FOUNDATIONS OF NONLINEAR NETWORK THEORY

PART II: LOSSLESSNESS

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FOUNDATIONS OF NONLINEAR NETWORK THEORY

PART II: LOSSLESSNESS

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ABSTRACT

This paper is the second in a two part series [1] which aims to provide a rigorous foundation in the nonlinear domain for the two energy-based concepts which are fundamental to network theory: passivity and losslessness. We hope to clarify the way they enter into both the state-space and the input-output viewpoints. Our definition of losslessness is inspired by that of a "conservative system" in classical mechanics, and we use several examples to compare it with other concepts of losslessness found in the literature. We show in detail how our definition avoids the anomalies and contradictions which many current definitions produce. This concept of losslessness has the desirable property of being preserved under interconnections, and we extend it to one which is representation independent as well. Applied to five common classes of n-ports, it allows us to define explicit criteria for losslessness in terms of the state and

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output equations. In particular we give a rigorous justification for the various equivalent criteria in the linear case. And we give a canonical network realization for a large class of lossless systems.

I. Introduction

This paper completes our two-part series [1] on energy-based concepts which are fundamental to nonlinear network theory. Our motivation for writing this second part is the little recognized fact that losslessness, like passivity, has been given a number of conflicting definitions [2-4,12,18] in the modern network theory literature. And as before, we believe that the problem arises from the long period in which "network theory" meant essentially "linear network theory," since the various concepts nearly coincide in the linear case.

Unlike its counterpart on passivity [1], this paper differs significantly from the theory given in reference [4]. When applied to nonlinear n-ports, the theory in reference [4] defines losslessness only for <u>passive</u> n-ports. Other authors [12], [18] would define a lossless n-port to be a passive n-port which satisfies certain additional conditions. In this two-part series, we treat passivity and losslessness as independent concepts. As a result of this viewpoint, a more complete theory emerges. The definition of losslessness given in this paper classifies a negative linear capacitor as lossless--a very sensible classification--whereas other approaches are either incapable of classifying this active element as lossy or lossless, or they classify it as lossy.

Our definition of losslessness is similar to, but less restrictive than, the concept of a "conservative system" in classical mechanics [5]. Roughly speaking, we say that a system is lossless if the energy required to travel between any two points of the state space is independent of the path taken. This seems to us the most basic concept possible, and it is quite different from many definitions found in the literature, which are based on equations such as

$$\int_0^\infty \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle \, dt = 0 \tag{1.1}$$

as in [2], or

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle \, dt = 0 \tag{1.2}$$

as in [3]. We will show by means of examples that expressions of this sort must be viewed as criteria for losslessness rather than as definitions of the concept. The relation between the basic definition and these and other criteria is the subject of Section II.

Notice that the above expressions are purely input-output in character since they involve only the admissible pairs $\{\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\}$, whereas our definition of losslessness relies on a state-space description of the n-port. This distinction will play a central role in the next two sections. For example, with losslessness defined as path independence of the energy, it is clear that an element such as an ideal 1-volt d.c. voltage source is lossy, at least so long as we view it as a resistive element. But we could also choose to view it as a nonlinear capacitor defined by $v(q) \equiv 1$, and in that case it would of course be lossless. This raises the disturbing possibility that our concept of losslessness relies critically on the equations we choose to describe an n-port rather than reflecting in a straightforward way the physical behavior of the n-port itself. In fact, we show in Section III that this is not a trivial anomoly: given any n-port N with a (not necessarily lossless) state representation S, we can construct a lossless state representation S' which is equivalent to S. Hence, N always has at least one lossless state representation. If we say, "a lossless n-port is an n-port with a lossless state representation," then every n-port is lossless and the definition means nothing at all. In Section III we show that if there exists a lossless state representation for N which satisfies a certain observability requirement, then (essentially) all state representations for N are lossless. This result allows us to formulate a meaningful definition of losslessness for an n-port, and it completely resolves the anomoly described above.

In Section IV we show that the internal energy function[1] for a passive n-port becomes unique in the lossless case. And in Section V we derive explicit criteria for losslessness in terms of the state and output equations of several special classes of n-ports. In particular we show that the criteria (1.1) and (1.2) are equivalent to losslessness in the linear, time-invariant, finite dimensional case, which explains why they are often invoked as definitions. And Section VI is devoted to a canonical network realization of lossless n-ports which becomes possible under certain assumptions.

In this paper, n-ports will be mathematically modeled by state representations (a complete list of our technical assumptions and definitions is given in Section II of [1]). Briefly, a state representation is a set of state, output, voltage, and current equations

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{f}\big(\underset{\sim}{x}(t),\underset{\sim}{u}(t)\big) \tag{1.3}$$

$$\underset{\sim}{y}(t) = \underset{\sim}{g}\big(\underset{\sim}{x}(t),\underset{\sim}{u}(t)\big) \tag{1.4}$$

$$\underset{\sim}{v}(t) = \underset{\sim}{V}\big(\underset{\sim}{x}(t),\underset{\sim}{u}(t)\big) \tag{1.5a}$$

$$\underset{\sim}{i}(t) = \underset{\sim}{I}\left(\underset{\sim}{x}(t),\underset{\sim}{u}(t)\right) \tag{1.5b}$$

where $\underset{\sim}{f}(\cdot,\cdot)$, $\underset{\sim}{g}(\cdot,\cdot)$, $\underset{\sim}{V}(\cdot,\cdot)$, and $\underset{\sim}{I}(\cdot,\cdot)$ are all continuous functions defined on $\Sigma \times U \subset R^m \times R^n$ ($\Sigma \triangleq$ state space, $U \triangleq$ set of admissible input values). The inputs $\underset{\sim}{u}(\cdot)$ belong to a set $\mathcal{U}$ of functions mapping $R^+ \triangleq [0,\infty)$ to U. For each input $\underset{\sim}{u}(\cdot)$ and each initial state $\underset{\sim}{x}(0)$, we assume the existence and uniqueness of the solution to (1.3) over the time interval R^+. The power input function is defined by $p(\underset{\sim}{x},\underset{\sim}{u}) \triangleq \langle \underset{\sim}{V}(\underset{\sim}{x},\underset{\sim}{u}), \underset{\sim}{I}(\underset{\sim}{x},\underset{\sim}{u})\rangle$, we assume that $t \to p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))$ is locally L^1 for every input-trajectory pair. The energy consumed by an input-trajectory pair $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[0,T]$ is the quantity $\int_0^T p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt$--note that this quantity can be positive, negative, or zero.

We will continue to make the blanket assumption that $\mathcal{U}$ is translation invariant and closed under concatenation [1, defs. 6 and 7]; but unlike [1], we will no longer repeat these assumptions explicitly when a theorem or lemma requires them.

II. Five N-Port Attributes Associated with Losslessness

Five characteristics of an n-port which are frequently associated with losslessness are, in rough order from the most obvious to the most subtle:

1. zero energy required to drive the state around any closed path,

2. the existence of a scalar function of the state which "tracks" the energy entering the ports,

3. all the energy which enters the ports can be recovered at the ports,

4. the total energy entering the ports over the time interval $[0,\infty)$

is always zero, and,

5. the average power entering the ports over the time inverval $[0,T]$ is always zero in the limit as $T \to \infty$.

Note that properties 1 and 2 involve state-space ideas, while 3-5 are purely input-output in character. Although properties 1, 2, 4, and 5 have all been used by various authors to define losslessness, only property 3 means literally "no loss of energy."

We will give a detailed discussion of these properties in subsections 2.1 through 2.5, and we will mention here only the major conclusions. It might appear on first reading that these five concepts and losslessness itself are simply different ways of saying the same thing. But it is rare in systems theory for input-output and state-space concepts to coincide exactly without restrictive assumptions, and this case is no exception. The major conclusion which emerges from this section (indeed, our motivation for writing it) is that <u>not one</u> of these five notions is known to be strictly equivalent to losslessness, defined as path-independence of the energy. The first two <u>will</u> turn out equivalent to losslessness under the additional assumption of complete controllability [1, def. 13], but the last three will <u>not</u> be unless very restrictive assumptions are imposed.

Relationships weaker than equivalence certainly do exist, though. It is not hard to see, for example, that losslessness and complete controllability imply property 3. And we will present a more stringent set of assumptions under which property 5 implies losslessness.

The following definition is a rigorous statement of the concept of losslessness as path independence.

Definition 2.1. A state representation S is defined to be lossless if the following condition[1] holds for every pair of states $\underset{\sim}{x}_a$, $\underset{\sim}{x}_b$ in Σ. For any two input-trajectory pairs $\{\underset{\sim}{u}_1(\cdot),\underset{\sim}{x}_1(\cdot)\}|[0,T_1],\{\underset{\sim}{u}_2(\cdot),\underset{\sim}{x}_2(\cdot)\}|[0,T_2]$ from $\underset{\sim}{x}_a$ to $\underset{\sim}{x}_b$, the energy consumed [1, def. 8] by $\{\underset{\sim}{u}_1(\cdot),\underset{\sim}{x}_1(\cdot)\}|[0,T_1]$ equals the energy consumed by $\{\underset{\sim}{u}_2(\cdot),\underset{\sim}{x}_2(\cdot)\}|[0,T_2]$. A state representation which is not lossless is defined to be lossy.

Note that Definition 2.1 does not require that there exist two or more input-trajectory pairs between every pair of states $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$: there may exist only one input-trajectory pair between $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$, or none at all. Also, a state representation which has no more than one input-trajectory pair between every pair of states is lossless by default.

As we discussed in the introduction, this notion of losslessness is dependent upon the particular state representation we choose for an n-port. For this reason we will initially consider losslessness to be an attribute of a state representation S rather than of an n-port N. We will show later, in subsection 3.1, that we can rid ourselves of this dependence on S under certain reasonable assumptions and define losslessness directly as an attribute of N. In the next two subsections we will discuss the concepts of cyclo-losslessness and conservative potential energy functions, which suffer from this same dependence on S. In subsection 3.2 we will give conditions under which they can be made representation independent as well.

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[1] Since U is translation invariant [1, Def. 6] and the state equations are independent of time, there is no loss of generality in assuming that both trajectories pass through $\underset{\sim}{x}_a$ at $t = 0$. And because of our standing assumption [1, Section II] that $t \to p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))$ is locally L^1, the energy consumed over any finite time interval is always finite.

2.1. Cyclo-Losslessness

We will say that a state representation is cyclo-lossless if the energy required to drive the system around any closed path in its state-space is zero. The following definition says this a bit more formally.

Definition 2.2. A state representation S is defined to be cyclo-lossless if for every input-trajectory pair $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ and every $T \geq 0$ such that $\underset{\sim}{x}(0) = \underset{\sim}{x}(T)$, the energy consumed by $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[0,T]$ is zero.

This is essentially the definition of a conservative system in classical mechanics [5], and it is slightly less restrictive than the definition of cyclo-losslessness given by Hill and Moylan [18].

Like losslessness itself, cyclo-losslessness is not a pure input-output concept but depends upon the particular state representation we choose. The ideal voltage source, for example, is cyclo-lossless when considered as a capacitor but not when considered as a resistor. To see that losslessness and cyclo-losslessness are not entirely equivalent concepts, consider the following example.

Example 2.1. If the current-controlled 2-port in Fig. 1 is given the obvious state representation in terms of q_1 and q_2, it will be lossy because of the resistor. But it is cyclo-lossless "by default," because $(q_1(0),q_2(0)) = (q_1(T),q_2(T))$ is possible only if we don't excite port #1 over the interval [0,T].

Nonetheless, there is a very strong relationship between the two concepts as the following lemma shows.

Lemma 2.1. Let S denote a state representation. Then the following three statements are true:

a) If S is lossless, then S is cyclo-lossless.

b) If S is completely controllable and if there exists a state $\underset{\sim}{x}_0 \in \Sigma$ for which every input-trajectory pair $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[0,T]$ with $\underset{\sim}{x}(0) = \underset{\sim}{x}(T) = \underset{\sim}{x}_0$ consumes zero energy, then S is lossless.

c) If S is cyclo-lossless and completely controllable, then S is lossless.

Lemma 2.1 is fairly obvious, but a rigorous formal proof is given in Appendix A. In essence, the lemma says that losslessness and cyclo-losslessness are equivalent concepts for completely controllable systems. Statement b) of the lemma will be utilized in our proof of results for linear systems.

2.2. Conservative Potential Energy Functions

A conservative potential energy function is a scalar function defined on the state space, which increases along trajectories at the same rate that energy enters the ports. The following definition just says the same thing more precisely.

Definition 2.3. A function $\phi : \Sigma \to \mathbb{R}$ is defined to be a conservative potential energy function for a state representation S if

$$\phi\left(\underset{\sim}{x}(t_2)\right) - \phi\left(\underset{\sim}{x}(t_1)\right) = \int_{t_1}^{t_2} p\left(\underset{\sim}{x}(t),\underset{\sim}{u}(t)\right) dt \tag{2.1}$$

for all input-trajectory pairs $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ and all $0 \le t_1 \le t_2 < \infty$.

It is evident that every state representation with a conservative potential energy function is lossless, and that any two conservative potential energy functions for a given state representation can differ only by an additive constant on any region of Σ reachable from a given point $\underset{\sim}{x} \in \Sigma$. Note that any nonnegative conservative potential energy function is also an internal energy function [1, Def. 23].

Like losslessness and cyclo-losslessness, the concept of a conservative potential energy function is not purely input-output in character, but involves the state space in a fundamental way. The ideal 1-volt d.c. voltage source, for example, has the conservative potential energy function $\phi(q) = q$ if we view it as a capacitor; but there is no conservative potential energy function for this system if we view it as a resistor.

In this section we will be content to define conservative potential energy functions in terms of a given state representation S. In subsection 3.2 we will discuss the conditions under which a conservative potential energy function can be assigned to an n-port N, independent of our choice for S.

The following simple lemma shows that under a certain reachability assumption, every lossless state representation has a conservative potential energy function. We do not know whether this conclusion holds without such an assumption.

<u>Lemma 2.2</u>. Suppose a state representation S is lossless and that there exists some state $\hat{\underset{\sim}{x}} \in \Sigma$ such that all of Σ is reachable [1, Def. 12] from $\hat{\underset{\sim}{x}}$. And let $\psi(\underset{\sim}{x})$ represent the energy required to drive the state from $\hat{\underset{\sim}{x}}$ to any point $\underset{\sim}{x} \in \Sigma$. Then $\psi : \Sigma \to \mathbb{R}$ is a conservative potential energy function for this state representation.

The proof is in Appendix A. Since the reachability assumption in Lemma 2.2 is always satisfied by completely controllable systems, it follows that losslessness, cyclo-losslessness, and the existence of a conservative potential energy function are all equivalent concepts for completely controllable state representations.

We haven't required or assumed that a conservative potential energy function ϕ be continuous, much less differentiable. But in those cases where ϕ is continuously differentiable, it is possible to rephrase (2.1) in differential form as follows.

Lemma 2.3. Let S denote a state representation, and suppose that Σ is an open subset of $\mathbb{R}^m$. Suppose further that $\mathcal{U}$ satisfies the following mild technical assumption: for each $\underset{\sim}{u}_0 \in U$, there exists an input $\underset{\sim}{u}(\cdot) \in \mathcal{U}$ such that $\underset{\sim}{u}(0) = \underset{\sim}{u}_0$ and $\underset{\sim}{u}(\cdot)$ is continuous at $t = 0$. Then a C^1 function $\phi : \Sigma \to \mathbb{R}$ is a conservative potential energy function for S $\Leftrightarrow$

$$\langle \underset{\sim}{\nabla}\phi(\underset{\sim}{x}), \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u}) \rangle = p(\underset{\sim}{x},\underset{\sim}{u}) \tag{2.1a}$$

for all $(\underset{\sim}{x},\underset{\sim}{u}) \in \Sigma \times U$.

The proof is in Appendix A.

Note carefully that a conservative potential function need not be differentiable at all. It is an open question whether ϕ will be differentiable even when $\underset{\sim}{f}(\cdot,\cdot)$ and $p(\cdot,\cdot)$ are C^∞. (We have discussed a related question at length in [1, example 7].) Therefore the existence of a function satisfying (2.1a) is not known to be a necessary condition for losslessness, even for completely controllable systems where $\underset{\sim}{f}(\cdot,\cdot)$ and $p(\cdot,\cdot)$ are C^∞.

2.3. Energetically Reversible Systems

A third property associated with losslessness is the property of being an "ideal energy reservoir," i.e. that all energy pumped into the system through its ports can be recovered at a later time. This is a genuine input-output property; therefore, if a state representation for an n-port $\mathcal{N}$ has this property then all state representations for $\mathcal{N}$ will have this property.

Definition 2.4. A state representation S is defined to be energetically reversible if the following condition holds for each $\underset{\sim}{x} \in \Sigma$. For every admissible pair $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ with initial state $\underset{\sim}{x}$ and every $T \geq 0$, there exists an admissible pair $\{\underset{\sim}{v}'(\cdot),\underset{\sim}{i}'(\cdot)\}$ with the same initial state $\underset{\sim}{x}$, and a $T' \geq T$, such that

i) $\{\underset{\sim}{v}(t),\underset{\sim}{i}(t)\} = \{\underset{\sim}{v}'(t),\underset{\sim}{i}'(t)\}, \ \forall t \in [0,T]$

ii) $$\int_0^{T'} \langle \underset{\sim}{v}'(t),\underset{\sim}{i}'(t) \rangle dt = 0. \tag{2.2}$$

An n-port is defined to be energetically reversible if it has an energetically reversible state representation.

Condition i) and the requirement that $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ and $\{\underset{\sim}{v}'(\cdot),\underset{\sim}{i}'(\cdot)\}$ have the same initial state imply that $\{\underset{\sim}{v}'(\cdot),\underset{\sim}{i}'(\cdot)\}|(T,\infty)$ is a development of the port voltages and currents in time which remains possible for N at the moment T, after the waveforms $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}|[0,T]$ have been observed. In the light of this observation, (2.2) means that all the energy deposited in N over the interval [0,T] can be recovered over some interval (T,T'].

An n-port is energetically reversible if, from the viewpoint of the outside world, no energy can ever disappear or be lost inside it. For this reason we were once tempted to adopt Def. 2.4 as our definition of losslessness. But we have decided to define losslessness as path independence of the energy instead, since the latter concept corresponds more closely to the standard electrical engineering usage of the term.

While it may seem natural to associate energetic reversibility with losslessness, the former property is neither a necessary nor a sufficient condition for the latter. For example, a 2-terminal resistor whose constitutive relation (v-i curve) contains points in both the first and second quadrants is a lossy element which is energetically reversible.

And the 1-port in the following example is lossless but not energetically reversible.

Example 2.2. The 1-port in Fig. 2 has the following state representation:

$$\dot{q} = \frac{1}{2}(i+|i|)$$

$$v = \begin{cases} q, & \text{if } i \geq 0, \\ 0, & \text{if } i < 0. \end{cases}$$

This 1-port is clearly lossless, but it is not energetically reversible because of the ideal diode in series with the capacitor. (Note that this example violates our technical assumptions because the port voltage is not a continuous function of q and i. This violation does not arise if one makes the artificial (but permissible) restriction $i \geq 0$.)

In spite of Example 2.2, there is a strong connection between the state dependent property of losslessness and the input-output property of energetic reversibility, as the following lemma shows.

Lemma 2.4. Suppose that a state representation S is lossless and completely controllable. Then it is energetically reversible.

The proof is in Appendix A.

2.4. The Zero Total Energy Property

The zero total energy property is the term we have adopted to express conditions of the type

$$\int_0^\infty p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt = 0 \tag{2.3}$$

where appropriate restrictions may be placed on the input-trajectory pairs $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ for which (2.3) is required to hold. The zero energy idea is rather appealing in the usual case that $\underset{\sim}{u}(\cdot)$ and $\underset{\sim}{y}(\cdot)$ are a hybrid pair [1, Def. 3]. For then (2.3) becomes

$$\int_0^\infty \langle \underset{\sim}{u}(t), \underset{\sim}{y}(t) \rangle dt = 0 \tag{2.4}$$

and has the straightforward geometric interpretation that $\underset{\sim}{u}(\cdot)$ and $\underset{\sim}{y}(\cdot)$ are orthogonal in the Hilbert space $L^2(\mathbb{R}^+ \to \mathbb{R}^n)$. In other words, if $\mathcal{U}$ and $\mathcal{Y}$ are contained in L^2, then (2.4) says that the n-port acts as an operator which maps each input waveform $\underset{\sim}{u}(\cdot)$ into the subspace of L^2 orthogonal to $\underset{\sim}{u}(\cdot)$. In this guise the zero total energy property appears as a generalization to function spaces of the idea of a nonenergic n-port [6], one for which $\underset{\sim}{v}(t)$ and $\underset{\sim}{i}(t)$ are orthogonal vectors in $\mathbb{R}^n$ at each instant t.

There are many possible versions of the zero total energy property, depending upon the conditions we place on $\underset{\sim}{u}(\cdot)$ and $\underset{\sim}{x}(\cdot)$ or $\underset{\sim}{u}(\cdot)$ and $\underset{\sim}{y}(\cdot)$. Since no single version is really definitive for our purposes, we will describe some of the most significant variations and their relation to losslessness.

A version of the zero total energy property was proposed in [2] as the definition of losslessness in both the linear and the nonlinear case. In the language of this paper, the definition in [2] can be paraphrased as follows. "An n-port $\mathcal{N}$ is lossless if

$$\int_0^\infty \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle dt = 0 \tag{2.5}$$

holds for all admissible pairs $\{\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\}$ in $L^2(\mathbb{R}^+ \to \mathbb{R}^n)$ so long as there is no energy stored in $\mathcal{N}$ at T = 0." This conception of losslessness is adequate as a criterion in the linear theory, but the following example shows that it is inappropriate for nonlinear systems.

Example 2.3. Consider the 1-port capacitor with the constitutive relation

$$v(q) = \begin{cases} q & , q \leq 0 \\ \sin q, & 0 \leq q \leq \pi \\ 0 & , q \geq \pi \end{cases}$$

shown in Fig. 3a. If we give it the usual state representation for a capacitor, with q as the state variable, then it is clearly lossless (Def. 2.1). In fact, it has properties 1), 2), and 3) listed at the beginning of this section, and property 5 holds also if $\{v(\cdot),i(\cdot)\}$ is bounded. But to see that it doesn't satisfy the definition in [2], consider the following signal pair, shown in Fig. 3b:

$$i(t) = \begin{cases} 1, & 0 \leq t \leq \pi \\ 0, & \text{otherwise} \end{cases} \qquad v(t) = \begin{cases} \sin(t), & 0 \leq t \leq \pi \\ 0 & , \text{otherwise.} \end{cases}$$

This is an admissible pair if the initial state is $q(0) = 0$, and it is clearly in L^2. The "stored energy" is initially zero in this case, but the total energy entering the ports is 2 joules. Thus the definition in [2] would have to classify this capacitor as lossy, which is contrary to the intuitive view that a 1-port charge-controlled capacitor with a continuous constitutive relation ought to be lossless.

Nonetheless, the following two lemmas show that there is a definite relation between losslessness, as we define it, and certain versions of the zero total energy property.

Lemma 2.5. Suppose a state representation S is lossless and completely controllable. Then S has a conservative potential energy function ϕ, and we suppose further that ϕ is continuous. Under these conditions,

$$\lim_{T\to\infty} \int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle dt = 0 \qquad (2.6)$$

for all (not necessarily L^2) admissible pairs $\{\underset{\sim}{v}(\cdot) = \underset{\sim}{V}(\underset{\sim}{x}(\cdot),\underset{\sim}{u}(\cdot)), \underset{\sim}{i}(\cdot) = \underset{\sim}{I}((\underset{\sim}{x}(\cdot),\underset{\sim}{u}(\cdot))\}$ such that $\lim_{t\to\infty} \underset{\sim}{x}(t) = \underset{\sim}{x}(0)$. Furthermore, (2.5) holds for all L^2 admissible pairs such that $\lim_{t\to\infty} \underset{\sim}{x}(t) = \underset{\sim}{x}(0)$.

The proof is in Appendix A. The difference between equations (2.5) and (2.6) is a technical point based on the definition of the Lebesgue integral [7]. Because of our standing assumption that $t \to \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle$ is locally L^1, the integral in (2.6) will necessarily exist for each finite value of T. But the integral in (2.5) exists only if the positive and negative parts of $\langle \underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot) \rangle$ individually yield finite values when integrated over all of $\mathbb{R}^+$, a mathematically stronger assumption which explains our requirement that in that case $\underset{\sim}{v}(\cdot)$, $\underset{\sim}{i}(\cdot) \in L^2(\mathbb{R}^+ \to \mathbb{R}^n)$.

<ins>Lemma 2.6</ins>. Suppose that a state representation S is lossless. Then

$$\int_0^{nT} p\left(\underset{\sim}{x}(t), \underset{\sim}{u}(t)\right) dt = 0$$

for all input-trajectory pairs $\{\underset{\sim}{u}(\cdot), \underset{\sim}{x}(\cdot)\}$ such that $\underset{\sim}{x}(\cdot)$ is a periodic function with period T, and for each integer $n \geq 0$.

Lemma 2.6 follows immediately from statement a) of Lemma 2.1.

Note that the versions of the zero total energy property invoked in these two particular lemmas are not purely input-output in character since they include restrictions on the state-space trajectory $\underset{\sim}{x}(\cdot)$.

2.5. <ins>The Zero Average Power Property</ins>

<ins>Definition 2.5</ins>. A state representation is defined to have the <ins>zero average power property</ins> if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle dt = 0 \tag{2.7}$$

for every admissible pair $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ such that $\underset{\sim}{v}(\cdot)$ and $\underset{\sim}{i}(\cdot)$ are bounded functions. An n-port is defined to have the <u>zero average power property</u> if it has a state representation with the zero average power property.

Since Definition 2.5 involves only the admissible pairs of a system, it is purely input-output in character. Therefore if an n-port N has the zero average power property, then <u>all</u> state representations for N have the zero average power property.

This property and variations on it have been commonly associated with losslessness in the literature on linear network theory. It has even been proposed as a definition of losslessness for nonlinear algebraic n-ports [3]. But we shall present examples, admittedly somewhat contrived, which show that the zero average power property is neither a necessary nor a sufficient condition for losslessness in general.

Our stipulation that (2.7) need only hold when $\underset{\sim}{v}(\cdot)$ and $\underset{\sim}{i}(\cdot)$ are bounded requires some explanation. In keeping with the traditions of linear circuit theory, we would certainly want to say that a 1-farad capacitor, for example, has the zero average power property. But (2.7) doesn't hold for all admissible pairs of a 1-farad capacitor, as we can see by considering the admissible pair $\{i(t) = 1, v(t) = t\}$. We could eliminate this particular admissible pair from consideration by requiring that $v(\cdot)$, or the state-space trajectory $x(\cdot) = q(\cdot)$, or both be bounded. It turns out that a sensible general theory emerges only if we require boundedness of $\underset{\sim}{v}(\cdot)$ and $\underset{\sim}{i}(\cdot)$ but not of $\underset{\sim}{x}(\cdot)$. A detailed discussion of this point is given in Appendix B.

<u>Example 2.4.</u>

To produce a voltage-controlled state representation for the 1-port in Fig. 4, we define f by

$$f(v_c,v) = \begin{cases} v-v_c, & v \geq v_c \\ 0 & , v < v_c. \end{cases}$$

Then f is continuous, and the voltage-controlled state equations are $\dot{v}_c = f(v_c,v)$, $i = f(v_c,v)$. Since we are only interested in bounded admissible pairs, we can take $\mathcal{U} = L^\infty(\mathbb{R}^+ \to \mathbb{R})$.. It is easy to see that this is a lossy state representation.

To show that Example 2.4 has the zero average power property, let $\{v(\cdot),i(\cdot)\}$ be any bounded admissible pair. Then there exists a finite constant $M > 0$ such that $|v(t)| \leq M$ and $|i(t)| \leq M$ for all t. Since $f(v_c,v) \geq 0$ always, it follows that $i(t) \geq 0$ for all t and $v_c(\cdot)$ is monotonically increasing. If $v_c(t) = v_c(0)$ for all t, then $i(t) = 0$ for a.a. $t \in \mathbb{R}^+$ and (2.7) is trivially satisfied. Now suppose that $v_c(\cdot)$ is not constant. Then it is obvious from the circuit shown in Fig. 4 that $v_c(t) \leq M$ for all $t \in \mathbb{R}^+$. To prove this assertion rigorously, suppose that $v_c(t_0) > M$ for some $t_0 \in \mathbb{R}^+$. Define $\alpha \triangleq \sup\{t \geq t_0 : v_c(t) > M\}$. By the continuity of $v_c(\cdot)$, $\alpha > t_0$; and by the definition of α, $v_c(t) > M$ for all $t \in [t_0,\alpha)$. But whenever $v_c(t) > M$, it must be constant (because $f(v_c(t),v(t)) = 0$; i.e., no current can flow through the ideal diode). Thus $v_c(\cdot)$ is constant on the interval $[t_0,\alpha)$. If $\alpha < \infty$, then, by continuity, $v_c(\alpha) = \lim_{\substack{t\to\alpha \\ t<\alpha}} v_c(t) = v_c(t_0) > M$, and so there exists an $\varepsilon > 0$ such that $v_c(\alpha+\varepsilon) > M$, which contradicts the definition of α. Therefore $v_c(\cdot)$ is constant on the interval $[t_0,\infty)$, and a similar argument shows that $v_c(\cdot)$ is constant on the interval $[0,t_0]$. These facts contradict the assumption that $v_c(\cdot)$ is not constant; hence, $v_c(t) \leq M$ for all t. Now

$$\left|\frac{1}{T}\int_0^T v(t)i(t)dt\right| \le \frac{1}{T}\int_0^T |v(t)||i(t)|dt$$

$$\le \frac{M}{T}\int_0^T |i(t)|dt = \frac{M}{T}\int_0^T i(t)dt = \frac{M}{T}\left(v_c(T)-v_c(0)\right)$$

$$\le \frac{M}{T}\left(M-v_c(0)\right) \to 0 \text{ as } T \to \infty. \tag{2.8}$$

This shows that (2.7) is satisfied; so Example 2.4 has the zero average power property, as claimed.

The previous example showed that a system with the zero average power property need not be lossless. The next example exhibits a lossless system which does not have the zero average power property.

Example 2.5. The capacitive constitutive relation $v(q) = q/\left(1+|q|\right)$ is drawn in Fig. 5. This system is clearly lossless; in fact $\phi(q) = |q| - \ln(1+|q|)$ is a conservative potential energy function. But it doesn't have the zero average power property, as we can see by considering the bounded admissible pair $\{i(t) = 1,\ v(t) = t/(1+t)\}$ for which the limiting value of the average input power is 1 as $T \to \infty$.

In the previous example the input and output were bounded functions, but the state $q(\cdot)$ was not. At first glance we might think that the problem could be resolved by amending Definition 2.5 so that we only consider bounded admissible pairs $\{\underset{\sim}{u}(\cdot),\underset{\sim}{y}(\cdot)\}$ for which the state trajectory $\underset{\sim}{x}(\cdot)$ is also a bounded function. We discuss this topic in depth in Appendix B, where we show that such an amendment would not resolve this apparent anomaly. Nonetheless, if we do place restrictions on the state space trajectory $\underset{\sim}{x}(\cdot)$, we can establish certain relationships between losslessness and a certain sort of zero average power condition. The following two lemmas are elementary.

Lemma 2.7. Suppose the state representation S is lossless and completely controllable, and that its state space Σ is all of $\mathbb{R}^m$. Then S has a conservative potential energy function ϕ, and we suppose further that ϕ is continuous. Under these conditions, (2.7) holds for all admissible pairs $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\} = \{\underset{\sim}{V}(\underset{\sim}{x}(\cdot),\underset{\sim}{u}(\cdot)),\underset{\sim}{I}(\underset{\sim}{x}(\cdot),\underset{\sim}{u}(\cdot))\}$ such that $\underset{\sim}{x}(\cdot)$ is bounded.

The proof is given in Appendix A.

Lemma 2.8. If a state representation S is lossless, then (2.7) holds for all admissible pairs $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\} = \{\underset{\sim}{V}(\underset{\sim}{x}(\cdot),\underset{\sim}{u}(\cdot)),\underset{\sim}{I}(\underset{\sim}{x}(\cdot),\underset{\sim}{u}(\cdot))\}$ such that $\underset{\sim}{x}(\cdot)$ is a periodic function.

The proof is given in Appendix A.

These two lemmas do not yet show a relationship between losslessness and the zero average power property as in Definition 2.5, because they require additional information about the state trajectory $\underset{\sim}{x}(\cdot)$. Can we find a connection between losslessness and the purely input-output statement of the zero average power property, one which holds for nonlinear n-ports and nonperiodic inputs and trajectories? Examples 2.4 and 2.5 place rather restrictive bounds on possible theorems in this area, but Lemma 2.8 suggests that we might have some success if we could find a way to reduce the general case to the periodic case. In linear circuit theory the Fourier transform does exactly that, but we must find another approach for nonlinear systems. First, we need the following technical definition, the terms of which are illustrated in Fig. 6.

