GENERAL STUDY OF DISCRETE-TIME CONVOLUTION CONTROL SYSTEMS

by

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Abstract

In this report, we formulate a general class of linear time-invariant discrete-time distributed systems; and we study in depth these systems from the control system design point of view. We consider both single-input single-output (SISO) and multi-input multi-output (MIMO) systems, investigate their analytic properties, and establish design procedures for these systems.

The input-output (I/O) behavior of a linear time-invariant system is specified by its transfer function: For a causal lumped SISO system, its transfer function is a proper rational function. Such rational transfer functions have been extensively studied as functions of a complex variable, which led to many important control theory results (e.g. Nyquist theory, Bode plot, etc.). Vidyasagar pointed out that a proper rational function can be expressed as a ratio of two elements in some algebra other than the algebra of polynomials (e.g. the algebra of proper "stable" rational functions): This observation has led to a broad effort to investigate the relationship between the important properties of linear time-invariant systems and their algebraic structures. Extending this idea to continuous-time (distributed) convolution systems, F. M. Callier and C. A. Desoer (1978) developed an algebra of transfer

functions $\hat{B}(\sigma_0)$ for describing their transfer functions: here every element of $\hat{B}(\sigma_0)$ is expressed as a ratio of two elements in an algebra $\hat{A}_-(\sigma_0)$, the subalgebra of causal σ_0-stable transfer functions.

In this research, we study discrete-time (distributed) convolution systems, by making full use of the algebraic tools that have proved to be useful in the study of the other system representations, we develop a commutative algebra of transfer functions, $\tilde{b}(\rho_0)$, for a general class of SISO discrete-time convolution systems, which covers <u>sampled distributed</u>-systems and, of course, lumped systems as a special case. Each element of $\tilde{b}(\rho_0)$ is formulated as a ratio of two elements in an algebra $\tilde{\ell}_{1-}(\rho_0)$ of causal ρ_0-stable transfer functions. We demonstrate that $\tilde{\ell}_{1-}(\rho_0)$ is indeed a Euclidean ring; we give necessary and sufficient conditions for coprimeness between elements in $\tilde{\ell}_{1-}(\rho_0)$; and we study the concepts of poles and zeros for elements in $\tilde{b}(\rho_0)$. In contrast to the existing theory on transfer functions corresponding to ℓ_1-sequences, the algebra $\tilde{b}(\rho_0)$ includes both <u>stable</u> and <u>unstable</u> systems; and since $\rho_0 < 1$, this formulation allows us to study the dominant poles <u>inside</u> the unit disc of the complex plane.

With the SISO theory well established, we study MIMO systems whose transfer functions are matrices with elements in $\tilde{b}(\rho_0)$, and we establish the matrix fraction representation theory. Consequently, matrix multiplication introduces many additional problems: commutativity is lost, zero divisors are present, and the ring structure is lost in the case of nonsquare matrices.

We then investigate in detail many results of MIMO $\tilde{b}(\rho_0)$-systems that have similar counterparts in the other system descriptions: In particular, we obtain the dynamic interpretation of <u>poles</u> and transmission zeros. We consider interconnections of such MIMO systems, with feedback as a special case. We introduce the notion of characteristic functions to study the overall stability of any such interconnection (an idea similar to but not identical with that of characteristic polynomial); and we obtain necessary and sufficient conditions for ℓ_p-stability, $\forall p \in [1,\infty]$. The matrix fraction representation also allows us to obtain procedures for designing feedback systems with controllers to achieve stabilization (analogous to arbitrary closed-loop eigenvalue assignment), asymptotic tracking and disturbance rejection; finally, for the case of stable square plants (which can be obtained from an unstable one by the stabilization procedure), we are able to achieve complete decoupling with detailed pole assignment and finite settling-time, subject to, of course, the limitations imposed by the plant transmission zeros outside the open unit disc.

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GENERAL STUDY OF DISCRETE-TIME CONVOLUTION CONTROL SYSTEMS[†]

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1. Introduction

Consider a discrete-time convolution system whose weighting sequence is obtained by sampling the impulse response of a continuous-time linear time-invariant distributed system. Such a sampled system cannot, in general, be represented by a rational z-transfer function. In this paper, we develop a general theory to cover such cases. Our approach includes the rational transfer functions as a special case, and in many instances, the analysis exhibits some resemblance with the existing techniques for the rational case.

In Section 2, we develop a model for a class of such systems, whose transfer functions are elements of an algebra denoted by $\tilde{b}(\rho_0)$. The model encompasses both stable and unstable systems in the input-output (I/O) context. We discuss some properties of the poles and zeros for such systems, and we give some examples to demonstrate this more general model of system description. We consider in Section 3 multi-input multi-output (MIMO) systems whose transfer functions are matrices with elements in $\tilde{b}(\rho_0)$: we examine the notion of coprimeness (left- and right-coprime) and derive the matrix fraction representation theory for these systems. In Section 4, we consider the poles and define (transmission) zeros for MIMO systems and exhibit their dynamic interpretations; an example is given to demonstrate the claimed properties of the transmission zeros. Interconnected systems are considered in Section 5: here we introduce the notion of characteristic function for studying I/O stability. As a special case of interconnected systems, feedback systems and their I/O stability are studied in Section 6. In Section 7, we study the problem of controller design for feedback systems to satisfy specifications on stabilization, tracking and disturbance rejection; an example is provided to demonstrate the step-by-step procedure to obtain the controller transfer function. In Section 8, we extend the findings of [Des 5] to study feedback decoupling when the given plant is square and stable. We conclude this paper by some discussions in Section 9.

Notation

Let $\mathbb{R}$ ($\mathbb{C}$) be the field of all real (complex) numbers; let $\mathbb{N} := \{0,1,2,\ldots\}$ be the set of all natural numbers, and $\mathbb{N}^* := \mathbb{N}\setminus\{0\} = \{1,2,3,\ldots\}$ be the set of all positive integers. We denote by $\mathbb{R}^{\mathbb{N}}$ (respectively $\mathbb{C}^{\mathbb{N}}$) the set of all real (resp. complex) sequences on $\mathbb{N}$, i.e. $\mathbb{R}^{\mathbb{N}}$ (resp. $\mathbb{C}^{\mathbb{N}}$) $:= \{(g(0),g(1),g(2),\ldots) \mid g(k) \in \mathbb{R}$ (resp. $\mathbb{C}$), $\forall k \in \mathbb{N}\}$; and we denote by $\mathbb{R}_z^{\mathbb{N}}$ (resp. $\mathbb{C}_z^{\mathbb{N}}$) the subset of all <u>z-transformable</u> sequences in $\mathbb{R}^{\mathbb{N}}$ (resp. $\mathbb{C}^{\mathbb{N}}$), i.e. $g \in \mathbb{R}^{\mathbb{N}}$ (resp. $\mathbb{C}^{\mathbb{N}}$) belongs to $\mathbb{R}_z^{\mathbb{N}}$ (resp. $\mathbb{C}_z^{\mathbb{N}}$) if and only if the series $\sum_{k=0}^{\infty} g(k)z^{-k}$ converges for some $z \in \mathbb{C}$. For any $k \in \mathbb{N}$, we define $\delta_k \in \mathbb{C}_z^{\mathbb{N}}$ as the complex sequence on $\mathbb{N}$ with $\delta_k(k) = 1$, and $\delta_k(i) = 0$ $\forall i \neq k$. Let the superscript $\tilde{}$ denote z-transforms: if $g \in \mathbb{C}_z^{\mathbb{N}}$, then $\tilde{g}(z) := \sum_{k=0}^{\infty} g(k)z^{-k}$ is defined for $z \in \mathbb{C}$ wherever the series converges; if $S \subset \mathbb{C}_z^{\mathbb{N}}$, then $\tilde{S} := \{\tilde{g} \mid g \in S\} \subset \tilde{\mathbb{C}}_z^{\mathbb{N}}$. For any <u>nonzero</u> $g \in \mathbb{C}_z^{\mathbb{N}}$, we define the <u>order</u> [Kuc 1] of its z-transform $\tilde{g}$ as $\mathrm{ord}(\tilde{g}) :=$ index of the first nonzero component of g. Let $\mathbb{C}[z]$ be the ring of all polynomials in the complex variable z with coefficients in $\mathbb{C}$, $\mathbb{C}(z)$ be the field of all rational functions, and $\mathbb{C}_p(z)$ be the subset of all <u>proper</u> elements of $\mathbb{C}(z)$. The spaces of n-tuples and matrices are specified by superscripts in the usual manner, e.g. $\mathbb{C}^n$, $\mathbb{R}(z)^{m\times m}, \ldots$. Let θ_n

be the zero element of $\mathbb{C}^n$. For $a \in \mathbb{C}[z]$, let ∂a denote the degree of a; if $v \in \mathbb{C}[z]^n$, then ∂v denotes the maximum degree of the components of v. For $M \in \mathbb{C}[z]^{n_0 \times n_i}$, we denote by $\partial_{c_i} M$ the i^{th} column degree of M (i.e. the maximum degree of the components in the i^{th} column of M). Similarly, for $a \in \mathbb{C}[z^{-1}]$, a polynomial in z^{-1}, we denote by $\bar{\partial} a$ the degree of a as a polynomial in z^{-1}. Let $K \in \mathbb{C}$ be open: for $f\colon K \to \mathbb{C}$, $Z[f]$ denotes the set of zeros of f; for $F\colon K \to \mathbb{C}^{n_0 \times n_i}$, $P[F]$ denotes the set of poles of all components of F. Let $\rho_0 > 0$; we denote by $D(\rho_0) := \{z \in \mathbb{C} \mid |z| < \rho_0\}$ the open disc with radius ρ_0 about the origin in the complex plane.

2. $\tilde{b}(\rho_0)$, the Class of Transfer Functions

2.1 Convolution systems

The I/O characterization of discrete-time causal convolution systems is most conveniently done through their weighting sequences.

When an input sequence $u \in \mathbb{C}^{\mathbb{N}}$ is applied to a causal convolution system with weighting sequence $h \in \mathbb{C}^{\mathbb{N}}$, the output sequence $y = (y(k))_{k=0}^{\infty} := h*u$ is given by the convolution formula

$$y(k) = \sum_{i=0}^{k} h(i)u(k-i) = \sum_{i=0}^{k} h(k-i)u(i) \quad \forall k \in \mathbb{N} \qquad (2.1)$$

where the summing variable i represents the age variable.

For any $p \in [1,\infty]$, let $|\cdot|_p$ be the usual norm defined on the normed space $\ell_p \subset \mathbb{C}^{\mathbb{N}}$. It is a well-known fact [Des 1, p.244] that if $h \in \ell_1$, then, $\forall p \in [1,\infty]$,

$$u \in \ell_p \Rightarrow y := h*u \in \ell_p \ ,$$

and in fact, $\forall p \in [1,\infty]$,

$$|h*u|_p \leq |h|_1 |u|_p \ . \qquad (2.2)$$

The relationship, however, lacks the useful information of how fast the sequence y decays, even when u is zero except for a finite number of components.†

Suppose now that the sequence h satisfies the stronger condition that, for some $\rho_0 \in [0,1[$, the sequence $\bar{h}$ defined by $\bar{h} := (h(k)\rho_0^{-k})_{k=0}^{\infty}$ belongs to ℓ_1. If the input sequence u has

†The sequence $g \in \mathbb{C}^{\mathbb{N}}$ is said to decay to 0 exponentially at a rate (at least) μ iff $\exists \mu > 1$, $\exists M > 0$ such that $|g(i)| \leq M(1/\mu)^i \ \forall i \in \mathbb{N}$.

finite support (i.e. there exists least $N \in \mathbb{N}$ such that $u(k) = 0$ $\forall k > N$), then for any $k \geq N$,

$$
\begin{aligned}
|y(k)| &= \Big|\sum_{i=0}^{k} h(i)u(k-i)\Big| \\
&\leq \sum_{i=0}^{k} |h(i)||u(k-i)| \\
&= \sum_{i=k-N}^{k} |h(i)||u(k-i)| && \text{since } u(i) = 0 \text{ for } i > N \\
&\leq |u|_\infty \sum_{i=k-N}^{k} |h(i)| && \text{since } |u|_\infty = \max_{0\leq i\leq N} |u(i)| \\
&\leq |u|_\infty \, \rho_0^{k-N} \sum_{i=k-N}^{k} |h(i)|\rho_0^{-i}
\end{aligned}
$$

$$
|y(k)| \leq |u|_\infty |\bar{h}|_1 \rho_0^{k-N} \quad \forall k \geq N. \tag{2.3}
$$

Hence, the output y decays exponentially to 0 at a rate at least ρ_0^{-1}, where the constant† M may depend on N (from (2.3), we may take $M := \max\{|y(0)|, |y(1)|\rho_0^{-1}, \ldots, |y(N-1)|\rho_0^{-(N-1)};\ |u|_\infty |\bar{h}|_1 \rho_0^{-N}\}$).

2.2 The Class of Sequences $\ell_1(\rho_0)$

The preceding discussion leads us to consider the class of weighted sequences

$$
\ell_1(\rho_0) := \{g \in \mathbb{C}^{\mathbb{N}} \,|\, \sum_{k=0}^{\infty} |g(k)|\rho_0^{-k} < \infty\} \subset \mathbb{C}_z^{\mathbb{N}} \tag{2.4}
$$

where typically $\rho_0 \in [0,1[$. The properties of this class of sequences are given below. Detailed proofs of these properties and the properties in the next subsection, namely 2.3, are given in Appendix A.

†See footnote of previous page.

$\ell_1(\rho_0)$ is a complex vector space. It forms a complete normed space with the norm $\|\cdot\|_{\rho_0}: \ell_1(\rho_0) \to \mathbb{R}_+$ defined by

$$\|g\|_{\rho_0} := \sum_{k=0}^{\infty} |g(k)|\rho_0^{-k} \quad \forall g \in \ell_1(\rho_0) \ . \tag{2.5}$$

(2.2.1) If we choose as multiplication in $\ell_1(\rho_0)$ the convolution operator, $\ell_1(\rho_0)$ is a commutative Banach algebra with neutral element (unit, multiplicative unit, multiplicative identity) $\delta_0 := (1,0,0,\ldots)$.

(2.2.2) For $0 \le \rho_1 < \rho_0$, $\ell_1(\rho_1) \subset \ell_1(\rho_0)$.

(2.2.3) $\ell_1(\rho_0)$ has no divisors of zero and is thus an integral domain (entire ring [Lan 1]).

(2.2.4) For any $g \in \ell_1(\rho_0)$,

(i) the series $\sum_{k=0}^{\infty} g(k)z^{-k}$ converges absolutely for all $z \in D(\rho_0)^c$ and is bounded there by $\|g\|_{\rho_0}$;

(ii) $\forall \varepsilon > 0$, it converges uniformly in $D(\rho_0+\varepsilon)^c$, hence $\tilde{g}(\cdot)$ is analytic in $\overline{D(\rho_0)}^c$;

(iii) as $|z| \to \infty$, $\tilde{g}(z) \to g(0)$.

(2.2.5) $\tilde{\ell}_1(\rho_0)$ is a commutative algebra of functions analytic in $\overline{D(\rho_0)}^c$ and bounded in $D(\rho_0)^c$, with pointwise addition and multiplication, with neutral element $\tilde{\delta}_0(z) = 1 \ \ \forall |z| \ge \rho_0$, and with no divisors of zero.

(2.2.6) <u>Inversion Theorem</u>

$$g \in \ell_1(\rho_0) \text{ has an inverse in } \ell_1(\rho_0) \tag{2.6}$$

$$\Leftrightarrow \inf_{|z|\geq\rho_0} |\tilde{g}(z)| > 0 \tag{2.7}$$

$$\Leftrightarrow \text{(i)} \quad g(0) \neq 0 \tag{2.8}$$

$$\text{(ii)} \quad \tilde{g}(z) \neq 0 \quad \forall |z| \geq \rho_0 \qquad \square$$

Note that if $h \in \ell_1(\rho_0)$ is the inverse of g, then $\tilde{h}(z) = 1/\tilde{g}(z)$. The next property will be useful for proving the coprimeness condition of (2.3.6).

(2.2.7) Given f, g in the Banach algebra $\ell_1(\rho_0)$. $\exists u,v \in \ell_1(\rho_0)$ such that

$$u*f + v*g = \delta_0 \ , \tag{2.10a}$$

or equivalently, $$(\tilde{u}\tilde{f} + \tilde{v}\tilde{g})(z) = 1 \quad \forall |z| \geq \rho_0 \tag{2.10b}$$

$$\Leftrightarrow \inf_{|z|\geq\rho_0} |(\tilde{f}(z),\tilde{g}(z))| > 0 \tag{2.11}$$

$$\Leftrightarrow \text{(i)} \quad |(f(0),g(0))| > 0 \tag{2.12}$$

$$\text{(ii)} \quad |(\tilde{f}(z),\tilde{g}(z))| > 0 \quad \forall |z| \geq \rho_0$$

where $|\cdot|$ is any norm on $\mathbb{C}^2$. $\square$

2.3 The Class of Sequences $\ell_{1-}(\rho_0)$

For $\rho_0 > 0$, typically $\rho_0 \leq 1$, we define a class of complex sequences on $\mathbb{N}$ by

$$\ell_{1-}(\rho_0) := \bigcup_{0\leq\rho_1<\rho_0} \ell_1(\rho_1) \subset \mathbb{C}_z^{\mathbb{N}} \ . \tag{2.14}$$

Note that $\ell_{1-}(\rho_0) \subset \ell_1(\rho_0)$, in view of definition (2.14) and Property (2.2.2). $\square$

(2.3.1) $\ell_{1-}(\rho_0)$ is a normed commutative subalgebra of $\ell_1(\rho_0)$ with norm $\|\cdot\|_{\rho_0}$, with neutral element δ_0, and with no divisors of zero. Similarly $\tilde{\ell}_{1-}(\rho_0)$ is a commutative pointwise-product subalgebra of $\tilde{\ell}_1(\rho_0)$, with neutral element $\tilde{\delta}_0(z) = 1 \;\; \forall |z| \geq \rho_0$, and with no divisors of zero. Consequently, $\ell_{1-}(\rho_0)$ and $\tilde{\ell}_{1-}(\rho_0)$ are both integral domains.

(2.3.2) If $g \in \ell_{1-}(\rho_0)$, then

(i) $\tilde{g}(\cdot)$ is analytic in $\overline{D(\rho_g)}^c$ for some $\rho_g < \rho_0$; in particular, it is analytic in $D(\rho_0)^c$;

(ii) $\tilde{g}(\cdot)$ is bounded on $D(\rho_g)^c \supset D(\rho_0)^c$;

(iii) $\tilde{g}(\cdot)$ has a finite number of zeros in $D(\rho_0)^c$.

(2.3.3) $g \in \ell_{1-}(\rho_0)$ has an inverse in $\ell_{1-}(\rho_0)$ (2.15)

$$\Leftrightarrow \inf_{|z|\geq\rho_0} |\tilde{g}(z)| > 0 \tag{2.16}$$

$$\Leftrightarrow \text{(i)} \;\; g(0) \neq 0 \tag{2.17}$$

$$\text{(ii)} \;\; \tilde{g}(z) \neq 0 \;\; \forall |z| \geq \rho_0 .$$

(2.3.4) $\tilde{\ell}_{1-}(\rho_0)$ is a <u>Euclidean ring</u> (hence a principal ideal ring [Sig 1, p.133]), with a <u>gauge</u> [Sig 1, p.132] [Her 1, p.143] (or stathm [McD 1, p.30]) $\gamma: \tilde{\ell}_{1-}(\rho_0)\backslash\{0\} \to \mathbb{N}$ defined for all nonzero $\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$ by

$$\gamma(\tilde{g}) := \text{ord}(\tilde{g}) + \text{number of zeros of } \tilde{g} \text{ in } D(\rho_0)^c\text{, counting multiplicities.} \tag{2.18}$$

The Euclidean algorithm is given in Procedure A.1 of Appendix A.

