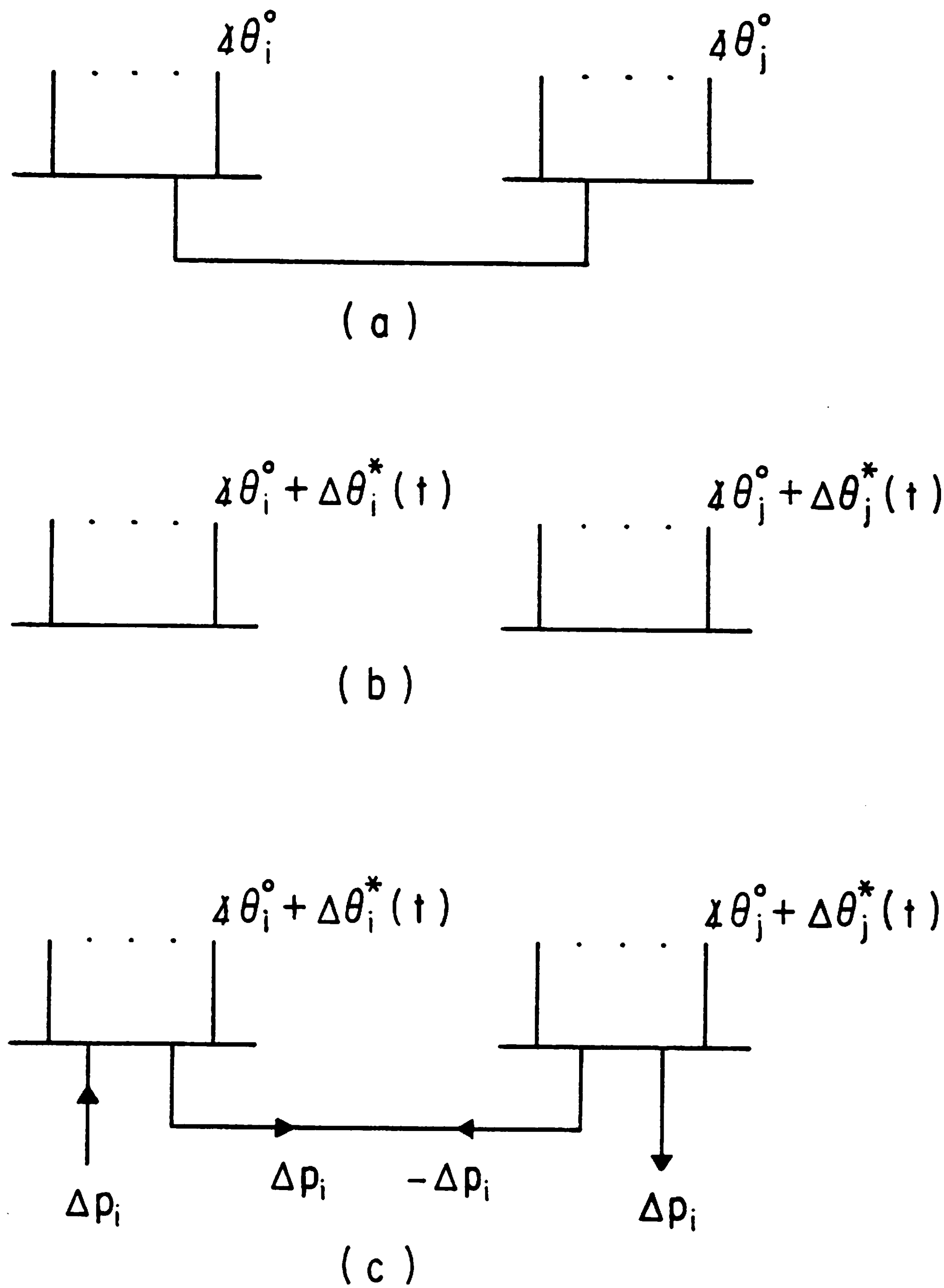

Fig. 1. Modeling of line switching as changes in power injections at the load buses without network modification.

(a) The connection and phase angles in the line before switching.
(b) The removal of the line and postfault phase angles.
(c) The removal of the line can be represented as changes in power injection with line connected.