Definition 2.6. Given $\underset{\sim}{u}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$, we let $\underset{\sim}{u}(\cdot)|[0,T)$ denote the restriction of $\underset{\sim}{u}(\cdot)$ to the interval $[0,T)$, $T > 0$. Given $\underset{\sim}{u}(\cdot)$ and $T > 0$ we say that $\underset{\sim}{w}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ is <u>the periodic extension</u> of $\underset{\sim}{u}(\cdot)|[0,T)$ if for each $t \in \mathbb{R}^+$, $\underset{\sim}{w}(t) = \underset{\sim}{u}(t-nT)$, where n is that unique nonnegative integer such that $t-nT \in [0,T)$. (See Fig. 6.) Finally, we say that $\mathcal{U}$

is closed under periodic extension if for each $\underline{u}(\cdot) \in \mathcal{U}$ and each $T > 0$, the periodic extension of $\underline{u}(\cdot)|[0,T)$ is also an element of $\mathcal{U}$.

Although "closure under periodic extension" bears a superficial resemblance to "closure under concatenation" [1,Def.7], it is actually a quite different concept. The essential difference is that closure under concatenation means one can piece together two (and hence any finite number) of different waveforms, whereas closure under periodic extension means one can piece together a segment of any single waveform an infinite number of times with itself. Consider Fig. 6 again. The waveform $\underline{u}(\cdot)$ in Fig. 6a belongs to all the spaces $L^p(\mathbb{R}^+ \to \mathbb{R})$, $1 \leq p \leq \infty$, since it is bounded and vanishes outside some finite interval. On the other hand the periodic extension of $\underline{u}(\cdot)|[0,T)$, shown in Fig. 6c, is in L^∞ but not in L^r, $1 \leq r < \infty$. Thus while all the L^p spaces are closed under concatenation, only L^∞ is closed under periodic extension.

The following theorem gives the relation between the input-output property of zero average power and the state-space property of losslessness.

Theorem 2.1. We are given an n-port $\mathcal{N}$ with state representation S satisfying the following assumptions:

i) S is completely controllable,

ii) $\mathcal{U}$ is closed under periodic extension,

iii) each waveform in $\mathcal{U}$ is bounded on every compact interval $[0,T]$,

and

iv) $\underline{V}(\cdot,\cdot)$ and $\underline{I}(\cdot,\cdot)$ are bounded on every bounded subset of $\Sigma \times U$.

Under these conditions, if S has the zero average power property then S is lossless.

Remark. Assumptions iii) and iv) are rather technical and not very restrictive. For example if $\mathcal{U}$ contains only piecewise-continuous waveforms then iii) is satisfied automatically, and if $\Sigma = \mathbb{R}^m$ and $U = \mathbb{R}^n$ then iv) is satisfied automatically because we have assumed that $\underset{\sim}{V}(\cdot,\cdot)$ and $\underset{\sim}{I}(\cdot,\cdot)$ are continuous [1, section II]. Assumptions i) and ii), on the other hand, are essential. The theorem fails without assumption i), as we see from Example 2.4, Fig. 4. It also fails without assumption ii), for consider the (admittedly artificial) example of a 1 ohm resistor with u = i where we make the very special choice $\mathcal{U} = L^2(\mathbb{R}^+ \to \mathbb{R})$. This is a lossy system, but it has the zero average power property as a result of $\mathcal{U}$ being L^2, a space which is not closed under periodic extension.

Proof of Theorem 2.1. The proof proceeds by contradiction. We will assume the system has the zero average power property and satisfies assumptions i)-iv) but is lossy. A contradiction will emerge.

If it is lossy, then there exist two states $\underset{\sim}{x}_a$, $\underset{\sim}{x}_b$ in Σ and two input-trajectory pairs $\{\underset{\sim}{u}_1(\cdot),\underset{\sim}{x}_1(\cdot)\}|[0,T_1]$, $\{\underset{\sim}{u}_2(\cdot),\underset{\sim}{x}_2(\cdot)\}|[0,T_2]$ from $\underset{\sim}{x}_a$ to $\underset{\sim}{x}_b$ such that $E_1 \neq E_2$, where E_1 is the energy consumed by $\{\underset{\sim}{u}_1(\cdot),\underset{\sim}{x}_1(\cdot)\}|[0,T_1]$ and E_2 is the energy consumed by $\{\underset{\sim}{u}_2(\cdot),\underset{\sim}{x}_2(\cdot)\}|[0,T_2]$ [1, Def. 8]. (See Fig. 7.)

Since the system is completely controllable, there is an input-trajectory pair $\{\underset{\sim}{u}_3(\cdot),\underset{\sim}{x}_3(\cdot)\}|[0,T_3]$ from $\underset{\sim}{x}_b$ to $\underset{\sim}{x}_a$, and we let E_3 be the energy consumed by $\{\underset{\sim}{u}_3(\cdot),\underset{\sim}{x}_3(\cdot)\}|[0,T_3]$. And since $E_1 \neq E_2$, either $E_1+E_3 \neq 0$, or $E_2+E_3 \neq 0$, or both. For definiteness, suppose $E_1+E_3 \neq 0$. Let $\underset{\sim}{u}_4(\cdot)$ consist of $\underset{\sim}{u}_1(\cdot)$ followed by $\underset{\sim}{u}_3(\cdot)$, i.e.

$$\underset{\sim}{u}_4(t) = \begin{cases} \underset{\sim}{u}_1(t), & 0 \leq t \leq T_1 \\ \underset{\sim}{u}_3(t-T_1), & t > T_1. \end{cases}$$

Since $\mathcal{U}$ is closed under concatenation, $\underset{\sim}{u}_4(\cdot) \in \mathcal{U}$. And since the state equations are time-invariant, $\{\underset{\sim}{u}_4(\cdot),\underset{\sim}{x}_4(\cdot)\}$ is an input-trajectory pair, where

$$\underline{x}_4(t) = \begin{cases} \underline{x}_1(t), & 0 \le t \le T_1 \\ \underline{x}_3(t-T_1), & t > T_1. \end{cases}$$

(Note that $\underline{x}_1(T_1) = \underline{x}_3(0)$). Then $\underline{x}_4(T_1) = \underline{x}_b$ and $\underline{x}_4(0) = \underline{x}_4(T_1+T_3) = \underline{x}_a$, so $\underline{x}_4(\cdot)$ passes once around a loop. And the energy consumed by $\{\underline{u}_4(\cdot),\underline{x}_4(\cdot)\}|[0,T_1+T_3]$ is $E_1+E_3 \gtrless 0$.

To complete the construction of a contradiction, we just drive $\underline{x}$ around the loop forever. More formally, let $\hat{\underline{u}}_4(\cdot)$ be the periodic extension of $\underline{u}_4(\cdot)|[0,T_1+T_3)$. Since $\mathcal{U}$ is closed under periodic extension, $\hat{\underline{u}}_4(\cdot) \in \mathcal{U}$. And since the state equations are time invariant, $\{\hat{\underline{u}}_4(\cdot),\hat{\underline{x}}_4(\cdot)\}$ is a valid input-trajectory pair if $\hat{\underline{x}}_4(\cdot)$ is the periodic extension of $\underline{x}_4(\cdot)|[0,T_1+T_3)$. This furnishes our contradiction, since

$$\frac{1}{n(T_1+T_3)}\int_0^{n(T_1+T_3)} p\big(\hat{\underline{x}}_4(t),\hat{\underline{u}}_4(t)\big)dt = \frac{n(E_1+E_3)}{n(T_1+T_3)} \qquad (2.9)$$

$$= \frac{E_1+E_3}{T_1+T_3} \gtrless 0$$

for every positive integer n. In order to prove that (2.9) genuinely contradicts our assumption that the system has the zero average power property, as in Def. 2.5, we must verify that $\underline{V}\big(\hat{\underline{x}}_4(\cdot),\hat{\underline{u}}_4(\cdot)\big)$ and $\underline{I}\big(\hat{\underline{x}}_4(\cdot),\hat{\underline{u}}_4(\cdot)\big)$ are bounded. Since $\hat{\underline{x}}_4(\cdot)$ is continuous and periodic it is bounded. And $\hat{\underline{u}}_4(\cdot)$ is bounded by assumption iii), since it is also periodic. Therefore $\underline{V}\big(\hat{\underline{x}}_4(\cdot),\hat{\underline{u}}_4(\cdot)\big)$ and $\underline{I}\big(\hat{\underline{x}}_4(\cdot),\hat{\underline{u}}_4(\cdot)\big)$ are bounded by assumption iv). ◻

<u>Corollary</u>. If a system satisfies the assumptions of Theorem 2.1 and has the zero average power property, then it is energetically reversible.

This follows from Theorem 2.1 and the fact that a lossless, completely controllable state representation is energetically reversible (Lemma 2.4).

III. Representation Independence and Closure

In subsection 3.1 we define the term "total observability" for state representations, a concept which is essentially the same as the usual "complete observability" in system theory. Our main result is to prove that losslessness is a genuine physical property of an n-port, independent of the particular state representation we choose for it, so long as we restrict ourselves to totally observable state representations. In subsection 3.2 we give related results for cyclo-losslessness and conservative potential energy functions. And in subsection 3.3 we will make precise the idea that an interconnection of lossless n-ports is itself lossless.

3.1. Losslessness, Total Observability, and Equivalent State Representations

The example of a d.c. voltage source, which is lossless when viewed as a capacitor but lossy when viewed as a resistor, raises a serious question about the physical significance of our definition of losslessness. Is losslessness a genuine physical property of an n-port, or is it merely an artifact of the particular state representation we choose for it? The following example shows how pervasive an issue this is.

Example 3.1. Given any n-port N with a state representation S consisting of the equations $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u})$, $\underset{\sim}{y} = g(\underset{\sim}{x},\underset{\sim}{u})$ and some specification for U, $\mathcal{U}$ and Σ, it is possible to create a lossless state representation S' for N as follows. We augment the state space by one dimension, defining $\Sigma' \triangleq \Sigma \times \mathbb{R}$, and then we add an artificial state variable e(t) which measures the total energy which has entered the ports over the interval [0,T]. The state of the new system is $(\underset{\sim}{x},e)$, and its equations are

$$\begin{cases} \dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u}) \\ \dot{e} = p(\underset{\sim}{x},\underset{\sim}{u}) \end{cases}$$

$$\underset{\sim}{y} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{u}).$$

The new state representation S' is obviously lossless because the energy required to travel between two states is now just the difference in their last coordinate. But S' is definitely peculiar because the artificial state variable e is not directly represented in the output $\underset{\sim}{y}$, which depends on $\underset{\sim}{x}$ and $\underset{\sim}{\mu}$ alone. The state representation of a d.c. voltage source as a capacitor has this same peculiarity -- its "charge" doesn't affect its output. By weeding out these "unobservable" state representations, we will be able to attach a definite physical meaning to losslessness after all.

Definition 3.1. Let S and S^* be two (not necessarily distinct) state representations. State $\underset{\sim}{x}$ of S and state $\underset{\sim}{x}^*$ of S^* are defined to be equivalent if the set of admissible pairs of S with initial state $\underset{\sim}{x}$ is identical to the set of admissible pairs of S^* with initial state $\underset{\sim}{x}^*$. S is defined to be state-observable if the equivalence of any two states $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ of S implies that $\underset{\sim}{x}_1 = \underset{\sim}{x}_2$.

In other words, S is state-observable if and only if the following condition is satisfied: if $\underset{\sim}{x}_1 \neq \underset{\sim}{x}_2$, then $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ are not equivalent. State-observability as defined above is essentially the standard notion of (complete) observability from system theory [11], the only difference being that it is stated in terms of admissible pairs, rather than input-output pairs. We have given it the name "state-observability" in order to distinguish it from the concept of "input-observability," which will be defined shortly. First, however, some discussion on equivalent state representations is in order.

<u>Definition 3.2</u>. Two state representations, S and S^*, are defined to be <u>equivalent</u> if for any state $\underset{\sim}{x}$ of S there exists an equivalent state $\underset{\sim}{x}^*$ of S^*, and conversely, for any state $\underset{\sim}{x}^*$ of S^* there exists an equivalent state $\underset{\sim}{x}$ of S.

This is essentially the definition of equivalence given by Desoer [11]. Definition 3.2 is less restrictive than the definition of equivalence given in Part I of this series [1, Def. 19, p. 29]. The reason we are changing our definition of equivalence is to clear up a vague point in Part I. We consider two state representations to be (equally valid) mathematical models for the same n-port if and only if they are equivalent according to Def. 3.2: this is implicit from the discussion throughout this paper and its counterpart on passivity. An illustration is afforded by our recurrent example of a 1-volt d.c. source, which has both resistive and capacitive state representations. Definition 3.2 properly classifies these state representations as equivalent, whereas Definition 19 in [1] does not. The same comment applies to the two state representations S and S' in Example 3.1.

Another vague point in Part I was that we never explicitly stated how we view an n-port within the framework of our theory. This situation is rectified by the following statement: An n-port is identified with an equivalence class [7] of state representations, where the equivalence relation is given by Definition 3.2. When we say that an n-port N "has" a state representation S (or that S is a state representation "for" N), we mean that S is an element of the equivalence class which is identified with N.

When we say that a property is representation independent, we mean that if a state representation S has that property, then all state representations equivalent to S have that property also. It is easy to see that the theorem for representation independence of passivity [1, Theorem 8] remains valid with the less restrictive form of equivalence given in Definition 3.2. In Part I we defined an n-port to be passive if it has a passive state representation; thus, by representation independence, all state representations for a passive n-port are passive.

Although a new form of equivalence has been introduced in Definition 3.2, the concept of equivalence given in Part I [1, Def. 19] will continue to be of interest to us. In order to avoid confusion, we shall henceforth refer to it as "bijective equivalence." Formally, we have the following definition.

<u>Definition 3.3</u>. Two state representations, S_1 and S_2, are defined to be <u>bijectively equivalent</u> if there exists a bijective map $\underset{\sim}{b} : \Sigma_1 \to \Sigma_2$ such that for each $\underset{\sim}{x} \in \Sigma_1$, the class of admissible pairs of S_1 with initial state $\underset{\sim}{x}$ is identical to the class of admissible pairs of S_2 with initial state $\underset{\sim}{b}(\underset{\sim}{x})$.

<u>Lemma 3.1</u>. Suppose S_1 and S_2 are bijectively equivalent state representations. Then S_1 is state-observable $\Leftrightarrow$ S_2 is state-observable.

The proof is given in Appendix A.

<u>Definition 3.4</u>. A state representation S is <u>input-observable</u> if the following condition holds for any two input-trajectory pairs $\{\underset{\sim}{u}_1(\cdot),\underset{\sim}{x}_1(\cdot)\}$, $\{\underset{\sim}{u}_2(\cdot),\underset{\sim}{x}_2(\cdot)\}$ with a common initial state $\underset{\sim}{x}_1(0) = \underset{\sim}{x}_2(0)$. If $\underset{\sim}{u}_1(t') \neq \underset{\sim}{u}_2(t')$ at some time $t' \geq 0$, then $\{\underset{\sim}{V}(\underset{\sim}{x}_1(t),\underset{\sim}{u}_1(t)),\ \underset{\sim}{I}(\underset{\sim}{x}_1(t),\underset{\sim}{u}_1(t))\} \neq \{\underset{\sim}{V}(\underset{\sim}{x}_2(t),\underset{\sim}{u}_2(t)),\underset{\sim}{I}(\underset{\sim}{x}_2(t),\underset{\sim}{u}_2(t))\}$ for some $t \in [0,t']$.

Input observability means that to any admissible pair $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ with a given initial state $\underset{\sim}{x}_0$, there corresponds exactly one input waveform $\underset{\sim}{u}(\cdot)$. In conjunction with our assumption that solutions are unique, it implies that to any admissible pair $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ with a given initial state $\underset{\sim}{x}_0$ there corresponds a unique input-trajectory pair $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$. We have defined this concept only in order to state our lemmas and theorems in a rigorously correct way; it is always satisfied in any practical case. For example, all hybrid and transmission representations are automatically input-observable because the inputs are a subset of the port voltages and currents. In these cases, the inequality in Definition 3.4 will be satisfied at $t = t'$.

If we make the modest technical assumption that for each $\underset{\sim}{u}_0 \in U$ there exists a $\underset{\sim}{u}(\cdot) \in \mathcal{U}$ such that $\underset{\sim}{u}(0) = \underset{\sim}{u}_0$, then input observability implies that the mapping $\underset{\sim}{u} \to \{\underset{\sim}{V}(\underset{\sim}{x},\underset{\sim}{u}),\underset{\sim}{I}(\underset{\sim}{x},\underset{\sim}{u})\}$ from U to $\mathbb{R}^n \times \mathbb{R}^n$ is injective for each fixed $\underset{\sim}{x} \in \Sigma$. We do not know whether this condition is sufficient for input observability.

Definition 3.5. A state representation is defined to be totally observable if it is both state-observable and input-observable.

Before proceeding to the next lemma, a few technical comments are in order. Let S_1 and S_2 be two equivalent state representations, and suppose that S_2 is state-observable. Then, by the definition of equivalence, for each state $\underset{\sim}{x}_1$ of S_1 there exists a state $\underset{\sim}{x}_2$ of S_2 which is equivalent to S_1; moreover, because S_2 is state-observable, $\underset{\sim}{x}_2$ is unique. Thus there exists a unique map $\underset{\sim}{\alpha} : \Sigma_1 \to \Sigma_2$ such that for each state $\underset{\sim}{x}_1$ of S_1, $\underset{\sim}{\alpha}(\underset{\sim}{x}_1)$ is the unique state of S_2 which is equivalent to $\underset{\sim}{x}_1$. If, in addition, S_1 and S_2 are input-observable, then the map $\underset{\sim}{\alpha}(\cdot)$ "matches up" the entire state trajectories of those input-trajectory pairs which produce identical port voltage and current waveforms in the two systems. This is stated precisely in the following lemma.

Lemma 3.2. Let S_1 and S_2 be equivalent state representations, with S_1 input-observable and S_2 totally observable. Let $\underset{\sim}{\alpha} : \Sigma_1 \rightarrow \Sigma_2$ denote the unique map such that for each state $\underset{\sim}{x}$ of S_1, $\underset{\sim}{\alpha}(\underset{\sim}{x})$ is the (necessarily unique) state of S_2 which is equivalent to $\underset{\sim}{x}$. Let $T > 0$ be any time and let $\{\underset{\sim}{u}_1(\cdot),\underset{\sim}{x}_1(\cdot)\}|[0,T]$ and $\{\underset{\sim}{u}_2(\cdot),\underset{\sim}{x}_2(\cdot)\}|[0,T]$ be any input-trajectory pairs of S_1 and S_2, respectively, such that $\{\underset{\sim}{V}_1(\underset{\sim}{x}_1(t),\underset{\sim}{u}_1(t)),$ $\underset{\sim}{I}_1(\underset{\sim}{x}_1(t),\underset{\sim}{u}_1(t))\} = \{\underset{\sim}{V}_2(\underset{\sim}{x}_2(t),\underset{\sim}{u}_2(t)),\underset{\sim}{I}_2(\underset{\sim}{x}_2(t),\underset{\sim}{u}_2(t))\}$ for all $t \in [0,T]$, where $\underset{\sim}{V}_1(\cdot,\cdot)$, $\underset{\sim}{I}_1(\cdot,\cdot)$ are the readout maps for S_1, and $\underset{\sim}{V}_2(\cdot,\cdot)$, $\underset{\sim}{I}_2(\cdot,\cdot)$ are the readout maps for S_2. Under these conditions, if $\underset{\sim}{x}_2(0) = \underset{\sim}{\alpha}(\underset{\sim}{x}_1(0))$, then $\underset{\sim}{x}_2(t) = \underset{\sim}{\alpha}(\underset{\sim}{x}_1(t))$ for all $t \in [0,T]$.

The proof is given in Appendix A.

Theorem 3.1. Let S_1 and S_2 be equivalent state representations, with S_1 input-observable and S_2 totally observable. Under these conditions, if S_2 is lossless, then S_1 is lossless.

Proof. We will prove the equivalent statement S_1 lossy $\Rightarrow S_2$ lossy. Assume S_1 is lossy. Then there exist two states $\underset{\sim}{x}_a$, $\underset{\sim}{x}_b$ in Σ_1, two times T', $T'' \geq 0$, and two admissible pairs $\{\underset{\sim}{v}'(\cdot),\underset{\sim}{i}'(\cdot)\} = \{\underset{\sim}{V}_1(\underset{\sim}{x}_1'(\cdot),\underset{\sim}{u}_1'(\cdot)),$ $\underset{\sim}{I}_1(\underset{\sim}{x}_1'(\cdot),\underset{\sim}{u}_1'(\cdot))\}$ and $\{\underset{\sim}{v}''(\cdot),\underset{\sim}{i}''(\cdot)\} = \{\underset{\sim}{V}_1(\underset{\sim}{x}_1''(\cdot),\underset{\sim}{u}_1''(\cdot)),\underset{\sim}{I}_1(\underset{\sim}{x}_1''(\cdot),\underset{\sim}{u}_1''(\cdot))\}$ of S_1 such that $\underset{\sim}{x}_1'(0) = \underset{\sim}{x}_1''(0) = \underset{\sim}{x}_a$, $\underset{\sim}{x}_1'(T') = \underset{\sim}{x}_1''(T'') = \underset{\sim}{x}_b$ and $E' \neq E''$, where E' is the energy consumed [1, Def. 8] by $\{\underset{\sim}{u}_1'(\cdot),\underset{\sim}{x}_1'(\cdot)\}|[0,T']$ and E'' is the energy consumed by $\{\underset{\sim}{u}_1''(\cdot),\underset{\sim}{x}_1''(\cdot)\}|[0,T'']$ (see Fig. 8).

Now let $\underset{\sim}{\alpha} : \Sigma_1 \rightarrow \Sigma_2$ be the unique map which is defined in Lemma 3.2. Then $\{\underset{\sim}{v}'(\cdot),\underset{\sim}{i}'(\cdot)\}$ and $\{\underset{\sim}{v}''(\cdot),\underset{\sim}{i}''(\cdot)\}$ are admissible pairs of S_2 with initial state $\underset{\sim}{\alpha}(\underset{\sim}{x}_a)$. So there exist input-trajectory pairs $\{\underset{\sim}{u}_2'(\cdot),\underset{\sim}{x}_2'(\cdot)\}$ and $\{\underset{\sim}{u}_2''(\cdot),\underset{\sim}{x}_2''(\cdot)\}$ of S_2 such that $\{\underset{\sim}{v}'(\cdot),\underset{\sim}{i}'(\cdot)\}$ $= \{\underset{\sim}{V}_2(\underset{\sim}{x}_2'(\cdot),\underset{\sim}{u}_2'(\cdot)),\underset{\sim}{I}_2(\underset{\sim}{x}_2'(\cdot),\underset{\sim}{u}_2'(\cdot))\}$ and $\{\underset{\sim}{v}''(\cdot),\underset{\sim}{i}''(\cdot)\} = \{\underset{\sim}{V}_2(\underset{\sim}{x}_2''(\cdot),\underset{\sim}{u}_2''(\cdot)),$ $\underset{\sim}{I}_2(\underset{\sim}{x}_2''(\cdot),\underset{\sim}{u}_2''(\cdot))\}$. By Lemma 3.2, $\underset{\sim}{x}_2'(T') = \underset{\sim}{\alpha}(\underset{\sim}{x}_1'(T')) = \underset{\sim}{\alpha}(\underset{\sim}{x}_b)$ and

$x_2''(T'') = \alpha(x_1''(T'')) = \alpha(x_b)$. Thus $\{u_2'(\cdot), x_2'(\cdot)\}|[0,T']$ and $\{u_2''(\cdot), x_2''(\cdot)\}|[0,T'']$ are input-trajectory pairs of S_2 from $\alpha(x_a)$ to $\alpha(x_b)$. Since the energy consumed by the former is E' and the energy consumed by the latter is $E'' \neq E'$, S_2 is lossy. ¤

<u>Corollary</u>. Let S_1 and S_2 be equivalent, totally observable state representations. Under these conditions, S_1 is lossless $\Leftrightarrow$ S_2 is lossless.

If we restrict ourselves to totally observable state representations, the corollary tells us that losslessness is representation independent.

If an n-port N has a lossy state representation which satisfies the trivial requirement of input-observability, then N <u>cannot</u> have a lossless, totally observable state representation. This follows immediately from Theorem 3.1, and it allows us to formulate a meaningful definition of losslessness for an n-port.

<u>Definition 3.6</u>. An n-port N is <u>lossless</u> if there exists for N a totally observable state representation S which is lossless by Definition 2.1. An n-port which is not lossless is <u>lossy</u>.

Note that according to Definition 3.6, a <u>nonzero ideal d.c. voltage source is a lossy 1-port</u>. (To prove that this conclusion follows rigorously from Definition 3.6, suppose there existed a lossless totally observable state representation for such an ideal voltage source. Since an ideal voltage source is a resistor, the state space can contain at most a single point if the state representation is to be state-observable. Such a system is lossless only if power never enters or leaves the port. For a voltage source, this implies v = 0.)

<u>Lemma 3.3</u>. If an n-port N is lossless, then every input-observable state representation for N is lossless.

(Note, however, that if N is lossy, it does not follow that every input-observable state representation for N is lossy. The ideal 1 volt source is a good example.)

Proof. This follows immediately from Definition 3.6 and Theorem 3.1.

¤

3.2. Representation Independence for Cyclo-Losslessness and Conservative Potential Energy Functions

In Example 3.1, we showed that any n-port N with a state representation S has another state representation S' which is lossless (and non-observable). From its definition, it is easy to see that S' is cyclo-lossless as well and has a conservative potential energy function. Consequently, if we said "a cyclo-lossless n-port is an n-port with a cyclo-lossless state representation," then all n-ports would be cyclo-lossless and the definition would be meaningless. Analogous comments apply regarding the existence of a state representation for N which has a conservative potential energy function. In this subsection we exploit Lemma 3.2 to determine a way in which these properties can be viewed as being characteristic of the n-port itself. The results of this subsection show that cyclo-losslessness and the existence of conservative potential energy functions are representation independent properties when we restrict ourselves to totally observable state representations.

Lemma 3.4. Let S_1 and S_2 be equivalent state representations, with S_1 input-observable and S_2 totally observable. Under these conditions, if S_2 is cyclo-lossless, then S_1 is cyclo-lossless.

The proof is given in Appendix A.

According to Lemma 3.4, if N has a totally observable cyclo-lossless state representation, then all state representations for N are cyclo-lossless, provided they satisfy the trivial requirement of input-observability. This justifies the following definition.

Definition 3.7. An n-port N is defined to be cyclo-lossless if there exists for N a totally observable state representation S which is cyclo-lossless by Definition 2.2.

Lemma 3.5. Let S_1 and S_2 be equivalent state representations, with S_1 input-observable and S_2 totally observable. Let $\underline{\alpha} : \Sigma_1 \to \Sigma_2$ denote the unique map defined in Lemma 3.2. Under these conditions, if $\phi_2(\cdot)$ is a conservative potential energy function for S_2, then $\phi_1(\cdot) \triangleq (\phi_2 \circ \underline{\alpha})(\cdot)$ is a conservative potential energy function for S_1.

The proof is given in Appendix A.

Lemma 3.5 says that if an n-port N has a totally observable state representation with a conservative potential energy function, then all input-observable state representations for N will have a conservative potential energy function. This justifies the following definition.

Definition 3.8. An n-port N is defined to be a conservative potential energy n-port if there exists for N a totally observable state representation with a conservative potential energy function (Def. 2.3).

As for the other properties which were given formal definitions in Section II, we have already defined what it means for an n-port to be energetically reversible (Def. 2.4) or to have the zero average power property (Def. 2.5).

3.3. The Interconnection of Lossless N-Ports

Suppose $N_1,\ldots,N_k$ are lossless n-ports and N is created by interconnecting $N_1,\ldots,N_k$. Will N necessarily be lossless? If so, we would say that losslessness possesses the attribute of closure, a concept we have discussed in [1, subsection 5.3].

We would certainly expect an interconnection of lossless n-ports to be lossless, but a difficulty arises when we attempt a completely general proof. The problem is that N may not have a totally observable state representation (or any state representation at all, for that matter), even though $N_1,\dots,N_k$ do. We will not address that problem here, but in its absence the closure property is almost immediate.

<u>Lemma 3.6</u>. Suppose $N_1,\dots,N_k$ are n-ports with lossless state representations $S_1,\dots,S_k$ as in Definition 2.1. Suppose N, created by interconnecting $N_1,\dots,N_k$, has a state representation S with a state space Σ which is any subset of $\Sigma_1\times\dots\times\Sigma_k$. Then S is lossless.

Moreover, if S is totally observable, then N is lossless. The proof of Lemma 3.6 is given in Appendix A.

3.4. Distinct N-Ports Made from a Multiterminal Element

Distinct n-ports made from the same multiterminal element by the use of Excitation-Observation-Mode-Transformation of Type 1 (EOMT 1) and of Type 2 (EOMT 2), the concept of EOMT equivalence were introduced and discussed in [1, Section 5.2]; it was also shown there that passivity is preserved under EOMT equivalence. In the following we will show that similar results hold for losslessness as well, i.e. assuming $\{\underline{u}(\cdot),\underline{y}(\cdot)\}$ is an hybrid pair we will show that losslessness is also preserved when the roles of the inputs and outputs are reversed. But first we will need two technical lemmas which are very much similar in spirit to Lemmas 3.1 and 3.2.

<u>Lemma 3.7</u>. Let N with state representation S be EOMT equivalent to $\hat{N}$ with state representation $\hat{S}$. Then $\underset{\sim}{S}$ is state observable $\Leftrightarrow$ $\hat{S}$ is state observable.

<u>Proof</u>.

($\Leftarrow$) Let $\hat{S}$ be state observable, $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$ two distinct states in Σ, C_a and C_b the classes of input-output pairs with initial states $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$ respectively and, $\hat{C}_a$ and $\hat{C}_b$ the classes of input-output pairs with initial states $\hat{\underset{\sim}{x}}_a$ and $\hat{\underset{\sim}{x}}_b$ respectively where $\hat{\underset{\sim}{x}}_a = \underset{\sim}{b}(\underset{\sim}{x}_a)$ and $\hat{\underset{\sim}{x}}_b = \underset{\sim}{b}(\underset{\sim}{x}_b)$. As $\underset{\sim}{b}$ is the bijection in the definition of EOMT equivalence $\underset{\sim}{x}_a \neq \underset{\sim}{x}_b$ $\Rightarrow \hat{\underset{\sim}{x}}_a \neq \hat{\underset{\sim}{x}}_b \Rightarrow \hat{C}_a \neq \hat{C}_b$. The last implication being true since $\hat{S}$ is state observable. $\hat{C}_a \neq \hat{C}_b$ implies one or both of the following two statements.

(i) $\exists\{\hat{\underset{\sim}{u}}_a(\cdot),\hat{\underset{\sim}{y}}_a(\cdot)\} \in \hat{C}_a$ such that $\{\hat{\underset{\sim}{u}}_a(\cdot),\hat{\underset{\sim}{y}}_a(\cdot)\} \notin \hat{C}_b$

(ii) $\exists\{\hat{\underset{\sim}{u}}_b(\cdot),\hat{\underset{\sim}{y}}_b(\cdot)\} \in \hat{C}_b$ such that $\{\hat{\underset{\sim}{u}}_b(\cdot),\hat{\underset{\sim}{y}}_b(\cdot)\} \notin \hat{C}_a$

Suppose (i) holds and let $\{\underset{\sim}{u}_a(\cdot),\underset{\sim}{y}_a(\cdot)\}$ be the input-output pair with initial state $\underset{\sim}{x}_a$ which is EOMT related to $\{\hat{\underset{\sim}{u}}_a(\cdot),\hat{\underset{\sim}{y}}_a(\cdot)\}$ as required by (iii) of EOMT equivalence. Clearly $\{\underset{\sim}{u}_a(\cdot),\underset{\sim}{y}_a(\cdot)\} \in C_a$. Moreover $\{\underset{\sim}{u}_a(\cdot),\underset{\sim}{y}_a(\cdot)\} \notin C_b$ because otherwise, $\{\hat{\underset{\sim}{u}}_a(\cdot),\hat{\underset{\sim}{y}}_a(\cdot)\}$ would be in $\hat{C}_b$ by (iii) of EOMT equivalence and since both EOMT are nonsingular transformations. So, if (i) holds then S is state-observable. The proof in case (ii) holds is similar.

($\Rightarrow$) Same proof as for ($\Leftarrow$). ¤

<u>Lemma 3.8</u>. Let N with state representation S be totally state-observable and EOMT equivalent to $\hat{N}$ with $\hat{S}$. Then the input-output pair $\{\hat{\underset{\sim}{u}}(\cdot),\hat{\underset{\sim}{y}}(\cdot)\}$ of $\hat{N}$ is from $\hat{\underset{\sim}{x}}_a \in \hat{\Sigma}$ to $\hat{\underset{\sim}{x}}_b \in \hat{\Sigma}$ $\Leftrightarrow$ the EOMT related input-output pair

$\{\underset{\sim}{u}(\cdot),\underset{\sim}{y}(\cdot)\}$ of N is from $\underset{\sim}{x}_a \in \Sigma$ to $\hat{\underset{\sim}{x}}_b \in \Sigma$ where $\hat{\underset{\sim}{x}}_a = \underset{\sim}{b}(\underset{\sim}{x}_a)$ $\hat{\underset{\sim}{x}}_b = \underset{\sim}{b}(\underset{\sim}{x}_b)$ and $\underset{\sim}{b}$ is the bijection in the definition of EOMT equivalence.