Consequently, $\ell_{1-}(\rho_0)$ is a Euclidean ring (and thus a principal ideal ring) with the same gauge defined for $\tilde{\ell}_{1-}(\rho_0)$.

(2.3.5) Definition. Given f, g in the commutative Euclidean ring $\ell_{1-}(\rho_0)$. Then f, g are said to be ρ_0-coprime iff any greatest common divisor of f and g, denoted by gcd(f,g), is an invertible element of $\ell_{1-}(\rho_0)$ [Sig 1, p.142] [McL1, p.154].

$\tilde{f}$, $\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$ are also said to be ρ_0-coprime if and only if f, $g \in \ell_{1-}(\rho_0)$ are ρ_0-coprime.

(2.3.6) Given f, $g \in \ell_{1-}(\rho_0)$. f, g are ρ_0-coprime, (2.19)

$\Leftrightarrow \exists u,v \in \ell_{1-}(\rho_0)$ such that

$$u*f + v*g = \delta_0 \ , \tag{2.20a}$$

or equivalently,
$$(\tilde{u}\tilde{f} + \tilde{v}\tilde{g})(z) = 1 \quad \forall |z| \geq \rho_0, \tag{2.20b}$$

$$\Leftrightarrow \inf_{|z|\geq\rho_0} |(\tilde{f}(z),\tilde{g}(z))| > 0 \tag{2.21}$$

$$\Leftrightarrow \text{(i)} \quad |(f(0),g(0))| > 0 \tag{2.22}$$

$$\text{(ii)} \quad |(\tilde{f}(z),\tilde{g}(z))| > 0 \quad \forall |z| \geq \rho_0,$$

where $|\cdot|$ is any norm on $\mathbb{C}^2$. □

2.4 The Class of Sequences $\ell^{\infty}_{1-}(\rho_0)$

With $\ell_{1-}(\rho_0)$ defined above as in (2.14), we define a subset of it by

$$\ell^{\infty}_{1-}(\rho_0) := \{g \in \ell_{1-}(\rho_0) \mid \lim_{|z|\to\infty} \tilde{g}(z) = g(0) \neq 0\} \ . \tag{2.23}$$

Note that $\ell^{\infty}_{1-}(\rho_0)$ and $\tilde{\ell}^{\infty}_{1-}(\rho_0)$ are multiplicative subsets [Lan 1, p.66] [Zar 1, p.46] of $\ell_{1-}(\rho_0)$ and $\tilde{\ell}_{1-}(\rho_0)$, respectively. □

Remark 2.1. Consider Property (2.3.6). A necessary condition for f, $g \in \ell_{1-}(\rho_0)$ to be ρ_0-coprime is that at least one of them must belong to $\ell_{1-}^{\infty}(\rho_0)$. Under this condition, f and g are ρ_0-coprime iff $|(\tilde{f}(z),\tilde{g}(z))| > 0 \;\; \forall |z| \geq \rho_0$, i.e. $\tilde{f}$ and $\tilde{g}$ have no common zeros in $D(\rho_0)^c$. □

2.5 The Transfer Functions in $\tilde{b}(\rho_0)$

We now define a class of complex sequences on $\mathbb{N}$ whose z-transforms form the class of (stable or unstable) transfer functions we are concerned with.

Definition 2.1. Given the convolution algebra $\ell_{1-}(\rho_0)$ and the multiplicative subset $\ell_{1-}^{\infty}(\rho_0)$, $0 < \rho_0 \leq 1$, the algebra of fractions $\tilde{b}(\rho_0)$ [Zar 1, p.46] [Lan 1, p.66] is defined by

$$\begin{aligned}\tilde{b}(\rho_0) &:= [\tilde{\ell}_{1-}(\rho_0)][\tilde{\ell}_{1-}^{\infty}(\rho_0)]^{-1} \\ &= \{\tilde{g} = \tilde{n}/\tilde{d} \,|\, \tilde{n} \in \tilde{\ell}_{1-}(\rho_0),\ \tilde{d} \in \tilde{\ell}_{1-}^{\infty}(\rho_0)\}\ .\end{aligned} \tag{2.24}$$

Let $b(\rho_0)$ be the set of complex sequences on $\mathbb{N}$ defined as

$$b(\rho_0) := \{g \in \mathbb{C}^{\mathbb{N}} \,|\, \tilde{g} \in \tilde{b}(\rho_0)\} \subset \mathbb{C}_z^{\mathbb{N}}\ . \quad \square \tag{2.25}$$

Remark 2.2. (i) The z-transform is a linear bijective map from $b(\rho_0)$ onto $\tilde{b}(\rho_0)$. The definition (2.25) shows immediately that it is a linear map from $b(\rho_0)$ into $\tilde{b}(\rho_0)$. This map is bijective because every $\tilde{g} \in \tilde{b}(\rho_0)$ can be expressed as a Taylor series (necessarily unique) about infinity, thus specifying a unique sequence in $b(\rho_0) \subset \mathbb{C}^{\mathbb{N}}$. More precisely, $\tilde{g} = \tilde{n}/\tilde{d}$ where $\tilde{n} \in \tilde{\ell}_{1-}(\rho_0)$ and $\tilde{d} \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$ are both analytic and bounded in $D(\rho_0)^c$. Since $\tilde{d}$ has a finite number of zeros in $D(\rho_0)^c$ and $\lim_{|z|\to\infty} \tilde{d}(z) = d(0) \neq 0$, then $\exists \rho_d \geq \rho_0$ such that $\inf_{|z|\geq\rho_d} |\tilde{d}(z)| > 0$. Thus $\tilde{g}$ is analytic and bounded in the "annulus" given by $|z| \in [\rho_d, \infty[$. Hence $\forall |z| \geq \rho_d$, $\tilde{g}(z)$ can be expanded as a unique Laurent series

$$\tilde{g}(z) = \sum_{k=0}^{\infty} g(k) z^{-k} + \sum_{k=1}^{\infty} g'(k) z^{k} \tag{2.26}$$

where, $\forall \rho \geq \rho_d$,

$$g(k) = \frac{1}{2\pi j} \oint_{|z|=\rho} \tilde{g}(z) z^{k-1} dz \quad \forall k \in \mathbb{N} \tag{2.27}$$

and

$$g'(k) = \frac{1}{2\pi j} \oint_{|z|=\rho} \tilde{g}(z) z^{-k-1} dz \quad \forall k \in \mathbb{N}^* . \tag{2.28}$$

Since the value of the contour integral in (2.28) is independent of $\rho \geq \rho_d$, and since $\tilde{g}(z)$ is bounded by some $\tilde{g}_{max} < \infty$ in $D(\rho_d)^c$, hence for $k \geq 1$, as $\rho \to \infty$, the integral in (2.28) goes to zero. Hence $g'(k) = 0$ $\forall k \in \mathbb{N}^*$, and (2.26) represents $\tilde{g}$ as a power series in z^{-1}; thus $\tilde{g}$ specifies a unique sequence $(g(k))_{k=0}^{\infty}$ in $b(\rho_0)$.

(ii) From the proof of the preceding remark, it follows that if $\tilde{g}: D(\rho)^c \to \mathbb{C}$ is analytic and bounded on $D(\rho_g)^c$ for some $\rho_g > \rho$, then $\tilde{g} \in \tilde{\ell}_{1-}(\rho_g)$. □

(2.5.1) It is well known [Zar 1, p.46] [Lan 1, p.66] that $\tilde{b}(\rho_0)$ is a commutative <u>algebra</u> of fractions with pointwise sum and product, and neutral element given by $\tilde{\delta}_0(z) = 1$, $\forall |z| \geq \rho_0$. Consequently, $b(\rho_0)$ is a commutative convolution algebra of complex sequences on $\mathbb{N}$ with neutral element $\delta_0 := (1,0,0,\ldots)$.

(2.5.2) For any $\tilde{g} = \tilde{n}/\tilde{d} \in \tilde{b}(\rho_0)$ with $\tilde{n} \in \tilde{\ell}_{1-}(\rho_0)$ and $\tilde{d} \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$, since $\tilde{n}$, $\tilde{d}$ are analytic in $D(\rho_0)^c$ and both have only a finite number of zeros in $D(\rho_0)^c$, $\tilde{g}$ has a finite number of zeros in $D(\rho_0)^c$

and is analytic except for a finite number of poles in $D(\rho_0)^c$ (i.e. $\tilde{g}$ is meromorphic in $D(\rho_0)^c$). Moreover, $\tilde{g}$ is bounded at ∞ because $\lim_{|z|\to\infty} \tilde{g}(z) = n(0)/d(0)$ and $|n(0)| < \infty$, $d(0) \neq 0$. □

<u>Definition 2.2</u>. The pair $(\tilde{n},\tilde{d})$ is called a ρ_0-<u>representation</u> (ρ_0-r.) of $\tilde{g} \in \tilde{b}(\rho_0)$ iff

(i) $\tilde{n} \in \tilde{\ell}_{1-}(\rho_0)$, $\tilde{d} \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$,

(ii) $\tilde{g} = \tilde{n}/\tilde{d}$;

(iii) $\tilde{n}$, $\tilde{d}$ are ρ_0-coprime. □

<u>Lemma 2.1</u>. If $\tilde{g} \in \tilde{b}(\rho_0)$, then $\tilde{g}$ admits a ρ_0-representation. □

<u>Note</u>. The proofs of lemmas and theorems are relegated to Appendix B.

<u>Lemma 2.2</u>. Given $\tilde{g} \in \tilde{b}(\rho_0)$, let $(\tilde{n},\tilde{d})$ be one of its ρ_0-representations (whose existence is guaranteed by Lemma 2.1). Then for any $p \in D(\rho_0)^c$,

(i) $\tilde{g}$ has an m^{th} order <u>zero</u> at p iff $\tilde{n}$ has an m^{th} order zero at p;

(ii) $\tilde{g}$ has an m^{th} order <u>pole</u> at p iff $\tilde{d}$ has an m^{th} order zero at p. □

Recall that $\mathbb{C}_p(z)$ denotes the set of all proper rational functions in the complex variable z with coefficients in $\mathbb{C}$, and let

$$\mathcal{R}(\rho_0) := \mathbb{C}_p(z) \cap \tilde{\ell}_{1-}(\rho_0) \tag{2.29}$$

$$\mathcal{R}^{\infty}(\rho_0) := \mathcal{R}(\rho_0) \cap \tilde{\ell}^{\infty}_{1-}(\rho_0) = \mathbb{C}_p(z) \cap \tilde{\ell}^{\infty}_{1-}(\rho_0) \; . \tag{2.30}$$

It has been shown in [Mor 1] that $\mathcal{R}(\rho_0)$ is a principal ideal ring. In fact, $\mathcal{R}(\rho_0)$ is a Euclidean ring, with a <u>gauge</u>

$\gamma\colon \mathcal{R}(\rho_0)\backslash\{0\} \to \mathbb{N}$ defined as in (2.18) when $\mathcal{R}(\rho_0)$ is viewed as a subset of $\tilde{\ell}_{1-}(\rho_0)$; equivalently, for any nonzero $a \in \mathcal{R}(\rho_0)$,

$$\gamma(a) = \text{number of poles of } a \text{ in } D(\rho_0) - \text{number of zeros of } a \text{ in } D(\rho_0).$$

The Euclidean algorithm for $\mathcal{R}(\rho_0)$ is similar to the one for $R(\sigma_0)$ given in [Cal 2] [Cal 3], by noting that the role of $D(\rho_0)$ with respect to $\mathcal{R}(\rho_0)$ is the same as the role of $\mathring{\mathbb{C}}_{\sigma_0-}$ with respect to $R(\sigma_0)$.

<u>Definition 2.3</u>. A ρ_0-representation $(\tilde{n},\tilde{d})$ of $\tilde{g} \in \tilde{b}(\rho_0)$ is said to be <u>normalized</u> iff

(i) $\tilde{d} \in \mathcal{R}^{\infty}(\rho_0)$

(ii) $\lim_{|z|\to\infty} \tilde{d}(z) = 1$

(iii) $Z[\tilde{d}] \subset D(\rho_0)^c$ □

<u>Remark 2.3</u>. (i) Observe that $\tilde{d} \in \mathcal{R}^{\infty}(\rho_0)$ is a rational function whose numerator and denominator polynomials have the same degree, and all the poles of $\tilde{d}$ are inside the open disc $D(\rho_0)$. Hence a ρ_0-representation $(\tilde{n},\tilde{d})$ of $\tilde{g} \in \tilde{b}(\rho_0)$ is normalized if and only if $\tilde{d}(z)$ can be expressed as a finite product of rational factors of the form

$$(z-p)/(z-a) \tag{2.31}$$

where $p \in D(\rho_0)^c$ and $a \in D(\rho_0)$.

(ii) If $(\tilde{\tilde{n}},\tilde{\tilde{d}})$ is a ρ_0-representation of $\tilde{g} \in \tilde{b}(\rho_0)$ with $\tilde{\tilde{d}} \in \mathcal{R}^{\infty}(\rho_0)$, we can easily obtain a normalized ρ_0-representation $(\tilde{n},\tilde{d})$ of $\tilde{g} \in \tilde{b}(\rho_0)$ by adjusting the factors in $\tilde{\tilde{d}}$: more precisely, put $\tilde{\tilde{d}}(z)$ in the form

$$\tilde{\tilde{d}}(z) = d(0) \prod_{i=1}^{m_1} \frac{(z-p_i)}{(z-a_i)} \cdot \prod_{i=m_1+1}^{m_2} \frac{(z-p_i)}{(z-a_i)} \tag{2.32}$$

where $d(0) \in \mathbb{C}$, $d(0) \neq 0$; $a_i \in D(\rho_0)$, $i = 1,2,\ldots,m_2$; $p_i \in D(\rho_0)$, $i = 1,2,\ldots,m_1$; $p_i \in D(\rho_0)^c$, $i = m_1+1,\ldots,m_2$. Note that

$$\tilde{c} := d(0) \prod_{i=1}^{m_1} \frac{(z-p_i)}{(z-a_i)} \in \mathcal{R}^{\infty}(\rho_0) \tag{2.33}$$

is an invertible element of $\mathcal{R}(\rho_0) \subset \tilde{\ell}_{1-}(\rho_0)$, and $(\tilde{n},\tilde{d})$ given by

$$\tilde{n} := \tilde{\tilde{n}}\tilde{c}^{-1} \quad , \quad \tilde{d} := \tilde{\tilde{d}}\tilde{c}^{-1} \tag{2.34}$$

is a normalized ρ_0-representation of $\tilde{g}$. □

Theorem 2.1. If $\tilde{g} \in \tilde{b}(\rho_0)$, then $\tilde{g}$ admits a normalized ρ_0-representation. One such representation can be obtained by the following procedure.

Procedure 2.1. Normalized ρ_0-representation.

Given $\tilde{g} \in \tilde{b}(\rho_0)$

Step 1. Obtain a ρ_0-representation $(\tilde{\tilde{n}},\tilde{\tilde{d}})$ of $\tilde{g}$.

Step 2. Determine all ν not-necessarily-different zeros of $\tilde{\tilde{d}}$ in $D(\rho_0)^c$, call them p_α, $\alpha = 1,2,\ldots,\nu$.

Step 3. Let $\tilde{\tilde{d}} = \tilde{d}\tilde{c}$ where $\tilde{d}(z) := \prod_{\alpha=1}^{\nu} \frac{(z-p_\alpha)}{z}$ and we adopt the convention that $\prod_{\alpha=1}^{0} \frac{(z-p)}{z} = 1$. Note that $\tilde{c} := \tilde{\tilde{d}}/\tilde{d}$ is invertible in $\tilde{\ell}_{1-}(\rho_0)$. (Observe that $\tilde{d}$ can also be chosen to be $\tilde{d}(z) := \prod_{\alpha=1}^{\nu} \frac{(z-p_\alpha)}{(z-a_\alpha)}$ for any choice of $a_\alpha \in D(\rho_0)$, $\alpha = 1,2,\ldots,\nu$.)

Step 4. Define $\tilde{n} := \tilde{\tilde{n}}\tilde{c}^{-1} \in \tilde{\ell}_{1-}(\rho_0)$, then $(\tilde{n},\tilde{d})$ is a normalized ρ_0-representation of $\tilde{g}$.

Stop. □

Remark 2.4. $\mathcal{r}^{\infty}(\rho_0)$ is a multiplicative subset [Zar 1, p.46] [Lan 1, p.66] of both $\mathbb{C}_p(z)$ and $\tilde{\ell}_{1-}(\rho_0)$. In view of Theorem 2.1, we conclude that $\tilde{b}(\rho_0) = [\tilde{\ell}_{1-}(\rho_0)][\tilde{\ell}^{\infty}_{1-}(\rho_0)]^{-1} = [\tilde{\ell}_{1-}(\rho_0)][\mathcal{r}^{\infty}(\rho_0)]^{-1}$. □

Theorem 2.2. Let $g \in \mathbb{C}_z^{\mathbb{N}}$. Then

$$\tilde{g} \in \tilde{b}(\rho_0) \tag{2.35}$$

if and only if $\exists \rho_1 < \rho_0$ such that

$$\tilde{g}(z) = \tilde{r}(z) + \tilde{q}(z) \quad \forall z \in D(\rho_1)^c \tag{2.36}$$

where

(i) $\tilde{q} \in \tilde{\ell}_{1-}(\rho_0)$; (2.37)

(ii) $\tilde{r} \in \mathbb{C}_p(z)$ is strictly proper, and is zero if and only if $\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$; (2.38)

(iii) if $\tilde{g} \notin \tilde{\ell}_{1-}(\rho_0)$, then $\tilde{r}$ is the sum of the principal parts of the Laurent expansions of $\tilde{g}$ at its poles in $D(\rho_0)^c$, in particular: (2.39)

(a) all poles of $\tilde{r}$ are in $D(\rho_0)^c$, and (2.39a)

(b) $\tilde{g}$ has an m^{th} order pole at $p \in D(\rho_0)^c$ if and only if $\tilde{r}$ has an m^{th} order pole at $p \in D(\rho_0)^c$. (2.39b)

□

The proof of Theorem 2.2 is given in Appendix B. Sufficiency is proved by construction, and a procedure for obtaining a normalized ρ_0-representation $(\tilde{n},\tilde{d})$ for $\tilde{g}$ described by (2.36) through (2.39b) is given next.

Procedure 2.2. Normalized ρ_0-representation from $\tilde{g} = \tilde{r} + \tilde{q}$.

Given $\tilde{g}$ in the form (2.36)-(2.39b).

Step 1. If $\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$, i.e. $\tilde{r} \equiv 0$, set

$$\tilde{n} := \tilde{g} \ , \quad \tilde{d} \equiv 1 \tag{2.40}$$

and stop.

Step 2. Let $\tilde{r} =: n_r/d_r$ define a coprime factorization of $\tilde{r}$ (2.41)
in the ring of polynomials in z, with d_r monic.

Determine $\nu := \deg(d_r)$. (2.42)

Step 3. Define $(\tilde{n},\tilde{d})$ by

$$\tilde{d}(z) := d_r(z)/z^{\nu} \tag{2.43}$$

$$\tilde{n}(z) := [n_r(z) + \tilde{q}(z)d_r(z)]/z^{\nu} \qquad |z| \geq \rho_1 \tag{2.44}$$

and stop.

Observation. In both (2.40) and (2.43)-(2.44), $(\tilde{n},\tilde{d})$ is a normalized ρ_0-representation of $\tilde{g}$.

Remark 2.5. In step 3 of Procedure 2.2, instead of using z^{ν} as denominator of both $\tilde{n}(z)$ and $\tilde{d}(z)$, it can be generalized to be any ν^{th} order polynomial in the form

$$\prod_{\alpha=1}^{\nu} (z-a_{\alpha}) \tag{2.45}$$

where $a_{\alpha} \in D(\rho_0)$, $\alpha = 1,2,\ldots,\nu$. □

Theorem 2.3. Let $\tilde{g} \in \tilde{b}(\rho_0)$, and let $(\tilde{n},\tilde{d})$ and $(\tilde{\tilde{n}},\tilde{\tilde{d}})$ be two ρ_0-representations of $\tilde{g}$.