Proof. S is totally state-observable implies $\hat{S}$ is totally state observable by Lemma 3.7. Hence the lemma becomes symmetric in both directions. Therefore the proof will be done only in one direction.

($\Leftarrow$) Let $\{\underset{\sim}{u}(\cdot),\underset{\sim}{y}(\cdot)\}$ be from $\underset{\sim}{x}_a$ to $\underset{\sim}{x}_b$ and let C_a, C_b, C_c be respectively the sets of input-output pairs with initial states $\underset{\sim}{x}_a$, $\underset{\sim}{x}_b$, $\underset{\sim}{x}_c$ and $\hat{C}_a$, $\hat{C}_b$, $\hat{C}_c$ with $\hat{\underset{\sim}{x}}_a$, $\hat{\underset{\sim}{x}}_b$, $\hat{\underset{\sim}{x}}_c$. Then $\{\hat{\underset{\sim}{u}}(\cdot),\hat{\underset{\sim}{y}}(\cdot)\} \in \hat{C}_a$ by (iii) of EOMT equivalence. All there remains to show is that the final state of $\{\hat{\underset{\sim}{u}}(\cdot),\hat{\underset{\sim}{y}}(\cdot)\}$ is $\underset{\sim}{b}(\underset{\sim}{x}_b) = \hat{\underset{\sim}{x}}_b$. Suppose not, i.e. let the final state of $\{\hat{\underset{\sim}{u}}(\cdot),\hat{\underset{\sim}{y}}(\cdot)\}$ be $\hat{\underset{\sim}{x}}_c \neq \hat{\underset{\sim}{x}}_b$. Then, since $\hat{S}$ is state-observable, there exists an input-output pair $\{\hat{\underset{\sim}{u}}_c(\cdot),\hat{\underset{\sim}{y}}_c(\cdot)\} \in \hat{C}_c$ such that $\{\hat{\underset{\sim}{u}}_c(\cdot),\hat{\underset{\sim}{y}}_c(\cdot)\} \notin \hat{C}_b$. If $\{\underset{\sim}{u}_c(\cdot),\underset{\sim}{y}_c(\cdot)\}$ is the EOMT related input-output pair of N to $\{\hat{\underset{\sim}{u}}_c(\cdot),\hat{\underset{\sim}{y}}_c(\cdot)\}$, then by (iii) of EOMT equivalence $\{\underset{\sim}{u}_c(\cdot),\underset{\sim}{y}_c(\cdot)\} \notin C_b$. Therefore the concatenation of $\{\underset{\sim}{u}(\cdot),\underset{\sim}{y}(\cdot)\}$ with $\{\underset{\sim}{u}_c(\cdot),\underset{\sim}{y}_c(\cdot)\}$ is not in C_a whereas the concatenation of $\{\hat{\underset{\sim}{u}}(\cdot),\hat{\underset{\sim}{y}}(\cdot)\}$ with $\{\hat{\underset{\sim}{u}}_c(\cdot),\hat{\underset{\sim}{y}}_c(\cdot)\}$ is in $\hat{C}_a$; this contradicts the fact that N and $\hat{N}$ are EOMT equivalent. ¤

Theorem 3.2. Let N with state representation S be totally state-observable and EOMT equivalent to $\hat{N}$ with state representation $\hat{S}$. Then N is lossless $\Leftrightarrow$ $\hat{N}$ is lossless.

Proof. As S is state-observable $\Leftrightarrow$ $\hat{S}$ is state-observable by Lemma 3.7 the proof is symmetric for both directions.

($\Rightarrow$) Let $\hat{\underset{\sim}{x}}_a$ and $\hat{\underset{\sim}{x}}_b$ be any pair of states in $\hat{\Sigma}$ of $\hat{S}$, $\{\hat{\underset{\sim}{u}}_i(\cdot),\hat{\underset{\sim}{y}}_i(\cdot)\}$ for $i \in \{1,2\}$ two input-output pairs from $\hat{\underset{\sim}{x}}_a$ to $\hat{\underset{\sim}{x}}_b$ and $\hat{E}_i$ the energy consumed by the pair $\{\hat{\underset{\sim}{u}}_i(\cdot),\hat{\underset{\sim}{y}}_i(\cdot)\}$ for $i \in \{1,2\}$. By Lemma 3.8 and by EOMT equivalence there exists two states $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$ in Σ of S such that $\hat{\underset{\sim}{x}}_a = \underset{\sim}{b}(\underset{\sim}{x}_a)$, $\hat{\underset{\sim}{x}}_b = \underset{\sim}{b}(\underset{\sim}{x}_b)$ and two input-output pairs $\{\underset{\sim}{u}_i(\cdot),\underset{\sim}{y}_i(\cdot)\}$ for $i \in \{1,2\}$ which are EOMT related to $\{\hat{\underset{\sim}{u}}_i(\cdot),\hat{\underset{\sim}{y}}_i(\cdot)\}$. If E_i is the energy consumed by $\{\underset{\sim}{u}_i(\cdot),\underset{\sim}{y}_i(\cdot)\}$ for $i \in \{1,2\}$ then $E_1 = E_2$ since N is lossless. It was shown in [1, Theorem 9] that for EOMT related input-output pairs

$$\langle \hat{\underset{\sim}{u}}_i(t),\hat{\underset{\sim}{y}}_i(t)\rangle = \langle \underset{\sim}{u}_i(t),\underset{\sim}{y}_i(t)\rangle \text{ for all } t \geq 0$$

which implies

$$\hat{E}_1 = E_1 = E_2 = \hat{E}_2$$

proving that $\hat{N}$ is lossless. ¤

The following corollaries can be proved in exactly the same manner as Corollaries A, B, C to Theorem 9 in [1].

<u>Corollary A</u>. Suppose that the n-port $\hat{N}$ is a new orientation (partial or complete) of the n-port N which is totally state-observable and that N is EOMT equivalent to $\hat{N}$. Then, N is lossless $\Leftrightarrow$ $\hat{N}$ is lossless.

<u>Corollary B</u>. Suppose that the n-port $\hat{N}$ is obtained from N through a generalized datum-node transformation and that N is totally state-observable. Then, N is lossless $\Leftrightarrow$ $\hat{N}$ is lossless.

<u>Corollary C</u>. Let the n-port N be totally state-observable and suppose that N_k is obtained from N by successive applications of EOMT producing each time equivalent n-ports. Then, N is lossless $\Leftrightarrow$ N_k is lossless.

IV. <u>Passive Lossless N-Ports</u>

We showed by example in [1, Section VI] that the internal energy function [1, Def. 23] for a passive state representation is not uniquely determined in general, not even to within an additive constant. But lossless passive state representations do not have this indeterminacy at least provided we impose a controllability requirement. The following lemma was originally due to Willems [4].

<u>Lemma 4.1</u>. Let N denote an n-port which is lossless and passive, and let S denote an input-observable, completely controllable state representation for N. Then any internal energy function $E_I(\cdot)$ for S is also a conservative potential energy function for S.

In other words, the inequality in (6.1) of [1] becomes an equality for passive lossless systems. Since the conservative potential energy function is unique up to an additive constant, the internal energy is also unique to within an additive constant for these systems. Since Willems doesn't really prove this lemma, we have provided a rigorous proof in Appendix A.

<u>Corollary</u>. In addition to the assumptions of Lemma 4.1, suppose that N is strongly passive and $\underset{\sim}{x}^* \in \Sigma$ is a relaxed state of S. Let $E_{R\underset{\sim}{x}^*}(\underset{\sim}{x})$ represent the energy required to reach any state $\underset{\sim}{x}$ from $\underset{\sim}{x}^*$, as in [1, Def. 24]. Then $E_A(\underset{\sim}{x}) = E_{R\underset{\sim}{x}^*}(\underset{\sim}{x})$ for all $\underset{\sim}{x} \in \Sigma$, and S has exactly one internal energy function $E_I(\cdot)$ such that $E_I(\underset{\sim}{x}^*) = 0$, namely $E_I(\cdot) = E_A(\cdot) = E_{R\underset{\sim}{x}^*}(\cdot)$.

The corollary results from Lemma 4.1, the fact that $E_A(\cdot)$ and $E_{R\underset{\sim}{x}^*}(\cdot)$ are themselves internal energy functions, the uniqueness of conservative potential energy functions to within an additive constant, and the fact that $E_A(x^*) = E_{R\underset{\sim}{x}^*}(x^*) = 0$. The equality $E_I(\cdot) = E_A(\cdot)$ has the natural interpretation that for these lossless passive systems, all the internal energy is available at the ports.

It may be tempting to suppose the converse, i.e. that if the state representation for a strongly passive n-port satisfies $E_A(\cdot) = E_I(\cdot) = E_{R\underset{\sim}{x}^*}(\cdot)$ so that all its internal energy is available at the ports, then it must be lossless. But the 1-port in Fig.10 of [1] is a counterexample when G = 0. It is still lossy in that case, but all the energy stored in the capacitor is available at the ports in the limit of infinitely small input currents and infinitely long times. Lossy n-ports of this sort are of independent interest. They include as a special case the systems studied in classical thermodynamics [9].

V. Necessary and Sufficient Conditions for Losslessness of Several Classes of N-Ports

For the same classes of n-ports we studied in [1, Section IV], it is possible to find necessary and sufficient conditions for losslessness in terms of the state and output equations alone. With the exception of the first-order n-ports discussed in subsection 5.5, the basic assumption will be that $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a hybrid pair [1, Def. 3] so that the instantaneous input power is $\langle \underset{\sim}{u},\underset{\sim}{y} \rangle$, i.e. $p(\underset{\sim}{x},\underset{\sim}{u}) = \langle \underset{\sim}{u},\underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{u}) \rangle$. State representations of this sort are automatically input-observable, so total observability reduces to state-observability in this case.

5.1. Resistive N-Ports

We define a resistive state representation to be a state representation of the form

$$\begin{aligned} \dot{\underset{\sim}{x}} &= \underset{\sim}{0} \\ \underset{\sim}{y} &= \underset{\sim}{g}(\underset{\sim}{u}) \end{aligned} \qquad (5.1)$$

where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair, U is a nonempty subset of $\mathbb{R}^n$, $\mathcal{U}$ is the class of all functions $\underset{\sim}{u}(\cdot) : \mathbb{R}^+ \to U$ such that $t \to \langle \underset{\sim}{u}(t),\underset{\sim}{g}(\underset{\sim}{u}(t)) \rangle$ is locally L^1, and Σ is any nonempty subset of $\mathbb{R}^m$. By definition, a resistive n-port is an n-port with a resistive state representation. Thus, a resistive n-port is completely characterized by the instantaneous relation $\underset{\sim}{y}(t) = \underset{\sim}{g}(\underset{\sim}{u}(t))$ between the input $\underset{\sim}{u}(\cdot)$ and the output $\underset{\sim}{y}(\cdot)$.

Since the class of admissible pairs of a resistive state representation is independent of the initial state, the following lemma is obvious.

Lemma 5.1. Let S denote a resistive state representation. Then S is state-observable $\Leftrightarrow$ the state space Σ of S consists of exactly one point.

The next lemma gives losslessness criteria for resistive state representations and n-ports. Note that the criterion for the losslessness of an n-port applies regardless of whether the given state representation is state-observable.

Lemma 5.2. Let N denote a resistive n-port, and let S denote any resistive state representation for N. Then the following statements are true.

a) S is lossless $\Leftrightarrow \langle \underset{\sim}{u}, \underset{\sim}{g}(\underset{\sim}{u}) \rangle = 0$ for all $\underset{\sim}{u} \in U$.

b) N is lossless $\Leftrightarrow$ S is lossless.

The proof is given in Appendix A. Note that a lossless resistive n-port is passive; in fact, it is nonenergic [6].

5.2. Generalized Capacitive/Inductive N-Ports

By definition, a generalized capacitive/inductive (GCI) state representation is one of the form

$$\begin{aligned} \dot{\underset{\sim}{x}} &= \underset{\sim}{u} \\ \underset{\sim}{y} &= \underset{\sim}{g}(\underset{\sim}{x}) \end{aligned} \tag{5.2}$$

where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair, $\Sigma = U = \mathbb{R}^n$, $\mathcal{U} = L^1_{loc}(\mathbb{R}^+ \to \mathbb{R}^n)$, and $\underset{\sim}{g} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. We define a GCI n-port to be an n-port with a GCI state representation.[2]

[2]Note that our recurrent example of a 1-volt d.c. source is both a resistive 1-port and a GCI 1-port.

Lemma 5.3. Let S denote a GCI state representation. Then S is state-observable $\Leftrightarrow$ for any two distinct states $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \Sigma = \mathbb{R}^n$, there exists a vector $\underset{\sim}{w} \in \mathbb{R}^n$ such that $\underset{\sim}{g}(\underset{\sim}{x}_1+\underset{\sim}{w}) \neq \underset{\sim}{g}(\underset{\sim}{x}_2+\underset{\sim}{w})$.

In particular, a state representation of this form will not be state-observable if $\underset{\sim}{g}(\cdot)$ is a constant (this includes the case of a capacitive state representation for a 1-volt d.c. source). The proof is given in Appendix A.

Lemma 5.4. Let N denote an n-port with a GCI state representation S. Then the following statements are true.

a) S is lossless $\Leftrightarrow$

$$\underset{\sim}{g} = \underset{\sim}{\nabla}\phi, \tag{5.3}$$

where $\phi : \Sigma \to \mathbb{R}$ is a C^1 scalar function.

b) If N is lossless, then S is lossless.

c) If S is lossless and state-observable, then N is lossless.

The proof is given in Appendix A. Unlike Lemma 5.2, state-observability plays a genuine role in this case. The example of a capacitive state representation $\dot{q} = i$, $v = 1$ for a 1-volt d.c. source satisfies (5.3) with $\phi(q) = q$, but such a 1-port is not lossless.

The difference between statement a) of Lemma 5.4 and Theorem 4 of [1] is simply that ϕ need not be bounded from below in the present case. Therefore the following two corollaries are immediate.

Corollary. A passive GCI state representation is lossless.

Corollary. Let S denote a capacitive or inductive state representation in which $\underset{\sim}{g}(\cdot)$ is C^1. If S is lossless, then S is reciprocal.

5.3. Generalized N-Port Memristors

We define a generalized memristive state representation to be one of the form

$$\dot{\underset{\sim}{x}} = \underset{\sim}{u} \qquad (5.4)$$
$$\underset{\sim}{y} = \underset{\sim}{R}(\underset{\sim}{x})\underset{\sim}{u}$$

where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair, $\Sigma = U = \mathbb{R}^n$, $\underset{\sim}{R} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is continuous, and $\mathcal{U} = L^2_{loc}(\mathbb{R}^+ \to \mathbb{R}^n)$. An n-port with such a state representation is, by definition, a generalized n-port memristor.

Lemma 5.5. Let S denote a generalized memristive state representation. Then S is state-observable $\Leftrightarrow$ for any two distinct states $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \Sigma$, there exists a vector $\underset{\sim}{w} \in \mathbb{R}^n$ such that $\underset{\sim}{R}(\underset{\sim}{x}_1+\underset{\sim}{w}) \neq \underset{\sim}{R}(\underset{\sim}{x}_2+\underset{\sim}{w})$.

In particular, $\underset{\sim}{R}(\cdot)$ cannot be constant in a state-observable state representation of this kind. The proof is given in Appendix A.

Lemma 5.6. Let N denote an n-port with a generalized memristive state representation S. Then the following statements are true.

a) S is lossless $\Leftrightarrow$ $\underset{\sim}{R}(\underset{\sim}{x})$ is antisymmetric at each point $\underset{\sim}{x} \in \mathbb{R}^n$.

b) If N is lossless, then S is lossless.

c) If S is lossless and state-observable, then N is lossless.

It follows that a lossless generalized n-port memristor is nonenergic [6]. The proof of Lemma 5.6 is given in Appendix A.

If we enlarge the class of mathematical representations for n-ports to include dynamical systems [11], then the converse of statement b) is true. The proof proceeds by partitioning the state space Σ of S into equivalence classes, where the equivalence relation is given by Definition 3.1. Each equivalence class in Σ becomes the state for a new, totally observable dynamical system representation S_o for N [19, Lemma 5.1.6]. (The states of S_o are not points in $\mathbb{R}^m$, but rather subsets of $\mathbb{R}^m$; thus, S_o is not a state representation in the sense of [1, Def. 1], but it *is* a dynamical system.) Since all input-trajectory pairs of S consume zero energy over every time interval, S_0 is lossless; thus, $\mathcal{N}$ is lossless.

5.4. Linear N-Ports

By definition, a linear (time-invariant, finite dimensional, hybrid) state representation is one of the form

$$\dot{x} = Ax + Bu \tag{5.5a}$$

$$y = Cx + Du \tag{5.5b}$$

where u and y form a hybrid pair; $U = \mathbb{R}^n$ and $\Sigma = \mathbb{R}^m$; A, B, C, and D are real constant matrices of appropriate dimension; and $\mathcal{U} = L^2_{loc}(\mathbb{R}^+ \to \mathbb{R}^n)$. An n-port is defined to be linear if it has a linear state representation.

In the following theorem, the superscript "T" denotes the transpose of a matrix, i.e., M^T is the transpose of the matrix M. The symbol $\lambda(A)$ will denote the set of eigenvalues of the m×m matrix A, i.e., $\lambda(A) \triangleq \{s \in \mathbb{C} : \det(sI-A) = 0\}$.

Theorem 5.1. Let S denote a linear state representation as in (5.5). Let i) through vii) denote the following statements:

i) S is lossless.

ii) The hybrid matrix transfer function of S,

$$H(s) \triangleq C(sI-A)^{-1}B + D,$$

satisfies $H(j\omega) = -H^T(-j\omega)$ for all $\omega \in \mathbb{R}$ such that $j\omega \notin \lambda(A)$.

iii) The hybrid matrix transfer function of S satisfies $H(s) = -H^T(-s)$ for all $s \in \mathbb{C} \setminus \lambda(A)$.

iv) $\int_0^\infty \langle v(t), i(t) \rangle \, dt = 0$ for all L^2 admissible pairs of S with zero initial state.

v) $\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle v(t), i(t) \rangle dt = 0$ for all bounded admissible pairs of S.

vi) $D = -D^T$ (i.e., D is antisymmetric) and there exists a symmetric matrix K such that $KA = -A^T K$ (i.e., KA is antisymmetric) and $KB = C^T$.

vii) S has a quadratic conservative potential energy function $\phi : \Sigma \to \mathbb{R}$.

Then the following conclusions are valid:

a) vi) $\Leftrightarrow$ vii) $\Rightarrow$ i) $\Rightarrow$ ii) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv)

b) If S is completely controllable, then statements i) through vii) are equivalent.

The proof is given in Appendix C. Statement ii) is less restrictive than the traditional losslessness criterion for the hybrid matrix transfer function [4,12]. The traditional criterion is derived under the assumption that the state representation is passive, as well as lossless, and it includes the following additional conditions: *) all poles of $\underset{\sim}{H}(\cdot)$ lie on the imaginary axis, and **) the poles of $\underset{\sim}{H}(\cdot)$ are simple and the residue matrix at those poles is Hermitian and positive semidefinite. The hybrid scalar transfer function $H(s) = (s^4+s^2-1)/(s^5-s^3)$ does not satisfy *) or **), but it <u>does</u> satisfy statement iii). therefore it is the transfer function of a completely controllable state representation of the form (5.5) which is lossless, but not passive.

The simple example $\dot{x} = x$, $y = x$ satisfies statement ii) but is not lossless; therefore, ii) does not imply i) in the absence of complete controllability. This example also satisfies statement v); therefore, v) does not imply i) in the absence of complete controllability. We simply do not know whether i) implies v) in the absence of complete controllability. Likewise, we do not know whether i) implies vii) in the absence of complete controllability.

<u>Lemma 5.7</u>. Suppose that an n-port N has a completely controllable linear state representation S of the form (5.5). Then N is lossless $\Leftrightarrow$ S is lossless.

The proof is given in Appendix C.

5.5. First-Order N-Ports

A first-order state representation is one for which $\Sigma \subset \mathbb{R}$. An n-port which has a first-order state representation is called a first-order n-port.

For any state representation S, a state $\underset{\sim}{x}_0$ is called a singular state if $\underset{\sim}{f}(\underset{\sim}{x}_0, \underset{\sim}{u}) = 0$ for all $\underset{\sim}{u} \in U$. A state which is not singular is called a nonsingular state. If S is completely controllable, then all states of S are nonsingular.

Lemma 5.8. Suppose that an n-port N has a first-order state representation S. Under these conditions, the following statements are true.

a) S is lossless $\Leftrightarrow$ there exists a function $h : \Sigma \rightarrow \mathbb{R}$ (which is necessarily continuous at each nonsingular state) such that $p(x, \underset{\sim}{u}) = h(x)\, f(x, \underset{\sim}{u})$ for all $(x, \underset{\sim}{u}) \in \Sigma \times U$.

b) If N is lossless and S is input-observable, then S is lossless.

c) If S is lossless and totally observable, then N is lossless.

The proof is given in Appendix A.

Let S be a lossless, completely controllable first-order state representation, and let $h : \Sigma \rightarrow \mathbb{R}$ denote the function in statement a) of Lemma 5.8. Define $\phi : \Sigma \rightarrow \mathbb{R}$ by $\phi(x) \triangleq \int_{x_0}^{x} h(x')dx'$, where x_0 is any fixed point in Σ. Then $\phi(\cdot)$ is a C^1 function which satisfies $p(x, \underset{\sim}{u}) = \frac{d\phi(x)}{dx} f(x, \underset{\sim}{u})$ for all $(x, \underset{\sim}{u}) \in \Sigma \times U$. Hence, the existence of a C^1 conservative potential energy function is a necessary and sufficient losslessness condition for completely controllable first-order systems (cf. Lemma 2.3 and the remarks following it).

VI. The Realization of Lossless N-Ports and a Canonical Algebraic Form

6.1. Lossless Realizations

Our treatment will be based on the use of a C^1 conservative potential energy function, and will parallel quite closely the passive realization theory given in [1, section VII].

Consider the n-port N in Fig. 9 formed by connecting the capacitive m-port C to the resistive (n+m)-port R. It is assumed that C is charge-controlled and lossless; thus, by lemma 5.4, there exists a C^1 function $\psi : \mathbb{R}^m \to \mathbb{R}$ such that $\underset{\sim}{e} = \underset{\sim}{\nabla}\psi(\underset{\sim}{q})$. The constitutive relation of R is assumed to be defined by the continuous functions $\hat{\underset{\sim}{f}} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\hat{\underset{\sim}{g}} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\underset{\sim}{j} = \hat{\underset{\sim}{f}}(\underset{\sim}{e},\underset{\sim}{v}) \qquad (6.1)$$
$$\underset{\sim}{i} = \hat{\underset{\sim}{g}}(\underset{\sim}{e},\underset{\sim}{v}).$$

Substituting the equation $\dot{\underset{\sim}{q}} = \underset{\sim}{j}$ and the constitutive relation of C into (6.1), we obtain a state representation S for N with the following state and output equations:

$$\dot{\underset{\sim}{q}} = \hat{\underset{\sim}{f}}(\underset{\sim}{\nabla}\psi(\underset{\sim}{q}),\underset{\sim}{v}) \qquad (6.2)$$
$$\underset{\sim}{i} = \hat{\underset{\sim}{g}}(\underset{\sim}{\nabla}\psi(\underset{\sim}{q}),\underset{\sim}{v}).$$

Technical Assumptions

We assume throughout the remainder of this subsection that $U = \mathbb{R}^n$, $\Sigma = \mathbb{R}^m$, and that U satisfies the mild technical assumptions given in lemma 2.3. Also the phrase "R is lossless" will mean that R is lossless when its inputs are restricted to $\underset{\sim}{\nabla}\psi[\mathbb{R}^m] \times \mathbb{R}^n \subset \mathbb{R}^m \times \mathbb{R}^n$.

Lemma 6.1. The function $\psi(\cdot)$ is a conservative potential energy function for the state representation S defined above $\Leftrightarrow$ R is lossless.

To see this, recall from Definition 2.3 and lemma 2.3 that $\psi(\cdot)$ is a conservative potential energy function for this system if and only if

$$\langle \underset{\sim}{\nabla}\psi(\underset{\sim}{q}), \hat{\underset{\sim}{f}}(\underset{\sim}{\nabla}\psi(\underset{\sim}{q}),\underset{\sim}{v}) \rangle = \langle \underset{\sim}{v}, \hat{\underset{\sim}{g}}(\underset{\sim}{\nabla}\psi(\underset{\sim}{q}),\underset{\sim}{v}) \rangle \qquad (6.3)$$

for all $\underset{\sim}{q},\underset{\sim}{v} \in \mathbb{R}^m \times \mathbb{R}^n$. Since $\underset{\sim}{\nabla}\psi = \underset{\sim}{e}$, we can rewrite (6.3) as

$$\langle \underset{\sim}{v}, \hat{\underset{\sim}{g}}(\underset{\sim}{e},\underset{\sim}{v})\rangle + \langle \underset{\sim}{e}, -\hat{\underset{\sim}{f}}(\underset{\sim}{e},\underset{\sim}{v})\rangle = \langle \underset{\sim}{v}, \underset{\sim}{i}\rangle - \langle \underset{\sim}{e}, \underset{\sim}{j}\rangle = 0,$$

which is equivalent to the losslessness of $\mathcal{R}$ once the reference direction for $\underset{\sim}{j}$ is taken into account.

Definition 6.1. The n-port $\mathcal{N}$ in Fig. 9 is a realization of the state representation

$$\begin{aligned} \dot{\underset{\sim}{x}} &= \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{v}) \\ \underset{\sim}{i} &= \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{v}) \end{aligned} \tag{6.4}$$

with the technical assumptions listed above if

$$\begin{aligned} \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{v}) &= \hat{\underset{\sim}{f}}\big(\nabla\psi(\underset{\sim}{x}),\underset{\sim}{v}\big) \\ \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{v}) &= \hat{\underset{\sim}{g}}\big(\nabla\psi(\underset{\sim}{x}),\underset{\sim}{v}\big), \quad \forall(\underset{\sim}{x},\underset{\sim}{v}) \in \mathbb{R}^m \times \mathbb{R}^n. \end{aligned} \tag{6.5}$$

It is a lossless realization if $\mathcal{R}$ and $\mathcal{C}$ are both lossless.

We view the multiports $\mathcal{R}$ and $\mathcal{C}$ as given quantities -- we are not concerned with the difficult and unsolved problem of synthesizing these nonlinear multiports. It is clear that any voltage-controlled state representation S has a realization in which $\mathcal{C}$ is lossless and linear: if each port of $\mathcal{C}$ is a 1-farad capacitor, then $\nabla\psi(\underset{\sim}{q}) = \underset{\sim}{q}$ and we obtain a realization by choosing $\hat{\underset{\sim}{f}}(\cdot,\cdot) = \underset{\sim}{f}(\cdot,\cdot)$ and $\hat{\underset{\sim}{g}}(\cdot,\cdot) = \underset{\sim}{g}(\cdot,\cdot)$; in general, however, the resistive (m+n)-port $\mathcal{R}$ will not be lossless for such a realization.

The following theorem is an immediate consequence of the preceding lemma and definition.

Theorem 6.1. Suppose the state representation S, given in (6.4) along with the technical assumptions, is lossless and further that we have found a C^1 conservative potential energy function $\psi : \mathbb{R}^m \to \mathbb{R}$ such that (6.5) holds. Then the n-port in Fig. 9 is a lossless realization of S.

Since $\mathcal{C}$ is clearly lossless under these conditions, the point of theorem 6.1 is that $\mathcal{R}$ is lossless as well, precisely because $\psi(\cdot)$ is a conservative potential energy function. The problem with theorem 6.1 is of course that we do not generally know how to find $\hat{f}(\cdot,\cdot)$ and $\hat{g}(\cdot,\cdot)$ satisfying (6.5); we do not even know in general when they exist. The following corollary gives us one special case in which these problems do not arise.

Corollary. Suppose the state representation S, given in (6.4) along with the technical assumptions, is lossless and that there exists a C^1 conservative potential energy function $\psi : \mathbb{R}^m \to \mathbb{R}$ such that $\nabla\psi : \mathbb{R}^m \to \mathbb{R}^m$ is 1-1. Then S has a lossless realization as in Fig. 9.

In this case we can simply construct $\hat{f}(\cdot,\cdot)$ and $\hat{g}(\cdot,\cdot)$ as follows:

$$\hat{f}\big(\nabla\psi(x),v\big) = f[(\nabla\psi)^{-1}\circ\big(\nabla\psi(x)\big),v]$$

$$\hat{g}\big(\nabla\psi(x),v\big) = g[(\nabla\psi)^{-1}\circ\big(\nabla\psi(x)\big),v].$$

For simplicity, we have discussed only voltage-controlled state representations in this subsection. Actually, analogous results hold for any state representation in which u and y form a hybrid pair.

Theorem 6.1 and its corollary show that the recovery of a C^1 conservative potential energy function from a given lossless state representation S is an important first step toward obtaining a lossless realization of S.

6.2. A General Algebraic Form for the State Equations of Lossless Systems

The lossless realization of Theorem 6.1 and Fig. 9 suggests a more explicit general algebraic form for the state and output equations of a lossless n-port. Our attention will focus on the resistive (n+m)-port R. In lemma 5.2 it was shown that every lossless resistive k-port is nonenergic. And two of the authors have shown in [6] that there is a certain canonical form for the constitutive relation of nonenergic resistive elements: if $u,y \in \mathbb{R}^k$ are a hybrid pair and we let $\mathbb{R}_A^{k\times k}$ denote the class of all real k×k antisymmetric matrices, then the constitutive relation $y = g(u)$

of a nonenergic k-port resistor can be written [6] in the form

$$\underset{\sim}{y} = [\underset{\sim}{A}(\underset{\sim}{u})]\underset{\sim}{u}, \tag{6.6}$$

where $\underset{\sim}{A}(\cdot) : \mathbb{R}^{k} \to \mathbb{R}_{A}^{k\times k}$.

Since $\underset{\sim}{y} \triangleq (-\underset{\sim}{j},\underset{\sim}{i})$ and $\underset{\sim}{u} \triangleq (\underset{\sim}{e},\underset{\sim}{v})$ are a hybrid pair for R, the constitutive relation (6.1) can be written in the form of (6.6). Partitioning the antisymmetric matrix $\underset{\sim}{A}$ into blocks corresponding to the partitioning of $\underset{\sim}{u}$ and $\underset{\sim}{y}$, we have

$$\begin{pmatrix} -\underset{\sim}{j} \\ \hline \underset{\sim}{i} \end{pmatrix} = \left[\begin{array}{c|c} -\underset{\sim}{A}(\underset{\sim}{e},\underset{\sim}{v}) & -\underset{\sim}{B}(\underset{\sim}{e},\underset{\sim}{v}) \\ \hline \underset{\sim}{B}^{T}(\underset{\sim}{e},\underset{\sim}{v}) & \underset{\sim}{C}(\underset{\sim}{e},\underset{\sim}{v}) \end{array}\right] \begin{pmatrix} \underset{\sim}{e} \\ \hline \underset{\sim}{v} \end{pmatrix}$$

or, upon eliminating the minus sign from $\underset{\sim}{j}$,

$$\begin{pmatrix} \underset{\sim}{j} \\ \hline \underset{\sim}{i} \end{pmatrix} = \left[\begin{array}{c|c} \underset{\sim}{A}(\underset{\sim}{e},\underset{\sim}{v}) & \underset{\sim}{B}(\underset{\sim}{e},\underset{\sim}{v}) \\ \hline \underset{\sim}{B}^{T}(\underset{\sim}{e},\underset{\sim}{v}) & \underset{\sim}{C}(\underset{\sim}{e},\underset{\sim}{v}) \end{array}\right] \begin{pmatrix} \underset{\sim}{e} \\ \hline \underset{\sim}{v} \end{pmatrix}, \tag{6.7}$$

where $\underset{\sim}{A}(\cdot) : \mathbb{R}^{m+n} \to \mathbb{R}_{A}^{m\times m}$, $\underset{\sim}{C}(\cdot) : \mathbb{R}^{m+n} \to \mathbb{R}_{A}^{n\times n}$, $\underset{\sim}{B}(\cdot) : \mathbb{R}^{m+n} \to \mathbb{R}^{m\times n}$.

Equation (6.7) is an explicit form in which (6.1) can always be expanded, so long as R is nonenergic. Substituting this expansion into (6.4) and (6.5), we have the following form of state and output equations for a voltage-controlled lossless n-port:

$$\begin{aligned} \dot{\underset{\sim}{x}} &= [\underset{\sim}{A}(\nabla\psi(\underset{\sim}{x}),\underset{\sim}{v})](\nabla\psi(\underset{\sim}{x})) + [\underset{\sim}{B}(\nabla\psi(\underset{\sim}{x}),\underset{\sim}{v}) \; (\underset{\sim}{v}) \\ \underset{\sim}{i} &= [\underset{\sim}{B}^{T}(\nabla\psi(\underset{\sim}{x}),\underset{\sim}{v})](\nabla\psi(\underset{\sim}{x})) + [\underset{\sim}{C}(\nabla\psi(\underset{\sim}{x}),\underset{\sim}{v})](\underset{\sim}{v}) \end{aligned} \tag{6.8}$$

where $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ are matrix-valued functions whose dimensions are given beneath (6.7), and $\underset{\sim}{A}$ and $\underset{\sim}{C}$ are antisymmetric. In order to have a compact statement of this result which repeats all the necessary assumptions involved, we summarize this development as a theorem.

<u>Theorem 6.2</u>. Suppose the voltage-controlled state representation S, given in (6.4) along with the technical assumptions, is lossless and

further that we have found a C^1 conservative potential energy function $\psi : \mathbb{R}^m \to \mathbb{R}$ for S such that (6.5) holds. Then the equations (6.4) can always be written in the form given in (6.8).

There is an obvious extension of theorem 6.2 to the case in which S is not necessarily voltage-controlled but $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are any hybrid pair.

The following example illustrates the recovery and the use of the conservative potential energy function in realizing an n-port.

<u>Example 6.1</u>.

Consider a 1-port with state equations

$$\begin{aligned} \dot{x}_1 &= x_2^4 \\ \dot{x}_2 &= -x_1^2 x_2^2 - 2v_1 \\ i_1 &= -2x_2^2 \end{aligned} \qquad (6.9)$$

Note that this 1-port is uncontrollable since $\dot{x}_1(\cdot) \geq 0$. For any input-output pair from $\underset{\sim}{x}_o$ to $\underset{\sim}{x}$ the energy consumed can be obtained as follows

$$\int_{\substack{0 \\ \underset{\sim}{x}_o \to \underset{\sim}{x}}}^{t} v_1(\tau)\, i_1(\tau) d\tau = \int_{\substack{0 \\ \underset{\sim}{x}_o \to \underset{\sim}{x}}}^{t} [\dot{x}_2(\tau) + x_1^2(\tau) x_2^2(\tau)] x_2^2(\tau) d\tau = \int_o^t (x_2^2 \dot{x}_2 + x_1^2 \dot{x}_1) dt$$

$$= x_2^3/3 - x_{20}^3/3 + x_1^3/3 - x_{10}^3/3$$

where $\underset{\sim}{x}_o \triangleq [x_{10}, x_{20}]^T$ and $\underset{\sim}{x}(\cdot) \triangleq [x_1(\cdot), x_2(\cdot)]^T$ is the state trajectory corresponding to the input $v_1(\cdot)$. Thus, the conservative potential energy function can be taken as $\psi(\underset{\sim}{x}) = x_2^3/3 + x_1^3/3$ with $\underset{\sim}{\nabla}\psi(\underset{\sim}{x}) = [x_1^2, x_2^2]$ and now, rewriting the state eqs. (6.9) in the form of (6.8) yields:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & x_2^2 \\ -x_2^2 & 0 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} v_1 \\ i_1 &= [0 \;\; -2] \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} \end{aligned} \qquad (6.10)$$

In accordance with equations (6.1) and (6.2), the constitutive relation of the capacitive 2-port is characterized by $\underset{\sim}{\nabla}\psi$ and the constitutive relation of the algebraic 3-port can be obtained from (6.10) as

$$\begin{bmatrix} j_1 \\ j_2 \end{bmatrix} = \begin{bmatrix} 0 & e_2 \\ -e_2 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} v_1$$

$$i_1 = [0 \ -2] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} .$$

Allowing a hybrid formulation for the resistive 3-port, another realization can be given as shown in Fig. 10. Two comments are in order; although $\underset{\sim}{\nabla}\psi$ is not bijective it is still possible to recover the constitutive relation of the algebraic (n+m)-port, which shows why the assumption that $\underset{\sim}{\nabla}\psi$ is bijective has not been made in Theorem 6.1 and, as ψ is not bounded from below, any realization of this n-port has to be active [1, Theorem 4].

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APPENDICES

Appendix A - Proofs of Lemmas

Proof of Lemma 2.1.

Proof of a). To prove statement a), we assume that S is lossless. To show cyclo-losslessness, let $\underset{\sim}{x}^*$ be any state in Σ, $T \geq 0$ any time and $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[0,T]$ any input-trajectory pair from $\underset{\sim}{x}^*$ to $\underset{\sim}{x}^*$, i.e. such that $\underset{\sim}{x}(0) = \underset{\sim}{x}(T) = \underset{\sim}{x}^*$. Then $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[0,0]$ is also [1, Def. 8] an input-trajectory pair from $\underset{\sim}{x}^*$ to $\underset{\sim}{x}^*$, and the energy consumed by $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[0,0]$ is zero. Therefore the energy consumed by $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[0,T]$ must also be zero, since the system is lossless.

Proof of b) and c). First it will be shown that the hypotheses of statement b) imply that S is cyclo-lossless. The proof will then be completed by showing that statement c) is true.

Hence, assume the hypotheses of b). Let $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[0,T]$ be any input-trajectory pair with $\underset{\sim}{x}(0) = \underset{\sim}{x}(T)$, and let E denote the energy consumed by $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[0,T]$. Define $\underset{\sim}{x}' \triangleq \underset{\sim}{x}(0) = \underset{\sim}{x}(T)$. By complete controllability, there exist input-trajectory pairs $\{\underset{\sim}{u}_1(\cdot),\underset{\sim}{x}_1(\cdot)\}|[0,T_1]$ and $\{\underset{\sim}{u}_2(\cdot),\underset{\sim}{x}_2(\cdot)\}|[0,T_2]$ from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}'$ and from $\underset{\sim}{x}'$ to $\underset{\sim}{x}_0$, respectively, where $\underset{\sim}{x}_0$ is the state mentioned in statement b). Let E_1 and E_2 denote the energy consumed by $\{\underset{\sim}{u}_1(\cdot),\underset{\sim}{x}_1(\cdot)\}|[0,T_1]$ and $\{\underset{\sim}{u}_2(\cdot),\underset{\sim}{x}_2(\cdot)\}|[0,T_2]$, respectively. Define $\underset{\sim}{u}_3(\cdot)$ by

$$\underset{\sim}{u}_3(t) \triangleq \begin{cases} \underset{\sim}{u}_1(t), & 0 \leq t \leq T_1 \\ \underset{\sim}{u}_2(t-T_1), & t > T_1. \end{cases}$$

Then $\underset{\sim}{u}_3(\cdot) \in \mathcal{U}$, since $\mathcal{U}$ is closed under concatenation. Define $\underset{\sim}{x}_3(\cdot)$ by

$$\underset{\sim}{x}_3(t) \triangleq \begin{cases} \underset{\sim}{x}_1(t), & 0 \leq t \leq T_1 \\ \underset{\sim}{x}_2(t-T_1), & t > T_1. \end{cases}$$

Since the state equations are time-invariant and $\underset{\sim}{x}_1(T_1) = \underset{\sim}{x}_2(0)$, $\{\underset{\sim}{u}_3(\cdot), \underset{\sim}{x}_3(\cdot)\}$ is a valid input-trajectory pair. Note that the energy consumed by $\{\underset{\sim}{u}_3(\cdot),\underset{\sim}{x}_3(\cdot)\}|[0,T_1+T_2]$ is $E_1+ E_2$; moreover, $\underset{\sim}{x}_3(0) = \underset{\sim}{x}_3(T_1+T_2) = \underset{\sim}{x}_0$. Thus, $E_1 + E_2 = 0$. Now define

$$\underset{\sim}{u}_4(t) \triangleq \begin{cases} \underset{\sim}{u}_1(t), & 0 \le t \le T_1 \\ \underset{\sim}{u}(t-T_1), & T_1 < t \le T_1 + T \\ \underset{\sim}{u}_2(t-T_1-T), & t > T_1 + T. \end{cases}$$

Then $\underset{\sim}{u}_4(\cdot) \in \mathcal{U}$, since $\mathcal{U}$ is closed under concatenation. Define $\underset{\sim}{x}_4(\cdot)$ by

$$\underset{\sim}{x}_4(t) \triangleq \begin{cases} \underset{\sim}{x}_1(t), & 0 \le t \le T_1 \\ \underset{\sim}{x}(t-T_1), & T_1 < t \le T_1 + T \\ \underset{\sim}{x}_2(t-T_1-T), & t > T_1 + T. \end{cases}$$

Then $\{\underset{\sim}{u}_4(\cdot),\underset{\sim}{x}_4(\cdot)\}$ is a valid input-trajectory pair. The energy consumed by $\{\underset{\sim}{u}_4(\cdot),\underset{\sim}{x}_4(\cdot)\}|[0,T_1+T_2+T]$ is $E_1 + E_2 + E$; moreover, $\underset{\sim}{x}_4(0) = \underset{\sim}{x}_4(T_1+T_2+T) = \underset{\sim}{x}_0$. Thus $E_1 + E_2 + E = 0$. But it has already been shown that $E_1 + E_2 = 0$; hence, $E = 0$. This shows that S is cyclo-lossless.

It only remains to prove statement c). Assume that S is cyclo-lossless and completely controllable. We will show that a contradiction emerges if S is not lossless. If it isn't lossless, then there exist two states $\underset{\sim}{x}_a$, $\underset{\sim}{x}_b$, two admissible pairs $\{\underset{\sim}{u}_5(\cdot),\underset{\sim}{x}_5(\cdot)\}$, $\{\underset{\sim}{u}_6(\cdot),\underset{\sim}{x}_6(\cdot)\}$, and two times T_5, $T_6 \ge 0$ such that $\{\underset{\sim}{u}_5(\cdot),\underset{\sim}{x}_5(\cdot)\}|[0,T_5]$ and $\{\underset{\sim}{u}_6(\cdot), \underset{\sim}{x}_6(\cdot)\}|[0,T_6]$ are input-trajectory pairs from $\underset{\sim}{x}_a$ to $\underset{\sim}{x}_b$ but $E_5 \ne E_6$, where E_5 is the energy consumed by $\{\underset{\sim}{u}_5(\cdot),\underset{\sim}{x}_5(\cdot)\}|[0,T_5]$ and E_6 is the energy consumed by $\{\underset{\sim}{u}_6(\cdot),\underset{\sim}{x}_6(\cdot)\}|[0,T_6]$. Since the system is completely controllable, there exists an input-trajectory pair $\{\underset{\sim}{u}_7(\cdot),\underset{\sim}{x}_7(\cdot)\}|[0,T_7]$ from $\underset{\sim}{x}_b$ to $\underset{\sim}{x}_a$. We let E_7 be the energy consumed by $\{\underset{\sim}{u}_7(\cdot),\underset{\sim}{x}_7(\cdot)\}|[0,T_7]$.

Since $E_5 \neq E_6$, either $E_5 + E_7 \neq 0$ or $E_6 + E_7 \neq 0$, or both. For definiteness, suppose $E_5 + E_7 \neq 0$. We define $\underset{\sim}{u}_8(\cdot)$ by

$$\underset{\sim}{u}_8(t) \triangleq \begin{cases} \underset{\sim}{u}_5(t), & 0 \leq t \leq T_5 \\ \underset{\sim}{u}_7(t-T_5), & t > T_5. \end{cases}$$

Then $\underset{\sim}{u}_8(\cdot) \in \mathcal{U}$ since $\mathcal{U}$ is closed under concatenation. We define $\underset{\sim}{x}_8(\cdot)$ by

$$\underset{\sim}{x}_8(t) \triangleq \begin{cases} \underset{\sim}{x}_5(t), & 0 \leq t \leq T_5 \\ \underset{\sim}{x}_7(t-T_5), & t > T_5. \end{cases}$$

Since the state equations are time-invariant, $\{\underset{\sim}{u}_8(\cdot),\underset{\sim}{x}_8(\cdot)\}$ is a valid input-trajectory pair. And the energy consumed by $\{\underset{\sim}{u}_8(\cdot),\underset{\sim}{x}_8(\cdot)\}|[0,T_5+T_7]$ is $E_5 + E_7 \neq 0$. Since $\underset{\sim}{x}_8(0) = \underset{\sim}{x}_8(T_5+T_7) = \underset{\sim}{x}_a$, the system is not cyclo-lossless, contradicting our assumption. ¤

Proof. of Lemma 2.2: Since $\sum$ is reachable from $\hat{\underset{\sim}{x}}$, $\psi(\cdot)$ is defined on all of $\sum$. And since $t \rightarrow p(t)$ is assumed to be locally L^1 [1, Def. 5 and standing asspt. #4], $\psi(\underset{\sim}{x})$ is finite at each $\underset{\sim}{x} \in \sum$.

Given any two points $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ of $\sum$, the energy consumed by an input-trajectory pair from $\underset{\sim}{x}_1$ to $\underset{\sim}{x}_2$ is a function of $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ only, since S is lossless, and can be written $E(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$. To show that $\psi(\cdot)$ is a conservative potential energy function, we must show that $E(\underset{\sim}{x}_1,\underset{\sim}{x}_2) = \psi(\underset{\sim}{x}_2) - \psi(\underset{\sim}{x}_1)$ for any two points $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ such that $\underset{\sim}{x}_2$ is reachable from $\underset{\sim}{x}_1$ (see Fig. A.1).

Let $\{\underset{\sim}{u}_1(\cdot),\underset{\sim}{x}_1(\cdot)\}|[0,T_1]$ be an input-trajectory pair from $\hat{\underset{\sim}{x}}$ to $\underset{\sim}{x}_1$, $\{\underset{\sim}{u}_2(\cdot),\underset{\sim}{x}_2(\cdot)\}|[0,T_2]$ an input-trajectory pair from $\hat{\underset{\sim}{x}}$ to $\underset{\sim}{x}_2$, and $\{\underset{\sim}{u}_3(\cdot),\underset{\sim}{x}_3(\cdot)\}|[0,T_3]$ an input-trajectory pair from $\underset{\sim}{x}_1$ to $\underset{\sim}{x}_2$. Let E_1, E_2 and E_3 be the energy consumed by these input-trajectory pairs, respectively.

Then $E_1 = \psi(\underset{\sim}{x}_1)$, $E_2 = \psi(\underset{\sim}{x}_2)$, and $E_3 = E(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$. Let $\underset{\sim}{u}_4(\cdot)$ and $\underset{\sim}{x}_4(\cdot)$ be defined by

$$\underset{\sim}{u}_4(t) = \begin{cases} \underset{\sim}{u}_1(t), & 0 \le t \le T_1 \\ \underset{\sim}{u}_3(t-T_1), & t > T_1 \end{cases} \qquad \underset{\sim}{x}_4(t) = \begin{cases} \underset{\sim}{x}_1(t), & 0 \le t \le T_1 \\ \underset{\sim}{x}_3(t-T_1), & t > T_1. \end{cases}$$

Then $\underset{\sim}{u}_4(\cdot) \in \mathcal{U}$ since $\mathcal{U}$ is closed under concatenation, and $\{\underset{\sim}{u}_4(\cdot),\underset{\sim}{x}_4(\cdot)\}$ is an input-trajectory pair of S because the state equations are time-invariant. The energy consumed by $\{\underset{\sim}{u}_4(\cdot),\underset{\sim}{x}_4(\cdot)\}|[0,T_1+T_3]$ is $E_1 + E_3 = \psi(\underset{\sim}{x}_1) + E(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$. Since $\{\underset{\sim}{u}_4(\cdot),\underset{\sim}{x}_4(\cdot)\}|[0,T_1+T_3]$ is an input-trajectory pair from $\hat{\underset{\sim}{x}}$ to $\underset{\sim}{x}_2$, this must equal $\psi(\underset{\sim}{x}_2)$, i.e. $\psi(\underset{\sim}{x}_1) + E(\underset{\sim}{x}_1,\underset{\sim}{x}_2) = \psi(\underset{\sim}{x}_2)$. Rearranging this equation yields $E(\underset{\sim}{x}_1,\underset{\sim}{x}_2) = \psi(\underset{\sim}{x}_2) - \psi(\underset{\sim}{x}_1)$, as claimed. ⌑

Proof of Lemma 2.3

($\Leftarrow$) If we integrate (2.1a) along any input-trajectory pair $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$, the result is (2.1).

($\Rightarrow$) Let $(\underset{\sim}{x}_0,\underset{\sim}{u}_0)$ be an arbitrary point of $\sum \times \mathcal{U}$, and $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ be an input-trajectory pair such that $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$, $\underset{\sim}{u}(0) = \underset{\sim}{u}_0$, and $\underset{\sim}{u}(\cdot)$ is continuous at $t = 0$. Then since $\phi(\cdot)$ is C^1, $\underset{\sim}{u}(\cdot)$ is continuous at $t = 0$, and $\underset{\sim}{x}(\cdot)$ is C^1 at $t = 0$, we have

$$\langle \nabla\phi(\underset{\sim}{x}_0),\underset{\sim}{f}(\underset{\sim}{x}_0,\underset{\sim}{u}_0)\rangle = \frac{d\phi(\underset{\sim}{x}(t))}{dt}\bigg|_{t=0}$$

$$= \lim_{t\to 0^+} \frac{\psi(\underset{\sim}{x}(t))-\psi(\underset{\sim}{x}(0))}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt = p(\underset{\sim}{x}_0,\underset{\sim}{u}_0).$$

Since $\underset{\sim}{x}_0$ and $\underset{\sim}{u}_0$ were arbitrary, this concludes the proof. ⌑

Proof of Lemma 2.4. Suppose we are given an arbitrary admissible pair

$\{v(\cdot), i(\cdot)\}$ with some initial state $\hat{x}$ and an arbitrary time $T \geq 0$. Then there exists an input-trajectory pair $\{u(\cdot), x(\cdot)\}|[0,T]$ such that $x(0) = \hat{x}$ and $\{v(\cdot), i(\cdot)\}|[0,T] = \{V(x(\cdot), u(\cdot)), I(x(\cdot), u(\cdot))\}|[0,T]$. Since S is completely controllable, there exists a return path from $x(T)$ to $x(0) = \hat{x}$, i.e., a time T_1 and an input-trajectory pair $\{u_1(\cdot), x_1(\cdot)\}|[0,T_1]$ from $x(T)$ to $x(0)$. Since $\mathcal{U}$ is closed under concatenation, the input $u'(\cdot)$ given by

$$u'(t) = \begin{cases} u(t), & 0 \leq t \leq T \\ u_1(t-T), & t > T \end{cases}$$

is in $\mathcal{U}$. Let $\{u'(\cdot), x'(\cdot)\}$ be the input-trajectory pair such that $x'(0) = x(0) = \hat{x}$. Then $x'(T) = x(T)$ and, since the state equation is time invariant, $x'(T+T_1) = x(0) = \hat{x}$. Define $T' \triangleq T + T_1$ and $\{v'(\cdot), i'(\cdot)\}|[0,T']$ $\triangleq \{V(x'(\cdot), u'(\cdot)), I(x'(\cdot), u'(\cdot))\}|[0,T']$. Then, since $x'(T') = x'(0)$ and every lossless state representation is cyclo-lossless,

$$\int_0^{T'} \langle v'(t), i'(t) \rangle \, dt = 0.$$

Since $\{v'(\cdot), i'(\cdot)\} = \{v(\cdot), i(\cdot)\}$ on $[0,T]$ by construction, this concludes the proof. ¤

<u>Proof of Lemma 2.5</u>. By Definition 2.3,

$$\int_0^{T} \langle v(t), i(t) \rangle \, dt = \phi(x(T)) - \phi(x(0)).$$

Since $\phi(\cdot)$ is continuous and $\lim_{t\to\infty} x(t) = x(0)$ by assumption,

$$\lim_{T\to\infty}\{\phi(x(T)) - \phi(x(0))\} = \phi(\lim_{T\to\infty} x(T)) - \phi(x(0)) = \phi(x(0)) - \phi(x(0)) = 0,$$

which proves the first assertion.

In the second part we assume $\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot) \in L^2$, which implies that $\langle \underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\rangle \in L^1$ and hence that the integral in (2.5) exists. For each integer $n \geq 1$, define $h_n : \mathbb{R}^+ \to \mathbb{R}$ by

$$h_n(t) \triangleq \begin{cases} \langle \underset{\sim}{v}(t),\underset{\sim}{i}(t)\rangle, & 0 \leq t \leq n, \\ 0, & t > n. \end{cases}$$

Since $\langle \underset{\sim}{v}(t),\underset{\sim}{i}(t)\rangle = \lim_{n\to\infty} h_n(t)$ for all t, the Lebesgue Dominated Convergence theorem [7, p.88] can be applied to obtain

$$\int_0^\infty \langle \underset{\sim}{v}(t),\underset{\sim}{i}(t)\rangle\, dt = \lim_{n\to\infty}\int_0^\infty h_n(t)dt = \lim_{n\to\infty}\int_0^n \langle \underset{\sim}{v}(t),\underset{\sim}{i}(t)\rangle\, dt = 0.$$

¤

Proof of Lemma 2.7. Since ϕ is continuous, it is bounded on every bounded subset of Σ. Since $\underset{\sim}{x}(\cdot)$ is bounded on $\mathbb{R}^+$, it follows that $t \to \phi(\underset{\sim}{x}(t))$ is bounded on $\mathbb{R}^+$. If M is an upper bound on $\phi(\underset{\sim}{x}(t))$, then $|\phi(\underset{\sim}{x}(t_1))-\phi(\underset{\sim}{x}(t_2))| \leq 2M$, $\forall\, t_1, t_2 \in \mathbb{R}^+$. Therefore

$$\left|\frac{1}{T}\int_0^T p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt\right| = \frac{1}{T}\left|\phi(\underset{\sim}{x}(t))-\phi(\underset{\sim}{x}(0))\right| \leq \frac{2M}{T} \to 0 \text{ as } T \to \infty.$$ ¤

Proof of Lemma 2.8. Let $T_0 > 0$ denote the period of $\underset{\sim}{x}(\cdot)$, i.e., for any nonnegative integer k, $\underset{\sim}{x}(t+kT_0) = \underset{\sim}{x}(t)$ for all $t \in \mathbb{R}^+$. Consider the continuous function $E : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$E(t) \triangleq \int_0^t \langle \underset{\sim}{v}(\tau),\underset{\sim}{i}(\tau)\rangle\, d\tau.$$

Since S is lossless, it is cyclo-lossless (Lemma 2.1); therefore,

$$E(t+kT_0)-E(t) = \int_t^{t+kT_0} \langle \underset{\sim}{v}(\tau),\underset{\sim}{i}(\tau)\rangle\, d\tau = 0$$

for all $t \in \mathbb{R}^+$. This shows that $E(\cdot)$ is periodic with period T_0; since $E(\cdot)$ is also continuous, there must exist a finite number $E_m > 0$ such that $|E(t)| \leq E_m$ for all $t \in \mathbb{R}^+$. Thus

$$\left| \frac{1}{T} \int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle \, dt \right| = \left| \frac{1}{T} E(T) \right| \leq \frac{Em}{T} \to 0 \text{ as } T \to \infty. \qquad \square$$

Proof of Lemma 3.1.

($\Leftarrow$) Suppose S_2 is state-observable, and let $\underset{\sim}{b} : \sum_1 \to \sum_2$ be the bijection which appears in the definition of bijective equivalence. Let $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$ be any two distinct states in $\sum_1$. Let C_{1a} be the class of admissible pairs of S_1 with initial state $\underset{\sim}{x}_a$, C_{1b} the admissible pairs of S_1 with initial state $\underset{\sim}{x}_b$, C_{2a} the pairs of S_2 with initial state $\underset{\sim}{b}(\underset{\sim}{x}_a)$, and C_{2b} the pairs of S_2 with initial state $\underset{\sim}{b}(\underset{\sim}{x}_b)$.

Since $\underset{\sim}{x}_a \neq \underset{\sim}{x}_b$ and $\underset{\sim}{b}$ is a bijection, $\underset{\sim}{b}(\underset{\sim}{x}_a) \neq \underset{\sim}{b}(\underset{\sim}{x}_b)$. And since S_2 is state observable, this implies $C_{2a} \neq C_{2b}$. But by the definition of bijective equivalence, $C_{1a} = C_{2a}$ and $C_{1b} = C_{2b}$. Therefore $C_{1a} \neq C_{1b}$, and since $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$ were arbitrary, this implies that S_1 is state observable.

($\Rightarrow$) The assumptions are completely symmetric in S_1 and S_2. $\square$

Proof of Lemma 3.2. The proof proceeds by contradiction. We will assume that there exist input-trajectory pairs $\{\underset{\sim}{u}_1(\cdot), \underset{\sim}{x}_1(\cdot)\}$ of S_1 and $\{\underset{\sim}{u}_2(\cdot), \underset{\sim}{x}_2(\cdot)\}$ of S_2 such that $\{\underset{\sim}{V}_1(\underset{\sim}{x}_1(t), \underset{\sim}{u}_1(t)), \underset{\sim}{I}_1(\underset{\sim}{x}_1(t), \underset{\sim}{u}_1(t))\} = \{\underset{\sim}{V}_2(\underset{\sim}{x}_2(t), \underset{\sim}{u}_2(t)), \underset{\sim}{I}_2(\underset{\sim}{x}_2(t), \underset{\sim}{u}_2(t))\}$ for all $t \in [0,T]$, and $\underset{\sim}{x}_2(0) = \underset{\sim}{\alpha}(\underset{\sim}{x}_1(0))$, but $\underset{\sim}{x}_2(t') \neq \underset{\sim}{\alpha}(\underset{\sim}{x}_1(t'))$ for some $t' \in (0,T]$ (see Fig. A.2). Then we will show that a contradiction emerges.

Suppose $\underset{\sim}{x}_2(t') \neq \underset{\sim}{\alpha}(\underset{\sim}{x}_1(t'))$, as shown in the figure. Since S_2 is state-observable, the class of admissible pairs of S_2 with initial state $\underset{\sim}{x}_2(t')$ is not identical to the class of admissible pairs of S_2 with initial state $\underset{\sim}{\alpha}(\underset{\sim}{x}_1(t'))$. There are two ways this can happen. We discuss them separately below and show that a contradiction emerges in either case. For later use we define $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}|[0,t'] = \{\underset{\sim}{V}_1(\underset{\sim}{x}_1(\cdot), \underset{\sim}{u}_1(\cdot)),\underset{\sim}{I}_1(\underset{\sim}{x}_1(\cdot),\underset{\sim}{u}_1(\cdot))\}|[0,t'] = \{\underset{\sim}{V}_2(\underset{\sim}{x}_2(\cdot),\underset{\sim}{u}_2(\cdot)),\underset{\sim}{I}_2(\underset{\sim}{x}_2(\cdot),\underset{\sim}{u}_2(\cdot))\}|[0,t']$. Then $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}|[0,t']$ is an admissible pair of S_1 with initial state $\underset{\sim}{x}_1(0)$ and an admissible pair of S_2 with initial state $\underset{\sim}{x}_2(0) = \underset{\sim}{\alpha}(\underset{\sim}{x}_1(0))$.

<u>Case 1</u>. There exists an admissible pair $\{\underset{\sim}{v}^*(\cdot),\underset{\sim}{i}^*(\cdot)\}$ of S_2 with initial state $\underset{\sim}{x}_2(t')$ which is not an admissible pair of S_2 with initial state $\underset{\sim}{\alpha}(\underset{\sim}{x}_1(t'))$.

Define $\{\hat{\underset{\sim}{v}}^*(\cdot),\hat{\underset{\sim}{i}}^*(\cdot)\}$ by

$$\{\hat{\underset{\sim}{v}}^*(t),\hat{\underset{\sim}{i}}^*(t)\} = \begin{cases}\{\underset{\sim}{v}(t),\underset{\sim}{i}(t)\}, & 0 \le t < t' \\ \{\underset{\sim}{v}^*(t-t'),\underset{\sim}{i}^*(t-t')\}, & t \ge t'.\end{cases} \tag{A.1}$$

We claim that $\{\hat{\underset{\sim}{v}}^*(\cdot),\hat{\underset{\sim}{i}}^*(\cdot)\}$ is an admissible pair of S_2 with initial state $\underset{\sim}{\alpha}(\underset{\sim}{x}_1(0))$. To see this, first note that there exists an input-trajectory pair $\{\underset{\sim}{u}_2^*(\cdot),\underset{\sim}{x}_2^*(\cdot)\}$ of S_2 with initial state $\underset{\sim}{x}_2^*(0) = \underset{\sim}{x}_2(t')$ such that $\{\underset{\sim}{v}^*(\cdot),\underset{\sim}{i}^*(\cdot)\} = \{\underset{\sim}{V}_2(\underset{\sim}{x}_2^*(\cdot),\underset{\sim}{u}_2^*(\cdot)),\underset{\sim}{I}_2(\underset{\sim}{x}_2^*(\cdot),\underset{\sim}{u}_2^*(\cdot))\}$. Define $\hat{\underset{\sim}{u}}_2^*(\cdot)$ by

$$\hat{\underset{\sim}{u}}_2^*(t) = \begin{cases}\underset{\sim}{u}_2(t), & 0 \le t < t' \\ \underset{\sim}{u}_2^*(t-t'), & t \ge t',\end{cases} \tag{A.2}$$

(we defined $\underset{\sim}{u}_2(\cdot)$ in the statement of Lemma 3.2). Let $\hat{\underset{\sim}{x}}_2^*(\cdot)$ be the state-space trajectory of S_2 such that $\{\hat{\underset{\sim}{u}}_2^*(\cdot),\hat{\underset{\sim}{x}}_2^*(\cdot)\}$ is an input-trajectory pair of S_2 with initial state $\hat{\underset{\sim}{x}}_2^*(0) = \underset{\sim}{x}_2(0) = \underset{\sim}{\alpha}(\underset{\sim}{x}_1(0))$. Then $\{\hat{\underset{\sim}{v}}^*(\cdot), \hat{\underset{\sim}{i}}^*(\cdot)\} = \{\underset{\sim}{V}_2(\hat{\underset{\sim}{x}}_2^*(\cdot),\hat{\underset{\sim}{u}}_2^*(\cdot)),\underset{\sim}{I}_2(\hat{\underset{\sim}{x}}_2^*(\cdot),\hat{\underset{\sim}{u}}_2^*(\cdot))\}$, which proves our claim that

$\{\hat{v}^*(\cdot), \hat{i}^*(\cdot)\}$ defined in (A.1) is an admissible pair of S_2 with initial state $\alpha(x_1(0))$.

By the definition of $\alpha(\cdot)$, $\{\hat{v}^*(\cdot), \hat{i}^*(\cdot)\}$ is also an admissible pair of S_1 with initial state $x_1(0)$, so there exists an input-trajectory pair $\{\hat{u}_1^*(\cdot), \hat{x}_1^*(\cdot)\}$ of S_1 with initial state $\hat{x}_1^*(0) = x_1(0)$ such that

$$\{\hat{v}^*(\cdot), \hat{i}^*(\cdot)\} = \{V_1(\hat{x}_1^*(\cdot), \hat{u}_1^*(\cdot)), I_1(\hat{x}_1^*(\cdot), \hat{u}_1^*(\cdot))\}. \tag{A.3}$$

Since $\{\hat{v}^*(t), \hat{i}^*(t)\} = \{v(t), i(t)\}$ for all $t \in [0,t')$ and since S_1 is input-observable, it follows that $\hat{u}_1^*(t) = u_1(t)$ for all $t \in [0,t')$. And by our assumption [1, Section II] of unique solutions, $\hat{x}_1^*(t) = x_1(t)$ for all $t \in [0,t')$. Finally, since $\hat{x}_1^*(\cdot)$ and $x_1(\cdot)$ are continuous on $[0,t']$ and are equal on $[0,t')$, they must be equal at t' as well, i.e.

$$\hat{x}_1^*(t') = x_1(t'). \tag{A.4}$$

We have already shown that $\{\hat{v}^*(\cdot), \hat{i}^*(\cdot)\}$ is an admissible pair of S_1 with initial state $x_1(0)$, and from (A.3) and (A.4) we can conclude that its state at time t' is $x_1(t')$. Referring to (A.1) and remembering that the class of admissible pairs is translation invariant by our assumptions, we see that $\{v^*(\cdot), i^*(\cdot)\}$ must be an admissible pair of S_1 with <u>initial</u> state $x_1(t')$. So by the definition of $\alpha(\cdot)$, $\{v^*(\cdot), i^*(\cdot)\}$ must also be an admissible pair of S_2 with initial state $\alpha(x_1(t'))$, contrary to the assumption with which we began Case 1.

<u>Case 2</u>. The other alternative is that there exists an admissible pair $\{v^{**}(\cdot), i^{**}(\cdot)\}$ of S_2 with initial state $\alpha(x_1(t'))$ which is not an admissible pair of S_2 with initial state $x_2(t')$.