U.t.c.

(i) $\exists \tilde{h} \in \tilde{\ell}_{1-}(\rho_0)$, invertible in $\tilde{\ell}_{1-}(\rho_0)$, such that

$$\tilde{\tilde{n}} = \tilde{n}\tilde{h} \ ,$$

$$\tilde{\tilde{d}} = \tilde{d}\tilde{h}$$

(ii) if, in addition, the representations are both normalized, then $\tilde{h}$ is rational with all poles and zeros in $D(\rho_0)$, in particular,

(a) if $\tilde{d} \equiv 1$, then $\tilde{h} \equiv 1$

(b) if $\tilde{d} \not\equiv 1$, let

$$\tilde{r} =: n_r/d_r \tag{2.46}$$

be a coprime polynomial factorization of $\tilde{r}$ given in (2.36)-(2.39b) with d_r being a monic polynomial, then

$$\tilde{d} = d_r/n_h \ , \quad \tilde{\tilde{d}} = d_r/d_h \ , \quad \tilde{h} = n_h/d_h \tag{2.47}$$

with d_r, n_h, d_h monic polynomials in $\mathbb{C}[z]$ of the same degree, and such that n_h, d_h have zeros only in $D(\rho_0)$. □

Remark 2.6. From Theorem 2.3, if $(\tilde{n},\tilde{d})$ is any normalized ρ_0-representation of $\tilde{g} \in \tilde{b}(\rho_0)$, then $\tilde{d} = d_r/p$, for some monic polynomial p of the same degree as d_r and has zeros only in $D(\rho_0)$. By (2.43)-(2.45) in Procedure 2.2, we can write

$$\tilde{g} = \tilde{n}/\tilde{d} := [(n_r + \tilde{q}d_r)/p]/(d_r/p) \tag{2.48}$$

and p appears as a common divisor (polynomial in $\mathbb{C}[z]$) in defining $\tilde{n}$ and $\tilde{d}$. Hence the choice of p does not affect $\tilde{g}$, and p could thus be called a scaling polynomial in defining the normalized ρ_0-representation: the restrictions of p being that it is a monic polynomial with $\partial(p) = \partial(d_r)$ and $Z[p] \subset D(\rho_0)$. Consequently, we conclude that a normalized ρ_0-representation is unique up to a scaling polynomial. □

Theorem 2.4. Let $\tilde{g} \in \tilde{b}(\rho_0)$. Then

$$\tilde{g} \text{ is an invertible element of } \tilde{b}(\rho_0) \tag{2.49}$$

if and only if

$$g(0) = \lim_{|z|\to\infty} \tilde{g}(z) \neq 0 . \tag{2.50}$$

□

2.6 Examples. $g = (g(k))_{k=0}^{\infty} \in \mathbb{C}^{\mathbb{N}}$.

(2.6.1)
$$\begin{aligned} g_1(0) &:= 1 \\ g_1(1) &:= -\frac{1}{2} \\ g_1(k) &:= -\frac{1}{2}\prod_{m=2}^{k}\frac{2m-3}{2m}, \quad k = 2,3,4,\ldots . \end{aligned} \tag{2.51}$$

By [Dwi 1, formula 5.3],

$$\sqrt{1-x} = \sum_{k=0}^{\infty} g_1(k)x^k \quad \forall |x| \leq 1 . \tag{2.52}$$

Hence evaluating (2.52) at $x = 1$, we obtain

$$\sum_{k=0}^{\infty} g_1(k) = 0 . \tag{2.53}$$

Note that $g_1(0) = 1$ and $g_1(k) < 0$ for $k = 1,2,\ldots,$ hence by (2.53)

$$\sum_{k=1}^{\infty} g_1(k) = -1 \quad \text{and} \quad \sum_{k=1}^{\infty} |g_1(k)| = 1 ,$$

i.e.

$$\sum_{k=0}^{\infty} |g_1(k)| = 2 . \tag{2.54}$$

Therefore,

$$g_1 \in \ell_1(1) . \tag{2.55}$$

By Property (2.2.4), the series defining $\tilde{g}_1$ converges absolutely in $D(\rho_1)^c$; hence using (2.52), we obtain

$$\tilde{g}_1(z) = \sum_{k=0}^{\infty} g_1(k) z^{-k} = \sqrt{\frac{z-1}{z}} \quad \forall z \in D(1)^c \; . \tag{2.56}$$

However, $\tilde{g}_1$ in (2.56) is not analytic at $z = 1$, hence

$$g_1 \notin \ell_{1-}(1) \; . \tag{2.57}$$

Since $g_1(0) \neq 0$, thus

$$g_1 \in \ell_{1-}^{\infty}(\rho_0) \subset \ell_{1-}(\rho_0) \quad \forall \rho_0 > 1 \; . \tag{2.58}$$

(2.6.2) Consider the slight variation of example (2.6.1):

$$\tilde{g}_2(z) := \sqrt{\frac{z-0.5}{z}} \; , \qquad z \in D(0.5)^c \; ; \tag{2.59}$$

then $\tilde{g}_2 \in \tilde{\ell}_1(0.5)$, and $\tilde{g}_2 \notin \tilde{\ell}_{1-}(0.5)$; but $\tilde{g}_2 \in \tilde{\ell}_{1-}^{\infty}(\rho_0) \subset \tilde{\ell}_{1-}(\rho_0)$ for all $\rho_0 > 0.5$.

(2.6.3) For any fixed $a \in \mathbb{C}$, consider

$$g_3(k) = a^k/k! \; , \quad k = 0,1,2,\ldots \; . \tag{2.60}$$

Hence, for all $z \neq 0$,

$$\tilde{g}_3(k) := \sum_{k=0}^{\infty} \frac{a^k}{k!} z^{-k} = e^{az^{-1}} \; , \tag{2.61}$$

i.e.

$$\sum_{k=0}^{\infty} |g_3(k)| \rho_0^{-k} = e^{|a|\rho_0^{-1}} < \infty \quad \forall \rho_0 > 0 \; ; \tag{2.62}$$

furthermore, by noting that $g_3(0) = 1$, hence nonzero, we conclude that

$$g_3 \in \ell_{1-}^{\infty}(\rho_0) \subset \ell_{1-}(\rho_0) \subset \ell_1(\rho_0) \quad \forall \rho_0 > 0 \; . \tag{2.63}$$

(2.6.4) Let

$$\begin{aligned} g_4(0) &:= 0 \\ g_4(k) &= \frac{1}{k} \; , \quad k = 1,2,\ldots \; . \end{aligned} \tag{2.64}$$

By [Han 1, (5.13.4)],

$$\sum_{k=1}^{\infty} \frac{1}{k}x^k = -\ln(1-x) \quad \forall |x| \leq 1, \quad x \neq 1, \tag{2.65}$$

i.e.
$$\sum_{k=0}^{\infty} |g_4(k)|\rho_0^{-k} = -\ln(1-\rho_0^{-1}) < \infty, \quad \forall \rho_0 > 1. \tag{2.66}$$

Hence,

$$g_4 \in \ell_{1-}(\rho_0) \subset \ell_1(\rho_0) \ , \quad \forall \rho_0 > 1; \tag{2.67}$$

but since the series in (2.65) does not converge for $x = 1$,

$$g_4 \notin \ell_1(1) \ . \tag{2.68}$$

By (2.65) and using the absolute-convergence property as in example (2.6.1), we conclude that

$$\tilde{g}(z) = -\ln(\frac{z-1}{z}) \ , \quad \forall z \in \overline{D(1)}^c \ . \tag{2.69}$$

(2.6.5) Let
$$\begin{aligned} g_5(0) &:= 0 \\ g_5(k) &:= \sum_{i=1}^{k} \frac{1}{i} \ , \quad k = 1,2,\ldots \ . \end{aligned} \tag{2.70}$$

Note that the sequence of positive numbers $(g_5(k))_{k=0}^{\infty}$ is <u>unbounded</u>, hence

$$g_5 \notin \ell_1(1) \ . \tag{2.71}$$

By [Han 1, (5.13.21)],

$$\sum_{k=1}^{\infty} (\sum_{i=1}^{k} \frac{1}{i})x^k = \frac{1}{x-1}\ln(1-x) \ , \quad \forall |x| < 1, \tag{2.72}$$

i.e.
$$\sum_{k=0}^{\infty} |g_5(k)|\rho_0^{-k} = \frac{1}{\rho_0^{-1}-1}\ln(1-\rho_0^{-1}) < \infty \ , \quad \forall \rho_0 > 1. \tag{2.73}$$

Hence,

$$g_5 \in \ell_{1-}(\rho_0) \subset \ell_1(\rho_0) \ , \quad \forall \rho_0 > 1. \tag{2.74}$$

Using similar arguments as before, we have

$$\tilde{g}_5(z) = \frac{z}{1-z}\ln\left(\frac{z-1}{z}\right) , \quad \forall z \in \overline{D(1)^c} . \tag{2.75}$$

(2.6.6)

$$\begin{aligned} g_6(0) &:= 0 \\ g_6(k) &:= \frac{1}{k^2} , \quad k = 1,2,\ldots \end{aligned} \tag{2.76}$$

By [Han 1, (5.12.43)],

$$\sum_{k=1}^{\infty} \frac{1}{k^2} x^k = -\int_0^x t^{-1}\ln(1-t)dt , \quad \forall |x| \leq 1 . \tag{2.77}$$

In particular,

$$\sum_{k=0}^{\infty} |g_6(k)| = \sum_{k=1}^{\infty} \frac{1}{k^2} = -\int_0^1 t^{-1}\ln(1-t)dt = \frac{\pi^2}{6} ; \tag{2.78}$$

hence

$$g_6 \in \ell_1(1) . \tag{2.79}$$

However, the series in (2.77) does not converge for $|x| > 1$, hence

$$g_6 \notin \ell_1(\rho_0) , \quad \forall \rho_0 < 1 , \tag{2.80}$$

and thus

$$g_6 \notin \ell_{1-}(1) . \tag{2.81}$$

(2.6.7)

$$\begin{aligned} g_7(0) &:= 0 \\ g_7(k) &:= \frac{1}{k(k-1)} , \quad k = 1,2,\ldots \end{aligned} \tag{2.82}$$

By [Han 1, (5.9.16)],

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} x^k = 1 + \left(\frac{1}{x} - 1\right)\ln(1-x) , \quad |x| \leq 1 , \tag{2.83}$$

i.e.

$$\sum_{k=0}^{\infty} |g_7(k)| \rho_0^{-k} = 1 + \left(\frac{1}{\rho_0^{-1}} - 1\right)\ln(1-\rho_0^{-1}) < \infty , \quad \forall \rho_0 \geq 1 . \tag{2.84}$$

The case when $\rho_0 = 1$ is best calculated by using [Han 1, (5.9.17)]

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} ; \tag{2.85}$$

hence
$$\sum_{k=0}^{\infty} |g_7(k)| = \lim_{n\to\infty} (1 - \frac{1}{n+1}) = 1 < \infty , \tag{2.86}$$

and so $g_7 \in \ell_1(1)$.

However, since the series in (2.83) does not converge for $|x| > 1$,

$$g_7 \notin \ell_1(\rho_0) \quad \forall \rho_0 < 1 . \tag{2.87}$$

By (2.83) and the absolute convergence property as before,

$$\tilde{g}_7(z) = 1 + (z-1)\ln(\frac{z-1}{z}) , \quad \forall z \in D(1)^c . \tag{2.88}$$

(2.6.8) Recall that $\mathcal{R}(\rho_0) \subset \tilde{\ell}_{1-}(\rho_0)$; thus

$$z \mapsto \frac{z-1}{z} \in \tilde{\ell}^{\infty}_{1-}(\rho_0) \subset \tilde{\ell}_{1-}(\rho_0) , \quad \forall \rho_0 > 0 , \tag{2.89}$$

furthermore, it is also an invertible element of $\tilde{\ell}_1(\rho_0)$, $\forall \rho_0 > 1$. (2.90)

Using (2.89) and example (2.6.2), if

$$\tilde{g}(z) := \frac{z}{z-1} \cdot \sqrt{\frac{z-0.5}{z}} , \quad \forall z \in D(0.5)^c , \tag{2.91}$$

then
$$g \in b(\rho_0) , \quad \forall \rho_0 > 0.5 ; \tag{2.92}$$

note that $\tilde{g}$ has a pole at $z = 1$. Finally, from (2.90),

$$g \in \ell_{1-}(\rho_0) \subset \ell_1(\rho_0) , \quad \forall \rho_0 > 1 . \tag{2.93}$$

3. Matrix Fraction Representation Theory

From this point on, we are concerned with multi-input multi-output (MIMO) convolution systems whose transfer functions are matrices with elements in $\tilde{\mathbb{C}}_z^{\mathbb{N}}$, $\tilde{\ell}_1(\rho_0)$, $\tilde{\ell}_{1-}(\rho_0)$ or $\tilde{b}(\rho_0)$.

Observe that $(\tilde{\mathbb{C}}_z^{\mathbb{N}})^{n\times n}$, $\tilde{\ell}_1(\rho_0)^{n\times n}$, $\tilde{\ell}_{1-}(\rho_0)^{n\times n}$ and $\tilde{b}(\rho_0)^{n\times n}$ are all algebras with a pointwise sum and a non-commutative (pointwise) product, with unit I_n.

Lemma 3.1. $\tilde{G} \in \tilde{\ell}_1(\rho_0)^{n\times n}$ (respectively $\tilde{\ell}_{1-}(\rho_0)^{n\times n}$) is invertible in $\tilde{\ell}_1(\rho_0)^{n\times n}$ (resp. $\tilde{\ell}_{1-}(\rho_0)^{n\times n}$) if and only if

$$\inf_{|z|\geq\rho_0} |\det \tilde{G}(z)| > 0 \tag{3.1}$$

i.e. $\det \tilde{G}$ is invertible in $\tilde{\ell}_1(\rho_0)$ (resp. $\tilde{\ell}_{1-}(\rho_0)$).

Comment. Such $\tilde{G}$ is called a unimodular matrix in $\tilde{\ell}_1(\rho_0)^{n\times n}$ (resp. $\tilde{\ell}_{1-}(\rho_0)^{n\times n}$). □

Lemma 3.2. $\tilde{G} \in \tilde{b}(\rho_0)^{n\times n}$ is invertible in $\tilde{b}(\rho_0)^{n\times n}$ if and only if

$$\lim_{|z|\to\infty} \det \tilde{G}(z) \neq 0 \ , \tag{3.2}$$

i.e. $\det \tilde{G}$ is invertible in $\tilde{b}(\rho_0)$. □

The next lemma is a multi-input multi-output generalization of Theorem 2.2.

Lemma 3.3. Let $G \in (\mathbb{C}_z^{\mathbb{N}})^{n_0\times n_i}$. Then $\tilde{G} \in \tilde{b}(\rho_0)^{n_0\times n_i}$ if and only if for some $\rho_1 \in [0,\rho_0[$,

$$\tilde{G} = \tilde{R} + \tilde{Q} \ \text{ in } D(\rho_1)^c \tag{3.3}$$

where (i) $\tilde{Q} \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$;

(ii) $\tilde{R} \in \mathbb{C}_p(z)^{n_0 \times n_i}$ is strictly proper, and $\tilde{R} \equiv 0$ if and only if $\tilde{G} \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$;

(iii) if $\tilde{G} \notin \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$, then $\tilde{R} = (\tilde{r}_{ij})$ is the sum of the principal parts of the Laurent expansions of $\tilde{G} = (\tilde{g}_{ij})$ at its poles in $D(\rho_0)^c$; in particular, $\tilde{g}_{ij}$ has an m^{th} order pole at $p \in D(\rho_0)^c$ if and only if $\tilde{r}_{ij}$ has an m^{th} order pole at $p \in D(\rho_0)^c$. □

<u>Definition 3.1(r)</u>. Let $N_r \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$ and $\mathcal{D}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$. The pair $(N_r, \mathcal{D}_r)$ is said to be ρ_0-<u>right coprime</u> (ρ_0-<u>r.c.</u>) iff any greatest common right divisor (g.c.r.d.) of N_r and $\mathcal{D}_r$ in $\tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ [McD 1, p.35] is an invertible element of $\tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$.

<u>Definition 3.1(ℓ)</u>. Let $\mathcal{D}_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_0}$ and $N_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$. The pair $(\mathcal{D}_\ell, N_\ell)$ is said to be ρ_0-<u>left coprime</u> (ρ_0-<u>l.c.</u>) iff any greatest common left divisor (g.c.l.d.) of $\mathcal{D}_\ell$ and N_ℓ in $\tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_0}$ is an invertible element of $\tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_0}$. □

<u>Lemma 3.4(r)</u>. $N_r \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$ and $\mathcal{D}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ are ρ_0-r.c. if and only if $\exists U_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_0}$, $V_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ such that

$$U_r N_r + V_r \mathcal{D}_r \equiv I_{n_i} . \tag{3.4}$$

<u>Lemma 3.4(ℓ)</u>. $\mathcal{D}_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_0}$ and $N_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$ are ρ_0-l.c. if and only if $\exists V_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_0}$, $U_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_0}$ such that

$$N_\ell U_\ell + \mathcal{D}_\ell V_\ell \equiv I_{n_0} . \tag{3.5}$$

□

Definition 3.2(r). Let $G \in (\mathbb{C}_z^{\mathbb{N}})^{n_0 \times n_i}$. The pair $(N_r, \mathcal{D}_r)$ is said to be a ρ_0-right representation (ρ_0-r.r.) of $\tilde{G}$ iff $N_r \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$ and $\mathcal{D}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ such that

(i) $\tilde{G} = N_r \mathcal{D}_r^{-1}$

(ii) $(N_r, \mathcal{D}_r)$ is ρ_0-r.c.

(iii) $\det \mathcal{D}_r \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$.

Definition 3.2(ℓ). Let $G \in (\mathbb{C}_z^{\mathbb{N}})^{n_0 \times n_i}$. The pair $(\mathcal{D}_\ell, N_\ell)$ is said to be a ρ_0-left representation (ρ_0-l.r.) of $\tilde{G}$ iff $\mathcal{D}_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_0}$ and $N_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$ such that

(i) $\tilde{G} = \mathcal{D}_\ell^{-1} N_\ell$

(ii) $(\mathcal{D}_\ell, N_\ell)$ is ρ_0-l.c.

(iii) $\det \mathcal{D}_\ell \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$. □

Remark 3.1. By Cramer's rule and Definition 3.2(r) (respectively Definition 3.2(ℓ)), if $\tilde{G} \in (\tilde{\mathbb{C}}_z^{\mathbb{N}})^{n_0 \times n_i}$ admits a ρ_0-r.r. (respectively ρ_0-l.r.), then $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$. The next Theorem states that the converse is also true.

Theorem 3.1. If $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$, then $\tilde{G}$ admits a ρ_0-r.r. and a ρ_0-l.r. More precisely, there exist matrices with elements in $\tilde{\ell}_{1-}(\rho_0)$, namely

$$N_r,\ \mathcal{D}_r,\ \mathcal{U}_r,\ \mathcal{V}_r$$
$$N_\ell,\ \mathcal{D}_\ell,\ \mathcal{U}_\ell,\ \mathcal{V}_\ell$$

such that

(i) $(N_r, \mathcal{D}_r)$ is a ρ_0-r.r. of $\tilde{G}$

(ii) $(\mathcal{D}_\ell, N_\ell)$ is a ρ_0-l.r. of $\tilde{G}$

(iii)
$$\begin{array}{c c} & \begin{array}{cc} n_i & n_0 \end{array} \\ \begin{array}{c} n_i \\ n_0 \end{array} & \left[\begin{array}{c|c} V_r & U_r \\ \hline -N_\ell & D_\ell \end{array}\right] \end{array} \begin{array}{c} \begin{array}{cc} n_i & n_0 \end{array} \\ \left[\begin{array}{c|c} D_r & -U_\ell \\ \hline N_r & V_\ell \end{array}\right] \end{array} = \left[\begin{array}{c|c} I_{n_i} & 0 \\ \hline 0 & I_{n_0} \end{array}\right] \tag{3.6}$$

Remark 3.2. If we call the matrices on the left hand side of (3.6) W and W^{-1} respectively, then obviously W is an invertible element of $\tilde{\ell}_{1-}(\rho_0)^{(n_i+n_0)\times(n_i+n_0)}$. In particular, we can scale W and W^{-1} so that

$$\det W = \det W^{-1} \equiv 1 \ . \tag{3.7}$$

□

Theorem 3.1 can be proved easily by construction using the Euclidean algorithm for $\tilde{\ell}_{1-}(\rho_0)$. However, this is an unnecessarily difficult way to obtain a ρ_0-l.r. and ρ_0-r.r. We give instead a proof based on the following procedure for the general case when $\tilde{G} \notin \tilde{\ell}_{1-}(\rho_0)^{n_0\times n_i}$: this procedure uses the Euclidean algorithm for $r(\rho_0)$ instead of the one for $\tilde{\ell}_{1-}(\rho_0)$.

Procedure 3.1. ρ_0-r.r. and ρ_0-l.r.

Given $\tilde{G} \in \tilde{b}(\rho_0)^{n_0\times n_i}$, $\tilde{G} \notin \tilde{\ell}_{1-}(\rho_0)^{n_0\times n_i}$

Step 1. Find $\tilde{R}$, $\tilde{Q}$ according to Lemma 3.3 so that

$$\tilde{G} = \tilde{R} + \tilde{Q} \tag{3.8}$$

with $\tilde{R} \in r(\rho_0)^{n_0\times n_i}$, $\tilde{Q} \in \tilde{\ell}_{1-}(\rho_0)^{n_0\times n_i}$.

Step 2. Find $\hat{N}_r \in r(\rho_0)^{n_0\times n_i}$ and $\hat{D}_r \in r(\rho_0)^{n_i\times n_i}$ with $\det \hat{D}_r \in r^\infty(\rho_0)$ such that

$$\tilde{R} = \hat{N}_r \hat{D}_r^{-1} \tag{3.9}$$

e.g. set $\hat{D}_r = \mathrm{diag}[d_j]_{j=1}^{n_i}$ where for $j = 1,2,\ldots,n_i$, d_j is a j^{th}

column least common denominator of $\tilde{R}$ with respect to $\mathcal{R}(\rho_0)$.

Step 3. Consider the $(n_i+n_0)\times n_i$ full rank matrix

$$\hat{M} := \begin{array}{c c} & n_i \\ \begin{array}{c} n_i \\ n_0 \end{array} & \begin{bmatrix} \hat{D}_r \\ \hline \hat{N}_r \end{bmatrix} \end{array} \in \mathcal{R}(\rho_0)^{(n_i+n_0)\times n_i} . \tag{3.10}$$

By performing elementary row operations based on the Euclidean algorithm in the Euclidean ring $\mathcal{R}(\rho_0)$, bring $\hat{M}$ to "upper triangular" forms, i.e. find an $(n_i+n_0)\times(n_i+n_0)$ matrix $\bar{W}$ invertible in $\mathcal{R}(\rho_0)^{(n_i+n_0)\times(n_i+n_0)}$ and a full rank upper triangular matrix $\bar{R}\in\mathcal{R}(\rho_0)^{n_i\times n_i}$ such that

$$\begin{bmatrix} \bar{W} \end{bmatrix}\begin{bmatrix} \hat{M} \end{bmatrix} = \begin{array}{c c} & n_i \\ \begin{array}{c} n_i \\ n_0 \end{array} & \begin{bmatrix} \bar{R} \\ \hline 0 \end{bmatrix} \end{array} . \tag{3.11}$$

Observe that, as in the previous remark, $\bar{W}$ can be scaled so that $\det \bar{W} \equiv 1$.

Step 4. Partition $\bar{W}$ and $\bar{W}^{-1}$ into

$$\bar{W} = \begin{array}{c c} & \begin{array}{cc} n_i & n_0 \end{array} \\ \begin{array}{c} n_i \\ n_0 \end{array} & \left[\begin{array}{c|c} \bar{V}_r & \bar{U}_r \\ \hline -\bar{N}_\ell & \bar{D}_\ell \end{array}\right] \end{array} ; \qquad \bar{W}^{-1} = \begin{array}{c c} & \begin{array}{cc} n_i & n_0 \end{array} \\ \begin{array}{c} n_i \\ n_0 \end{array} & \left[\begin{array}{c|c} \bar{D}_r & -\bar{U}_\ell \\ \hline \bar{N}_r & \bar{V}_\ell \end{array}\right] \end{array} \tag{3.12}$$

Step 5. Define

$$\begin{array}{ll} D_r := \bar{D}_r & D_\ell := \bar{D}_\ell \\ N_r := \bar{N}_r + \tilde{Q}\bar{D}_r & N_\ell := \bar{N}_\ell + \bar{D}_\ell\tilde{Q} \\ V_r := \bar{V}_r - \bar{U}_r\tilde{Q} & V_\ell := \bar{V}_\ell - \tilde{Q}\bar{U}_\ell \\ U_r := \bar{U}_r & U_\ell := \bar{U}_\ell \end{array} \tag{3.13}$$

and stop. □

<u>Comments</u>. (i) The eight matrices in (3.12) with elements in $\mathcal{R}(\rho_0) \subset \tilde{\ell}_{1-}(\rho_0)$, namely

$$\bar{N}_r,\ \bar{\mathcal{D}}_r,\ \bar{\mathcal{U}}_r,\ \bar{\mathcal{V}}_r$$
$$\bar{N}_\ell,\ \bar{\mathcal{D}}_\ell,\ \bar{\mathcal{U}}_\ell,\ \bar{\mathcal{V}}_\ell$$

satisfy Theorem 3.1 with $\tilde{G} \leftarrow \tilde{R}$.

(ii) The eight matrices in (3.13) satisfy the conclusions of Theorem 3.1.

<u>Remark 3.3</u>. Observe that in Procedure 3.1, which is used in the Proof of Theorem 3.1, we actually obtain

$$\mathcal{D}_r \in \mathcal{R}(\rho_0)^{n_i \times n_i} \qquad \det \mathcal{D}_r \in \mathcal{R}^\infty(\rho_0)$$
$$\mathcal{D}_\ell \in \mathcal{R}(\rho_0)^{n_0 \times n_0} \qquad \det \mathcal{D}_\ell \in \mathcal{R}^\infty(\rho_0)$$

i.e. the denominator matrices of the ρ_0-r.r. $(N_r,\mathcal{D}_r)$ and the ρ_0-l.r. $(\mathcal{D}_\ell,N_\ell)$ are rational. □

The next corollary follows from Theorem 3.1 and Remark 3.1.

<u>Corollary 3.1a</u>. Let $G \in (\mathbb{C}_z^{\mathbb{N}})^{n_0 \times n_i}$; then

$$\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i} \tag{3.14}$$
$$\Leftrightarrow \tilde{G} \text{ admits a } \rho_0\text{-r.r. } (N_r,\mathcal{D}_r) \tag{3.15}$$
$$\Leftrightarrow \tilde{G} \text{ admits a } \rho_0\text{-l.r. } (\mathcal{D}_\ell,N_\ell) \tag{3.16}$$

□

<u>Remark 3.4</u>. In view of Corollary 3.1a, we have

$$b(\rho_0)^{n_0 \times n_i} = \{G \in (\mathbb{C}_z^{\mathbb{N}})^{n_0 \times n_i} \mid \tilde{G} \text{ admits a } \rho_0\text{-r.r. or } \rho_0\text{-l.r.}\}\ . \tag{3.17}$$

□

The following corollaries are the MIMO generalization of Remark 2.1.

Corollary 3.1b(r). Let $N_r \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$ and $\mathcal{D}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ with $\det \mathcal{D}_r \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$. Then $(N_r, \mathcal{D}_r)$ is ρ_0-r.c. if and only if

$$\operatorname{rank}\begin{bmatrix} \mathcal{D}_r(z) \\ \hdashline N_r(z) \end{bmatrix} = n_i \quad \forall z \in D(\rho_0)^c . \tag{3.18}$$

Corollary 3.1b(ℓ). Let $N_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$ and $\mathcal{D}_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_0}$ with $\det \mathcal{D}_\ell \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$. Then $(\mathcal{D}_\ell, N_\ell)$ is ρ_0-l.c. if and only if

$$\operatorname{rank}\left[\mathcal{D}_\ell(z) \;\middle|\; N_\ell(z)\right] = n_0 \quad \forall z \in D(\rho_0)^c . \tag{3.19}$$

□

In view of Corollary 3.1b(r) (Corollary 3.1b(ℓ)), we present next an algorithm to obtain a ρ_0-r.r. (respectively, ρ_0-l.r.) for $\tilde{G}$ given by (3.3) and (3.8) that does not use the Euclidean algorithm in $r(\rho_0)$ (which is used in Steps 2 and 3 of Procedure 3.1).

Procedure 3.2(r). ρ_0-r.r. for $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$

Given

$$\tilde{G} = \tilde{R} + \tilde{Q} \tag{3.20}$$

as in Lemma 3.3.

Step 1. Find $\bar{N}_r \in \mathbb{C}[z]^{n_0 \times n_i}$, $\bar{D}_r \in \mathbb{C}[z]^{n_i \times n_i}$ such that

(i) $\tilde{R} = \bar{N}_r \bar{D}_r^{-1}$

(ii) $(\bar{N}_r, \bar{D}_r)$ is right coprime in the ring of polynomials $\mathbb{C}[z]$

(iii) $\det \bar{D}_r \not\equiv 0$.

Step 2. Find $M \in \mathbb{C}[z]^{n_i \times n_i}$ unimodular such that

$$D_r := \bar{D}_r M \tag{3.21}$$

is column-reduced [Wol 1, Thm. 2.5.7]. Let

$$N_r := \bar{N}_r M \ . \tag{3.22}$$

Now $N_r D_r^{-1}$ is also a right coprime factorization of $\tilde{R}$.

Step 3. For $i = 1,2,\ldots,n_i$, let

$$\gamma_i := \partial_{c_i}[D_r] \tag{3.23}$$

and let $\pi_i \in \mathbb{C}[z]$ be defined by

$$\pi_i(z) := z^{\gamma_i} \tag{3.24}$$

Define
$$S := \operatorname{diag}(\pi_i)_{i=1}^{n_i} \in \mathbb{C}[z]^{n_i \times n_i} \ . \tag{3.25}$$

Step 4. Define

$$\bar{N}_r := N_r S^{-1} \in r(\rho_0)^{n_0 \times n_i} \tag{3.26}$$

$$\bar{D}_r := D_r S^{-1} \in r(\rho_0)^{n_i \times n_i} \ . \tag{3.27}$$

Comment. $(\bar{N}_r, \bar{D}_r)$ is a ρ_0-r.r. of $\tilde{R}$ with elements in $r(\rho_0)$.

Step 5. Define

$$N_r := \bar{N}_r + \tilde{Q}\bar{D}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i} \tag{3.28}$$

$$D_r := \bar{D}_r \in r(\rho_0)^{n_i \times n_i} \subset \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i} \ . \tag{3.29}$$

Comment. (N_r, D_r) is a ρ_0-r.r. of $\tilde{G}$.

Procedure 3.2(ℓ). ρ_0-l.r. of $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$

A ρ_0-l.r. (D_ℓ, N_ℓ) of $\tilde{G}$ in (3.20) can be obtained through obvious modification of Procedure 3.2(r). □

Theorem 3.2. Let $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$. Then for any ρ_0-l.r. $(\mathcal{D}_\ell, \mathcal{N}_\ell)$ of $\tilde{G}$ and any ρ_0-r.r. $(\mathcal{N}_r, \mathcal{D}_r)$ of $\tilde{G}$, there exist matrices with elements in $\tilde{\ell}_{1-}(\rho_0)$, namely

$$\mathcal{U}_r,\ \mathcal{V}_r,\ \mathcal{U}_\ell,\ \mathcal{V}_\ell$$

such that

$$\begin{array}{c} \\ n_i \\ n_0 \end{array}\!\!\begin{array}{c} \begin{array}{cc} n_i & n_0 \end{array} \\ \left[\begin{array}{c|c} \mathcal{V}_r & \mathcal{U}_r \\ \hline -\mathcal{N}_\ell & \mathcal{D}_\ell \end{array}\right] \end{array} \begin{array}{c} \begin{array}{cc} n_i & n_0 \end{array} \\ \left[\begin{array}{c|c} \mathcal{D}_r & -\mathcal{U}_\ell \\ \hline \mathcal{N}_r & \mathcal{V}_\ell \end{array}\right] \end{array} = \left[\begin{array}{c|c} I_{n_i} & 0 \\ \hline 0 & I_{n_0} \end{array}\right]. \tag{3.30}$$

□

The next corollaries follow immediately from Theorem 3.1 and Theorem 3.2.

Corollary 3.2(ℓ). Let $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$. Then for any ρ_0-l.r. $(\mathcal{D}_\ell, \mathcal{N}_\ell)$ of $\tilde{G}$, there exist matrices with elements in $\tilde{\ell}_{1-}(\rho_0)$, namely

$$\mathcal{U}_\ell,\ \mathcal{V}_\ell;\ \mathcal{N}_r,\ \mathcal{D}_r,\ \mathcal{U}_r,\ \mathcal{V}_r$$

such that

(i) $(\mathcal{N}_r, \mathcal{D}_r)$ is a ρ_0-r.r. of $\tilde{G}$.

(ii) Equation (3.30) holds.

Corollary 3.2(r). A statement similar to Corollary 3.2(ℓ) holds by interchanging the terms "ρ_0-l.r." and "ρ_0-r.r.", and by interchanging the subscripts "ℓ" and "r". □

The following theorem is an MIMO generalization of Theorem 2.3.

Theorem 3.3. Let $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$ and let $(\mathcal{N}_r, \mathcal{D}_r)$ and $(\mathcal{N}_r', \mathcal{D}_r')$ be two ρ_0-r.r.'s of $\tilde{G}$ (respectively, let $(\mathcal{D}_\ell, \mathcal{N}_\ell)$ and $(\mathcal{D}_\ell', \mathcal{N}_\ell')$ be two ρ_0-l.r.'s of $\tilde{G}$). Under these conditions, there exists a unimodular matrix

$$R \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i} \tag{3.31}$$

(respectively $L \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_0}$) such that

$$\mathcal{D}_r = \mathcal{D}_r' R \ , \qquad N_r = N_r' R \tag{3.32}$$

(respectively $\mathcal{D}_\ell = L\mathcal{D}_\ell'$, $N_\ell = LN_\ell'$). $\square$

4. Poles, Zeros and Their Dynamic Interpretation

4.1 McMillan Degree of Poles, Smith and McMillan Forms

Consider a proper rational function matrix $\tilde{R} \in \mathbb{C}_p(z)^{n_0 \times n_i}$. It is well known that the McMillan degree of $\tilde{R}$ is the degree of the characteristic polynomial $\chi(z) := \det(zI-A)$ of a minimal realization (A,B,C,D) of $\tilde{R}$. Since the zeros of the characteristic polynomial χ are the poles of $\tilde{R}$, we henceforth define the McMillan degree of $p \in \mathbb{C}$ as a pole of $\tilde{R}$ to be the order of p as a zero of χ. Noting that the McMillan degrees defined for $\tilde{R} \in \mathbb{C}_p(z)^{n_0 \times n_i}$ are identical to those defined for the strictly proper rational function matrix $\tilde{R}_0$ defined by $\tilde{R}_0(z) := \tilde{R}(z) - \tilde{R}(\infty)$, we consider the following:

Lemma 4.1. Let $\tilde{R} \in \mathbb{C}_p(z)^{n_0 \times n_i}$ be strictly proper, with partial fraction expansion given by

$$\tilde{R}(z) = \sum_{\alpha=1}^{\nu} \sum_{i=1}^{m_\alpha} \frac{Z_{\alpha i}}{(z-p_\alpha)^i} . \tag{4.1}$$

For $\alpha = 1,2,\ldots,\nu$, the McMillan degree of p_α as a pole of $\tilde{R}$ is equal to the rank r_α of the matrix

$$H_\alpha := \begin{bmatrix} Z_{\alpha 1} & Z_{\alpha 2} & \cdots & Z_{\alpha m_\alpha} \\ Z_{\alpha 2} & Z_{\alpha 3} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ Z_{\alpha m_\alpha} & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{(m_\alpha n_0)\times(m_\alpha n_i)} . \tag{4.2}$$

□

In view of this lemma, we give the definition of McMillan degrees of poles for matrix transfer functions in $\tilde{b}(\rho_0)^{n_0 \times n_i}$ as follows:

Definition 4.1. Let $p \in D(\rho_0)^c$ be a pole of $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$, and let the principal part of the Laurent expansion of $\tilde{G}$ at p be given by

$$\tilde{G}_p(z) = \sum_{i=1}^{m} \frac{Z_i}{(z-p)^i} . \tag{4.3}$$

The McMillan degree of p as a pole of $\tilde{G}$ is defined as the rank of the matrix

$$H := \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_m \\ Z_2 & Z_3 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ Z_m & 0 & & 0 \end{bmatrix} \in \mathbb{C}^{(mn_0)\times(mn_i)} . \tag{4.4}$$

□

Remark 4.1. (i) If $p \in D(\rho_0)^c$ is a pole of $\tilde{R} \in \mathbb{C}_p(z)^{n_0 \times n_i} \subset \tilde{b}(\rho_0)^{n_0 \times n_i}$, then, by Lemma 4.1, its McMillan degree as defined in Definition 4.1 agrees with the definition discussed at the beginning of this section.

(ii) For $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$, let $\tilde{R} \in \mathbb{C}_p(s)^{n_0 \times n_i}$ be given as in Lemma 3.3, i.e. $\tilde{R}$ is the sum of the principal parts of the Laurent expansions of $\tilde{G}$ at its poles in $D(\rho_0)^c$. Then the McMillan degree of $p \in D(\rho_0)^c$ as a pole of $\tilde{G}$ is equal to the McMillan degree of p as a pole of $\tilde{R}$. □

Recall that if (N_r, D_r) (respectively (D_ℓ, N_ℓ)) is a right coprime (respectively left coprime) polynomial matrix factorization of $\tilde{R} \in \mathbb{C}_p(z)^{n_0 \times n_i}$, then $\det D_r$ (respectively $\det D_\ell$) is equal to the characteristic polynomial of any minimal representation of $\tilde{R}$ modulo a nonzero constant factor: hence the McMillan degree of the pole p of $\tilde{R}$ is the order of p as a zero of $\det D_r$ (respectively $\det D_\ell$). The next theorem contains a generalization of this result.

Theorem 4.1. Let $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$, with a ρ_0-r.r. $(N_r, \mathcal{D}_r)$ and a ρ_0-l.r. $(\mathcal{D}_\ell, N_\ell)$. Under these conditions,

(a) $p \in D(\rho_0)^c$ is a pole of $\tilde{G}$ $\Leftrightarrow$ $\det \mathcal{D}_r(p) = 0$ $\Leftrightarrow$ $\det \mathcal{D}_\ell(p) = 0$

(b) If $p \in D(\rho_0)^c$ is a pole of $\tilde{G}$, then the order of p as a zero of $\det \mathcal{D}_r$ (respectively of $\det \mathcal{D}_\ell$) is its McMillan degree.

(c) There exists $\tilde{r} \in \tilde{\ell}_{1-}(\rho_0)$ invertible in $\tilde{\ell}_{1-}(\rho_0)$ such that

$$\det \mathcal{D}_r = \tilde{r} \cdot \det \mathcal{D}_\ell \; . \tag{4.5}$$

□

We study next the Smith and McMillan forms, as these concepts are closely related to the notion of McMillan degree (see Theorem 4.3 below), and the notion of transmission zeros (to be discussed in subsection 4.3).