By the definition of $\alpha(\cdot)$, $\{v^{**}(\cdot), i^{**}(\cdot)\}$ is also an admissible pair of S_1 with initial state $\underset{\sim}{x}_1(t')$. Define $\{\hat{\underset{\sim}{v}}^{**}(\cdot),\hat{\underset{\sim}{i}}^{**}(\cdot)\}$ by

$$\{\hat{\underset{\sim}{v}}^{**}(\cdot),\hat{\underset{\sim}{i}}^{**}(\cdot)\} \triangleq \begin{cases}\{\underset{\sim}{v}(t),\underset{\sim}{i}(t)\}, & 0 \leq t < t' \\ \{\underset{\sim}{v}^{**}(t-t'),\underset{\sim}{i}^{**}(t-t')\}, & t \geq t'.\end{cases} \tag{A.5}$$

We claim that $\{\hat{\underset{\sim}{v}}^{**}(\cdot),\hat{\underset{\sim}{i}}^{**}(\cdot)\}$ is an admissible pair of S_1 with initial state $\underset{\sim}{x}_1(0)$. In case 1 we made a similar claim about (A.1), but concerning S_2, and provided a proof in the subsequent paragraph. The proof is entirely analogous here, so we will omit it.

By the definition of $\underset{\sim}{\alpha}(\cdot)$, $\{\hat{\underset{\sim}{v}}^{**}(\cdot),\hat{\underset{\sim}{i}}^{**}(\cdot)\}$ is also an admissible pair of S_2 with initial state $\underset{\sim}{\alpha}(\underset{\sim}{x}_1(0))$. So there is an input-trajectory pair $\{\hat{\underset{\sim}{u}}_2^{**}(\cdot),\hat{\underset{\sim}{x}}_2^{**}(\cdot)\}$ of S_2 with initial state $\hat{\underset{\sim}{x}}_2^{**}(0) = \underset{\sim}{x}_2(0) = \underset{\sim}{\alpha}(\underset{\sim}{x}_1(0))$ such that $\{\hat{\underset{\sim}{v}}^{**}(\cdot),\hat{\underset{\sim}{i}}^{**}(\cdot)\} = \{\underset{\sim}{V}_2(\hat{\underset{\sim}{x}}_2^{**}(\cdot),\hat{\underset{\sim}{u}}_2^{**}(\cdot)),\underset{\sim}{I}_2(\hat{\underset{\sim}{x}}_2^{**}(\cdot),\hat{\underset{\sim}{u}}_2^{**}(\cdot))\}$. And since S_2 is input-observable, $\hat{\underset{\sim}{u}}_2^{**}(t) = \underset{\sim}{u}_2(t)$, $\forall t \in [0,t')$. As in Case 1, we can conclude from the uniqueness of solutions and the continuity of trajectories that $\hat{\underset{\sim}{x}}_2^{**}(t') = \underset{\sim}{x}_2(t')$. Therefore $\{\underset{\sim}{v}_2^{**}(\cdot),\underset{\sim}{i}_2^{**}(\cdot)\}$ is an admissible pair of S_2 with initial state $\underset{\sim}{x}_2(t')$, contrary to assumption. ¤

<u>Proof of Lemma 3.4</u>. Suppose S_2 is cyclo-lossless. Let $T \geq 0$ be any time, $\underset{\sim}{x}^* \in \sum_1$ any state, and $\{\underset{\sim}{u}_1(\cdot),\underset{\sim}{x}_1(\cdot)\}|[0,T]$ any input-trajectory pair of S_1 from $\underset{\sim}{x}^*$ to $\underset{\sim}{x}^*$. Define $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}|[0,T] = \{\underset{\sim}{V}_1(\underset{\sim}{x}_1(\cdot),\underset{\sim}{u}_1(\cdot)),\underset{\sim}{I}_1(\underset{\sim}{x}_1(\cdot),\underset{\sim}{u}_1(\cdot))\}|[0,T]$, where $\underset{\sim}{V}_1(\cdot,\cdot)$ and $\underset{\sim}{I}_1(\cdot,\cdot)$ are the readout maps of S_1. Let $\underset{\sim}{\alpha} : \sum_1 \to \sum_2$ be the mapping which is defined in Lemma 3.2; hence, there exists an input-trajectory pair $\{\underset{\sim}{u}_2(\cdot),\underset{\sim}{x}_2(\cdot)\}|[0,T]$ of S_2 with initial state $\underset{\sim}{x}_2(0) = \underset{\sim}{\alpha}(\underset{\sim}{x}_1(0)) = \underset{\sim}{\alpha}(\underset{\sim}{x}^*)$ such that $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}|[0,T] = \{\underset{\sim}{V}_2(\underset{\sim}{x}_2(\cdot),\underset{\sim}{u}_2(\cdot)),\underset{\sim}{I}_2(\underset{\sim}{x}_2(\cdot),\underset{\sim}{u}_2(\cdot))\}|[0,T]$. By Lemma 3.2 $\underset{\sim}{x}_2(T) = \underset{\sim}{\alpha}(\underset{\sim}{x}_1(T))$, so $\{\underset{\sim}{x}_2(\cdot),\underset{\sim}{u}_2(\cdot)\}|[0,T]$ is an input-trajectory pair from $\underset{\sim}{\alpha}(\underset{\sim}{x}^*)$ to $\underset{\sim}{\alpha}(\underset{\sim}{x}^*)$.

Since S_2 is cyclo-lossless, the energy consumed by $\{\underline{x}_2(\cdot),\underline{u}_2(\cdot)\}|[0,T]$ is zero. And since $\{\underline{x}_1(\cdot),\underline{u}_1(\cdot)\}|[0,T]$ produces the same port voltages and currents, it must also consume zero energy. ⌑

Proof of Lemma 3.5. Let $T \geq 0$ be any time and $\{\underline{u}_1(\cdot),\underline{x}_1(\cdot)\}$ any input-trajectory pair of S_1. Define $\{\underline{v}(\cdot),\underline{i}(\cdot)\}|[0,T] = \{\underline{V}_1(\underline{x}_1(\cdot),\underline{u}_1(\cdot)), \underline{I}_1(\underline{x}_1(\cdot),\underline{u}_1(\cdot))\}|[0,T]$. By the definition of $\underline{\alpha}(\cdot)$, there exists an input-trajectory pair $\{\underline{u}_2(\cdot),\underline{x}_2(\cdot)\}$ of S_2 with initial state $\underline{x}_2(0) = \underline{\alpha}(\underline{x}_1(0))$ such that $\{\underline{v}(\cdot),\underline{i}(\cdot)\}|[0,T] = \underline{V}_2(\underline{x}_2(\cdot),\underline{u}_2(\cdot)),\underline{I}_2(\underline{x}_2(\cdot),\underline{u}_2(\cdot))\}|[0,T]$. And by lemma 3.2, $\underline{x}_2(T) = \underline{\alpha}(\underline{x}_1(T))$. Since $\phi_2(\cdot)$ is a conservative potential energy function for S_2, the energy consumed by $\{\underline{u}_2(\cdot),\underline{x}_2(\cdot)\}|[0,T]$ is $\phi_2(\underline{x}_2(T)) - \phi_2(\underline{x}_2(0)) = \phi_2(\underline{\alpha}(\underline{x}_1(T))) - \phi_2(\underline{\alpha}(\underline{x}_1(0)))$. Since $\{\underline{u}_1(\cdot), \underline{x}_1(\cdot)\}|[0,T]$ produces the same port voltages and currents as $\{\underline{u}_2(\cdot), \underline{x}_2(\cdot)\}|[0,T]$, the energy it consumes must also equal $\phi_2(\underline{\alpha}(\underline{x}_1(T))) - \phi_2(\underline{\alpha}(\underline{x}_1(0)))$. ⌑

Proof of Lemma 3.6. Let $\underline{x}_a = (\underline{x}_{1a},\ldots,\underline{x}_{ka})$ and $\underline{x}_b = (\underline{x}_{1b},\ldots,\underline{x}_{kb})$ be any two states in $\sum$, $\{\underline{u}'(\cdot),\underline{x}'(\cdot)\}|[0,T']$ and $\{\underline{u}''(\cdot),\underline{x}''(\cdot)\}|[0,T'']$ any two input-trajectory pairs of S from $\underline{x}_a$ to $\underline{x}_b$, and E' and E" the energy consumed by $\{\underline{u}'(\cdot),\underline{x}'(\cdot)\}|[0,T']$ and $\{\underline{u}''(\cdot),\underline{x}''(\cdot)\}|[0,T'']$, respectively. Let E_j' be the energy which enters the ports of N_j while S_j traverses the path $\underline{x}_j'(\cdot)$ in $\sum_j$, and E_j'' be the energy which enters its ports while it traverses $\underline{x}_j''(\cdot)$.

Then $E' = \sum_{j=1}^{k} E_j'$ and $E'' = \sum_{j=1}^{k} E_j''$ by Tellegen's theorem. And $E_j' = E_j''$, $1 \leq j \leq k$, because S_j is a lossless state representation for N_k and $\{\underline{x}_j'(\cdot)\}|[0,T']$ and $\{\underline{x}_j''(\cdot)\}|[0,T'']$ have the same endpoints. Therefore $E' = E''$. ⌑

Proof of Lemma 4.1. Since N is passive, S is passive. Since N is lossless and S is input-observable, S is lossless as well (Lemma 3.3). Let $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$ be any two states in Σ. By complete controllability, there exists an input-trajectory pair of S from $\underset{\sim}{x}_a$ to $\underset{\sim}{x}_b$. Since S is lossless, the energy consumed by an input-trajectory pair from $\underset{\sim}{x}_a$ to $\underset{\sim}{x}_b$ is a function of $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$ only: we write it as $E_c(\underset{\sim}{x}_a,\underset{\sim}{x}_b)$. By Def.23 of [1], an internal energy function $E_I(\cdot)$ must satisfy $E_I(\underset{\sim}{x}_b) - E_I(\underset{\sim}{x}_a) \leq E_c(\underset{\sim}{x}_a,\underset{\sim}{x}_b)$. Since U is closed under concatenation and every lossless n-port is cyclo-lossless, $E_c(\underset{\sim}{x}_a,\underset{\sim}{x}_b) = - E_c(\underset{\sim}{x}_b,\underset{\sim}{x}_a)$. Interchanging the roles of $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$ in the definition of an internal energy function, we must have $E_I(\underset{\sim}{x}_a) - E_I(\underset{\sim}{x}_b) \leq E_c(\underset{\sim}{x}_b,\underset{\sim}{x}_a) = - E_c(\underset{\sim}{x}_a,\underset{\sim}{x}_b)$, or $E_I(\underset{\sim}{x}_b) - E_I(\underset{\sim}{x}_a) \geq E_c(\underset{\sim}{x}_a,\underset{\sim}{x}_b)$. These two inequalities imply that $E_I(\underset{\sim}{x}_b) - E_I(\underset{\sim}{x}_a) = E_c(\underset{\sim}{x}_a,\underset{\sim}{x}_b)$ for all $\underset{\sim}{x}_a,\underset{\sim}{x}_b \in \Sigma$, which is just the definition of a conservative potential energy function. ¤

Proof of Lemma 5.2.

Proof of a).

($\Rightarrow$) Suppose S is lossless. For every $(\underset{\sim}{x}_0,\underset{\sim}{u}_0) \in \Sigma \times U$, $\{\underset{\sim}{u}_0,\underset{\sim}{x}_0\}$ is an input-trajectory pair of S. Since the state trajectory is constant, $\{\underset{\sim}{u}_0,\underset{\sim}{x}_0\}|[0,T]$ must consume zero energy for all $T \geq 0$. But this energy is just $\int_0^T \langle \underset{\sim}{u}_0,\underset{\sim}{g}(\underset{\sim}{u}_0)\rangle\, dt = \langle \underset{\sim}{u}_0,\underset{\sim}{g}(\underset{\sim}{u}_0)\rangle\, T$. Thus $\langle \underset{\sim}{u}_0,\underset{\sim}{g}(\underset{\sim}{u}_0)\rangle = 0$.

($\Leftarrow$) If $\langle \underset{\sim}{u},\underset{\sim}{g}(\underset{\sim}{u})\rangle = 0$ for all $\underset{\sim}{u} \in U$, then the energy consumed by all input-trajectory pairs is the same over every time interval; namely, zero. Thus S is lossless.

Proof of b).

($\Rightarrow$) This follows from Lemma 3.3.

($\Leftarrow$) Suppose S is lossless. Then S is equivalent (Def. 3.2) to a resistive state representation $\hat{S}$ whose state space consists of a single point; thus, $\hat{S}$ is state-observable (Lemma 5.1). The function $\underline{g}(\cdot)$ is the same for both S and $\hat{S}$; it follows from statement a) that $\hat{S}$ is lossless. We have constructed a lossless, state-observable(and hence totally observable) state representation $\hat{S}$ for N; by Def. 3.6, N is lossless. ∎

Proof of Lemma 5.3.

($\Leftarrow$) Given any two distinct states $\underline{x}_1 \neq \underline{x}_2$, and given $\underline{w}$ such that $\underline{g}(\underline{x}_1+\underline{w}) \neq \underline{g}(\underline{x}_2+\underline{w})$, consider the input-trajectory pairs $\{\underline{u}(t) \equiv \underline{w}, \underline{x}_1'(t) = \underline{x}_1 + \underline{w}t\}$, $\{\underline{u}(t) \equiv \underline{w}, \underline{x}_2'(t) = \underline{x}_2 + \underline{w}t\}$ and the corresponding input-output pairs [1, def. 9] $\{\underline{u}(t) \equiv \underline{w}, \underline{y}_1(t) = \underline{g}(\underline{x}_1+\underline{w}t)\}$, $\{\underline{u}(t) \equiv \underline{w}, \underline{y}_2(t) = \underline{g}(\underline{x}_2+\underline{w}t)\}$. Then $\underline{y}_1(1) \neq \underline{y}_2(1)$, and since for hybrid representations the map : $(\underline{u},\underline{y}) \rightarrow (\underline{v},\underline{i})$ is 1 - 1, we conclude that the class of admissible pairs of S with initial state $\underline{x}_1$ is distinct from the class of admissible pairs of S with initial state $\underline{x}_2$. So $\underline{x}_1$ and $\underline{x}_2$ are not equivalent (Def. 3.1); and since they were arbitrary, S is state-observable as claimed.

($\Rightarrow$) If S is state-observable, then for any two distinct states $\underline{x}_1$, $\underline{x}_2$ the class of admissible pairs of S with initial state $\underline{x}_1$ is distinct from the class of admissible pairs of S with initial state $\underline{x}_2$. And since for hybrid representations the map : $(\underline{u},\underline{y}) \rightarrow (\underline{v},\underline{i})$ is a bijection, the class of input-output pairs of S with initial state $\underline{x}_1$ must be distinct from the class of input-output pairs with initial state $\underline{x}_2$. Therefore there exists some $\underline{u}^*(\cdot) \in \mathcal{U}$ such that $\{\underline{u}^*(\cdot),\underline{y}_1(\cdot)\}$ is an input-output pair of S with initial state $\underline{x}_1$, $\{\underline{u}^*(\cdot),\underline{y}_2(\cdot)\}$ is an input-output pair of S with initial state $\underline{x}_2$, but $\underline{y}_1(t') \neq \underline{y}_2(t')$ for some $t' \geq 0$. Define $\underline{w}^* \in \mathbb{R}^n$ by

$$\underset{\sim}{w}^* \triangleq \int_0^{t'} \underset{\sim}{u}^*(t)dt.$$

Then $\underset{\sim}{g}(\underset{\sim}{x}_1+\underset{\sim}{w}^*) = \underset{\sim}{y}_1(t') \neq \underset{\sim}{y}_2(t') = \underset{\sim}{g}(\underset{\sim}{x}_2+\underset{\sim}{w}^*)$. ¤

Proof of Lemma 5.4.

Proof of a).

($\Leftarrow$) In this case ϕ is a conservative potential energy function for S.

($\Rightarrow$) If $\underset{\sim}{g}$ were the gradient of some scalar function, then that function must be C^1 because $\underset{\sim}{g}$ is continuous. The alternative is that $\underset{\sim}{g}$ is not the gradient of any scalar function. It follows [10, Theorem 7, p. 82] that there exists a point $\underset{\sim}{x}_0 \in \mathbb{R}^n$ and a piecewise C^1 curve $\underset{\sim}{\gamma} : [0,1] \to \mathbb{R}^n$ such that $\underset{\sim}{\gamma}(0) = \underset{\sim}{\gamma}(1) = \underset{\sim}{x}_0$ and

$$\int_0^1 \langle \dot{\underset{\sim}{\gamma}}(t), \underset{\sim}{g}(\underset{\sim}{\gamma}(t)) \rangle \, dt \neq 0.$$

Then $\{\dot{\underset{\sim}{\gamma}}(\cdot), \underset{\sim}{\gamma}(\cdot)\}|[0,1]$ is an input-trajectory pair from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}_0$, and the energy consumed by $\{\dot{\underset{\sim}{\gamma}}(\cdot), \underset{\sim}{\gamma}(\cdot)\}|[0,1]$ is nonzero. We have shown that S is not cyclo-lossless; hence, S is not lossless (Lemma 2.1).

Proof of b).

This follows from lemma 3.3 and the fact that S is input-observable by assumption.

Proof of c).

This is just Def. 3.6. ¤

Proof of Lemma 5.5.

($\Leftarrow$) Given any two distinct states $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, then for some integers $1 \leq i, j \leq n$ and some $\underset{\sim}{w} \in \mathbb{R}^n$ we have $R_{ij}(\underset{\sim}{x}_1+\underset{\sim}{w}) \neq R_{ij}(\underset{\sim}{x}_2+\underset{\sim}{w})$. Let $\underset{\sim}{\varepsilon}_j \in \mathbb{R}^n$ be the j-th element in the standard ordered basis for $\mathbb{R}^n$. We define the input waveform $\underset{\sim}{u}^*(\cdot) \in \mathcal{U}$ by

$$u^*(t) = \begin{cases} w, & 0 \le t \le 1 \\ \varepsilon_j, & t > 1. \end{cases}$$

Let $x_1'(\cdot)$ and $x_2'(\cdot)$ be the state space trajectories which result from applying the input $u^*(\cdot)$ with initial states x_1 and x_2 respectively, and $y_1(\cdot)$ and $y_2(\cdot)$ the corresponding outputs. Then for all $t > 1$, $y_1(t)$ is just the j-th column of $[R(x_1'(t))]$ and $y_2(t)$ is the j-th column of $[R(x_2'(t))]$. Since $R(\cdot)$, $x_1'(\cdot)$, and $x_2'(\cdot)$ are all continuous, $\lim_{t\to 1} x_1'(t) = x_1 + w$, $\lim_{t\to 1} x_2'(t) = x_2 + w$, and $R_{ij}(x_1+w) \neq R_{ij}(x_2+w)$, it follows that $(y_1(t))_i \neq (y_2(t))_i$ for all t in some interval $(1,1+\varepsilon)$. Therefore x_1 and x_2 are not equivalent. And since they were arbitrary, S is state-observable.

($\Rightarrow$) If S is state-observable, then, as we argued in the proof of Lemma 5.3, for any two distinct states x_1, x_2 in $\sum$ there exists an input $u^*(\cdot)$ such that $\{u^*(\cdot), y_1(\cdot)\}$ is an input-trajectory pair with initial state x_1, $\{u^*(\cdot), y_2(\cdot)\}$ is an input-trajectory pair with initial state x_2, but $y_1(t') \neq y_2(t')$ for some $t' \geq 0$. Let $x_1'(\cdot)$ and $x_2'(\cdot)$ be the corresponding state-space trajectories. Since $[R(x_1'(t'))]u^*(t') = y_1(t') \neq y_2(t') = [R(x_2'(t'))]u^*(t')$, it follows that $R(x_1'(t')) \neq R(x_2'(t'))$. If we define $w^* \in \mathbb{R}^n$ by

$$w^* \triangleq \int_0^{t'} u^*(t)dt,$$

then $R(x_1+w^*) = R(x_1'(t')) \neq R(x_2'(t')) = R(x_2+w^*)$. ∎

<u>Proof of Lemma 5.6.</u>

<u>Proof of a).</u>

($\Leftarrow$) If $R(x)$ is always antisymmetric, then $u^T(t)R(x(t))u(t)$ is always zero. Therefore the energy consumed along any trajectory depends only on the endpoints because it is always zero.

($\Rightarrow$) Suppose S is lossless but for some $\underset{\sim}{x}^* \in \Sigma$, $(\underset{\sim}{R}(\underset{\sim}{x}^*)+\underset{\sim}{R}^T(\underset{\sim}{x}^*)) \neq \underset{\sim}{0}$. Then there is a vector $\underset{\sim}{u}_0 \in \mathbb{R}^n$ such that $\underset{\sim}{u}_0^T[\underset{\sim}{R}(\underset{\sim}{x}^*)]\underset{\sim}{u}_0 = c \neq 0$, and since $\underset{\sim}{R}(\cdot)$ is continuous there is an $\varepsilon > 0$ such that $\|\underset{\sim}{x} - \underset{\sim}{x}^*\| \leq \varepsilon \Rightarrow \left|\underset{\sim}{u}_0^T[\underset{\sim}{R}(\underset{\sim}{x})]\underset{\sim}{u}_0 - c\right| < |c|/2$. Define $\underset{\sim}{u}'(\cdot)$ by $\underset{\sim}{u}'(t) = \frac{\varepsilon \underset{\sim}{u}_0}{\|\underset{\sim}{u}_0\|} \cos t$ and $\underset{\sim}{x}'(\cdot)$ by $\underset{\sim}{x}'(t) = \underset{\sim}{x}^* + \frac{\varepsilon \underset{\sim}{u}_0}{\|\underset{\sim}{u}_0\|} \sin t$. Then $\{\underset{\sim}{u}'(\cdot),\underset{\sim}{x}'(\cdot)\}|[0,2\pi]$ is an input-trajectory pair from $\underset{\sim}{x}^*$ to $\underset{\sim}{x}^*$, and $\|\underset{\sim}{x}'(t)-\underset{\sim}{x}^*\| \leq \varepsilon$. So the energy consumed by $\{\underset{\sim}{u}'(\cdot), \underset{\sim}{x}'(\cdot)\}|[0,2\pi]$ is

$$\frac{\varepsilon^2}{\|\underset{\sim}{u}_0\|^2} \int_0^{2\pi} \left(\underset{\sim}{u}_0^T\left[\underset{\sim}{R}\left(\underset{\sim}{x}'(t)\right)\right] \underset{\sim}{u}_0\right)\cos^2 t \, dt \neq 0,$$

so the system is not cyclo-lossless and hence not lossless.

<u>Proof of b).</u>

This is just Lemma 3.3.

<u>Proof of c).</u>

This is just Def. 3.6. ¤

<u>Proof of Lemma 5.8.</u>

<u>Proof of a).</u>

First we shall prove that a function $h : \Sigma \to \mathbb{R}$ which satisfies $p(x,\underset{\sim}{u}) = h(x)f(x,\underset{\sim}{u})$ everywhere is continuous at each nonsingular state. If a state x_0 is nonsingular, then there exists an input value $\underset{\sim}{u}_0$ and a neighborhood $N(x_0)$ of x_0 such that $f(x,\underset{\sim}{u}_0) \neq 0$ for all $x \in N(x_0)$. Thus $h(x) = p(x,\underset{\sim}{u}_0)/f(x,\underset{\sim}{u}_0)$ for all $x \in N(x_0)$, which shows that $h(\cdot)$ is continuous at x_0, since p and f are continuous by assumption.

($\Rightarrow$) Suppose that S is lossless. Let $D \triangleq \{(x,\underset{\sim}{u}) \in \Sigma \times U : f(x,\underset{\sim}{u}) \neq 0\}$ and define $\hat{h} : D \to \mathbb{R}$ by

$$\hat{h}(x,\underset{\sim}{u}) \triangleq \frac{p(x,\underset{\sim}{u})}{f(x,\underset{\sim}{u})} \ .$$

We begin by proving that $\hat{h}(x,\underset{\sim}{u})$ depends only on the first variable x. To obtain a contradiction, suppose that there exist $(x_0,\underset{\sim}{u}_1)$, $(x_0,\underset{\sim}{u}_2) \in D$ such that $\hat{h}(x_0,\underset{\sim}{u}_1) \neq \hat{h}(x_0,\underset{\sim}{u}_2)$. Then two cases arise.

Case 1. $\mathrm{sgn}\big(f(x_0,\underset{\sim}{u}_1)\big) = \mathrm{sgn}\big(f(x_0,\underset{\sim}{u}_2)\big)$.[3] Assume that $f(x_0,\underset{\sim}{u}_1) > 0$ and $f(x_0,\underset{\sim}{u}_2) > 0$ (similar arguments apply in the other case). By continuity, there exists a $\delta > 0$ such that

$$\begin{aligned}
&f(x,\underset{\sim}{u}_1) > 0 \quad \forall x \in [x_0,x_0+\delta] \\
&f(x,\underset{\sim}{u}_2) > 0 \quad \forall x \in [x_0,x_0+\delta] \\
&\hat{h}(x,\underset{\sim}{u}_1) \neq \hat{h}(x,\underset{\sim}{u}_2) \quad \forall x \in [x_0,x_0+\delta].
\end{aligned}$$

Let $\{\underset{\sim}{u}_1,x_1(\cdot)\}|[0,T_1]$ be an input-trajectory pair from x_0 to $x_0 + \delta$, then

$$\int_0^{T_1} p(x_1(t),\underset{\sim}{u}_1)dt = \int_0^{T_1} \frac{p\big(x_1(t),\underset{\sim}{u}_1\big)}{f\big(x_1(t),u_1\big)}\,\dot{x}_1(t)dt$$

$$= \int_0^{T_1} \hat{h}\big(x_1(t),\underset{\sim}{u}_1\big)\dot{x}_1(t)dt = \int_{x_0}^{x_0+\delta} \hat{h}(x,\underset{\sim}{u}_1)dx. \tag{A.6}$$

The use of the Change of Variables theorem [17] is justified because $x_1 : [0,T_1] \to \mathbb{R}$ is C^1 and the mapping $x \to \hat{h}(x,\underset{\sim}{u}_1)$ is defined and continuous on $x_1([0,T_1])$. Similarly, let $\{\underset{\sim}{u}_2,x_2(\cdot)\}|[0,T_2]$ denote an input-trajectory pair from x_0 to $x_0 + \delta$, then

$$\int_0^{T_2} p\big(x_2(t),\underset{\sim}{u}_2\big)dt = \int_{x_0}^{x_0+\delta} \hat{h}(x,\underset{\sim}{u}_2)dx. \tag{A.7}$$

[3]The function $\mathrm{sgn} : \mathbb{R} \to \mathbb{R}$ is defined by $\mathrm{sgn}(x) \triangleq 1$ if $x > 0$, $\triangleq -1$ if $x < 0$, $\triangleq 0$ if $x = 0$.

Since $x \mapsto \hat{h}(x,u_1)$ and $x \mapsto \hat{h}(x,u_2)$ are continuous and unequal everywhere in $[x_0,x_0+\delta]$, either $\hat{h}(x,u_1)>\hat{h}(x,u_2)$ or else $\hat{h}(x,u_1)<\hat{h}(x,u_2)$ everywhere in $[x_0,x_0+\delta]$. In either case the integrals on the right hand side of (A.6) and (A.7) are unequal, violating the assumption of losslessness.

Case 2. $\operatorname{sgn}\big(f(x_0,\underset{\sim}{u}_1)\big) = -\operatorname{sgn}\big(f(x_0,\underset{\sim}{u}_2)\big)$. For definiteness, assume that $f(x_0,\underset{\sim}{u}_1) > 0$ and $f(x_0,\underset{\sim}{u}_2) < 0$. By continuity, there exists a $\delta > 0$ such that

$$\begin{aligned}
&f(x,\underset{\sim}{u}_1) > 0 && \forall x \in [x_0,x_0+\delta] \\
&f(x,\underset{\sim}{u}_2) < 0 && \forall x \in [x_0,x_0+\delta] \\
&\hat{h}(x,\underset{\sim}{u}_1) \neq \hat{h}(x,\underset{\sim}{u}_2) && \forall x \in [x_0,x_0+\delta].
\end{aligned}$$

Let $\{\underset{\sim}{u}(\cdot),x(\cdot)\}\,|\,[0,T_2]$ be an input-trajectory pair from x_0 to x_0 with the following property: $\exists\ T_1 \in (0,T_2)$ such that $\underset{\sim}{u}(t) = \underset{\sim}{u}_1$ for $t \in [0,T_1]$, $\underset{\sim}{u}(t) = \underset{\sim}{u}_2$ for $t \in (T_1,T_2]$, and $x(T_1) = x_0 + \delta$. Thus

$$\begin{aligned}
\int_0^{T_2} p\big(x(t),\underset{\sim}{u}(t)\big)dt &= \int_0^{T_1} p\big(x(t),\underset{\sim}{u}_1\big)dt + \int_{T_1}^{T_2} p\big(x(t),\underset{\sim}{u}_2\big)dt \\
&= \int_{x_0}^{x_0+\delta} \hat{h}(x,\underset{\sim}{u}_1)dx + \int_{x_0+\delta}^{x_0} \hat{h}(x,\underset{\sim}{u}_2)dx \\
&= \int_{x_0}^{x_0+\delta} [\hat{h}(x,\underset{\sim}{u}_1)-\hat{h}(x,\underset{\sim}{u}_2)]dx. \qquad \text{(A.8)}
\end{aligned}$$

Since the integrand on the right-hand side of (A.8) is continuous and nonzero at every point of the interval $[x_0,x_0+\delta]$, it follows that the integral is nonzero. This violates the assumption of losslessness.

Thus $\hat{h}(x,\underset{\sim}{u})$ depends only on x. Let $pr_\Sigma(D)$ denote the projection of D onto Σ, i.e. $pr_\Sigma(D) = \{x\in\Sigma \,|\, f(x,\underset{\sim}{u}) \neq 0 \text{ for some } \underset{\sim}{u}\in U\}$. And let $\hat{\underset{\sim}{u}}: pr_\Sigma(D) \to U$ assign to each x in $pr_\Sigma(D)$ any value $\hat{u}(x)$ such that $f(x,\hat{u}(x)) \neq 0$. Then define $h(x) \triangleq \hat{h}(x,\hat{u}(x))$, and note that

$$h(x) = \hat{h}(x,\underset{\sim}{u}) = \frac{p(x,\underset{\sim}{u})}{f(x,\underset{\sim}{u})} \text{ for all } (x,\underset{\sim}{u}) \in D.$$

Note that $pr_\Sigma(D)$ is precisely the set of all nonsingular states. We shall define $h(\cdot)$ arbitrarily at the singular states. In order to show that $p(x,\underset{\sim}{u}) = h(x)f(x,\underset{\sim}{u})$ at <u>all</u> $(x,\underset{\sim}{u}) \in \Sigma \times U$, it only remains to show that $f(x,\underset{\sim}{u}) = 0 \Rightarrow p(x,\underset{\sim}{u}) = 0$. Thus, let $(x_0,\underset{\sim}{u}_0) \in \Sigma \times U$ be such that $f(x_0,\underset{\sim}{u}_0) = 0$. Then $\{\underset{\sim}{u}_0,x_0\}|[0,T]$ is a valid input-trajectory pair for all $T \geq 0$. By losslessness,

$$0 = \int_0^T p(x_0,\underset{\sim}{u}_0)dt = p(x_0,\underset{\sim}{u}_0)T$$

for all $T \geq 0$. Thus $p(x_0,\underset{\sim}{u}_0) = 0$.

($\Leftarrow$) Suppose that there exists a function $h : \Sigma \to \mathbb{R}$ such that $p(x,\underset{\sim}{u}) = h(x)f(x,\underset{\sim}{u})$ for all $(x,\underset{\sim}{u}) \in \Sigma \times U$. Let $\{\underset{\sim}{u}_1(\cdot),x_1(\cdot)\}|[0,T_1]$ and $\{\underset{\sim}{u}_2(\cdot), x_2(\cdot)\}|[0,T_2]$ be any input-trajectory pairs for which $x_1(0) = x_2(0) \triangleq a$ and $x_1(T_1) = x_2(T_2) \triangleq b$. We will show that S is lossless by showing that the energy consumed by $\{\underset{\sim}{u}_1(\cdot),x_1(\cdot)\}|[0,T_1]$ equals the energy consumed by $\{\underset{\sim}{u}_2(\cdot),x_2(\cdot)\}|[0,T_2]$. There are three cases which arise.

<u>Case 1</u>. a is singular. Then a = b and both state trajectories are constant. We have

$$\int_0^{T_1} p\big(x_1(t),\underset{\sim}{u}_1(t)\big)dt = \int_0^{T_1} h(a)\dot{x}_1(t)dt = 0$$

$$\int_0^{T_2} p\big(x_2(t),\underset{\sim}{u}_2(t)\big)dt = \int_0^{T_2} h(a)\dot{x}_2(t)dt = 0$$

since $\dot{x}_1(t) \equiv \dot{x}_2(t) \equiv 0$.