Smith Form [McD 1, p.40][McL 1, p.361][Sig 1, p.370]:

Definition 4.2. Given $N_1, N_2 \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$. N_1 and N_2 are said to be equivalent iff there exist unimodular matrices $L \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_0}$, $R \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ such that

$$N_1 = L N_2 R \; .$$

□

Remark 4.2. Throughout this paper, we say that $N \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$ (or $\tilde{b}(\rho_0)^{n_0 \times n_i}$) has normal rank r iff $\operatorname{rank}[N(z)] = r$ for almost all $z \in D(\rho_0)^c$. □

Theorem 4.2 [McD 1, p.40][McL 1, p.361][Sig 1, p.370]. Given $N \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$, with normal rank r. Then N is equivalent to a matrix $S[N] \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$ which satisfies

$$S[N] = \begin{array}{c} \\ r \\ \\ n_0-r \end{array} \overset{\begin{array}{cc} r & n_i-r \end{array}}{\left[\begin{array}{cccc|c} n_1 & & & & \\ & n_2 & & & 0 \\ & & \ddots & & \\ & & & n_r & \\ \hline & & 0 & & 0 \end{array}\right]} \tag{4.6}$$

where $n_i | n_{i+1}$, $i = 1,2,\ldots,r-1$. □

Definition 4.3. $S[N]$ in Theorem 4.2 is called the Smith form of N. □

Remark 4.3. (i) In Theorem 4.2, $n_i | n_{i+1}$ means that n_{i+1} is a multiple of n_i as elements of the ring $\tilde{\ell}_{1-}(\rho_0)$.

(ii) In general, the Smith form $S[N]$ is not unique. To avoid any confusion, however, we can assume that a fixed normalization procedure has been chosen so that $S[N]$ is unique: for instance, if we have obtained a Smith form $S[N]$ as in (4.6), we can use Fact A.1 of Appendix A to decompose

$$n_i = n_{iu} \cdot n_{is} , \quad i = 1,2,\ldots,r ,$$

such that the leading coefficient of n_{iu} (lowest z^{-1}-degree term) is 1. Then

$$S[N] = \begin{array}{c} \\ r \\ \\ n_0-r \end{array} \overset{\begin{array}{cc} r & n_i-r \end{array}}{\left[\begin{array}{cccc|c} n_{1u} & & & & \\ & n_{2u} & & & 0 \\ & & \ddots & & \\ & & & n_{ru} & \\ \hline & & 0 & & 0 \end{array}\right]} \overset{\begin{array}{cc} r & n_i-r \end{array}}{\left[\begin{array}{cccc|c} n_{1s} & & & & \\ & n_{2s} & & & 0 \\ & & \ddots & & \\ & & & n_{rs} & \\ \hline & & 0 & & I_{n_i-r} \end{array}\right]} . \tag{4.7}$$

On the right hand side of (4.7), if we call the left factor $S_u[N]$ and the right factor $S_s[N]$, then $S_u[N]$ is a uniquely normalized Smith form of N, and $S_s[N]$ is a unimodular matrix in $\tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ that can be absorbed by the definition of Smith form. □

McMillan Form:

Given $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$. Let $d \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$ be a least common multiple of all denominators (obtained from the ρ_0-r's) of all elements of $\tilde{G}$; and let $N := d\tilde{G} \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i}$. With the Smith form $S[N]$ of N defined through (4.6), we calculate

$$\tilde{G} = \frac{N}{d} = L\,\frac{S[N]}{d}\,R$$

$$= L \begin{array}{c c} & \begin{array}{cc} r & n_i - r \end{array} \\ \begin{array}{c} r \\ \\ n_0 - r \end{array} & \left[\begin{array}{cccc|c} \frac{\varepsilon_1}{\psi_1} & & & & \\ & \frac{\varepsilon_2}{\psi_2} & & & 0 \\ & & \ddots & & \\ & & & \frac{\varepsilon_r}{\psi_r} & \\ \hdashline & & 0 & & 0 \end{array}\right] \end{array} R \tag{4.8}$$

where $\frac{\varepsilon_i}{\psi_i}$ is a ρ_0-r. of $\frac{n_i}{d}$, $i = 1,2,\ldots,r$. The second factor in (4.8),

$$M[\tilde{G}] := \begin{array}{c c} & \begin{array}{cc} r & n_i - r \end{array} \\ \begin{array}{c} r \\ \\ n_0 - r \end{array} & \left[\begin{array}{cccc|c} \frac{\varepsilon_1}{\psi_1} & & & & \\ & \frac{\varepsilon_2}{\psi_2} & & & 0 \\ & & \ddots & & \\ & & & \frac{\varepsilon_r}{\psi_r} & \\ \hdashline & & 0 & & 0 \end{array}\right] \end{array} \in \tilde{b}(\rho_0)^{n_0 \times n_i} \tag{4.9}$$

is called the <u>McMillan form</u> of $\tilde{G}$.

Note that $\varepsilon_i | \varepsilon_{i+1}$, $\psi_{i+1} | \psi_i$, $i = 1,2,\ldots,r-1$. □

<u>Lemma 4.2</u>. Given the McMillan form $M[\tilde{G}]$ of $\tilde{G}$ in (4.8) and (4.9), let

$$E := \begin{array}{c} \\ r \\ n_0-r \end{array} \begin{array}{c} \begin{array}{cc} r & n_i-r \end{array} \\ \left[\begin{array}{cccc|c} \varepsilon_1 & & & & \\ & \varepsilon_2 & & & 0 \\ & & \ddots & & \\ & & & \varepsilon_r & \\ \hline & & 0 & & 0 \end{array}\right] \end{array} \in \tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_i} \qquad (4.10)$$

$$\Psi_r := \left[\begin{array}{cccc|c} \psi_1 & & & & \\ & \psi_2 & & & 0 \\ & & \ddots & & \\ & & & \psi_r & \\ \hline & & 0 & & I_{n_i-r} \end{array}\right] \in \tilde{\ell}_{1-}^{\infty}(\rho_0)^{n_i \times n_i} \qquad (4.11)$$

$$\Psi_\ell := \left[\begin{array}{cccc|c} \psi_1 & & & & \\ & \psi_2 & & & 0 \\ & & \ddots & & \\ & & & \psi_r & \\ \hline & & 0 & & I_{n_0-r} \end{array}\right] \in \tilde{\ell}_{1-}^{\infty}(\rho_0)^{n_0 \times n_0} . \qquad (4.12)$$

Then

$$(LE, R^{-1}\Psi_r) \text{ is a } \rho_0\text{-r.r. of } \tilde{G} \qquad (4.13)$$

and

$$(\Psi_\ell L^{-1}, ER) \text{ is a } \rho_0\text{-l.r. of } \tilde{G} . \qquad (4.14)$$

<u>Remark 4.4</u>. If $(N_r, \mathcal{D}_r)$ is any ρ_0-r.r. of $\tilde{G}$, then it is immediate from Theorem 3.3 that

$$N_r = LE, \quad \mathcal{D}_r = R^{-1}\Psi_r \ \ \underline{\text{modulo}} \text{ a unimodular matrix in } \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$$

on the right.

Similarly, if $(\mathcal{D}_\ell, \mathcal{N}_\ell)$ is any ρ_0-l.r. of $\tilde{G}$, then

$\mathcal{D}_\ell = \Psi_\ell L^{-1}$, $N_\ell = ER$ modulo a unimodular matrix in $\tilde{\ell}_{1-}(\rho_0)^{n_0 \times n_0}$

on the left. □

Theorem 4.3. Given the McMillan form $M[\tilde{G}]$ of $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$ in (4.8) and (4.9). Let $\tilde{\chi}_G := \prod_{i=1}^{r} \psi_i \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$. Under these conditions,

(a) $p \in D(\rho_0)^c$ is a pole of $\tilde{G}$ if and only if $\tilde{\chi}_G(p) = 0$;

(b) if $p \in D(\rho_0)^c$ is a pole of $\tilde{G}$, then the order of p as a zero of $\tilde{\chi}_G$ is its McMillan degree. □

4.2 Dynamic Interpretation of Poles

Given that $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$, $\tilde{G}$ is a meromorphic function in $D(\rho_1)^c$ (for some $\rho_1 \in [0,\rho_0[$), and $\tilde{G}$ may have at most a finite number of poles in $D(\rho_0)^c$. The following theorem gives a dynamic interpretation of such poles.

Theorem 4.4. Let $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$. Then $p \in D(\rho_0)^c$ is a pole of $\tilde{G}$ if and only if there exists an input sequence

$$e \in \ell_{1-}(\rho_1)^{n_i} \tag{4.15}$$

for some $\rho_1 \in]0,\rho_0[$, such that the output sequence $y := G*e$ satisfies

$$y(k) = \gamma \cdot p^k + h(k) \quad \forall k \in \mathbb{N} \tag{4.16}$$

where $\gamma \in \mathbb{C}^{n_0}$ is nonzero, and

$$h := (h(k))_{k=0}^{\infty} \in \ell_{1-}(\rho_2)^{n_0} \tag{4.17}$$

for some $\rho_2 \in]0,\rho_0[$. □

Remark 4.5. (i) As k increases towards $+\infty$, $\gamma \cdot p^k$ is the dominant term in the output (4.16): indeed, by (4.17) $h(k)$ is at most $O(\rho_2^k)$ whereas $\gamma \cdot p^k$ is $O(|p|^k)$, and $\rho_2 < \rho_0 \leq |p|$. So for k large, $|h(k)| \ll |\gamma| \cdot |p|^k = |\gamma \cdot p^k|$. Similarly, by (4.15), the output $y(k)$ also dominates the input $e(k)$.

(ii) Note that, from the proof of the theorem, both the input e and the vector γ depend on $\tilde{G}$: The point is that the input is carefully chosen so that p is the only $D(\rho_0)^c$-pole of $\tilde{G}$ excited by the input.

(iii) The proof uses a ρ_0-r.r. of $\tilde{G}$. A slightly more involved proof can be obtained with a ρ_0-l.r. of $\tilde{G}$.

(iv) In the lumped case, the input sequence e can be chosen so that e and h are identically zero except for a finite number of indices (see continuous-time analog in [Des 3; Thm. III]). □

4.3 Zeros and Their Dynamic Interpretation

Let $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$, with a ρ_0-r.r. $(\mathcal{N}_r, \mathcal{D}_r)$, i.e.

$$\tilde{G} = \mathcal{N}_r \mathcal{D}_r^{-1} \tag{4.18}$$

and a ρ_0-l.r. $(\mathcal{D}_\ell, \mathcal{N}_\ell)$, i.e.

$$\tilde{G} = \mathcal{D}_\ell^{-1} \mathcal{N}_\ell . \tag{4.19}$$

Lemma 4.3. For any $z \in D(\rho_0)^c$,

$$\operatorname{rank}[\mathcal{N}_r(z)] = \operatorname{rank}[\mathcal{N}_\ell(z)] . \tag{4.20}$$

□

Definition 4.4. Assume that

$$N_\ell \text{ (equivalently } N_r\text{) has full normal rank, i.e. } \min(n_0,n_i). \quad (4.21)$$

Then $z_0 \in D(\rho_0)^c$ is called a (transmission) zero of $\tilde{G}$ iff

$$\operatorname{rank}[N_\ell(z_0)] < \min(n_0,n_i) \quad (4.22)$$

(equivalently, $\operatorname{rank}[N_r(z_0)] < \min(n_0,n_i)$). □

Remark 4.6. (i) In view of Lemma 4.3, the notion of transmission zero is a property of the matrix transfer function $\tilde{G}$, independent of any particular choice of matrix fraction representation.

(ii) Note that $\tilde{G}$ can have a pole and a transmission zero at the same point $z_0 \in D(\rho_0)^c$.

(iii) Let $M[\tilde{G}]$ be the McMillan form of $\tilde{G}$ as in (4.8)-(4.14). Then $z_0 \in D(\rho_0)^c$ is a zero of $\tilde{G}$ if and only if z_0 is a zero of ε_i, for some $i \in \{1,2,\ldots,\min(n_0,n_i)\}$.

(iv) If assumption (4.21) is not satisfied, we can always ignore some redundant input or output, and consider a smaller matrix transfer function for which (4.21) is satisfied. Then the following theorems can be applied to this reduced matrix transfer function. □

Theorem 4.5. Let $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$, with $n_0 \geq n_i$.

(a) If $z_0 \in D(\rho_0)^c$ is a zero of $\tilde{G}$, then there exists a nonzero $\xi \in \mathbb{C}^{n_i}$ and a sequence

$$m \in \ell_{1-}(\rho_1)^{n_i} \text{ for some } \rho_1 \in]0,\rho_0[\quad (4.23)$$

such that the input sequence $e \in (\mathbb{C}^{\mathbb{N}})^{n_i}$ described by

$$e(k) = \xi z_0^k + m(k) \quad \forall k \in \mathbb{N} \tag{4.24}$$

produces an output sequence $y \in (\mathbb{C}^{\mathbb{N}})^{n_0}$ (i.e. $y = G*e$) such that

$$y \in \ell_{1-}(\rho_2)^{n_0} \tag{4.25}$$

for some $\rho_2 \in]0,\rho_0[$.

(b) If $v \in D(\rho_0)^c$ is neither a pole nor a zero of $\tilde{G}$, then for all nonzero vectors $\xi \in \mathbb{C}^{n_i}$, the input sequence $e \in (\mathbb{C}^{\mathbb{N}})^{n_i}$ described by

$$e(k) = \xi v^k \ , \quad k \in \mathbb{N} \tag{4.26}$$

produces an output sequence $y \in (\mathbb{C}^{\mathbb{N}})^{n_0}$ which contains the <u>nonzero</u> term

$$\tilde{G}(v)\xi v^k \ . \tag{4.27}$$

□

<u>Remark 4.7</u>. Consider part (a) of the theorem:

(i) In the lumped case, we can prove that the sequences m and y can be chosen to be identically zero except for a finite number of indices (see continuous-time analog in [Des 3; Thm. I]).

(ii) For k large, since $|z_0| \geq \rho_0$ and since (4.23) holds, the term ξz_0^k in (4.24) is the <u>dominant term in the input</u> sequence (indeed, $\rho_1 < \rho_0 \leq |z_0|$); furthermore, this term also dominates the output sequence (since $\rho_2 < \rho_0 \leq |z_0|$). In this sense, we still have the interpretation that the zero blocks the transmission of the term $(\xi z_0^k)_{k=0}^{\infty}$. The purpose of m in the input is to prevent any contribution in y of any of the $D(\rho_0)^c$-poles of $\tilde{G}$. □

<u>Theorem 4.6</u>. Let $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$ with $n_0 \leq n_i$. If $z_0 \in D(\rho_0)^c$ is a zero of $\tilde{G}$, then there is a nonzero $\eta \in \mathbb{C}^{n_0}$ such that for <u>all</u>

$\xi \in \mathbb{C}^{n_i}$ there is $\tilde{m} \in \mathcal{R}(\rho_0)^{n_i}$ so that the input sequence described by

$$e(k) = \xi z_0^k + m(k) \quad \forall k \in \mathbb{N} \tag{4.28}$$

produces an output sequence y (i.e. $y = G*e$) that satisfies

$$\eta^* y \in \ell_{1-}(\rho_1) \tag{4.29}$$

for some $\rho_1 \in]0,\rho_0[$, where $\eta^* y := (\eta^* y(k))_{k=0}^{\infty}$. □

Remark 4.8. Theorem 4.5 (which applies to cases where $n_0 \geq n_i$) asserts that for some ξ, the input $e(k) = \xi z_0^k + m(k)$ produces an output y which does not have a term in z_0^k, i.e. the sequence $(z_0^k)_{k=0}^{\infty}$ is blocked for those ξ's. Theorem 4.6 (where $n_0 \leq n_i$) allows any ξ and asserts that, in some direction dictated by η, y does not contain any term in z_0^k. □

4.4 Example

This example demonstrates Theorem 4.5(a) with a multi-input multi-output transfer function $\tilde{G} \in \tilde{b}(\rho_0)^{2\times 2}$, where $\rho_0 := 0.55$, defined by

$$\tilde{G}(z) := \left[\begin{array}{c|c} \frac{1}{(z-1)} + \frac{1}{z}e^{-2z^{-1}} & \frac{1}{(z-4)} + \frac{5}{(2z-1)}e^{1-3z^{-1}} \\ \hline 0 & \frac{1}{(z-2)} - \frac{3}{z}e^{1-3z^{-1}} \end{array}\right], \quad z \in D(\rho_0)^c. \tag{4.30}$$

Note that the set of poles of $\tilde{G}$ in $D(\rho_0)^c$ is

$$P[\tilde{G}] = \{1,2,4\} \quad . \tag{4.31}$$

A ρ_0-l.r. $(\mathcal{D}_\ell, N_\ell)$ of $\tilde{G}$ is given by

$$\mathcal{D}_\ell(z) := \left[\begin{array}{c|c} \frac{(z-1)(z-4)}{z^2} & 0 \\ \hline 0 & \frac{(z-2)}{z} \end{array}\right] \tag{4.32}$$

$$N_\ell(z) := \left[\begin{array}{c|c} \frac{(z-4)[z+(z-1)e^{-2z^{-1}}]}{z^3} & \frac{(z-1)[2z-1+5(z-4)e^{1-3z^{-1}}]}{z^2(2z-1)} \\ \hline 0 & \frac{z-3(z-2)e^{1-3z^{-1}}}{z^2} \end{array}\right]. \tag{4.33}$$

$(\mathcal{D}_\ell, N_\ell)$ are ρ_0-l.c.: indeed, they satisfy

$$(N_\ell U_\ell + \mathcal{D}_\ell V_\ell)(z) = I_2 \,, \quad z \in D(\rho_0)^c \tag{4.34}$$

with U_ℓ, $V_\ell \in \tilde{\ell}_{1-}(\rho_0)^{2\times 2}$ described by

$$U_\ell(z) := \left[\begin{array}{c|c} \frac{(3z^3-4z^2-32z+32)}{3z^3} & \frac{2(z-1)}{z^2} \\ \hline \frac{16(z-2)(z+4)}{3z^2} & -\frac{2(z-4)}{z} \end{array}\right] \tag{4.35}$$

and

$$V_\ell(z) := \left[\begin{array}{c|c} \frac{(3z^3-4z^2-32z+32)}{3z^3}(z-e^{-2z^{-1}}) + \frac{80(z-2)(z+4)}{3z^2(2z-1)}e^{1-3z^{-1}} & \frac{2(z-1)}{z^3}(z-e^{-2z^{-1}}) + \frac{10(z-4)}{z(2z-1)}e^{1-3z^{-1}} \\ \hline -\frac{16(z+4)}{3z^3}[z+3(z-2)e^{1-3z^{-1}}] & \frac{(z+4)}{z} - \frac{6(z-4)}{z^2}e^{1-3z^{-1}} \end{array}\right]. \tag{4.36}$$

Observe that

$$N_{\ell}(3) = \begin{bmatrix} -\frac{(3+2e^{-2/3})}{27} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.37}$$

hence, by definition, $\tilde{G}$ has a zero at $z = 3$. Consequently, we choose

$$\xi := [0 \quad 1]^T \in \mathbb{C}^2 \tag{4.38}$$

which satisfies (B.65). Next, as in (B.68), we define

$$\begin{aligned}\tilde{m}(z) &:= -U_{\ell}(z)N_{\ell}(z)\xi\cdot\frac{z}{(z-3)} \\ &= \begin{bmatrix} -\frac{(z-1)[(2z-1)(z-2)(3z-4)(z+4)+(15z^4-116z^3+10z^2+764z-640)e^{1-3z^{-1}}]}{3z^4(z-3)(2z-1)} \\ -\frac{2[(2z-1)(5z^3+20z^2-80z+64)+(z-2)(z-4)(58z^2+111z-160)e^{1-3z^{-1}}]}{3z^3(z-3)(2z-1)} \end{bmatrix}.\end{aligned} \tag{4.39}$$

Note that $\tilde{m}$ is analytic at $z = 3$, and $m \in \ell_1(\rho_1)^2$ with $\rho_1 := 0.51$. With the input

$$\tilde{e}(z) := \xi\cdot\frac{z}{(z-3)} + \tilde{m}(z) \tag{4.40}$$

defined as in (4.24), the output is

$$\tilde{y}(z) := \tilde{G}(z)\tilde{e}(z) = [\tilde{y}_1(z) \quad \tilde{y}_2(z)]^T \tag{4.41}$$

where

$$\begin{aligned}\tilde{y}_1(z) = \frac{1}{3z^5(z-3)(2z-1)^2}&[(z-1)(2z-1)^2(z-2)(3z-4)(z+4)(z-e^{-2z^{-1}}) \\ &+ z(2z-1)(15z^5-181z^4-74z^3+1554z^2-2044z+640)e^{1-3z^{-1}} \\ &- (z-1)(2z-1)(15z^4-116z^3+10z^2+764z-640)e^{1-5z^{-1}} \\ &- 10z^2(z-2)(z-4)(58z^2+111z-160)e^{2-6z^{-1}}]\end{aligned}$$

and

$$\tilde{y}_2(z) = \frac{(z-4)[z-3(z-2)e^{1-3z^{-1}}][(2z-1)(3z-4)(z+4)-2(58z^2+111z-160)e^{1-3z^{-1}}]}{3z^4(z-3)(2z-1)}.$$

Observe that $y \in \ell_1(\rho_2)^2$, with $\rho_2 := 0.51$, and $\tilde{y}$ is analytic at $z = 1,2,3,4$: note that $1,2,4 \in P[\tilde{G}]$ and $z = 3$ is a pole of the input $\tilde{e}$.

5. Interconnected Systems and Characteristic Functions

In order to discuss the stability of interconnected systems, we introduce the notion of characteristic functions. Basically the technique is very simple: we illustrate it by an example. In this process, we state without formal proof some properties that hold for more general interconnections.

Example 5.1. Consider the system depicted by Fig. 5-1. All transfer functions are matrices with elements in $\tilde{b}(\rho_0)$ for some $\rho_0 \in]0,1[$: $\tilde{G}_p$ is the plant transfer function, $\tilde{G}_i$ is the inner-loop feedback, $\tilde{G}_c$ is a precompensator, and $\tilde{G}_o$ is the outerloop feedback. The vectors $\tilde{u}_p$, $\tilde{u}_i$, $\tilde{u}_c$ and $\tilde{u}_o$ are the respective exogenous input signals to the summing nodes of these subsystems, and $\tilde{y}_p$, $\tilde{y}_i$, $\tilde{y}_c$ and $\tilde{y}_o$ are the respective outputs of these subsystems. Let

$$(\mathcal{N}_{pr},\mathcal{D}_{pr}) \text{ be a } \rho_0\text{-r.r. of } \tilde{G}_p \,, \tag{5.1}$$

$$(\mathcal{N}_{cr},\mathcal{D}_{cr}) \text{ be a } \rho_0\text{-r.r. of } \tilde{G}_c \,, \tag{5.2}$$

$$(\mathcal{D}_{i\ell},\mathcal{N}_{i\ell}) \text{ be a } \rho_0\text{-l.r. of } \tilde{G}_i \,, \tag{5.3}$$

and

$$(\mathcal{D}_{o\ell},\mathcal{N}_{o\ell}) \text{ be a } \rho_0\text{-l.r. of } \tilde{G}_o \,. \tag{5.4}$$

Let us denote by $\tilde{\xi}$ the list of output vectors from all the $\mathcal{D}^{-1}$ matrices with the appropriate subscripts, as depicted by Fig. 5-2. Define

$$\tilde{u} := \begin{bmatrix} \tilde{u}_p \\ \hdashline \tilde{u}_c \\ \hdashline \tilde{u}_i \\ \hdashline \tilde{u}_o \end{bmatrix} \qquad \tilde{y} := \begin{bmatrix} \tilde{y}_p \\ \hdashline \tilde{y}_c \\ \hdashline \tilde{y}_i \\ \hdashline \tilde{y}_o \end{bmatrix} \qquad \tilde{\xi} := \begin{bmatrix} \tilde{\xi}_p \\ \hdashline \tilde{\xi}_c \\ \hdashline \tilde{\xi}_i \\ \hdashline \tilde{\xi}_o \end{bmatrix} . \tag{5.5}$$

By equating the respective <u>input vector</u> of each $\mathcal{D}^{-1}$ matrix and the output vector of each subsystem, we describe the whole interconnected system (as in Fig. 5-2) by a set of equations in the form of

$$\mathcal{D}\tilde{\xi} = N_\ell \tilde{u} \tag{5.6a}$$

$$N_r \tilde{\xi} = \tilde{y} . \tag{5.6b}$$

Specifically for this particular example, we have

$$\left[\begin{array}{c:c:c:c} \mathcal{D}_{pr} & -N_{cr} & I & 0 \\ \hdashline 0 & \mathcal{D}_{cr} & 0 & I \\ \hdashline -N_{i\ell}N_{pr} & 0 & \mathcal{D}_{i\ell} & 0 \\ \hdashline -N_{o\ell}N_{pr} & 0 & 0 & \mathcal{D}_{o\ell} \end{array}\right] \begin{bmatrix} \tilde{\xi}_p \\ \hdashline \tilde{\xi}_c \\ \hdashline \tilde{\xi}_i \\ \hdashline \tilde{\xi}_o \end{bmatrix} = \left[\begin{array}{c:c:c:c} I & 0 & 0 & 0 \\ \hdashline 0 & I & 0 & 0 \\ \hdashline 0 & 0 & N_{i\ell} & 0 \\ \hdashline 0 & 0 & 0 & N_{o\ell} \end{array}\right] \begin{bmatrix} \tilde{u}_p \\ \hdashline \tilde{u}_c \\ \hdashline \tilde{u}_i \\ \hdashline \tilde{u}_o \end{bmatrix} \tag{5.7a}$$

$$\left[\begin{array}{c:c:c:c} N_{pr} & 0 & 0 & 0 \\ \hdashline 0 & N_{cr} & 0 & 0 \\ \hdashline 0 & 0 & I & 0 \\ \hdashline 0 & 0 & 0 & I \end{array}\right] \begin{bmatrix} \tilde{\xi}_p \\ \hdashline \tilde{\xi}_c \\ \hdashline \tilde{\xi}_i \\ \hdashline \tilde{\xi}_o \end{bmatrix} = \begin{bmatrix} \tilde{y}_p \\ \hdashline \tilde{y}_c \\ \hdashline \tilde{y}_i \\ \hdashline \tilde{y}_o \end{bmatrix} . \tag{5.7b}$$

Now consider the matrices $\mathcal{D}$, N_ℓ and N_r defined in (5.6) as they appear in (5.7). Using Corollary 3.1b(ℓ) and the ρ_0-l.c. property of $(\mathcal{D}_{i\ell}, N_{i\ell})$ and $(\mathcal{D}_{o\ell}, N_{o\ell})$, it is easy to see that the pair $(\mathcal{D}, N_\ell)$ in (5.7a) is ρ_0-l.c. Similarly, by Corollary 3.1b(r) and the ρ_0-r.c. property of $(N_{pr}, \mathcal{D}_{pr})$ and $(N_{cr}, \mathcal{D}_{cr})$, the pair $(N_r, \mathcal{D})$ is ρ_0-r.c. □

We now summarize the procedure for analyzing more general interconnected systems:

Procedure 5.1. Analysis of interconnected systems

Given: subsystems each described by a matrix transfer function $\tilde{G}_k$ with elements in $\tilde{b}(\rho_0)$; the input to the transfer function $\tilde{G}_k$ is the sum of an exogenous input $\tilde{u}_k$ and outputs (modulo sign) of conformable size from other subsystems; the output of $\tilde{G}_k$ is denoted by $\tilde{y}_k$ (see Fig. 5-3).

Step 1. For each subsystem $\tilde{G}_k$, find either a ρ_0-r.r. $(N_{kr}, \mathcal{D}_{kr})$ or a ρ_0-l.r. $(\mathcal{D}_{k\ell}, N_{k\ell})$.

Step 2. Denote by $\tilde{\xi}_k$ the output vector from each $\mathcal{D}_k^{-1}$ matrix.

Step 3. With the composite vectors $\tilde{u}$, $\tilde{y}$ and $\tilde{\xi}$ (defined as in (5.5)), equate the input vectors of each $\mathcal{D}_k^{-1}$ matrix and the output vector of each subsystem to get a description of the interconnected system in the form of (5.6), namely

$$\mathcal{D}\tilde{\xi} = N_\ell \tilde{u} \,, \quad \mathcal{D} \in \tilde{\ell}_{1-}(\rho_0)^{n\times n}, \quad N_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n\times n_i} \tag{5.8a}$$

$$N_r \tilde{\xi} = \tilde{y} \quad , \quad N_r \in \tilde{\ell}_{1-}(\rho_0)^{n_0\times n} \; . \tag{5.8b}$$

□

Then we have the following property:

Fact 5.1. $(\mathcal{D}, N_\ell)$ is ρ_0-l.c. and (5.9)

$(N_r, \mathcal{D})$ is ρ_0-r.c. (5.10)

□

Remark 5.1. In this formulation, there is an additive exogenous input to each subsystem, and the output of each subsystem can be observed. □

Assume a well-posedness condition[†] that $\det \mathcal{D} \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$. Then, by Cramer's rule,

$$\tilde{G}_{\ell} := \mathcal{D}^{-1} N_{\ell} \tag{5.11}$$

$$\tilde{G}_{r} := N_{r} \mathcal{D}^{-1} \tag{5.12}$$

and

$$\tilde{G} := N_{r} \mathcal{D}^{-1} N_{\ell} \tag{5.13}$$

are all matrices with elements in $\tilde{b}(\rho_0)$. In particular, $(N_r, \mathcal{D})$ is a ρ_0-r.r. of $\tilde{G}_r$, and $(\mathcal{D}, N_{\ell})$ is a ρ_0-l.r. of $\tilde{G}_{\ell}$.

<u>Definition 5.1</u>. We call $\tilde{\chi} := \det \mathcal{D} \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$ the <u>characteristic function</u> of the interconnected system described in Procedure 5.1. □

<u>Lemma 5.1</u>. $p \in D(\rho_0)^c$ is a <u>zero</u> of the characteristic function $\tilde{\chi}$ (5.14)

$$\Leftrightarrow \ p \in D(\rho_0)^c \text{ is a pole of } \tilde{G}_r \tag{5.15}$$

$$\Leftrightarrow \ p \in D(\rho_0)^c \text{ is a pole of } \tilde{G}_{\ell} \tag{5.16}$$

$$\Leftrightarrow \ p \in D(\rho_0)^c \text{ is a pole of } \tilde{G}\ . \tag{5.17}$$

□

Because of Lemma 5.1, the importance of the characteristic function $\tilde{\chi}$ is obvious by the dynamic interpretation of poles of $\tilde{G}$ in Theorem 4.4, and by the next Theorem.

<u>Theorem 5.1</u>. Consider the interconnected system described in Procedure 5.1. Let $\sigma \geq \rho_0$. The characteristic function $\tilde{\chi}$ has a zero p of absolute value σ if and only if there exist some $m \in \mathbb{N}^*$ and some

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[†]Note that if $\lim_{|z| \to \infty} \det \mathcal{D}(z) = 0$, then $\mathcal{D}^{-1}$ has a pole at infinity; then for some $\tilde{\ell}_{1-}(\rho_0)$-matrices U_{ℓ} and V_{ℓ}, $N_{\ell} U_{\ell} + \mathcal{D} V_{\ell} = I$, so $\tilde{G}_{\ell} U_{\ell} + V_{\ell} = \mathcal{D}^{-1}$ and $\tilde{G}_{\ell}$ has a pole at infinity: hence the map $u \mapsto \xi$ is noncausal.

input sequence u with support $\{0\}$ such that the corresponding output sequence $y = (y(k))_{k=0}^{\infty} := G*u$ includes a nonzero term which, for large k, is $O(k^{m-1}\sigma^k)$. □

Remark 5.2. In fact, a little more than Theorem 5.1 is proved: the zero p of the characteristic function $\tilde{\chi}$ in $D(\rho_0)^c$ corresponds to the mode p of the interconnected system which can be excited by some exogenous input, and observed at some subsystem output. □

Definition 5.2. Let $p \in [1,\infty]$. A map represented by a matrix transfer function $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$ is said to be ℓ_p-stable iff it takes ℓ_p-input sequences to ℓ_p-output sequences, and there exists some $k \in \mathbb{R}_+$ such that for all $u \in \ell_p$

$$\|G*u\|_p \leq k\|u\|_p .$$

(Note that k may depend on p.) □

Theorem 5.2. Consider an interconnected system described by Procedure 5.1 and assumed to be well-posed. For any $\rho \geq \rho_0$,

$$\tilde{G} \in \tilde{\ell}_1(\rho)^{n_0 \times n_i} \tag{5.18}$$

if and only if

$$\tilde{\chi}(z) \neq 0 \quad \forall z \in D(\rho)^c . \tag{5.19}$$

□

The next corollary follows from Theorem 5.2 and [Des 1, Thm. C.4.7].

Corollary 5.2a. Consider an interconnected system described by Procedure 5.1. Its input-output map represented by $\tilde{G}$ is ℓ_p-stable $\forall p \in [1,\infty]$ if and only if

$$\tilde{\chi}(z) \neq 0 \quad \forall z \in D(1)^c . \tag{5.20}$$

□

In such a case, we say that the <u>interconnected system is ℓ_p-stable</u>, $\forall p \in [1,\infty]$.

Applying Theorem 5.2 to a simple case leads to the following useful corollary.

<u>Corollary 5.2b</u>. Consider a system with input sequence u and output sequence y where $\tilde{y} = \tilde{G}\tilde{u}$; let $\tilde{G} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$, with a ρ_0-r.r. $(\mathcal{N}_r,\mathcal{D}_r)$ (respectively ρ_0-l.r. $(\mathcal{D}_\ell,\mathcal{N}_\ell)$). Then for any $\rho \geq \rho_0$

$$\tilde{G} \in \tilde{\ell}_1(\rho)^{n_0 \times n_i} \tag{5.21}$$

if and only if

$$\det \mathcal{D}_r(z) \neq 0 \quad \forall z \in D(\rho)^c \tag{5.22}$$

(respectively $\det \mathcal{D}_\ell(z) \neq 0 \;\; \forall z \in D(\rho)^c$). □

6. Feedback System Stability

We now apply the results developed in the preceding section to analyze the multi-input multi-output feedback system S depicted in Fig. 6-1. Let $\rho_0 \in]0,1[$.

(i) Let $\tilde{P} \in \tilde{b}(\rho_0)^{n_0 \times n_i}$ be the plant transfer function with input $u_p \in (\mathbb{C}_z^{\mathbb{N}})^{n_i}$ and output $y_p \in (\mathbb{C}_z^{\mathbb{N}})^{n_0}$; let $\tilde{C} \in \tilde{b}(\rho_0)^{n_i \times n_0}$ be the controller transfer function with input $u_c \in (\mathbb{C}_z^{\mathbb{N}})^{n_0}$ and output $y_c \in (\mathbb{C}_z^{\mathbb{N}})^{n_i}$.

(ii) $u_s \in (\mathbb{C}_z^{\mathbb{N}})^{n_0}$ is the system (reference) input and $w_p \in (\mathbb{C}_z^{\mathbb{N}})^{n_i}$ is the plant input disturbance.

(iii) $y_s = y_p$ is the system output and $e_s := u_s - y_s = u_c$ is the system error.

Observe that if an additive disturbance is present at the plant output, say w_0, then its effect on y_s is equivalent to an additional system input $-w_0$.

Next, we define the composite system input, output and error by

$$u := \begin{bmatrix} u_s \\ \hdashline w_p \end{bmatrix} \quad y := \begin{bmatrix} y_c \\ \hdashline y_s \end{bmatrix} = \begin{bmatrix} y_c \\ \hdashline y_p \end{bmatrix} \qquad e := \begin{bmatrix} e_s \\ \hdashline u_p \end{bmatrix} = \begin{bmatrix} u_c \\ \hdashline u_p \end{bmatrix} \tag{6.1}$$

where u, y and e are in $(\mathbb{C}_z^{\mathbb{N}})^{n_0+n_i}$. Then, from Fig. 6-1, the feedback system is described by

$$\tilde{u} = \begin{array}{c} \\ n_o \\ n_i \end{array} \!\!\! \begin{array}{c} \begin{array}{cc} n_o & n_i \end{array} \\ \left[\begin{array}{c:c} I_{n_o} & \tilde{P} \\ \hdashline -\tilde{C} & I_{n_i} \end{array}\right] \end{array} \tilde{e} \tag{6.2}$$

and

$$\tilde{y} = \begin{array}{c c} & \begin{array}{cc} n_o & n_i \end{array} \\ \begin{array}{c} n_i \\ n_o \end{array} & \left[\begin{array}{c|c} \tilde{C} & 0 \\ \hline 0 & \tilde{P} \end{array}\right] \end{array} \tilde{e} \ . \tag{6.3}$$

Let

$$\tilde{G} := \begin{array}{c c} & \begin{array}{cc} n_o & n_i \end{array} \\ \begin{array}{c} n_o \\ n_i \end{array} & \left[\begin{array}{c|c} 0 & \tilde{P} \\ \hline -\tilde{C} & 0 \end{array}\right] \end{array} \in \tilde{b}(\rho_0)^{(n_i+n_o)\times(n_i+n_o)} \tag{6.4}$$

$$J := \begin{array}{c c} & \begin{array}{cc} n_i & n_o \end{array} \\ \begin{array}{c} n_o \\ n_i \end{array} & \left[\begin{array}{c|c} 0 & I_{n_o} \\ \hline -I_{n_i} & 0 \end{array}\right] \end{array} \in \tilde{\ell}_{1-}(\rho_0)^{(n_i+n_o)\times(n_i+n_o)} \tag{6.5}$$

and observe that

$$\left[\begin{array}{c|c} \tilde{C} & 0 \\ \hline 0 & \tilde{P} \end{array}\right] = J^{-1}\tilde{G} \ . \tag{6.6}$$

Assume the <u>well-posedness</u> <u>condition</u> that

$$\lim_{|z|\to\infty} \det[I_{n_o} + \tilde{P}\tilde{C}](z) = \lim_{|z|\to\infty} \det[I_{n_i} + \tilde{C}\tilde{P}] \neq 0 \ . \tag{6.7}$$

Now the input-to-error transfer function $\tilde{H}_{eu}: \tilde{u} \mapsto \tilde{e}$ and input-to-output transfer function $\tilde{H}_{yu}: \tilde{u} \mapsto \tilde{y}$ satisfy

$$\tilde{H}_{eu} = (I + \tilde{G})^{-1} \tag{6.8}$$

$$J\tilde{H}_{yu} = \tilde{G}(I + \tilde{G})^{-1} = I - \tilde{H}_{eu} \ . \tag{6.9}$$

<u>Remark 6.1</u>. (i) By assumption (6.7), $\lim_{|z|\to\infty} \det[I_{n_o} + \tilde{P}\tilde{C}](z) \neq 0$; hence by Theorem 2.4, $\det[I_{n_o} + \tilde{P}\tilde{C}] = \det[I_{n_i} + \tilde{C}\tilde{P}]$ is an invertible element of $\tilde{b}(\rho_0)$; then by applying Cramer's rule to (6.8) and (6.9), we conclude that

$$\tilde{H}_{eu} \text{ and } \tilde{H}_{yu} \text{ belong to } \tilde{\mathcal{b}}(\rho_0)^{(n_i+n_o)\times(n_i+n_o)} . \tag{6.10}$$

(ii) For any $\rho \in \mathbb{R}_+$, due to (6.8), (6.9) and the closure properties of $\tilde{\ell}_1(\rho)$ under addition and multiplication

$$\tilde{H}_{eu} \in \tilde{\ell}_1(\rho)^{(n_i+n_o)\times(n_i+n_o)} \Leftrightarrow \tilde{H}_{yu} \in \tilde{\ell}_1(\rho)^{(n_i+n_o)\times(n_i+n_o)} . \tag{6.11}$$

□

$$\text{Let } (\mathcal{D}_{p\ell},\mathcal{N}_{p\ell}) \text{ be a } \rho_0\text{-l.r. of } \tilde{P} , \tag{6.12}$$

and

$$\text{let } (\mathcal{N}_{cr},\mathcal{D}_{cr}) \text{ be a } \rho_0\text{-r.r. of } \tilde{C} . \tag{6.13}$$

Then Procedure 5.1, Definition 5.1, and simple calculations show that

$$\tilde{\chi} := \det[\mathcal{D}_{p\ell}\mathcal{D}_{cr} + \mathcal{N}_{p\ell}\mathcal{N}_{cr}] \tag{6.14}$$

is an element of $\tilde{\ell}_{1-}(\rho_0)$, and is the <u>characteristic function</u> of the feedback system S; furthermore, by assumption (6.7), $\tilde{\chi} \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$.