<u>Case 2</u>. a and b are nonsingular. It follows that $x_1(t)$ is nonsingular for all $t \in [0,T_1]$ (otherwise, the condition $x_1(T_1) = b$ would be

impossible). Thus

$$\int_0^{T_1} p\big(x_1(t),\underset{\sim}{u}_1(t)\big)dt = \int_0^{T_1} h\big(x_1(t)\big)\dot{x}_1(t)dt = \int_a^b h(x)dx.$$

The use of the Change of Variables formula is justified because $x_1 : [0,T_1] \to \mathbb{R}$ is absolutely continuous and $h(\cdot)$ is continuous on $x_1([0,T_1])$ [20, pp. 95-96, Theorem I.4.42]. Likewise,

$$\int_0^{T_2} p\big(x_2(t),\underset{\sim}{u}_2(t)\big)dt = \int_a^b h(x)dx.$$

<u>Case 3</u>. a is nonsingular but b is singular. Assume $b > a$ (similar arguments apply when $b < a$). Suppose without loss of generality that $x_1(t) \neq b$ for $t \in [0,T_1)$. Then

$$\begin{aligned}\int_0^{T_1} p\big(x_1(t),\underset{\sim}{u}_1(t)\big)dt &= \lim_{\substack{T\to T_1\\ T<T_1}} \int_0^{T} p\big(x_1(t),\underset{\sim}{u}_1(t)\big)dt \\ &= \lim_{\substack{T\to T_1\\ T<T_1}} \int_a^{x_1(T)} h(x)dx \\ &= \lim_{\substack{z\to b\\ z<b}} \int_a^{z} h(x)dx.\end{aligned}$$

The first step follows since the integral is continuous on $[0,T_1]$. The second step follows from Case 2. The last step follows since $x_1(T) \to b$ as $T \to T_1$. Similarly,

$$\int_0^{T_2} p\big(x_2(t),\underset{\sim}{u}_2(t)\big)dt = \lim_{\substack{z\to b\\ z<b}} \int_a^{z} h(x)dx.$$

Hence, in all three cases,

$$\int_0^{T_1} p\left(x_1(t), \underset{\sim}{u}_1(t)\right)dt = \int_0^{T_2} p\left(x_2(t), \underset{\sim}{u}_2(t)\right)dt.$$

Proof of b).

This is just Lemma 3.3.

Proof of c).

This is just Def. 3.6. ∎

Appendix B - Defining the "Zero Average Power Property": What Should be Bounded?

In Def. 2.5 we have chosen to require that $\underset{\sim}{v}(\cdot)$ and $\underset{\sim}{i}(\cdot)$ be bounded before applying the criterion in (2.7), but we placed no such requirement of boundedness on the state space trajectory $\underset{\sim}{x}(\cdot)$. The purpose of this Appendix is to explain and defend this choice by considering the alternatives.

B.1). Two Alternative Definitions of the Zero Average Power Property

The two most obvious modifications of Def. (2.5) would be to require that (2.7) hold only when $\underset{\sim}{x}(\cdot)$ is bounded, or else only when $\underset{\sim}{x}(\cdot)$, $\underset{\sim}{v}(\cdot)$ and $\underset{\sim}{i}(\cdot)$ are all bounded. These modifications are formalized in the following alternative definitions.

Definition 2.5A. A state representation S is said to have version A of the zero average power property if (2.7) holds for all admissible pairs $\{\underset{\sim}{v}(\cdot) = \underset{\sim}{V}(\underset{\sim}{x}(\cdot),\underset{\sim}{u}(\cdot)),\underset{\sim}{i}(\cdot) = \underset{\sim}{I}(\underset{\sim}{x}(\cdot),\underset{\sim}{u}(\cdot))\}$ such that $\underset{\sim}{x}(\cdot)$ is bounded.

Definition 2.5B. A state representation S is said to have version B of the zero average power property if (2.7) holds for all admissible pairs $\{\underset{\sim}{v}(\cdot) = \underset{\sim}{V}(\underset{\sim}{x}(\cdot),\underset{\sim}{u}(\cdot)),\underset{\sim}{i}(\cdot) = \underset{\sim}{I}(\underset{\sim}{x}(\cdot),\underset{\sim}{u}(\cdot))\}$ such that $\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)$ and $\underset{\sim}{x}(\cdot)$ are bounded.

Note that the requirements of Def. 2.5B are weaker than those of Defs. 2.5 and 2.5A, since the class of admissible pairs to which we apply the limit test of (2.7) is smaller in Def. 2.5B than in the other two. In other words, if a system has the zero average power property as defined in either Def. 2.5 or 2.5A, then it automatically has that property as defined in Def. 2.5B.

B.2). Three Objections to These Two Alternatives

Both alternative definitions produce anomalies which do not arise with Def. 2.5 itself, and there are three separate reasons we have rejected Defs. 2.5A and 2.5B in favor of Def. 2.5.

Objection #1. Our first reason for choosing Def. 2.5 over the others is that only Def. 2.5 makes the zero average power property representation independent. It is clear that Def. 2.5 does have this property, since it is stated solely in terms of admissible pairs. To see that Defs. 2.5A and 2.5B do not, consider the following example.

Example B.1. Reconsider the capacitor in Fig. 5, which we discussed in Example 2.5. The natural state representation would be S_1, shown below; but the other state representation shown below, S_2, is an equally valid mathematical model for Fig. 5. In fact, we shall show later that S_1 and S_2 are bijectively equivalent (Def. 3.3).

S_1	S_2
$\dot{q} = i$	$\dot{w} = (1-\|w\|)^2 i$
$v = \frac{q}{1+\|q\|}$	$v = w$
$\Sigma_1 = \mathbb{R}$	$\Sigma_2 = (-1,1) \subset \mathbb{R}$

The main point here is that Σ_2 is bounded but Σ_1 is not. This Appendix concludes with a formal proof that S_1 and S_2 are bijectively equivalent, but the basic argument is quite simple. We obtained S_2 from S_1 by the following change of coordinates on the state space: $q \to w = q/(1+|q|)$. This explains the line $v = w$, and the line $\dot{w} = (1-|w|)^2 i$ follows from the chain rule application $\dot{w} = \frac{\partial w}{\partial q}\dot{q}$ with $\frac{\partial w}{\partial q}$ written in terms of w rather than q. And for technical completeness we suppose that $\mathcal{U} = L^1_{loc}(\mathbb{R}^+ \to \mathbb{R})$ for both S_1 and S_2.

Although they are equivalent, S_1 has versions A and B of the zero average power property while S_2 has neither.

For S_1, the reader can easily verify that $\phi(q) = |q| - \ell n(1+|q|)$ is a conservative potential energy function. And since

$$\int_0^T v(t)i(t)dt = \phi\big(q(T)\big) - \phi\big(q(0)\big)$$

and $\phi(\cdot)$ is bounded on any bounded subset of $\mathbb{R}$, it follows that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T v(t)i(t)dt = \lim_{T\to\infty} \frac{1}{T}\Big(\phi\big(q(T)\big)-\phi\big(q(0)\big)\Big) = 0$$

so long as $q(\cdot)$ is bounded. Therefore S_1 satisfies both versions A and B of the zero average power property, as claimed.

For S_2, consider the input-trajectory pair $\{i(t) \equiv 1, w(t) = t/(1+t)\}$ with output $v(t) = t/(1+t)$. The reader can quickly verify that $w(t) = t/(1+t)$ is in fact a solution of the state equation of S_2 when $i(t) \equiv 1$, as we claim. Since $i(\cdot)$, $v(\cdot)$ and $w(\cdot)$ are all bounded, the criteria of versions A and B are met. And since $i(t)\cdot v(t) \to 1$ as $t \to \infty$, the limiting value of the average power is 1 as $T \to \infty$. Therefore S_2 has neither version A nor version B of the zero average power property.

But S_1 and S_2 are equivalent; hence, versions A and B are not representation independent.

Objection #2. Our second objection to Defs. 2.5A and 2.5B is that under these definitions, those systems for which every trajectory is unbounded would gain the zero average power property by default. The following example shows how this can occur.

Example B.2. Consider the following state representation for a current-controlled 1-port:

$$\dot{x} = i^2 + 1$$

$$v = \arctan(x)$$

$$\Sigma = \mathbb{R},$$

where we can let $\mathcal{U} = L^2_{loc}(\mathbb{R}^+ \to \mathbb{R})$, although other choices for $\mathcal{U}$ wouldn't alter our conclusions. The point of this example is that $\dot{x} \geq 1$ for all time, so every trajectory is unbounded. Therefore the system has versions A and B of the zero average power property by default, since the class of admissible pairs for which we get to apply the test in (2.7) is empty. To see that this would be a bizarre classification for this system, consider the admissible pair $\{i(t) \equiv 1, v(t) = \arctan(2t)\}$, for which $i(t) \cdot v(t) \to \pi/2$ as $t \to \infty$. In contrast, this system does not have the zero average power property of Def. 2.5, as a result of the admissible pair mentioned above.

Objection #3. Our final reason for rejecting Defs. 2.5A and B is that they bring us the two problems mentioned above without offering a resolution of the major anomaly which arises from Def. 2.5: the fact that losslessness $\not\Rightarrow$ the zero average power property. We show below that this anomaly persists in all three definitions.

Example B.3. Corsider again the state representation S_2 in example B.1. It is easy to verify that it is lossless, since $\phi(w) = \frac{|w|}{1-|w|} + \ell n(1-|w|)$ is a conservative potential energy function for this system. But we showed in Example B.1 that it doesn't satisfy Def. 2.5A or 2.5B. Since Def. 2.5B is strictly weaker than Def. 2.5, S_2 doesn't satisfy Def. 2.5 either.

Therefore losslessness does not imply the zero average power property as represented in <u>any</u> of these three definitions. It has become unmistakably clear that this "anomaly" is fundamental to nonlinear circuit theory and doesn't arise from any defect in our definitions.

B.3). Proof that S_1 and S_2 in Example B.1 are Bijectively Equivalent

In order to prove that S_1 and S_2 are bijectively equivalent (Def. 3.3), we must exhibit the bijection $b : \sum_1 \to \sum_2$ and show that for any $q \in \sum_1$ the class of admissible pairs of S_1 with initial state q is identical to the class of admissible pairs of S_2 with initial state $w = b(q)$.

The function we propose is of course $b : q \to w = q/(1+|q|)$. The reader can easily verify the following facts about $b(\cdot)$, and we will use them without comment in the subsequent argument. First of all, $b : \mathbb{R} \to (-1,1)$ bijectively, and its inverse is given by $q = b^{-1}(w) = w/(1-|w|)$. Furthermore, despite the fact that $x \longrightarrow |x|$ is not differentiable at the origin, $b(\cdot)$ and $b^{-1}(\cdot)$ are both C^1 (although not C^2) everywhere, $b'(q) = 1/(1+|q|)^2$, and $(b^{-1})'(w) = 1/(1-|w|)^2$.

Let $q_1(0)$ be any state in $\sum_1$ and $\{i_1(\cdot),v_1(\cdot)\}$ be any admissible pair of S_1 with initial state $q_1(0)$. In order to prove that $\{i_1(\cdot),v_1(\cdot)\}$ is also an admissible pair of S_2 with initial state $w_1(0) = b\big(q_1(0)\big)$, we must exhibit a state space trajectory $\hat{w}_1(\cdot)$ of S_2 such that

i) $\hat{w}_1(0) = b\big(q_1(0)\big)$, i.e. the initial state is correct,

ii) $v_1(t) = \hat{w}_1(t)$, $\forall t \geq 0$, i.e. the output of S_2 is correct,

iii) $\dot{\hat{w}}_1(t) = \big(1-|\hat{w}_1(t)|\big)^2 i_1(t)$, i.e. $\hat{w}_1(\cdot)$ satisfies the state equation of S_2 with input $i_1(\cdot)$.

Requirement ii) uniquely determines our choice: $\hat{w}_1(\cdot) = v_1(\cdot)$. And then requirement ii) is satisfied trivially. Since the output equation of S_1 happens to be of the form $v = b(q)$, it is immediate that this choice of $\hat{w}_1(\cdot)$ satisfies requirement i) as well.

To check requirement iii), we first calculate that $\dot{\hat{w}}_1 = \dot{v}_1 = \frac{\partial v_1}{\partial q_1} \dot{q}_1$ $= \frac{1}{(1+|q_1|)^2} i_1$. Upon substituting $q_1 = \hat{w}_1/(1-|\hat{w}_1|)$ into this last expression, we have $\dot{\hat{w}}_1 = (1-|\hat{w}_1|)^2 i_1$, as desired. The proof that every admissible pair of S_2 with initial state $w(0)$ is also an admissible pair of S_1 with initial state $q_1(0) = b^{-1}(w_1(0))$ is similar and will be omitted. ¤

Appendix C - Proofs of the Results for Linear Systems

Even when it is applied to linear systems, the definition of losslessness given in this paper is less restrictive than the usual definition given in the linear systems theory literature [4,12]. For this reason we are providing complete, rigorous proofs for the results in subsection 5.4. First, however, it is necessary to define some terms and prove some preliminary lemmas.

C.1. Definitions. If $\underset{\sim}{w} \in \mathbb{C}^{p\times q}$ and if $\underset{\sim}{w} = \underset{\sim}{u} + j\underset{\sim}{v}$, where $\underset{\sim}{u},\underset{\sim}{v} \in \mathbb{R}^{p\times q}$, then, by definition, Re $\underset{\sim}{w} \triangleq \underset{\sim}{u}$ and Im $\underset{\sim}{w} \triangleq \underset{\sim}{v}$. The complex conjugate of $\underset{\sim}{w}$ is denoted by $\bar{\underset{\sim}{w}} \triangleq \underset{\sim}{u} - j\underset{\sim}{v}$, and $\underset{\sim}{w}^H \triangleq \bar{\underset{\sim}{w}}^T$.

Let S denote a linear state representation (5.5), where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a hybrid pair, $U = \mathbb{R}^n$, $\sum = \mathbb{R}^m$, and $\mathcal{U} = L^2_{loc}(\mathbb{R}^+ \to \mathbb{R}^n)$. The complexification of S, denoted $\hat{S}$, is the state representation with the same state and output equations as S, but with $\hat{U} = \mathbb{C}^n$, $\hat{\sum} = \mathbb{C}^m$, and $\hat{\mathcal{U}} = L^2_{loc}(\mathbb{R}^+ \to \mathbb{C}^n)$. Thus $\hat{S}$ is obtained from S simply by allowing the components of the input, output, and state to be complex-valued. If $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ is an admissible pair of $\hat{S}$ with initial state $\underset{\sim}{x}_0$, then, clearly, $\{\text{Re } \underset{\sim}{v}(\cdot),\text{Re } \underset{\sim}{i}(\cdot)\}$ (resp., $\{\text{Im } \underset{\sim}{v}(\cdot),\text{Im } \underset{\sim}{i}(\cdot)\}$) is an admissible pair of S with initial state Re $\underset{\sim}{x}_0$ (resp., Im $\underset{\sim}{x}_0$). The use of $\hat{S}$ instead of S; i.e., the use of complex-valued inputs,[4] outputs, and states; will greatly simplify the mathematical notation in the following proofs.

The energy consumed by an input-trajectory pair $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[0,T]$ of $\hat{S}$ is defined to be the quantity

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[4] A phrase such as "complex-valued input" is intended to mean "an input with complex-valued components." Such phrases will be used for brevity: their meaning should be clear.

$$\mathrm{Re} \int_0^T \underset{\sim}{v}^H(t)\underset{\sim}{i}(t)dt, \tag{C.1}$$

where $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ is the admissible pair corresponding to $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$. The state representation $\hat{S}$ is defined to be <u>lossless</u> if it satisfies Def. 2.1, with the energy consumed by an input-trajectory pair given by (C.1).

<u>C.2 Lemma</u>. S is lossless $\Longleftrightarrow$ $\hat{S}$ is lossless.

<u>Proof</u>.

($\Leftarrow$) Obvious, since the behavior of $\hat{S}$ when the input and initial state are real is the same as the behavior of S.

($\Rightarrow$) Let $\{\underset{\sim}{v}_1(\cdot),\underset{\sim}{i}_1(\cdot)\}|[0,T_1]$ be an admissible pair of $\hat{S}$ from $\underset{\sim}{x}_a$ to $\underset{\sim}{x}_b$ (i.e., the corresponding state trajectory $\underset{\sim}{x}_1(\cdot)$ satisfies $\underset{\sim}{x}_1(0) = \underset{\sim}{x}_a$ and $\underset{\sim}{x}_1(T_1) = \underset{\sim}{x}_b$) and let $\{\underset{\sim}{v}_2(\cdot),\underset{\sim}{i}_2(\cdot)\}|[0,T_2]$ be another admissible pair of $\hat{S}$ from $\underset{\sim}{x}_a$ to $\underset{\sim}{x}_b$. Then for $k = 1,2$, $\{\mathrm{Re}\ \underset{\sim}{v}_k(\cdot),\mathrm{Re}\ \underset{\sim}{i}_k(\cdot)\}|[0,T_k]$ is an admissible pair of S from Re $\underset{\sim}{x}_a$ to Re $\underset{\sim}{x}_b$, and $\{\mathrm{Im}\ \underset{\sim}{v}_k(\cdot),\ \mathrm{Im}\ \underset{\sim}{i}_k(\cdot)\}|[0,T_k]$ is an admissible pair of S from Im $\underset{\sim}{x}_a$ to Im $\underset{\sim}{x}_b$. Note that $\mathrm{Re}\ \underset{\sim}{v}^H\underset{\sim}{i} = \mathrm{Re}\ \underset{\sim}{v}^T\ \mathrm{Re}\ \underset{\sim}{i} + \mathrm{Im}\ \underset{\sim}{v}^T\ \mathrm{Im}\ \underset{\sim}{i}$. The losslessness of S implies that

$$\begin{aligned}\mathrm{Re} \int_0^{T_1} \underset{\sim}{v}_1^H(t)\underset{\sim}{i}_1(t)dt &= \int_0^{T_1} \mathrm{Re}\ \underset{\sim}{v}_1^T(t)\mathrm{Re}\ \underset{\sim}{i}_1(t)dt \\ &\quad + \int_0^{T_1} \mathrm{Im}\ \underset{\sim}{v}_1^T(t)\mathrm{Im}\ \underset{\sim}{i}_1(t)dt \\ &= \int_0^{T_2} \mathrm{Re}\ \underset{\sim}{v}_2^T(t)\mathrm{Re}\ \underset{\sim}{i}_2(t)dt \\ &\quad + \int_0^{T_2} \mathrm{Im}\ \underset{\sim}{v}_2^T(t)\mathrm{Im}\ \underset{\sim}{i}_2(t)dt\end{aligned}$$

$$= \text{Re} \int_0^{T_2} \underset{\sim}{v}_2^H(t)\underset{\sim}{i}_2(t)dt; \tag{C.2}$$

therefore $\hat{S}$ is lossless. ⌑

C.3. Lemma. S satisfies Statement iv) of Theorem 5.1 $\Longleftrightarrow \text{Re} \int_0^\infty \underset{\sim}{v}^H(t)\underset{\sim}{i}(t)dt = 0$ for all L^2 admissible pairs of $\hat{S}$ with zero initial state.

Proof.

($\Leftarrow$) Obvious.

($\Rightarrow$) Let $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ be an L^2 admissible pair of $\hat{S}$ with zero initial state. Then $\{\text{Re } \underset{\sim}{v}(\cdot), \text{Re } \underset{\sim}{i}(\cdot)\}$ and $\{\text{Im } \underset{\sim}{v}(\cdot), \text{Im } \underset{\sim}{i}(\cdot)\}$ are L^2 admissible pairs of S with zero initial state; thus

$$\text{Re} \int_0^\infty \underset{\sim}{v}^H(t)\underset{\sim}{i}(t)dt$$

$$= \int_0^\infty \text{Re } \underset{\sim}{v}^T(t)\text{Re } \underset{\sim}{i}(t)dt + \int_0^\infty \text{Im } \underset{\sim}{v}^T(t)\text{Im } \underset{\sim}{v}(t)dt = 0 \ .$$ ⌑

C.4 Lemma. A completely controllable linear state representation $S = \{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{D}\}$ is equivalent to a minimal[5] linear state representation $S_m = \{\underset{\sim}{A}_m, \underset{\sim}{B}_m, \underset{\sim}{C}_m, \underset{\sim}{D}\}$ (which has the same input and output variables as S); moreover, there exists a matrix $\underset{\sim}{P}$ such that if $\underset{\sim}{x}$ is any state of S, then $\underset{\sim}{x}_m \triangleq \underset{\sim}{P}\underset{\sim}{x}$ is the (necessarily unique) state of S_m which is equivalent to the state $\underset{\sim}{x}$ of S. (Equivalent states were defined in Def. 3.1; equivalent state representations were defined in Def. 3.2.)

Proof. The lemma follows from standard results in linear system

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[5] Recall that a linear state representation S is defined to be minimal if no linear state representation with the same transfer function as S has a state space of lower dimension than that of S. Equivalently, S is minimal if it is both completely controllable and completely observable [11, p. 181].

theory [11, Chap. 7, Theorem 7].

Proof of Assertion a) of Theorem 5.1. vi) $\Rightarrow$ vii). Define $\phi : \mathbb{R}^m \to \mathbb{R}$ by $\phi(\underset{\sim}{x}) \triangleq \frac{1}{2}\langle \underset{\sim}{x}, \underset{\sim}{K}\underset{\sim}{x}\rangle$, where $\underset{\sim}{K}$ is the matrix in statement vi). Then, for any input-trajectory pair $\{\underset{\sim}{u}(\cdot), \underset{\sim}{x}(\cdot)\}$ and any $T \geq 0$, we have

$$\phi\big(\underset{\sim}{x}(T)\big) - \phi\big(\underset{\sim}{x}(0)\big) = \int_0^T \frac{d\phi\big(\underset{\sim}{x}(t)\big)}{dt}\, dt$$

$$= \int_0^T \underset{\sim}{x}^T(t)\underset{\sim}{K}\dot{\underset{\sim}{x}}(t)dt = \int_0^T \underset{\sim}{x}^T(t)\underset{\sim}{K}\big(\underset{\sim}{A}\underset{\sim}{x}(t)+\underset{\sim}{B}\underset{\sim}{u}(t)\big)dt$$

$$= \int_0^T \underset{\sim}{x}^T(t)\underset{\sim}{C}^T\underset{\sim}{u}(t)dt = \int_0^T \underset{\sim}{u}^T(t)[\underset{\sim}{C}\underset{\sim}{x}(t)+\underset{\sim}{D}\underset{\sim}{u}(t)]dt$$

$$= \int_0^T \underset{\sim}{u}^T(t)\underset{\sim}{y}(t)dt$$

where $\underset{\sim}{y}(\cdot)$ is the output corresponding to $\{\underset{\sim}{u}(\cdot), \underset{\sim}{x}(\cdot)\}$. Therefore $\phi(\cdot)$ is a conservative potential energy function for S (Def. 2.3).

vii) $\Rightarrow$ vi). By assumption, there exists a matrix $\underset{\sim}{Q}$ such that $\phi(\underset{\sim}{x}) \triangleq \frac{1}{2}\langle \underset{\sim}{x}, \underset{\sim}{Q}\underset{\sim}{x}\rangle$ is a conservative potential energy function for S. Define $\underset{\sim}{K} \triangleq \frac{1}{2}(\underset{\sim}{Q}+\underset{\sim}{Q}^T)$; then $\underset{\sim}{K}$ is symmetric and $\phi(\underset{\sim}{x}) = \frac{1}{2}\langle \underset{\sim}{x}, \underset{\sim}{K}\underset{\sim}{x}\rangle$. Let $\{\underset{\sim}{u}(\cdot), \underset{\sim}{x}(\cdot)\}$ be any input-trajectory pair for S; then, for all $t \geq 0$,

$$\phi\big(\underset{\sim}{x}(t)\big) - \phi\big(\underset{\sim}{x}(0)\big) = \int_0^t \underset{\sim}{u}^T(\tau)[\underset{\sim}{C}\underset{\sim}{x}(\tau)+\underset{\sim}{D}\underset{\sim}{u}(\tau)]d\tau. \qquad (C.3)$$

Differentiating both sides of (C.3) and rearranging, one obtains for almost all $t \geq 0$,

$$\underset{\sim}{x}^T(t)\underset{\sim}{K}\underset{\sim}{A}\underset{\sim}{x}(t) + \underset{\sim}{x}^T(t)\big(\underset{\sim}{K}\underset{\sim}{B}-\underset{\sim}{C}^T\big)\underset{\sim}{u}(t) - \underset{\sim}{u}^T(t)\underset{\sim}{D}\underset{\sim}{u}(t) = 0. \qquad (C.4)$$

It is not hard to see that (C.4) will be satisfied for all input-trajectory pairs if and only if $\underset{\sim}{D}$ is antisymmetric, $\underset{\sim}{K}\underset{\sim}{A}$ is antisymmetric, and $\underset{\sim}{K}\underset{\sim}{B} = \underset{\sim}{C}^T$.

vii) $\Rightarrow$ i). This follows because every state representation with a conservative potential energy function is lossless (subsection 2.2).

i) $\Rightarrow$ ii). Let S be lossless. By Lemma C.2, $\hat{S}$ is lossless. Let $\omega_0 \in \mathbb{R}$ be such that $j\omega_0 \notin \lambda(\underset{\sim}{A})$, let $\underset{\sim}{w} \in \mathbb{C}^n$, let the input to $\hat{S}$ be $\underset{\sim}{u}(t) = \underset{\sim}{w}\, e^{j\omega_0 t}$ for $t \geq 0$, and let the state of $\hat{S}$ at time zero be $\underset{\sim}{x}(0) = (j\omega_0\underset{\sim}{I}-\underset{\sim}{A})^{-1}\underset{\sim}{B}\underset{\sim}{w}$. It is easy to verify [13] that the corresponding state trajectory $x(\cdot)$ is

$$\underset{\sim}{x}(t) = (j\omega_0\underset{\sim}{I}-\underset{\sim}{A})^{-1}\underset{\sim}{B}\underset{\sim}{w}\, e^{j\omega_0 t}, \quad t \geq 0; \tag{C.5}$$

moreover, the output is $\underset{\sim}{y}(t) = \underset{\sim}{H}(j\omega_0)\underset{\sim}{w}\, e^{j\omega_0 t}$ for $t \geq 0$. Eq. (C.5) shows that the state trajectory is periodic with period T, where $T = 2\pi/|\omega_0|$ if $\omega_0 \neq 0$, or T is any positive number if $\omega_0 = 0$. Note that

$$\begin{aligned} \mathrm{Re}[\underset{\sim}{w}^H\underset{\sim}{H}(j\omega_0)\underset{\sim}{w}]T &= \mathrm{Re}\int_0^T \underset{\sim}{w}^H\underset{\sim}{H}(j\omega_0)\underset{\sim}{w}\,dt \\ &= \mathrm{Re}\int_0^T \underset{\sim}{u}^H(t)\underset{\sim}{y}(t)dt = 0, \end{aligned} \tag{C.6}$$

where the last step follows from assertion a) of Lemma 2.1. Since $T > 0$, (C.6) shows that $0 = \mathrm{Re}[\underset{\sim}{w}^H\underset{\sim}{H}(j\omega_0)\underset{\sim}{w}] = \frac{1}{2}\underset{\sim}{w}^H[\underset{\sim}{H}(j\omega_0) + \underset{\sim}{H}^H(j\omega_0)]\underset{\sim}{w}$; from which it follows that $\underset{\sim}{H}(j\omega_0) + \underset{\sim}{H}^H(j\omega_0) = \underset{\sim}{0}$, because $\underset{\sim}{w} \in \mathbb{C}^n$ is arbitrary. Note that $\underset{\sim}{H}(-j\omega_0) = \bar{H}(j\omega_0)$ because $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, and $\underset{\sim}{D}$ are real matrices. Thus $\underset{\sim}{H}(j\omega_0) = -\underset{\sim}{H}^T(-j\omega_0)$.

ii) $\Rightarrow$ iii). The mapping $s \to \underset{\sim}{H}(s) + \underset{\sim}{H}^T(-s)$ is a matrix-valued function whose elements are holomorphic in $\mathbb{C} \setminus \lambda(\underset{\sim}{A})$; moreover, it vanishes on the set $\{s \in \mathbb{C} : \mathrm{Re}[s] = 0 \text{ and } s \notin \lambda(\underset{\sim}{A})\}$. From a standard result in complex analysis [14, Theorem 10.18], it follows that $\underset{\sim}{H}(s) + \underset{\sim}{H}^T(-s) = \underset{\sim}{0}$ for <u>all</u> $s \in \mathbb{C} \setminus \lambda(\underset{\sim}{A})$.

iii) $\Rightarrow$ iv). Let $\{\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\}$ be an L^2 admissible pair of S with zero initial state, and let $\{\underset{\sim}{u}(\cdot),\underset{\sim}{y}(\cdot)\}$ be the corresponding input-output pair of S. From standard results in the theory of Fourier integrals in the complex domain [15], it follows that $\underset{\sim}{u}(\cdot)$ and $\underset{\sim}{y}(\cdot)$ are Laplace transformable in the open right-half complex plane; moreover, their Laplace transforms, denoted $\underset{\sim}{U}(\cdot)$ and $\underset{\sim}{Y}(\cdot)$ respectively, are holomorphic there. For real σ and ω, define $\underset{\sim}{U}(j\omega) \triangleq \lim_{\sigma\to 0^+} \underset{\sim}{U}(\sigma+j\omega)$: this limit exists for almost all $\omega \in \mathbb{R}$, and the function $\omega \to \underset{\sim}{U}(j\omega)$ is the L^2-Fourier transform of $\underset{\sim}{u}(\cdot)$. [15]. Likewise, $\omega \to \underset{\sim}{Y}(j\omega) \triangleq \lim_{\sigma\to 0^+} \underset{\sim}{Y}(\sigma+j\omega)$ is the L^2-Fourier transform of $\underset{\sim}{y}(\cdot)$. If all poles of $\underset{\sim}{H}(\cdot)$ are in the open left-half plane, set $\sigma_0 = 0$; otherwise, let σ_0 be the maximum real part of the poles of $\underset{\sim}{H}(\cdot)$. It follows from the time-domain relation between $\underset{\sim}{u}(\cdot)$ and $\underset{\sim}{y}(\cdot)$ that $\underset{\sim}{Y}(s) = \underset{\sim}{H}(s)\underset{\sim}{U}(s)$ for $\mathrm{Re}[s] > \sigma_0$. Since $s \to \underset{\sim}{H}(s)\underset{\sim}{U}(s)$ is meromorphic in the open right-half plane and equal to the holomorphic function $s \to \underset{\sim}{Y}(s)$ for $\mathrm{Re}[s] > \sigma_0$, it follows that $\underset{\sim}{Y}(s) = \underset{\sim}{H}(s)\underset{\sim}{U}(s)$ for <u>all</u> s where $\underset{\sim}{H}(s)$ is defined in the open right-half plane [14].[6] Thus $\underset{\sim}{Y}(j\omega) = \underset{\sim}{H}(j\omega)\underset{\sim}{U}(j\omega)$ for almost all real ω. Parseval's theorem [15] gives

$$\int_0^\infty \langle \underset{\sim}{v}(t),\underset{\sim}{i}(t)\rangle\, dt = \int_0^\infty \langle \underset{\sim}{u}(t),\underset{\sim}{y}(t)\rangle\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle \overline{\underset{\sim}{U}(j\omega)},\underset{\sim}{Y}(j\omega)\rangle\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{\sim}{U}^H(j\omega)\underset{\sim}{H}(j\omega)\underset{\sim}{U}(j\omega)d\omega$$

$$= \frac{1}{2\pi}\,\mathrm{Re}\int_{-\infty}^{\infty} \underset{\sim}{U}^H(j\omega)\underset{\sim}{H}(j\omega)\underset{\sim}{U}(j\omega)d\omega$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} \underset{\sim}{U}^H(j\omega)[\underset{\sim}{H}(j\omega)+\underset{\sim}{H}^H(j\omega)]\underset{\sim}{U}(j\omega)d\omega = 0.$$

[6] Hence, the assumption that $\underset{\sim}{u}(\cdot)$ and $\underset{\sim}{y}(\cdot)$ belong to L^2 implies that the poles of $\underset{\sim}{H}(\cdot)$ in the open right-half plane (if any) are cancelled when the product $\underset{\sim}{H}(\cdot)\underset{\sim}{U}(\cdot)$ is formed.