<u>Theorem 6.1</u>. Consider a feedback system S described by (6.1)-(6.14). Then

(i) $p \in D(\rho_0)^c$ is a zero of $\tilde{\chi}$ (6.15)

$\Leftrightarrow$ $p \in D(\rho_0)^c$ is a pole of $\tilde{H}_{eu}$ (6.16)

$\Leftrightarrow$ $p \in D(\rho_0)^c$ is a pole of $\tilde{H}_{yu}$ (6.17)

(ii) the McMillan degree of $p \in D(\rho_0)^c$ as a pole of $\tilde{H}_{eu}$ and $\tilde{H}_{yu}$ are the same and are equal to the multiplicity of p as a zero of $\tilde{\chi}$. □

<u>Remark 6.2</u>. (i) By (6.11), Theorem 5.2 and [Des 1, Thm. C.4.7], $\tilde{H}_{eu}$ (equivalently $\tilde{H}_{yu}$) is ℓ_p-stable $\forall p \in [1,\infty]$ if and only if $\tilde{\chi}(z) \neq 0$ $\forall |z| \geq 1$.

(ii) As discussed in Section 2.1, if for some $\rho \in [0,1[$ and $\forall |z| \geq \rho$, $\tilde{\chi}(z) \neq 0$, then the map $u \mapsto (e,y)$ will take an input sequence with finite support to an output sequence that decays exponentially to $\theta_{2(n_i+n_o)}$ at a rate at least ρ^{-1}. $\square$

7. Compensator Design for Stabilization, Tracking and Disturbance Rejection

7.1 Preliminary Algebraic Result

Suppose

$$\tilde{G} \in \tilde{b}(\rho_0)^{n_o \times n_i}, \quad \text{with a } \rho_0\text{-l.r. } (\mathcal{D}_\ell, \mathcal{N}_\ell) \ . \tag{7.1}$$

Recall that by Corollary 3.2(ℓ) there exist six matrices with elements in $\tilde{\ell}_{1-}(\rho_0)$, namely,

$$\mathcal{U}_\ell,\ \mathcal{V}_\ell;\ \mathcal{N}_r,\ \mathcal{D}_r,\ \mathcal{U}_r,\ \mathcal{V}_r$$

such that

(i) $(\mathcal{N}_r, \mathcal{D}_r)$ is a ρ_0-r.r. of $\tilde{G}$ (7.2)

(ii)
$$\begin{array}{c c} & \begin{array}{cc} n_i & n_o \end{array} \\ \begin{array}{c} n_i \\ n_o \end{array} & \left[\begin{array}{c|c} \mathcal{V}_r & \mathcal{U}_r \\ \hline -\mathcal{N}_\ell & \mathcal{D}_\ell \end{array}\right] \end{array} \begin{array}{c} \begin{array}{cc} n_i & n_o \end{array} \\ \left[\begin{array}{c|c} \mathcal{D}_r & -\mathcal{U}_\ell \\ \hline \mathcal{N}_r & \mathcal{V}_\ell \end{array}\right] \end{array} = \left[\begin{array}{c|c} I_{n_i} & 0 \\ \hline 0 & I_{n_o} \end{array}\right] . \tag{7.3}$$

Let us call the two matrices on the left-hand side of (7.3) $\mathcal{W}$ and $\mathcal{W}^{-1}$ respectively.

<u>Lemma 7.1</u>. Given any $\mathcal{D} \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_o}$.

(a) The pair $X \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o}$, $Y \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_o}$ is a solution of

$$\mathcal{N}_\ell X + \mathcal{D}_\ell Y = \mathcal{D} \tag{7.4}$$

if and only if for some $N \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o}$

$$\begin{bmatrix} -X \\ \hline Y \end{bmatrix} = \mathcal{W}^{-1} \begin{bmatrix} N \\ \hline \mathcal{D} \end{bmatrix} \quad \text{i.e.} \quad \begin{cases} -X = \mathcal{D}_r N - \mathcal{U}_\ell \mathcal{D} \\ Y = \mathcal{N}_r N + \mathcal{V}_\ell \mathcal{D} \end{cases} \tag{7.5}$$

or equivalently,

$$\begin{bmatrix} N \\ - \\ \mathcal{D} \end{bmatrix} = \mathcal{W}\begin{bmatrix} -X \\ - \\ Y \end{bmatrix} \quad \text{i.e.} \quad \begin{cases} N = -V_r X + U_r Y \\ \mathcal{D} = N_\ell X + \mathcal{D}_\ell Y \end{cases} . \tag{7.6}$$

Furthermore,

$$(X,Y) \text{ is } \rho_0\text{-r.c.} \Leftrightarrow (N,\mathcal{D}) \text{ is a } \rho_0\text{-r.c.} \tag{7.7}$$

(b) If in addition

$$G(0) = \lim_{|z|\to\infty} \tilde{G}(z) = 0_{n_o \times n_i} \tag{7.8}$$

then

$$\det Y \in \tilde{\ell}^{\infty}_{1-}(\rho_0) \Leftrightarrow \det \mathcal{D} \in \tilde{\ell}^{\infty}_{1-}(\rho_0) . \tag{7.9}$$

□

7.2 Problem of Stabilization, Tracking and Disturbance Rejection

Consider the feedback structure depicted in Fig. 6-1. Suppose we are given a plant $\tilde{P} \in \tilde{b}(\rho_0)^{n_o \times n_i}$ for some $\rho_0 \in]0,1[$ and that

$$\text{(i)} \quad P \in (\mathbb{R}^{\mathbb{N}})^{n_o \times n_i} \subset (\mathbb{C}^{\mathbb{N}})^{n_o \times n_i} \tag{7.10}$$

$$\text{(ii)} \quad P(0) = \lim_{|z|\to\infty} \tilde{P}(z) = 0_{n_o \times n_i} . \tag{7.11}$$

Let $\quad (\mathcal{D}_{p\ell}, N_{p\ell})$ be a ρ_0-l.r. of $\tilde{P}$, with $\mathcal{D}_{p\ell} \in \mathcal{R}(\rho_0)^{n_o \times n_o}$. (7.12)

Reference signal sequences $u_s \in (\mathbb{R}^{\mathbb{N}})^{n_o}$ (to be tracked) are generated as follows: for some fixed $\phi_u \in \mathbb{R}[z]$ with $Z[\phi_u] \subset D(1)^c$,

$$\tilde{u}_s = \frac{v_u}{\phi_u} \tag{7.13}$$

where $v_u \in \mathbb{R}[z]^{n_o}$, with $\partial[v_u] \leq \partial[\phi_u]$, is arbitrary.

Disturbance signal sequences $w_p \in (\mathbb{R}^{\mathbb{N}})^{n_i}$ (to be rejected) are generated as follows: for some fixed $\phi_w \in \mathbb{R}[z]$ with $Z[\phi_w] \subset D(1)^c$,

$$\tilde{w}_p = \frac{v_w}{\phi_w} \tag{7.14}$$

where $v_w \in \mathbb{R}[z]^{n_i}$, with $\partial[v_w] \leq \partial[\phi_w]$, is arbitrary.

Define $\phi \in \mathbb{R}[z]$ and $q \in \mathbb{N}$ by

$$\phi := \text{monic l.c.m. of } \phi_u \text{ and } \phi_w \tag{7.16}$$

and

$$q := \partial\phi \,. \tag{7.17}$$

Let ϕ admit ν distinct zeros; let its α^{th} zero be z_α with multiplicity m_α. Then

$$\{z_1, z_2, \ldots, z_\nu\} = Z[\phi_u] \cup Z[\phi_w] \tag{7.18}$$

$$q = \sum_{\alpha=1}^{\nu} m_\alpha \tag{7.19}$$

and

$$z_\alpha \text{ is a zero of order } m_\alpha \text{ of } \phi$$
$$\Leftrightarrow \bar{z}_\alpha \text{ is a zero of order } m_\alpha \text{ of } \phi \,. \tag{7.20}$$

In addition, the maximal order of z_α as a pole of any element of $\tilde{u}_s$ and $\tilde{w}_p$ is m_α.

For tracking and disturbance rejection purposes, we assume for $\tilde{P} \in \tilde{b}(\rho_0)^{n_o \times n_i}$ that

$$n_i \geq n_o \tag{7.21}$$

$$\text{rank}[N_{p\ell}(z)] = n_o \quad \forall z \in Z[\phi_u] \cup Z[\phi_w] \,. \tag{7.22}$$

Remark 7.1. (i) To track n_o signals, (7.21) assures that at least as many plant inputs are available to facilitate the tracking. Furthermore, (7.22) assures that $\tilde{P}$ does not contain any transmission zeros in $Z[\phi_u]$, thus $\tilde{P}$ will not block the control signal required for asymptotic tracking.

(ii) To achieve asymptotic disturbance rejection, the disturbance input w_p has to be either asymptotically cancelled by the controller output y_c or blocked by some transmission zeros (of the plant $\tilde{P}$) that lie in $Z[\phi_w]$. However, since such transmission zeros are not preserved under plant perturbation, we cannot rely on them to achieve input disturbance rejection. □

<u>Stabilization, Tracking and Disturbance Rejection Problem (SP)</u>

Given data (7.10)-(7.22), and a finite list Λ of points in the annulus $\{z|\rho_0 \leq |z| < 1\}$ such that $\lambda \in \Lambda \Leftrightarrow \bar{\lambda} \in \Lambda$. Find a controller $\tilde{C} \in \tilde{b}(\rho_0)^{n_i \times n_o}$, with $C \in (\mathbb{R}^{\mathbb{N}})^{n_i \times n_o} \subset (\mathbb{C}^{\mathbb{N}})^{n_i \times n_o}$ such that for the feedback system S (6.1)-(6.14)

(a) $\tilde{H}_{eu}$ and $\tilde{H}_{yu}$ both are ℓ_p-stable $\forall p \in [1,\infty]$,

(b) the list of zeros of $\tilde{\chi}$ in $D(\rho_0)^c$, $Z[\tilde{\chi};D(\rho_0)^c]$, is exactly Λ;

(c) for any v_u and v_w satisfying (7.13) and (7.14) respectively, the reference signals u_s will be tracked asymptotically and the disturbances w_p will be rejected asymptotically: more specifically, there exists $\rho \in]0,1[$ such that

$$e_s(k) = o(\rho^k) \quad \text{as} \quad k \to \infty \ ;$$

(d) condition (c) holds for any perturbed plant $\tilde{\tilde{P}} \in \tilde{b}(\rho_0)^{n_o \times n_i}$ for which the feedback system S (described in (6.1)-(6.14)) still has $\tilde{H}_{eu}$ and $\tilde{H}_{yu}$ ℓ_p-stable, $\forall p \in [1,\infty]$. □

<u>Remark 7.2</u>. (i) By requiring $\tilde{C}$ to be in $\tilde{b}(\rho_0)^{n_i \times n_o}$, $\tilde{C}$ is bounded at infinity; hence the convolution operator C is causal.

(ii) By the restriction (7.11) $\lim_{|z| \to \infty} \tilde{P}(z) = 0_{n_o \times n_i}$, the well-posedness condition (6.7) is guaranteed.

(iii) By condition (b) of problem (SP) and Theorem 6.1, Λ is the list of (dominant) poles of $\tilde{H}_{eu}$ and $\tilde{H}_{yu}$ in $D(\rho_0)^c$.

(iv) Condition (c) of problem (SP) guarantees that the feedback system S is a servomechanism; furthermore, the system error e_s decays to zero at a rate at least ρ^{-1}.

(v) Condition (d) is a robustness condition that guarantees asymptotic tracking and disturbance rejection under plant perturbation, as long as the feedback system conditions (6.1)-(6.14) are satisfied and $\tilde{H}_{yu}$ and $\tilde{H}_{eu}$ are ℓ_p-stable $\forall p \in [1,\infty]$. □

7.3 Procedure for Controller Design

The problem (SP) is solved by obtaining a controller $\tilde{C}$ with the following procedure:

Procedure 7.1

Data: Plant $\tilde{P}$ with ρ_0-l.r. $(\mathcal{D}_{p\ell},\mathcal{N}_{p\ell})$; the polynomial $\phi \in \mathbb{R}[z]$; the list of dominant closed-loop poles Λ.

Step 1. Pick any $d \in \mathbb{R}[z]$ monic such that

$$\partial d = \partial\phi = q \quad \text{and} \quad d(z) \neq 0 \quad \forall z \in D(\rho_0)^c . \tag{7.23}$$

Comment. (i) A simple choice of d is given by $d(z) := z^q$.

(ii) $\frac{\phi}{d} \in \mathcal{A}^\infty(\rho_0) \cap \mathbb{R}(z) \subset \tilde{\ell}^\infty_{1-}(\rho_0)$; furthermore, ϕ and $\frac{\phi}{d}$ have the same list of zeros.

Step 2. Pick $\mathcal{D} \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_o}$ corresponding to a matrix sequence in $(\mathbb{R}^{\mathbb{N}})^{n_o \times n_o}$ such that

$$\det \mathcal{D} \in \tilde{\ell}^\infty_{1-}(\rho_0) \tag{7.24}$$

and such that the list of zeros of $\det \mathcal{D}$ in $D(\rho_0)^c$ is

$$Z[\det \mathcal{D};\ D(\rho_0)^c] = \Lambda\ . \tag{7.25}$$

<u>Comment</u>. In particular, we can choose $\mathcal{D} \in \mathcal{r}(\rho_0)^{n_o \times n_o}$.

<u>Step 3</u>. Observe that

$$\tilde{F} := \tilde{P}\frac{d}{\phi} \in \tilde{b}(\rho_0)^{n_o \times n_i} \tag{7.26}$$

with a ρ_0-l.r.

$$(\mathcal{D}_\ell, N_\ell) := (\mathcal{D}_{p\ell}\frac{\phi}{d}, N_{p\ell})\ . \tag{7.27}$$

Using Corollary 3.2(ℓ), find the six matrices with elements in $\tilde{\ell}_{1-}(\rho_0)$, corresponding to sequences in $\mathbb{R}^{\mathbb{N}}$, namely

$$U_\ell,\ V_\ell;\ N_r,\ \mathcal{D}_r,\ U_r,\ V_r \tag{7.28}$$

such that

$$\text{(i)}\quad (N_r, \mathcal{D}_r) \ \text{ is a } \rho_0\text{-r.r. of } \tilde{F} \tag{7.29}$$

$$\text{(ii)}\quad \begin{array}{cc} & \begin{array}{cc} n_i & n_o \end{array} \\ \begin{array}{c} n_i \\ n_o \end{array} & \left[\begin{array}{c|c} V_r & U_r \\ \hline -N_\ell & \mathcal{D}_\ell \end{array}\right] \end{array} \begin{array}{c} \begin{array}{cc} n_i & n_o \end{array} \\ \left[\begin{array}{c|c} \mathcal{D}_r & -U_\ell \\ \hline N_r & V_\ell \end{array}\right] \end{array} = \left[\begin{array}{c|c} I_{n_i} & 0 \\ \hline 0 & I_{n_o} \end{array}\right]\ . \tag{7.30}$$

<u>Step 4</u>. Solve, according to Lemma 7.1,

$$N_\ell X + \mathcal{D}_\ell Y = \mathcal{D} \tag{7.31}$$

for X and Y by (i) picking $N \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o}$ corresponding to a sequence in $(\mathbb{R}^{\mathbb{N}})^{n_i \times n_o}$ such that $(N, \mathcal{D})$ are ρ_0-r.c.; and (ii) setting

$$-X := \mathcal{D}_r N - U_\ell \mathcal{D} \tag{7.32}$$

$$Y := N_r N + V_\ell \mathcal{D}\ . \tag{7.33}$$

<u>Comment</u>. By Corollary 3.1b(r) the choice of N in (i) is equivalent to choosing N corresponding to a sequence in $(\mathbb{R}^{\mathbb{N}})^{n_i \times n_o}$ such that

$$\operatorname{rank}\begin{bmatrix} \mathcal{N}(z) \\ - - - \\ \mathcal{D}(z) \end{bmatrix} = n_0 \quad \forall z \in \Lambda \ . \tag{7.34}$$

Furthermore, (X,Y) is a ρ_0-r.r. of $XY^{-1} \in \tilde{b}(\rho_0)^{n_i \times n_o}$ by Lemma 7.1.

Step 5. Set

$$N_{cr} := X \ , \quad \mathcal{D}_{cr} := Y\frac{\phi}{d} \tag{7.35}$$

and

$$\tilde{C} := N_{cr}\mathcal{D}_{cr}^{-1} \ . \tag{7.36}$$

Stop □

Theorem 7.1. The controller $\tilde{C}$ constructed in Procedure 7.1

(i) belongs to $\tilde{b}(\rho_0)^{n_i \times n_o}$ with ρ_0-r.r. $(N_{cr},\mathcal{D}_{cr})$, corresponding to a matrix sequence in $(\mathbb{R}^{\mathbb{N}})^{n_i \times n_o}$;

(ii) solves problem (SP). □

Remark 7.3. It can be observed from the proof of Theorem 7.1 that the controller $\tilde{C}$ constructed in Procedure 7.1 (see (7.35), (7.36)) has created blocking zeros [Fer 1] at every point in $Z[\phi] = Z[\phi_u] \cup Z[\phi_w]$ for the transfer functions $\tilde{H}_{e_s u_s}$ from $\tilde{u}_s$ to $\tilde{e}_s$ and for the transfer functions $\tilde{H}_{e_s w_p}$ from $\tilde{w}_p$ to $\tilde{e}_s$.