The last step follows since $\underset{\sim}{H}(j\omega) + \underset{\sim}{H}^H(j\omega) = \underset{\sim}{H}(j\omega) + \underset{\sim}{H}^T(-j\omega) = \underset{\sim}{0}$.

iv) $\Rightarrow$ ii). Assume ii) is not satisfied. We shall show that iv) is not satisfied. If ii) is not satisfied, then there exists an $\omega_o \in \mathbb{R}$ such that $j\omega_o \notin \lambda(\underset{\sim}{A})$ and $\underset{\sim}{H}(j\omega_o) + \underset{\sim}{H}^H(j\omega_o) \neq \underset{\sim}{0}$. Choose $\underset{\sim}{w} \in \mathbb{C}^n$ such that $\underset{\sim}{w}^H[\underset{\sim}{H}(j\omega_o) + \underset{\sim}{H}^H(j\omega_o)]\,\underset{\sim}{w} \neq 0$. The elements of $\underset{\sim}{H}(\cdot)$ are continuous at $j\omega_o$ (because $j\omega_o \notin \lambda(\underset{\sim}{A})$); hence, there exists a $\Delta\omega \in (0,1)$ such that $j\omega \notin \lambda(\underset{\sim}{A})$ and $\underset{\sim}{w}^H[\underset{\sim}{H}(j\omega) + \underset{\sim}{H}^H(j\omega)]\underset{\sim}{w} \neq 0$ for all $\omega \in [\omega_o - \Delta\omega,\ \omega_o + \Delta\omega]$. Define $\sigma_o \triangleq \sqrt{1-(\Delta\omega)^2}$. Let $p_1, p_2, \ldots, p_k$ denote the poles of $\underset{\sim}{H}(\cdot)$ which lie in the closed right-half plane (if there are any), let m_i denote the multiplicity of p_i, and let M be any integer such that $M \geq \sum_{i=1}^{k} m_i$ (if $\underset{\sim}{H}(\cdot)$ has no poles in the closed right-half plane, set $m_i = 0$ for all i). Let the input to $\hat{S}$ be $\hat{\underset{\sim}{u}}(t) \triangleq (\frac{d}{dt} - p_1)^{m_1}(\frac{d}{dt} - p_2)^{m_2} \ldots (\frac{d}{dt} - p_k)^{m_k}(\frac{t^M}{M!})\exp[-(\sigma_o - j\omega_o)t]\underset{\sim}{w}$ for $t \geq 0$. Note that $\hat{\underset{\sim}{u}}(\cdot) \in L^2(\mathbb{R}^+ \to \mathbb{C}^n)$, and its Laplace transform is

$$\hat{\underset{\sim}{U}}(s) = \frac{(s-p_1)^{m_1}(s-p_2)^{m_2}\ldots(s-p_k)^{m_k}}{(s+\sigma_o - j\omega_o)^{M+1}}\underset{\sim}{w}.$$

If $\hat{\underset{\sim}{y}}(\cdot)$ denotes the output of $\hat{S}$ when the initial state is zero, then the Laplace transform of $\hat{\underset{\sim}{y}}(\cdot)$ is seen to be

$$\hat{\underset{\sim}{Y}}(s) = \frac{(s-p_1)^{m_1}(s-p_2)^{m_2}\ldots(s-p_k)^{m_k}}{(s+\sigma_o - j\omega_o)^{M+1}}\underset{\sim}{H}(s)\underset{\sim}{w}. \tag{C.7}$$

The numerator on the right-hand side of (C.7) cancels any poles of $\underset{\sim}{H}(\cdot)$ in the closed right-half plane[7]; thus, $\hat{\underset{\sim}{y}}(\cdot) \in L^2(\mathbb{R}^+ \to \mathbb{C}^n)$. If $\{\hat{\underset{\sim}{v}}(\cdot), \hat{\underset{\sim}{i}}(\cdot)\}$ denotes the admissible pair of $\hat{S}$ corresponding to the input-output pair $\{\hat{\underset{\sim}{u}}(\cdot), \hat{\underset{\sim}{y}}(\cdot)\}$, then an application of Parseval's theorem [15] yields

$$\mathrm{Re}\int_0^\infty \hat{\underset{\sim}{v}}^H(t)\ \hat{\underset{\sim}{i}}(t)dt = \mathrm{Re}\int_0^\infty \hat{\underset{\sim}{u}}^H(t)\ \hat{\underset{\sim}{y}}(t)dt$$

$$= \frac{1}{2\pi}\,\mathrm{Re}\int_{-\infty}^\infty \hat{\underset{\sim}{U}}^H(j\omega)\ \underset{\sim}{H}(j\omega)\ \hat{\underset{\sim}{U}}(j\omega)d\omega$$

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[7] And, by construction, $\underset{\sim}{H}(\cdot)$ has no poles at infinity.

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{\underline{U}}^H(j\omega)[\underline{H}(j\omega) + \underline{H}^H(j\omega)]\hat{\underline{U}}(j\omega)d\omega$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} \underline{w}^H[\underline{H}(j\omega) + \underline{H}^H(j\omega)]\underline{w}\,\frac{|j\omega-p_1|^{2m_1}|j\omega-p_2|^{2m_2}\cdots|j\omega-p_k|^{2m_k}}{[\sigma_o^2+(\omega-\omega_o)^2]^{M+1}}\,d\omega. \tag{C.8}$$

Define $r : \mathbb{R} \to \mathbb{R}$ by

$$r(\omega) \triangleq \underline{w}^H[\underline{H}(j\omega) + \underline{H}^H(j\omega)]\underline{w}\,|j\omega-p_1|^{2m_1}|j\omega-p_2|^{2m_2}\cdots|j\omega-p_k|^{2m_k}. \tag{C.9}$$

By the choice of ω_o, $\underline{w}$, and $\Delta\omega$, $r(\cdot)$ is continuous on the interval $J \triangleq [\omega_o-\Delta\omega, \omega_o+\Delta\omega]$ and $r(\omega) \neq 0$ for all $\omega \in J$; thus, $r(\cdot)$ is sign-definite on J. Substituting (C.9) and $\sigma_o^2 = 1 - (\Delta\omega)^2$ into (C.8), one obtains

$$\mathrm{Re}\int_0^{\infty} \hat{\underline{v}}^H(t)\hat{\underline{i}}(t)dt = \frac{1}{4\pi}\int_{-\infty}^{\omega_o-\Delta\omega} \frac{r(\omega)d\omega}{[1-(\Delta\omega)^2+(\omega-\omega_o)^2]^{M+1}}$$

$$+\frac{1}{4\pi}\int_{\omega_o+\Delta\omega}^{\infty} \frac{r(\omega)d\omega}{[1-(\Delta\omega)^2+(\omega-\omega_o)^2]^{M+1}} + \frac{1}{4\pi}\int_{\omega_o-\Delta\omega}^{\omega_o+\Delta\omega} \frac{r(\omega)d\omega}{[1-(\Delta\omega)^2+(\omega-\omega_o)^2]^{M+1}}. \tag{C.10}$$

Note that the denominator of the integrand in the first two integrals in (C.10) is greater than 1, thus the Lebesgue Dominated Convergence Theorem [14] shows that the first two integrals approach zero as $M \to \infty$. The denominator of the integrand in the last integral is less than 1 (but positive), and the numerator is sign-definite; hence, the magnitude of the last integral increases without bound as $M \to \infty$. Evidently, if M is chosen large enough, then $\{\hat{\underline{v}}(\cdot),\hat{\underline{i}}(\cdot)\}$ is an L^2 admissible pair of $\hat{S}$ with zero initial state such that $\mathrm{Re}\int_0^{\infty} \hat{\underline{v}}^H(t)\hat{\underline{i}}(t)dt \neq 0$. It follows from Lemma C.3 that iv) is not satisfied.

Proof of assertion b) of theorem 5.1

First, assume that S is completely controllable and statement ii) is true. Under these conditions, S is lossless. To see this, let T > 0, let $\{\underset{\sim}{u}^*(\cdot),\underset{\sim}{x}^*(\cdot)\}|[0,T]$ be any input-trajectory pair of S with $\underset{\sim}{x}^*(0) = \underset{\sim}{x}^*(T) = \underset{\sim}{0}$, and let $\underset{\sim}{y}^*(\cdot)$ denote the corresponding output. Define an input $\underset{\sim}{u}(\cdot)$ as follows:

$$\underset{\sim}{u}(t) \triangleq \begin{cases} \underset{\sim}{u}^*(t), & t \in [0,T] \\ \underset{\sim}{0}, & t > T. \end{cases}$$

Note that $\underset{\sim}{u}(\cdot) \in L^2(\mathbb{R}^+ \to \mathbb{R}^n)$. Let $\underset{\sim}{y}(\cdot)$ denote the output generated by $\underset{\sim}{u}(\cdot)$ when the initial state of S is zero. Thus

$$\underset{\sim}{y}(t) = \begin{cases} \underset{\sim}{y}^*(t), & t \in [0,T] \\ \underset{\sim}{0}, & t > T \end{cases}$$

(note that $\underset{\sim}{y}(t) = \underset{\sim}{0}$ for $t > T$ because both the input and the state are zero for $t > T$). Since $\underset{\sim}{u}(\cdot), \underset{\sim}{y}(\cdot) \in L^2(\mathbb{R}^+ \to \mathbb{R}^n)$, Parseval's theorem [15] can be applied with the following result:

$$\int_0^T \langle \underset{\sim}{u}^*(t),\underset{\sim}{y}^*(t)\rangle dt = \int_0^\infty \langle \underset{\sim}{u}(t),\underset{\sim}{y}(t)\rangle dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{\sim}{U}^H(j\omega)\, \underset{\sim}{H}(j\omega)\, \underset{\sim}{U}(j\omega)d\omega$$

$$= \frac{1}{2\pi}\,\mathrm{Re}\int_{-\infty}^{\infty} \underset{\sim}{U}^H(j\omega)\, \underset{\sim}{H}(j\omega)\, \underset{\sim}{U}(j\omega)d\omega$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} \underset{\sim}{U}^H(j\omega)[\underset{\sim}{H}(j\omega) + \underset{\sim}{H}^T(-j\omega)]\, \underset{\sim}{U}(j\omega)d\omega$$

$$= 0.$$

Thus any input-trajectory pair $\{\underset{\sim}{u}^*(\cdot),\underset{\sim}{x}^*(\cdot)\}|[0,T]$ with $\underset{\sim}{x}^*(0) = \underset{\sim}{x}^*(T) = \underset{\sim}{0}$ consumes zero energy: by assertion b) of lemma 2.1, S is lossless. What has been shown is that ii) ⇒ i) when S is completely controllable. This fact, combined with assertion a) of theorem 5.1, shows that statements i) through iv) are equivalent when S is completely controllable.

Now suppose that S is completely controllable and lossless; under these conditions, statement vii) is true. To prove this assertion, let $S_m = \{\underset{\sim}{A}_m, \underset{\sim}{B}_m, \underset{\sim}{C}_m, \underset{\sim}{D}\}$ denote a minimal state representation which is equivalent to $S = \{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{D}\}$ (lemma C.4). Since S and S_m are linear and equivalent, they are zero-state equivalent. Thus S_m satisfies statement iii), because S does. This implies that S_m is lossless, because it has already been shown that statements i) through iv) are equivalent under the assumption of complete controllability. The next step of the proof is to show that statement vi) is true when applied to S_m. To see this, note that S_m satisfies statement iii) because it is lossless; therefore

$$\underset{\sim}{C}_m(s\underset{\sim}{I}-\underset{\sim}{A}_m)^{-1}\underset{\sim}{B}_m + \underset{\sim}{D} = \underset{\sim}{B}_m^T(s\underset{\sim}{I}+\underset{\sim}{A}_m^T)^{-1}\underset{\sim}{C}_m^T - \underset{\sim}{D}^T \qquad \text{(C.11)}$$

for all $s \in \mathbb{C} \setminus \lambda(\underset{\sim}{A})$. Letting $s \to \infty$ in (C.11), we obtain $\underset{\sim}{D} = -\underset{\sim}{D}^T$; and so

$$\underset{\sim}{C}_m(s\underset{\sim}{I}-\underset{\sim}{A}_m)^{-1}\underset{\sim}{B}_m = \underset{\sim}{B}_m^T(s\underset{\sim}{I}+\underset{\sim}{A}_m^T)^{-1}\underset{\sim}{C}_m^T \qquad \text{(C.12)}$$

for all $s \in \mathbb{C} \setminus \lambda(\underset{\sim}{A})$. It follows from (C.12) that $\{\underset{\sim}{A}_m, \underset{\sim}{B}_m, \underset{\sim}{C}_m\}$ and $\{-\underset{\sim}{A}_m^T, \underset{\sim}{C}_m^T, \underset{\sim}{B}_m^T\}$ are both minimal realizations of the transfer function $\underset{\sim}{C}_m(s\underset{\sim}{I}-\underset{\sim}{A}_m)^{-1}\underset{\sim}{B}_m$. From a result in linear system theory [11, theorem 9, p. 184], there exists a unique invertible matrix $\underset{\sim}{Q}$ such that $-\underset{\sim}{A}_m^T = \underset{\sim}{Q}\underset{\sim}{A}_m\underset{\sim}{Q}^{-1}$, $\underset{\sim}{C}_m^T = \underset{\sim}{Q}\underset{\sim}{B}_m$, and $\underset{\sim}{B}_m^T = \underset{\sim}{C}_m\underset{\sim}{Q}^{-1}$. The reader can easily verify that the equations in vi) will be satisfied for S_m by choosing $\underset{\sim}{K} = \underset{\sim}{Q}$. To complete the proof that statement vi) is true when applied to S_m, it must be shown that $\underset{\sim}{Q}$ is symmetric. The reader can easily verify that $\underset{\sim}{Q}$ satisfies these three eqs. if and only if $\underset{\sim}{Q}^T$ does. Since the solution $\underset{\sim}{Q}$ is also unique, it follows that $\underset{\sim}{Q} = \underset{\sim}{Q}^T$. Since statement vi) is true when applied to

S_m, it follows from assertion a) and its proof that $\phi_m(\underset{\sim}{x}_m) \triangleq \frac{1}{2} \langle \underset{\sim}{x}_m, \underset{\sim}{K}\underset{\sim}{x}_m \rangle$ is a conservative potential energy function for S_m. Let $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ be an input-trajectory pair of S; from lemma 3.2, the corresponding (unique) input-trajectory pair of S_m is $\{\underset{\sim}{u}(\cdot),\underset{\sim}{P}\underset{\sim}{x}(\cdot)\}$, where $\underset{\sim}{P}$ is the matrix in lemma C.4. Thus, for any $T \geq 0$, the energy consumed by $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[0,T]$ is $\phi_m\big(\underset{\sim}{P}\underset{\sim}{x}(t)\big)-\phi_m\big(\underset{\sim}{P}\underset{\sim}{x}(0)\big)$; this implies that $\phi(\underset{\sim}{x}) \triangleq \frac{1}{2} \langle \underset{\sim}{x}, \underset{\sim}{P}^T\underset{\sim}{K}\underset{\sim}{P}\underset{\sim}{x} \rangle$ is a conservative potential energy function for S. What has been shown is that i) ⇒ vii) when S is completely controllable. This fact, combined with assertion a), shows that statements i), vi), and vii) are equivalent when S is completely controllable.

It remains to show the equivalence of statements i) and v) when S is completely controllable, so assume the latter. Theorem 2.1 will be utilized to prove v) ⇒ i), but note that theorem 2.1 cannot be applied directly to S because not every $u(\cdot) \in \mathcal{U} = L^2_{loc}(\mathbb{R}^+ \to \mathbb{R}^n)$ is bounded on every compact subset of $\mathbb{R}^+$. Let S^* denote a state representation which is identical to S except that the set of admissible input functions of S^*, denoted $\mathcal{U}^*$, is the set of piecewise continuous functions mapping $\mathbb{R}^+$ to $\mathbb{R}^n$. Theorem 2.1 shows that for S^*, v) ⇒ i). The proof that i) ⇒ ii) from assertion a) applies equally well to S^*. Thus we have the following relations: S satisfies v) ⇒ S^* satisfies v) ⇒ S^* satisfies i) ⇒ S^* satisfies ii) ⇔ S satisfies ii) ⇔ S satisfies i). This shows that if S is completely controllable and satisfies statement v), then S is lossless.

Now suppose that S is lossless and completely controllable. Let $S_m = \{\underset{\sim}{A}_m,\underset{\sim}{B}_m,\underset{\sim}{C}_m,\underset{\sim}{D}\}$ denote a miminal linear state representation which is equivalent to $S = \{\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}\}$ (lemma C.4). Let $\{\underset{\sim}{u}(\cdot),\underset{\sim}{y}(\cdot)\}$ be a bounded input-output pair of S; then, by equivalence, $\{\underset{\sim}{u}(\cdot),\underset{\sim}{y}(\cdot)\}$ is also an input-output pair of S_m, and $\underset{\sim}{x}_m(\cdot)$ will denote the corresponding (unique) state trajectory of S_m. Choose $\Delta t > 0$, and define

$$\underset{\sim}{N}_m(\Delta t) \triangleq \int_0^{\Delta t} e^{\underset{\sim}{A}_m^T s} \underset{\sim}{C}_m^T \underset{\sim}{C}_m e^{\underset{\sim}{A}_m s} ds. \tag{C.14}$$

Since S_m is completely observable, $\underset{\sim}{N}_m(\Delta t)$ is nonsingular [11, p. 176, theorem 5]; thus

$$\begin{aligned} \underset{\sim}{x}_m(t) &= [\underset{\sim}{N}_m(\Delta t)]^{-1} \underset{\sim}{N}_m(\Delta t) \underset{\sim}{x}_m(t) \\ &= [\underset{\sim}{N}_m(\Delta t)]^{-1} \int_0^{\Delta t} e^{\underset{\sim}{A}_m^T s} \underset{\sim}{C}_m^T \underset{\sim}{C}_m e^{\underset{\sim}{A}_m s} \underset{\sim}{x}_m(t) ds. \end{aligned} \tag{C.15}$$

Note that

$$\underset{\sim}{C}_m e^{\underset{\sim}{A}_m s} \underset{\sim}{x}_m(t) = \underset{\sim}{y}(t+s) - \int_t^{t+s} \underset{\sim}{C}_m e^{\underset{\sim}{A}_m(t+s-\tau)} \underset{\sim}{B}_m \underset{\sim}{u}(\tau) d\tau - \underset{\sim}{D}\underset{\sim}{u}(t+s). \tag{C.16}$$

Define

$$M_1 \triangleq \sup_{t \geq 0} \| \underset{\sim}{u}(t) \| < \infty \tag{C.17a}$$

$$M_2 \triangleq \sup_{t \geq 0} \| \underset{\sim}{y}(t) \| < \infty \tag{C.17b}$$

$$M_3(\Delta t) \triangleq \sup_{0 \leq s \leq \Delta t} \| \underset{\sim}{C}_m e^{\underset{\sim}{A}_m s} \underset{\sim}{B}_m \| < \infty \tag{C.17c}$$

$$M_4(\Delta t) \triangleq \sup_{0 \leq s \leq \Delta t} \| e^{\underset{\sim}{A}_m^T s} \underset{\sim}{C}_m^T \| < \infty. \tag{C.17d}$$

From (C.16) and (C.17), it follows that

$$\| \underset{\sim}{C}_m e^{\underset{\sim}{A}_m s} \underset{\sim}{x}_m(t) \| \leq M_2 + M_1 M_3(\Delta t) \Delta t + M_1 || \underset{\sim}{D} || < \infty. \tag{C.18}$$

for all (t,s) such that $t \geq 0$ and $0 \leq s \leq \Delta t$. Combining (C.15), (C.17d), and (C.18), one obtains

$$\| \underset{\sim}{x}_m(t) \| \leq \| [\underset{\sim}{N}_m(\Delta t)]^{-1} \| \; [M_2 + M_1 M_3(\Delta t) \Delta t + M_1 || \underset{\sim}{D} ||] M_4(\Delta t) \Delta t \tag{C.19}$$

for all $t \geq 0$. Thus $\underset{\sim}{x}_m(\cdot)$ is bounded. Now, since S_m is minimal and lossless, it has a continuous (in fact, quadratic) conservative potential energy function. If $\{\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\}$ denotes the admissible pair corresponding to $\{\underset{\sim}{u}(\cdot),\underset{\sim}{y}(\cdot)\}$, then it follows from lemma 2.7 that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \underset{\sim}{u}(t), \underset{\sim}{y}(t) \rangle dt = 0. \qquad ¤$$

Proof of Lemma 5.7

($\Rightarrow$) This follows immediately from lemma 3.3.

($\Leftarrow$) Suppose S is lossless. Then, since S is controllable, it is equivalent to a minimal linear state representation S_m (lemma C.4). By theorem 5.1, S_m is lossless. The minimal state representation S_m is completely observable [11]; therefore, it is state-observable (Def. 3.1). In summary, S_m is a lossless, state-observable hybrid state representation for N; therefore, N is lossless (Def. 3.6). ¤

FIGURE CAPTIONS

Fig. 1. A current-controlled 2-port which is cyclo-lossless but not lossless.

Fig. 2. A lossless 1-port which is not energetically reversible. Because of the diode in series with the capacitor, it "traps" all the energy which enters it.

Fig. 3. a. The constitutive relation of a nonlinear capacitor which is lossless and has properties 1, 2, 3 and 5 listed at the beginning of section II.

b. An L^2 admissible pair for this system for which the total energy is nonzero.

Fig. 4. A 1-port which has the zero average power property but is nevertheless lossy.

Fig. 5. This nonlinear capacitor is a lossless system which does not have the zero average power property.

Fig. 6. An illustration of def. 2.6 in the case that U is 1-dimensional.

a. A typical waveform $u(\cdot)$.

b. The restriction $u(\cdot)|[0,T)$.

c. The periodic extension of $u(\cdot)|[0,T)$.

Fig. 7. Figure for the proof of theorm 2.1. The trajectories $\underset{\sim}{x}_1(\cdot)$ and $\underset{\sim}{x}_2(\cdot)$ require different amounts of energy, i.e. $E_1 \neq E_2$. The existence of a return path $\underset{\sim}{x}_3(\cdot)$ is guaranteed by our assumption of complete controllability.

Fig. 8. Figure for the proof of theorem 3.1.

Fig. 9. Every voltage-controlled state representation has a realization of this form in which $\mathcal{C}$ is lossless. If $\mathcal{R}$ and $\mathcal{C}$ are both lossless we call it a lossless realization.

Fig. 10. A realization of the 1-port in example 6.1.

Appendices

Fig. A.1. Figure for the proof of lemma 2.2. The trajectory $\underset{\sim}{x}_4(\cdot)$ consists of $\underset{\sim}{x}_1(\cdot)$ followed by $\underset{\sim}{x}_3(\cdot)$.

Fig. A.2. Figure for the proof of lemma 3.2. We initially assume that $\underset{\sim}{\alpha}(\underset{\sim}{x}_1(t')) \neq \underset{\sim}{x}_2(t')$. The other trajectories are then used to show that this assumption results in a contradiction.

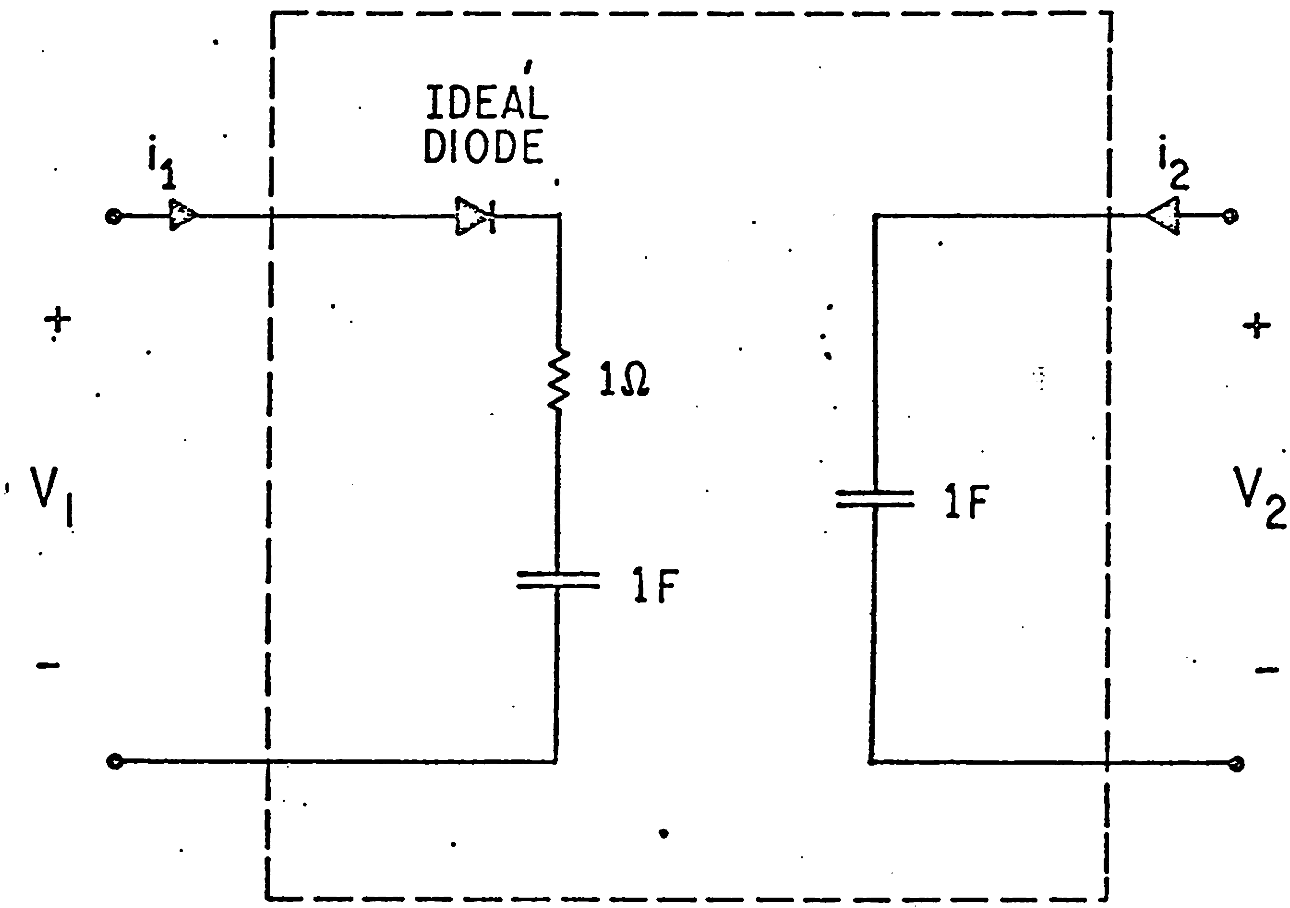


Fig. 1

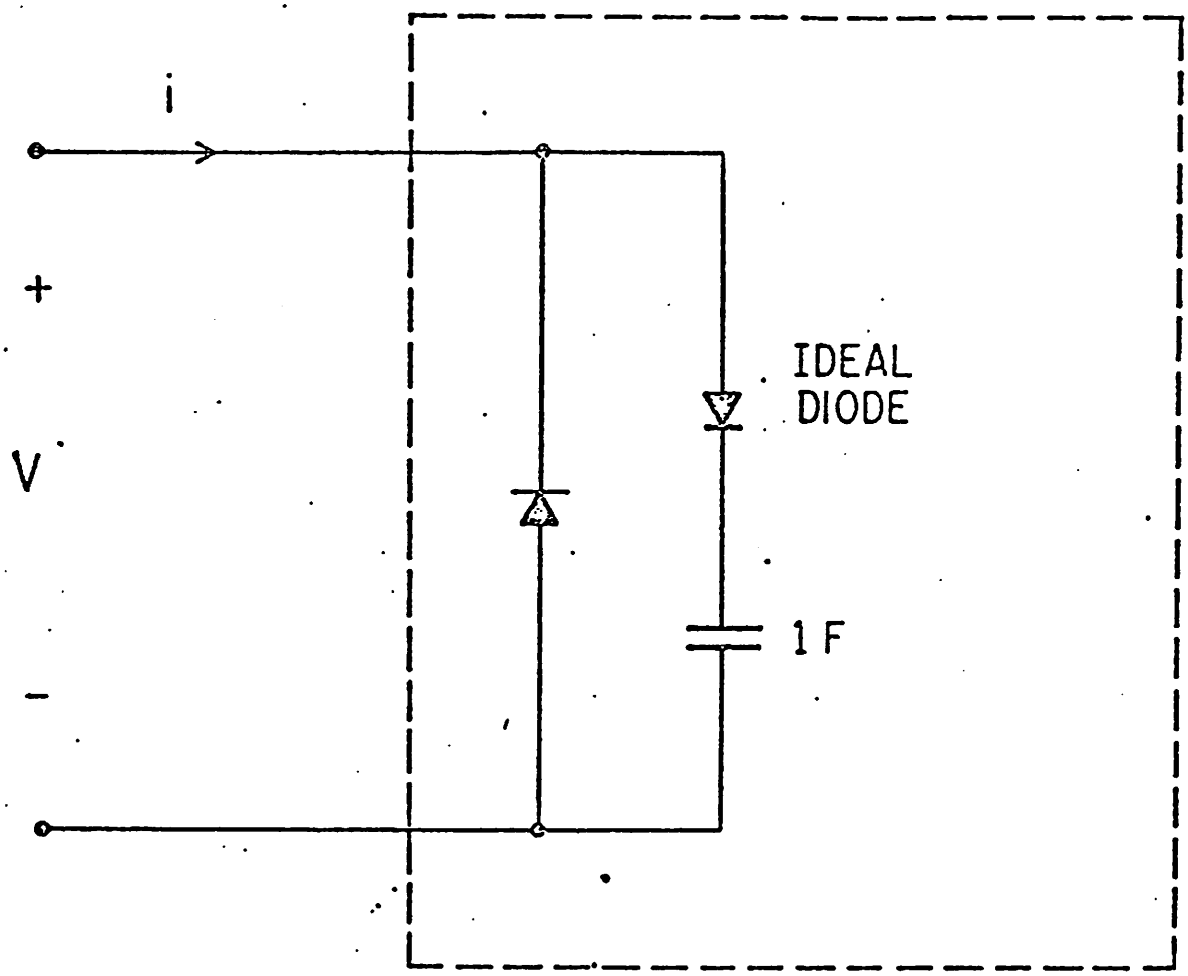


Fig. 2

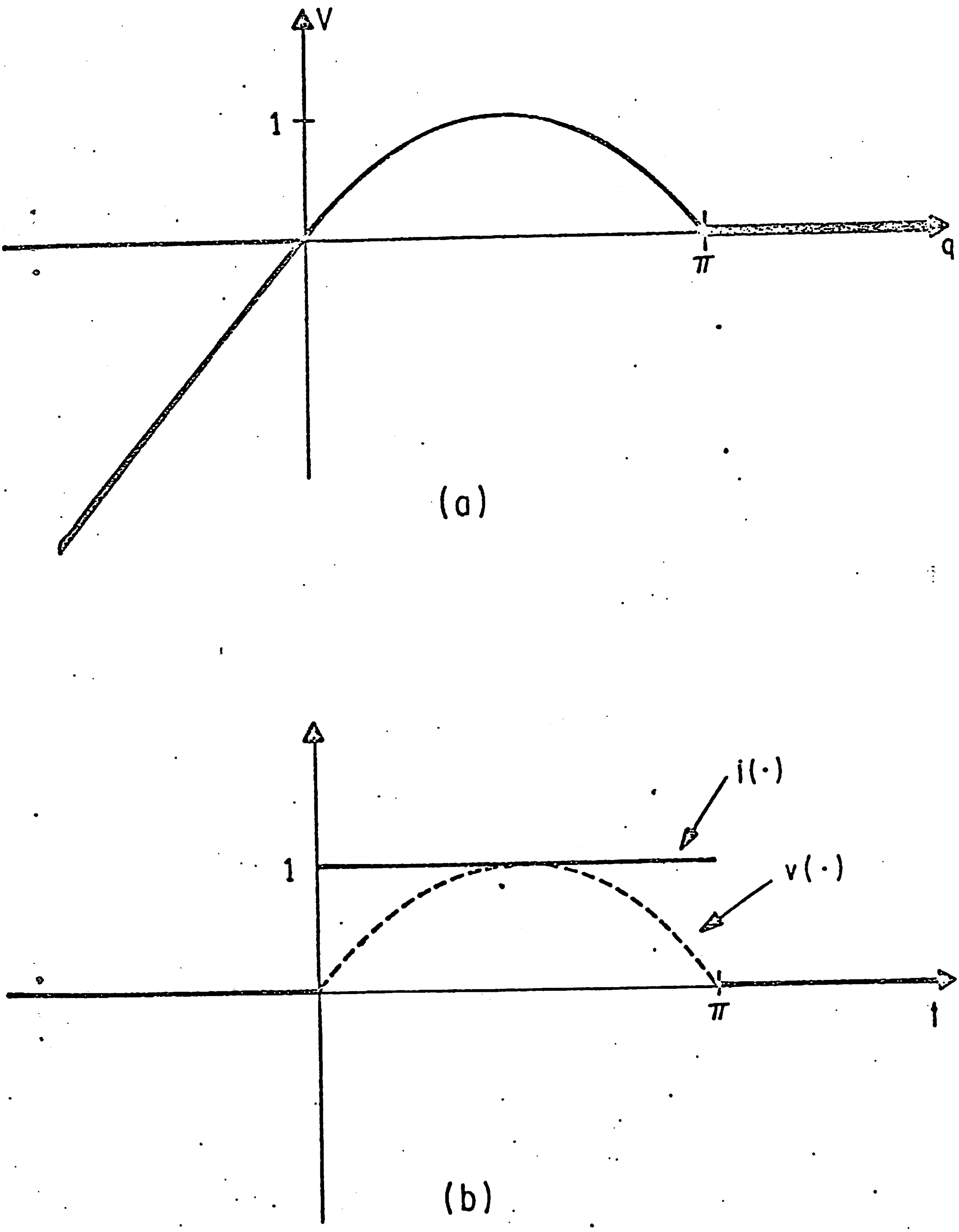


Fig. 3

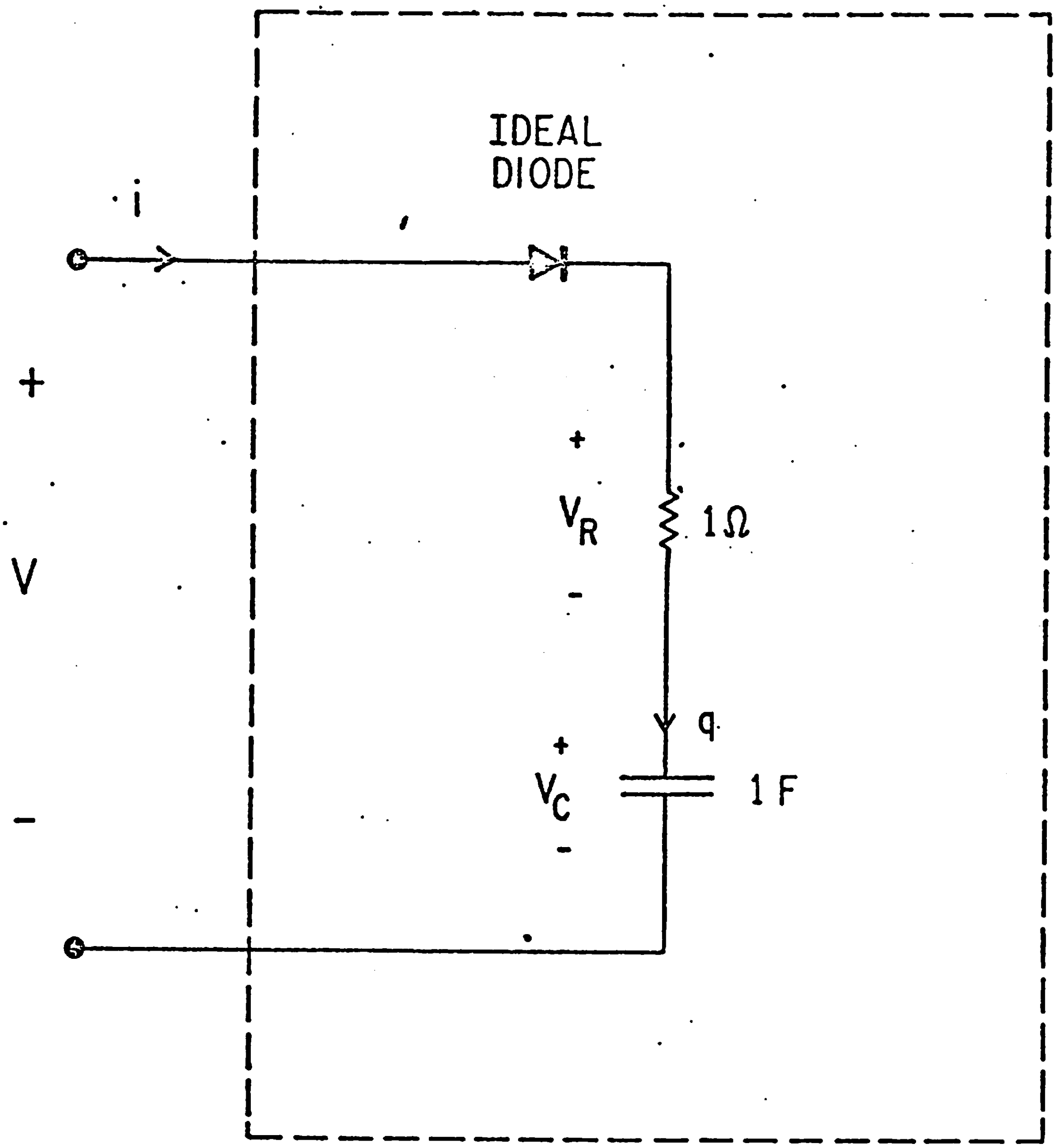


Fig. 4.

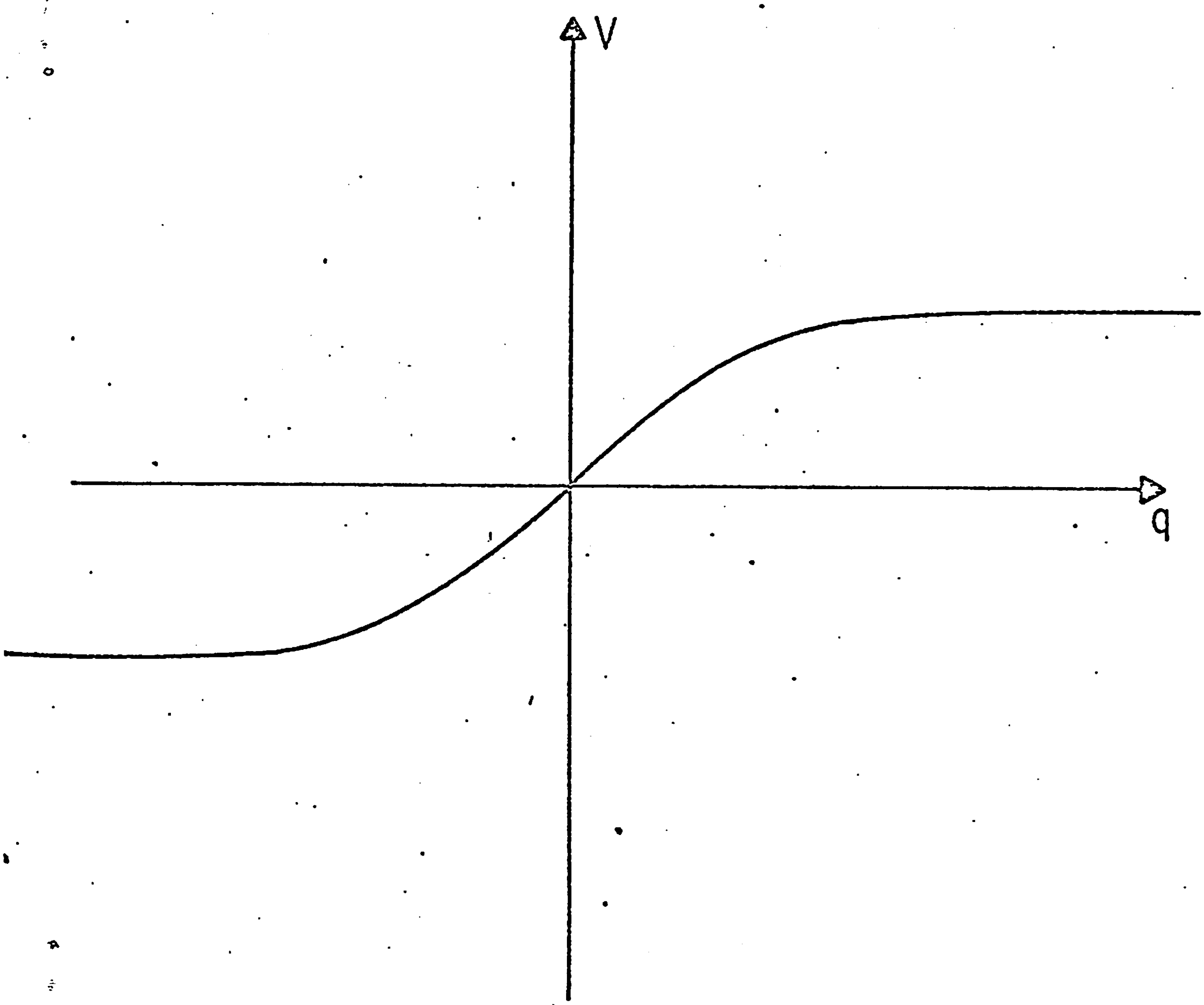


Fig. 5

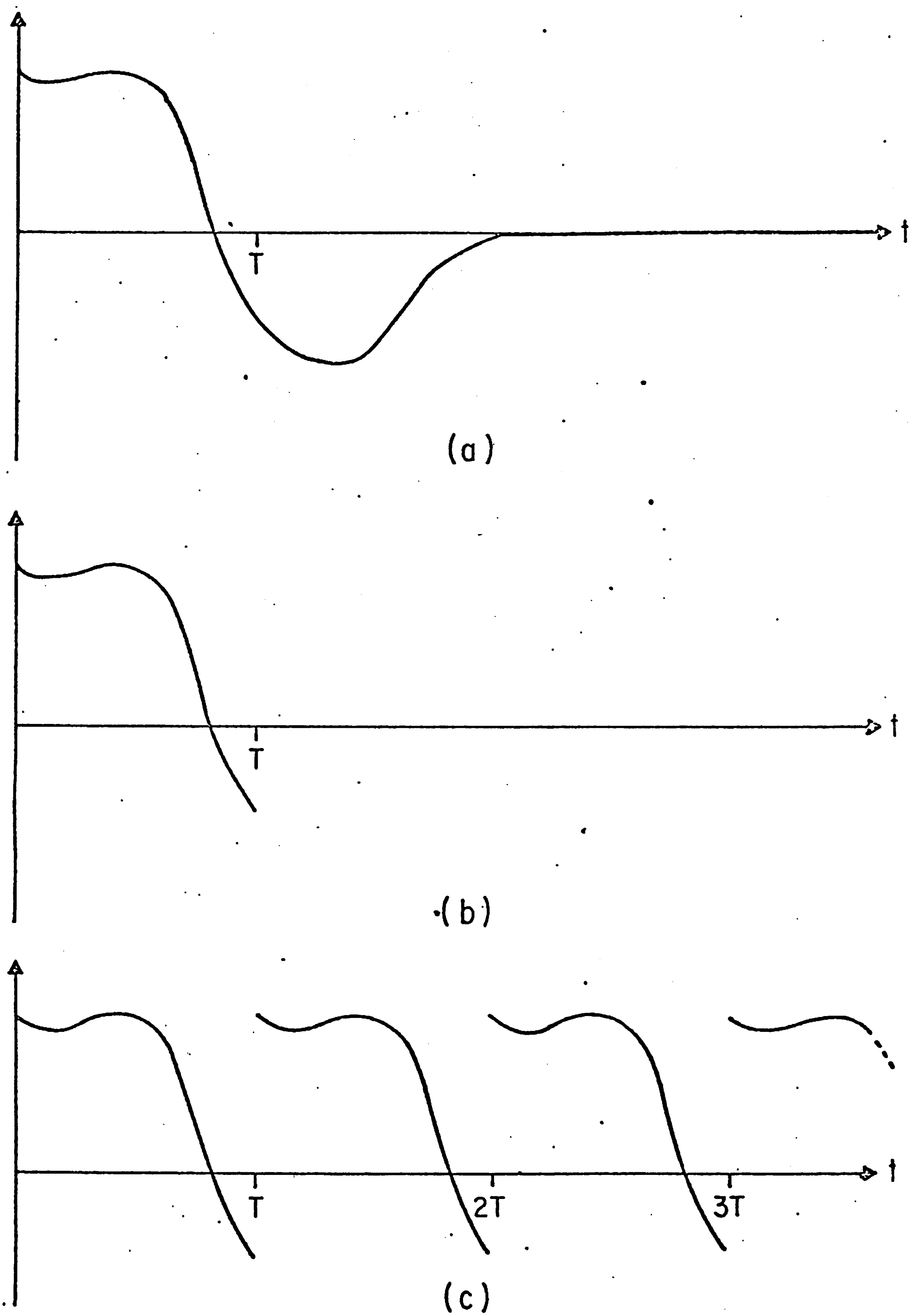
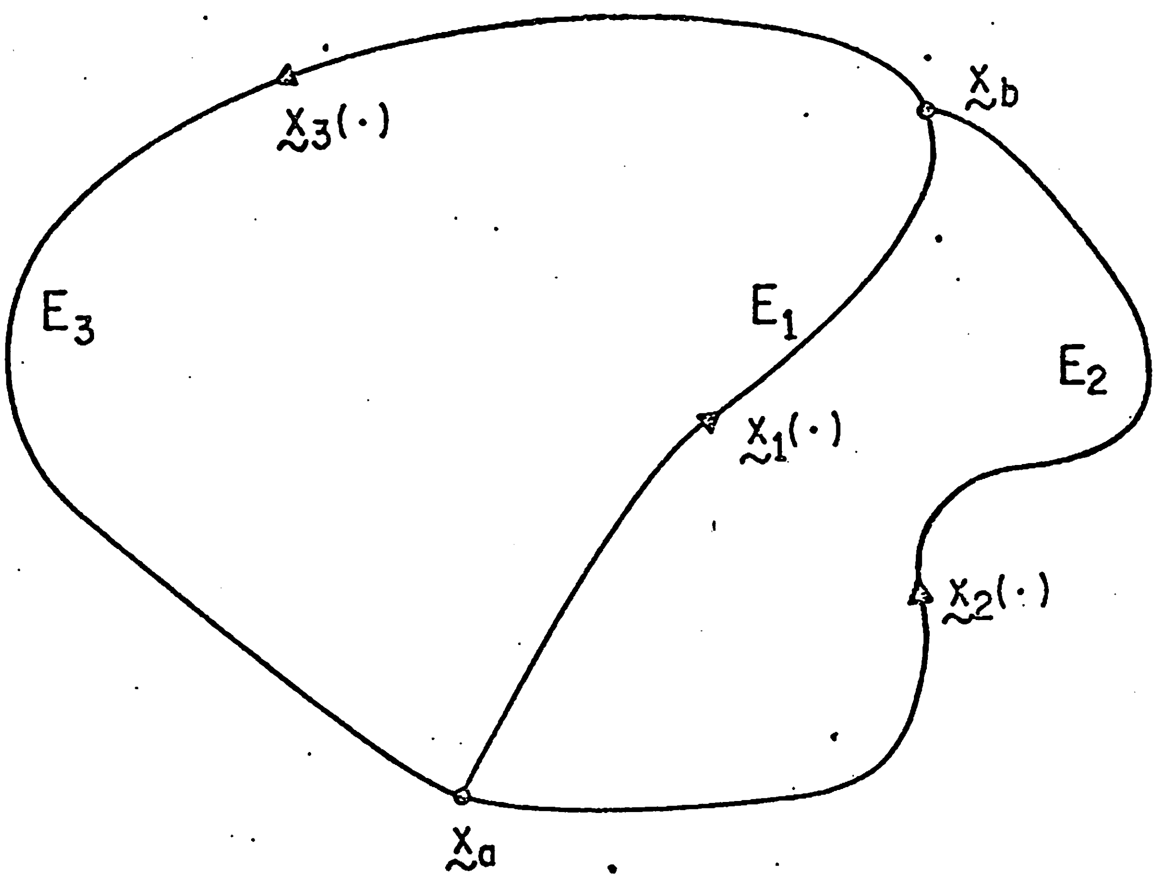


Fig. 6



$\underset{\sim}{X}_b$
$\underset{\sim}{X}_3(\cdot)$
E_3
E_1
E_2
$\underset{\sim}{X}_1(\cdot)$
$\underset{\sim}{X}_2(\cdot)$
$\underset{\sim}{X}_a$

Fig. 7

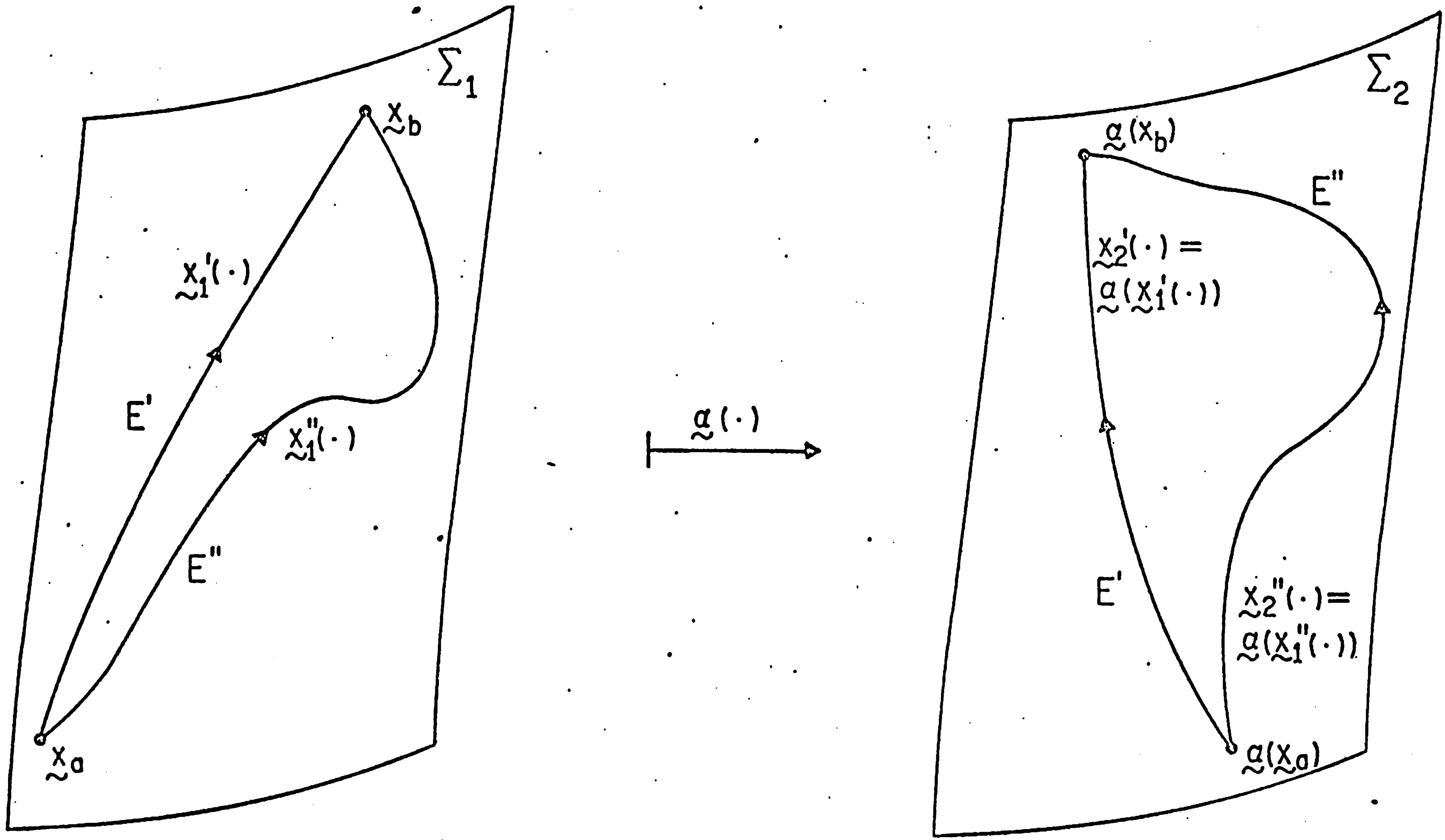
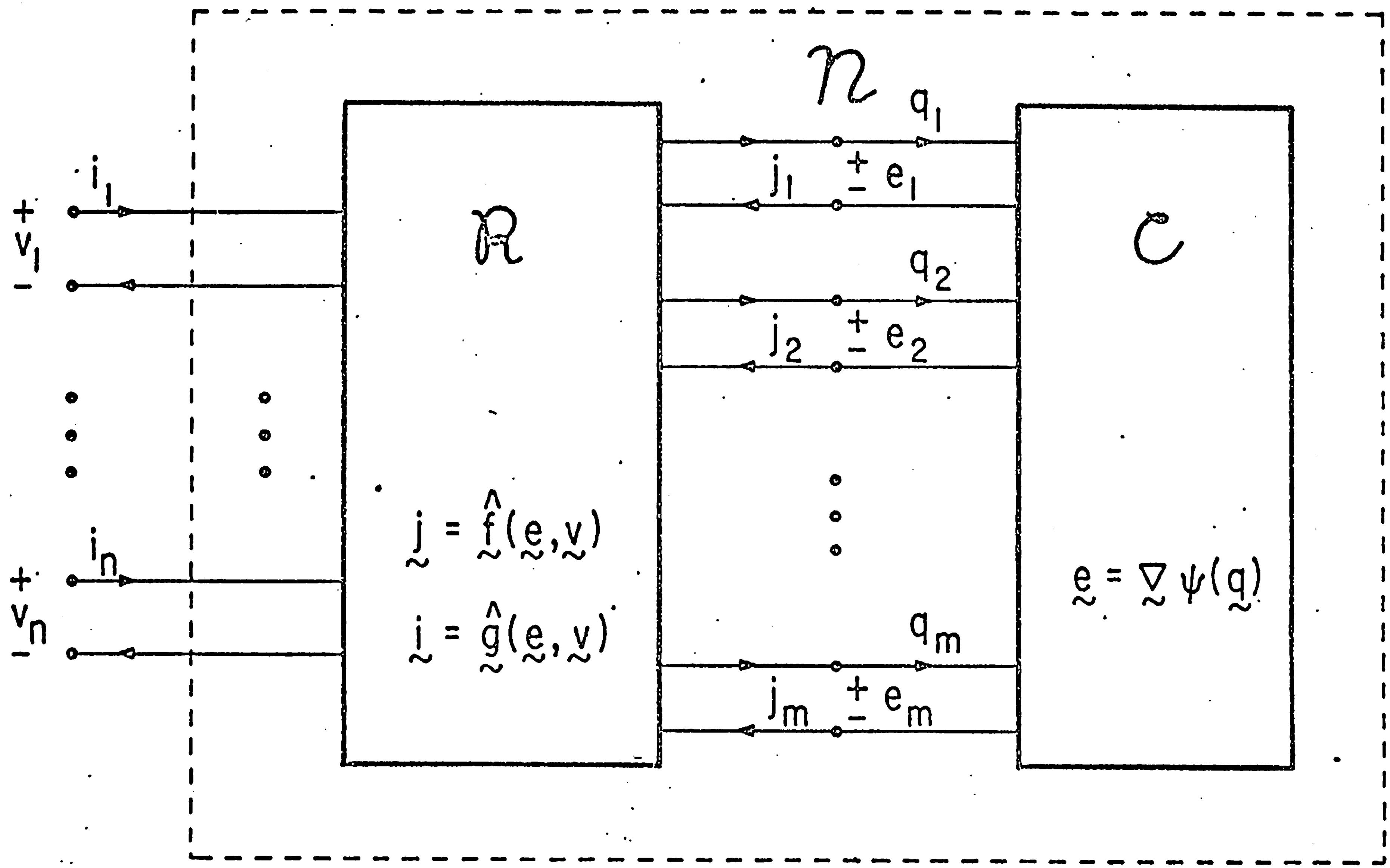
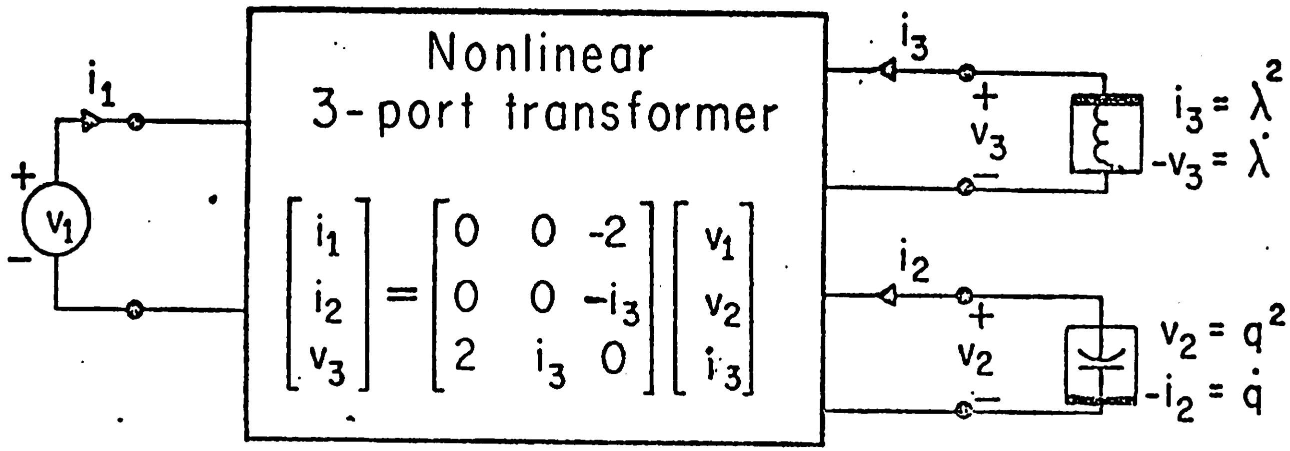


Fig. 8



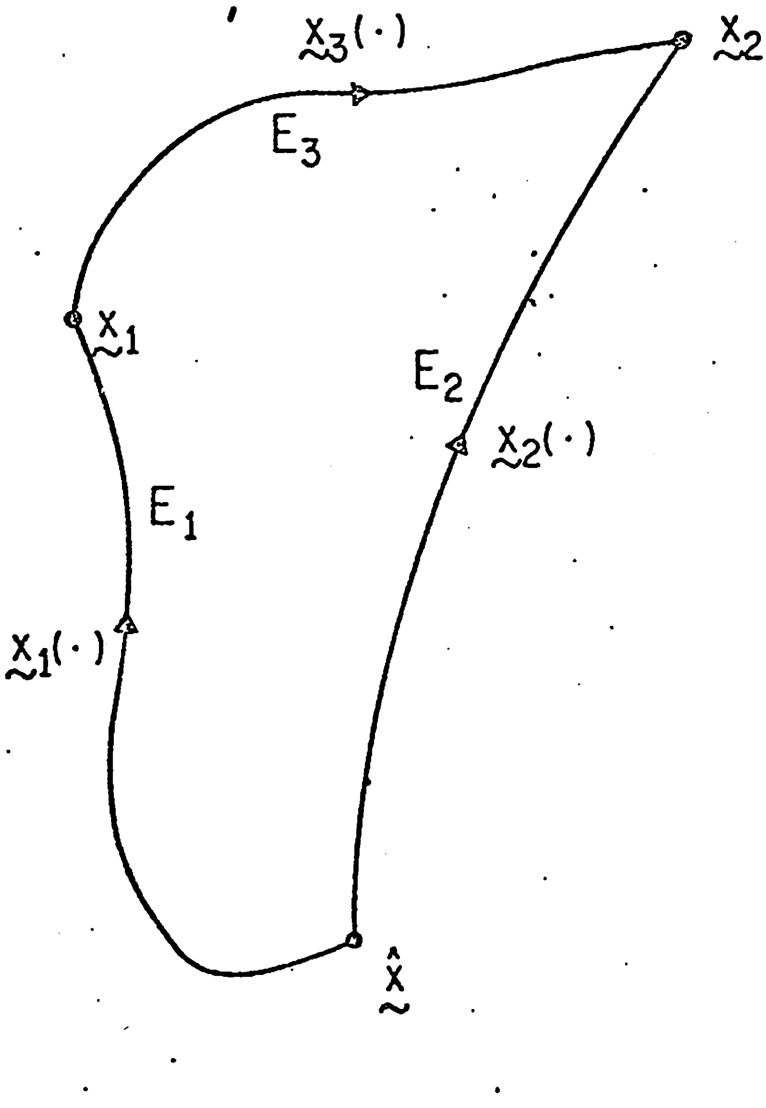
$\mathcal{N}$
$\mathcal{R}$
$\mathcal{C}$
i_1
$+ v_1 -$
i_n
$+ v_n -$
q_1
j_1
$+ e_1 -$
q_2
j_2
$+ e_2 -$
q_m
j_m
$+ e_m -$
$\underset{\sim}{j} = \hat{\underset{\sim}{f}}(\underset{\sim}{e}, \underset{\sim}{v})$
$\underset{\sim}{i} = \hat{\underset{\sim}{g}}(\underset{\sim}{e}, \underset{\sim}{v})$
$\underset{\sim}{e} = \underset{\sim}{\nabla} \psi(\underset{\sim}{q})$

Fig. 9



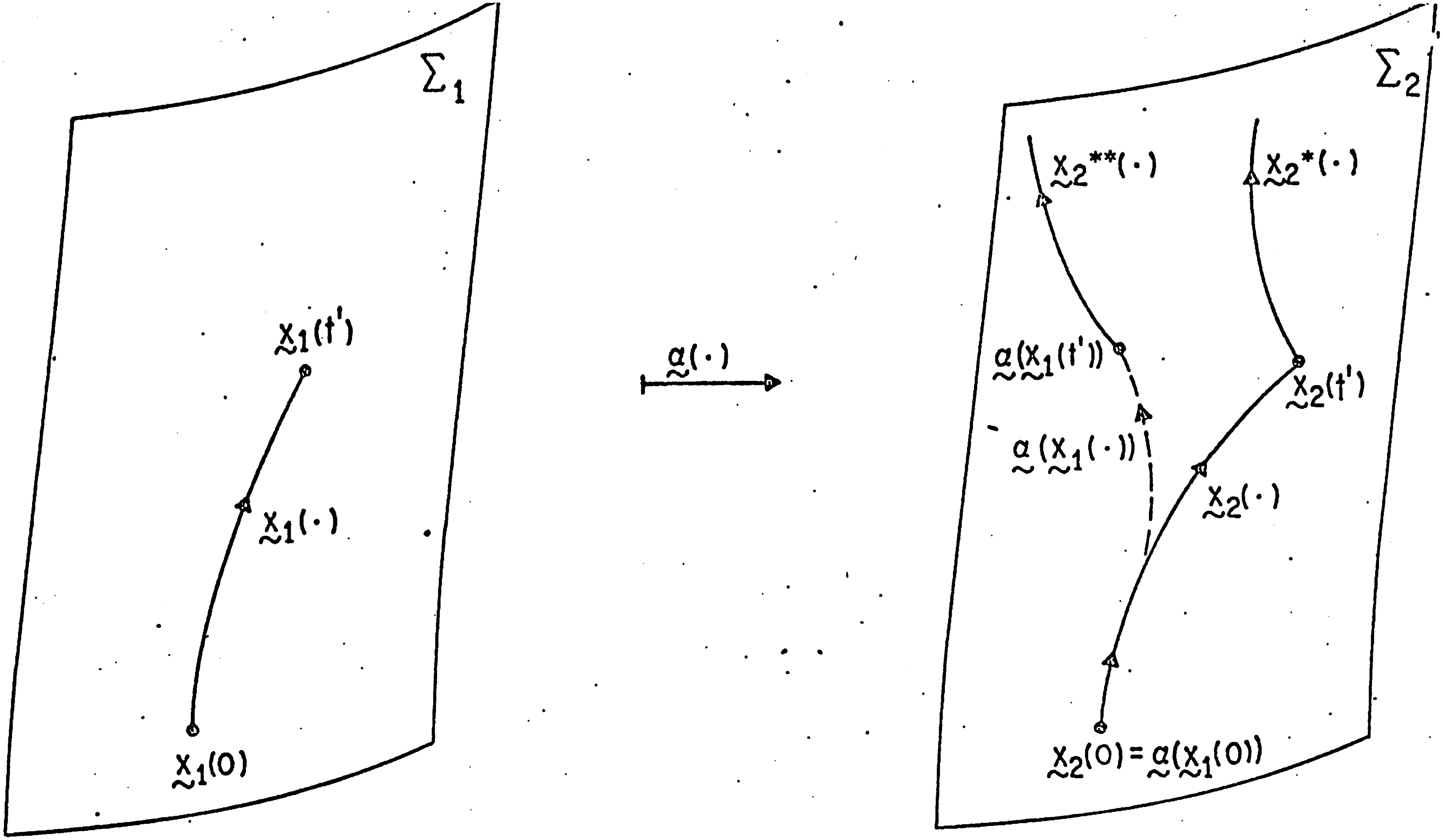
Nonlinear
3-port transformer
$\begin{bmatrix} i_1 \\ i_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & -i_3 \\ 2 & i_3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \dot{i}_3 \end{bmatrix}$
i_1
v_1
i_3
v_3
$i_3 = \lambda^2$
$-v_3 = \dot{\lambda}$
i_2
v_2
$v_2 = q^2$
$-i_2 = \dot{q}$

Fig. 10



$\underset{\sim}{x}_3(\cdot)$
$\underset{\sim}{x}_2$
E_3
$\underset{\sim}{x}_1$
E_2
$\underset{\sim}{x}_2(\cdot)$
E_1
$\underset{\sim}{x}_1(\cdot)$
$\hat{\underset{\sim}{x}}$

Fig. A.1



Σ_1
$\underset{\sim}{x}_1(t')$
$\underset{\sim}{x}_1(\cdot)$
$\underset{\sim}{x}_1(0)$
$\underset{\sim}{\alpha}(\cdot)$
Σ_2
$\underset{\sim}{x}_2^{**}(\cdot)$
$\underset{\sim}{x}_2^{*}(\cdot)$
$\underset{\sim}{\alpha}(\underset{\sim}{x}_1(t'))$
$\underset{\sim}{x}_2(t')$
$\underset{\sim}{\alpha}(\underset{\sim}{x}_1(\cdot))$
$\underset{\sim}{x}_2(\cdot)$
$\underset{\sim}{x}_2(0) = \underset{\sim}{\alpha}(\underset{\sim}{x}_1(0))$

Fig. A.2