7.4 Example

Data. The plant $\tilde{P} \in \tilde{b}(\rho_0)^{1\times 2}$, with $\rho_0 := 0.55$, is given by

$$\tilde{P}(z) := \left[\frac{3}{z-1} + \frac{5}{2z-1} \ \middle| \ \frac{2}{z-2} + \frac{1}{z}e^{1-2z^{-1}} \right] \tag{7.41}$$

which has a ρ_0-l.r. $(\mathcal{D}_{p\ell},N_{p\ell})$ described by

$$\mathcal{D}_{p\ell}(z) := (z-1)(z-2)/z^2 \tag{7.42}$$

$$N_{p\ell}(z) := \left[\frac{(z-2)(11z-8)}{z^2(2z-1)} \ \middle| \ \frac{(z-1)[2z+(z-2)e^{1+2z^{-1}}]}{z^3} \right] \ . \tag{7.43}$$

The polynomial ϕ and the list of dominant closed-loop poles Λ are given by

$$\phi(z) := z+2 \tag{7.44}$$

$$\Lambda := (0.6,-0.6) \; . \tag{7.45}$$

Step 1. Since $q := \partial\phi = 1$, we pick $d \in \mathbb{R}[z]$ as

$$d(z) := z \; . \tag{7.46}$$

Step 2. Choose $\mathcal{D} \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$ to have zeros at $z = 0.6, -0.6,$ as

$$\mathcal{D}(z) := \frac{(z+0.6)(z-0.6)}{z^2} \; . \tag{7.47}$$

Step 3. After defining $\tilde{F} := \tilde{P}\frac{d}{\phi} \in \tilde{b}(\rho_0)^{1\times 2}$, we obtain a ρ_0-l.r. $(\mathcal{D}_\ell, N_\ell)$ for $\tilde{F}$ described by

$$\mathcal{D}_\ell(z) := \mathcal{D}_{p\ell}(z) \cdot \frac{\phi(z)}{d(z)} = \frac{(z-1)(z-2)(z+2)}{z^3} \tag{7.48}$$

and
$$N_\ell(z) := N_{p\ell}(z) \tag{7.49}$$

which is given in (7.43). Next, we find the matrices N_r, $\mathcal{D}_r$, U_ℓ, V_ℓ that satisfy (7.29) and (7.30) (note that we do not need explicit knowledge of U_r and V_r in our computation):

$$N_r(z) := \left[\frac{(3z-2)(z-1)}{6z^2(2z-1)} - \frac{(5z-2)(z-2)}{6z^2(z+2)}(e^{1+2z^{-1}}-1) \quad \frac{(9z^4-2z^3+11z^2-16z+4)}{z^4(2z-1)} + \frac{(7z-2)(z-1)(z-2)}{z^4}(e^{1+2z^{-1}}-1)\right] \tag{7.50}$$

$$\mathcal{D}_r(z) := \begin{bmatrix} \frac{(3z-2)(z-1)}{6z^2} & -\frac{(3z-1)(z-1)(z-2)(z+2)}{z^4} \\ -\frac{(5z-2)(z-2)}{6z^2} & \frac{(7z-2)(z-1)(z-2)(z+2)}{z^4} \end{bmatrix} \tag{7.51}$$

$$\mathcal{U}_{\ell}(z) := \begin{bmatrix} -\dfrac{(3z-1)(z^2+4z-4)}{6z^3} \\ \dfrac{(7z-2)(z^2+4z-4)}{6z^3} \end{bmatrix} \tag{7.52}$$

$$\mathcal{V}_{\ell}(z) := \frac{(12z^4-3z^3+11z^2-16z+4)}{6z^3(2z-1)} - \frac{(7z-2)(z^2+4z-4)}{6z^3(z+2)}(e^{1+2z^{-1}}-1) \; . \tag{7.53}$$

Observe that these matrices are analytic in $D(0.55)^c$, despite some denominator term $(z+2)$.

Step 4. We choose $\mathcal{N} \in \tilde{\ell}_{1-}(\rho_0)^{2\times 1}$ by

$$\mathcal{N}(z) := \begin{bmatrix} \dfrac{(-0.42z^3-3.8z^2-1.08z+0.72)}{z^3} \\ 0 \end{bmatrix} . \tag{7.54}$$

Then we obtain a solution $(\mathcal{X},\mathcal{Y})$ of (7.31) by setting

$$-\mathcal{X}(z) := (\mathcal{D}_r\mathcal{N}-\mathcal{U}_{\ell}\mathcal{D})(z)$$

$$= \begin{bmatrix} \dfrac{(1.74z^2+1.7z-2.16)}{6z^2} \\ -\dfrac{(4.9z+12.04)}{6z} \end{bmatrix} \tag{7.55}$$

$$\mathcal{Y}(z) := (\mathcal{N}_r\mathcal{N}+\mathcal{V}_{\ell}\mathcal{D})(z)$$

$$= \frac{(12z^2-4.26z-2.62)}{6z(2z-1)} - \frac{(4.9z+12.04)}{6z(z+2)}(e^{1+2z^{-1}}-1) \; . \tag{7.56}$$

Step 5. By setting

$$\mathcal{N}_{cr} := \mathcal{X} \tag{7.57}$$

given in (7.55), and

$$\mathcal{D}_{cr}(z) := \mathcal{Y}(z)\frac{\phi(z)}{d(z)} \tag{7.58}$$
$$= \frac{(z+2)(12z^2-4.26z-2.62)}{6z^2(2z-1)} - \frac{(4.9z+12.04)}{6z^2}(e^{1+2z^{-1}}-1)\ ,$$

we obtain a controller $\tilde{C} := N_{cr}\mathcal{D}_{cr}^{-1} \in \tilde{b}(\rho_0)^{2\times 1}$ which has $(N_{cr},\mathcal{D}_{cr})$ as a ρ_0-r.r. Note that this controller $\tilde{C}$ solves problem (SP) with data (7.41)-(7.45); in particular, it is easy to check that $\mathcal{D}_{cr}$ has a zero at $z = -2$ (thus creating a blocking zero for $\tilde{H}_{e_s u_s}$ and $\tilde{H}_{e_s w_p}$ at $z = -2$), and

$$\tilde{\chi}(z) := (\mathcal{D}_{p\ell}\mathcal{D}_{cr} + N_{p\ell}N_{cr})(z) = \frac{(z-0.6)(z+0.6)}{z^2}\ . \qquad \square$$

8. Decoupling Feedback Design with Square Stable Plant

8.1 Preliminary Result and Additional Notations

In this section, we study again the MIMO unity feedback system depicted in Fig. 6-1. However, we now assume that the given plant matrix transfer function $\tilde{P}$ (with elements in $\tilde{b}(\rho_0)$ for some $\rho_0 \in]0,1[$) is square and ℓ_p-stable $\forall p \in [1,\infty]$, i.e.

$$\tilde{P} \in \tilde{b}(\rho_0)^{m\times m} \cap \tilde{\ell}_1^{m\times m} . \tag{8.1}$$

Observe that if the originally given plant does not satisfy these assumptions, we can apply the Stabilization Procedure 7.1 of Section 7 and consider the resulting stable square closed-loop system as our new plant $\tilde{P}$. For such $\tilde{P}$ in Fig. 6-1, we propose a design method such that the transfer function $\tilde{H}_{y_s u_s}$ from $\tilde{u}_s$ to $\tilde{y}_s$ is decoupled, with pole-zero assignment in each channel (subject to the constraint that every $D(1)^c$-zero of $\tilde{P}$ must remain a zero of $\tilde{H}_{y_s u_s}$, cf. continuous-time lumped-system analog in [Che 1]). The approach is based on the recent result obtained by Desoer and Chen [Des 5], which contains a refined stability theorem proposed by Zames [Zam 1].

In order to tie our description to the notations in [Des 5], we note that the algebra $\mathbb{A}$ is here $\tilde{b}(\rho_0)^{m\times m}$, and the radical $\mathbb{A}_s$ of $\mathbb{A}$ is $\tilde{b}_s(\rho_0)^{m\times m}$, where

$$\tilde{b}_s(\rho_0) := \{\tilde{g} \in \tilde{b}(\rho_0) \,\big|\, \lim_{|z|\to\infty} \tilde{g}(z) = g(0) = 0\} . \tag{8.2}$$

Since we consider ℓ_p-stability for all $p \in [1,\infty]$, we take the algebra $\mathbb{B}$ of stable maps to be $\tilde{\ell}_1^{m\times m}$; and $\mathbb{B}_s := \mathbb{A}_s \cap \mathbb{B}$ is hence given by $\tilde{\ell}_{1s}^{m\times m}$, where

$$\tilde{\ell}_{1s} := \{g \in \tilde{\ell}_1 \,|\, \lim_{|z|\to\infty} \tilde{g}(z) = g(0) = 0\} \quad . \tag{8.3}$$

(Note that while $\tilde{b}_s(\rho_0)$ is a radical of $\tilde{b}(\rho_0)^m$, $\tilde{\ell}_{1s}$ is not a radical of $\tilde{\ell}_1$.) The super-ring $\tilde{\mathbb{A}}$ of $\mathbb{A}$ is defined as $(\tilde{\mathbb{C}}_z^{\mathbb{N}})^{m\times m}$.

In the analysis, we need to extend the algebra $\tilde{\mathbb{C}}_z^{\mathbb{N}}$ to the field

$$\tilde{\mathbb{C}}_z^{\mathbb{Z}} := \{\tilde{h} \,|\, \tilde{h}(z) = z^k \tilde{g}(z), \ \tilde{g} \in \tilde{\mathbb{C}}_z^{\mathbb{N}}, \ k \in \mathbb{N}\} \quad . \tag{8.4}$$

Furthermore, we extend the definition of order to $\tilde{\mathbb{C}}_z^{\mathbb{Z}}$: for any nonzero $\tilde{h} \in \tilde{\mathbb{C}}_z^{\mathbb{Z}}$,

$$\mathrm{ord}(\tilde{h}) := k \ \text{ such that } \lim_{|z|\to\infty} z^k \tilde{h}(z) = \text{constant} \neq 0 \ , \tag{8.5}$$

i.e. $\mathrm{ord}(\tilde{h})$ picks out the first nonzero term of $\tilde{h}$, e.g. $\mathrm{ord}(\tilde{h}) = -2$ if $\tilde{h}(z) = h_{-2}z^2 + h_1 z^1 + h_0 + h_1 z^{-1} + \cdots$ with $h_{-2} \neq 0$. In addition, for $\tilde{H} = (\tilde{h}_{ij}) \in (\tilde{\mathbb{C}}_z^{\mathbb{Z}})^{m\times m}$,

$$\mathrm{ord}_{c_j}[\tilde{H}] := \min_i \mathrm{ord}[\tilde{h}_{ij}] \ . \tag{8.6}$$

Let $\rho_0 < 1$ and consider the feedback system of Fig. 6-1 with $\tilde{P} \in \tilde{b}(\rho_0)^{m\times m} \cap \tilde{\ell}_1^{m\times m}$, $\tilde{C} \in (\tilde{\mathbb{C}}_z^{\mathbb{N}})^{m\times m}$, and $\tilde{H}_{yu} \in \tilde{b}(\rho_0)^{2m\times 2m}$ as defined in (6.9) and rewritten here

$$\tilde{H}_{yu} = \left[\begin{array}{c|c} \tilde{C}(I+\tilde{P}\tilde{C})^{-1} & -\tilde{C}\tilde{P}(I+\tilde{C}\tilde{P})^{-1} \\ \hline \tilde{P}\tilde{C}(I+\tilde{P}\tilde{C})^{-1} & \tilde{P}(I+\tilde{C}\tilde{P})^{-1} \end{array}\right] . \tag{8.7}$$

By defining the transfer function from $\tilde{u}_s$ to $\tilde{y}_c$ as

$$\tilde{Q} := \tilde{H}_{y_c u_s} = \tilde{C}(I+\tilde{P}\tilde{C})^{-1} \ , \tag{8.8}$$

$\tilde{H}_{yu}$ in (8.7) can be rewritten as

$$\tilde{H}_{yu} = \left[\begin{array}{c|c} \tilde{Q} & -\tilde{Q}\tilde{P} \\ \hline \tilde{P}\tilde{Q} & \tilde{P}(I-\tilde{Q}\tilde{P}) \end{array}\right] \tag{8.9a}$$

and from (8.8),

$$\tilde{C} = \tilde{Q}(I-\tilde{P}\tilde{Q})^{-1} . \tag{8.9b}$$

We can now state a stability theorem analogous to [Des 5, Thm. 3.4].

Theorem 8.1. Consider the unity-feedback system of Fig. 6-1 with $\rho_0 < 1$, $\tilde{P}, \tilde{Q} \in \tilde{b}(\rho_0)^{m\times m}$, $\tilde{C} \in (\tilde{\mathbb{C}}_z^{\mathbb{N}})^{m\times m}$ and $\tilde{H}_{yu} \in \tilde{b}(\rho_0)^{2m\times 2m}$. Under these conditions,

(i) if

$$\tilde{P} \in \tilde{\ell}_1^{m\times m} , \tag{8.10}$$

then

$$\tilde{Q} \in \tilde{\ell}_1^{m\times m} \Leftrightarrow \tilde{H}_{yu} \in \tilde{\ell}_1^{2m\times 2m} \tag{8.11}$$

and

$$\tilde{Q} \in \tilde{\ell}_{1s}^{m\times m} \Leftrightarrow \tilde{H}_{yu} \in \tilde{\ell}_1^{2m\times 2m} \text{ and } \tilde{C} \in \tilde{b}_s(\rho_0)^{m\times m} ; \tag{8.12}$$

(ii) if

$$\tilde{P} \in \tilde{\ell}_{1s}^{m\times m} , \tag{8.13}$$

then

$$\tilde{Q} \in \tilde{\ell}_1^{m\times m} \Leftrightarrow \tilde{H}_{yu} \in \tilde{\ell}_1^{2m\times 2m} \text{ and } \tilde{C} \in \tilde{b}(\rho_0)^{m\times m} \tag{8.14}$$

and

$$\tilde{Q} \in \tilde{\ell}_{1s}^{m\times m} \Leftrightarrow \tilde{H}_{yu} \in \tilde{\ell}_{1s}^{2m\times 2m} \text{ and } \tilde{C} \in \tilde{b}_s(\rho_0)^{m\times m} . \tag{8.15}$$

□

Remark 8.1. (i) Note that for $\tilde{H}_{yu}$ to have elements in $\tilde{\mathbb{C}}_z^{\mathbb{N}}$, the transfer function $\tilde{C} \in (\tilde{\mathbb{C}}_z^{\mathbb{N}})^{m\times m}$ has to satisfy

$$\det[I + P(0)C(0)] \neq 0 .$$

(ii) Based on the equivalence condition (8.14), we propose a design procedure to achieve decoupling and pole-zero assignment of the feedback system. Note that had $\tilde{P}$ been the closed-loop system obtained through the stabilizing compensation of Section 7, it would satisfy condition (8.13) in view of assumption (7.11). (Note that the $\tilde{P}$ here and the $\tilde{P}$ in (7.11) are different.) Now, by (8.14), we have the following design capability.

Theorem 8.2. Suppose that we wish to design a unity-feedback system as shown in Fig. 6-1 with $\tilde{P} \in \tilde{b}(\rho_0)^{m\times m} \cap \tilde{\ell}_{1s}^{m\times m}$, $\tilde{C} \in (\tilde{\mathbb{C}}_z^{\mathbb{N}})^{m\times m}$ and $\tilde{H}_{yu} \in \tilde{b}(\rho_0)^{2m\times 2m}$. Then, for all $\tilde{H}_{y_s u_s} \in \tilde{\ell}_1^{m\times m}$ such that $\tilde{H}_{y_s u_s} = \tilde{P}\tilde{Q}$ for some $\tilde{Q} \in \tilde{\ell}_1^{m\times m}$, there exists a $\tilde{C} \in \tilde{b}(\rho_0)^{m\times m}$ for which

(i) the closed-loop system is ℓ_p-stable, $\forall p \in [1,\infty]$, and

(ii) the transfer function from u_s to y_s is described by the specific $\tilde{H}_{y_s u_s}$. □

8.2 Procedure for Decoupling Feedback Design

Decoupling Problem (DP). Given a plant $\tilde{P} \in \tilde{b}(\rho_0)^{m\times m} \cap \tilde{\ell}_{1s}^{m\times m} \cap (\tilde{\mathbb{R}}_z^{\mathbb{N}})^{m\times m}$ such that $\det \tilde{P} \not\equiv 0$ in $D(\rho_0)^c$, find a controller $\tilde{C} \in \tilde{b}(\rho_0)^{m\times m} \cap (\tilde{\mathbb{R}}_z^{\mathbb{N}})^{m\times m}$ such that

(i) the closed-loop unity feedback system in Fig. 6-1 is ℓ_p-stable, $\forall p \in [1,\infty]$;

(ii) the transfer function $\tilde{H}_{y_s u_s}$ representing the I/O map from u_s to y_s is a decoupled, proper rational function matrix;

(iii) in each diagonal element of $\tilde{H}_{y_s u_s}$, the poles and zeros (in addition to the $D(1)^c$-zeros imposed by $\tilde{P}$, see [Che 1]) can be specified by the designer. □

Procedure 8.1: Decoupling Feedback Design

Data. Plant $\tilde{P} \in \tilde{b}(\rho_0)^{m\times m} \cap \tilde{\ell}_{1s}^{m\times m} \cap (\tilde{\mathbb{R}}^{\mathbb{N}})^{m\times m}$, $\det \tilde{P} \equiv 0$ in $D(\rho_0)^c$.

Step 1. Obtain a ρ_0-r.r. of $\tilde{P}$

$$\tilde{P} = N_{pr} \mathcal{D}_{pr}^{-1} \tag{8.16}$$

where $N_{pr}, \mathcal{D}_{pr} \in \tilde{\ell}_{1-}(\rho_0)^{m\times m}$.

Step 2. Calculate

$$[\gamma_{ij}]_{m\times m} := N_{pr}^{-1} \in (\tilde{\mathbb{C}}_z^{\mathbb{Z}})^{m\times m} \,. \tag{8.17}$$

Step 3. For $j = 1,2,\ldots,m$, choose a polynomial $\hat{n}_j \in \mathbb{R}[z]$ of least degree such that for $i = 1,2,\ldots,m$,

$$\gamma_{ij}(\cdot)\hat{n}_j(\cdot) \in \tilde{\mathbb{C}}_z^{\mathbb{Z}} \tag{8.18}$$

is analytic in $D(1)^c$. (Comment: If $\tilde{P}$ has no $D(1)^c$-zeros, then we can pick $\hat{n}_j \equiv 1$, $\forall j$.)

Step 4. Choose polynomials n_j, $d_j \in \mathbb{R}[z]$, $j = 1,2,\ldots,m$, in

$$\tilde{H}_{y_s u_s} := \mathrm{diag}\left[\frac{\hat{n}_1 n_1}{d_1}, \frac{\hat{n}_2 n_2}{d_2}, \ldots, \frac{\hat{n}_m n_m}{d_m}\right] \tag{8.19}$$

such that for $j = 1,2,\ldots,m$,

(i) $Z[d_j] \subset D(1)$ $\quad$ (8.20)

(ii) the polynomial n_j can be chosen freely,

(iii) $\partial[d_j] \geq \partial[n_j] + \partial[\hat{n}_j] - \mathrm{ord}_{c_j}[\tilde{P}^{-1}]$. $\quad$ (8.21)

Step 5. Calculate the controller

$$\tilde{C} := \mathcal{D}_{pr} N_{pr}^{-1} \mathrm{diag}\left[\frac{\hat{n}_1 n_1}{d_1 - \hat{n}_1 n_1}, \frac{\hat{n}_2 n_2}{d_2 - \hat{n}_2 n_2}, \ldots, \frac{\hat{n}_m n_m}{d_m - \hat{n}_m n_m}\right] \,. \tag{8.22}$$

STOP. $\square$

Theorem 8.3. The controller $\tilde{C}$ in (8.22) solves Problem (DP). $\square$

Remark 8.2. (i) Equation (8.22) shows that a "stable" controller is always possible: indeed, after the polynomials $\hat{n}_j$ and n_j have been chosen, the polynomial d_j can always be found so that all zeros of $d_n - \hat{n}_j n_j$ lie inside $D(1)$.

(ii) Observe that it is not required that $\tilde{H}_{yu}$ of (8.7) be rational, but $\tilde{H}_{y_s u_s}$ can be made to be rational as in (8.19).

(iii) This procedure is not a direct application of [Des 5, Alg. 4.2], because we are not restricting n_j, $\hat{n}_j$ and d_j to $\tilde{\ell}_{1-}(\rho_0)$ or $\mathcal{R}(\rho_0)$; instead, we choose n_j, $\hat{n}_j$ and d_j to be polynomials and we are only restricting $\hat{n}_j n_j / d_j$ to belong to $\mathcal{R}(1)$ (Note: it is easier to work with polynomial n_j, $\hat{n}_j$ and d_j.). A direct modification of [Des 5, Alg. 4.2] can be obtained by letting $\hat{n}_j$, $n_j \in \mathcal{R}(\rho_0)$ and $d_j \in \mathcal{R}^{\infty}(\rho_0)$, and by replacing (8.21) by

$$\operatorname{ord}(\hat{n}_j n_j) \geq -\operatorname{ord}_{c_j}[\tilde{P}^{-1}] \geq 0 \ . \tag{8.23}$$

(iv) Since we are working with d_j in $\mathbb{R}[z]$ instead of in $\mathcal{R}^{\infty}(\rho_0)$, zeros of d_j need not be restricted to $D(\rho_0)^c$. In particular, if we put all these zeros at $z = 0$, then $\tilde{H}_{y_s u_s}$ is a transfer function corresponding to an I/O map with finite settling time.

(v) Let $[\pi_{ij}]_{m\times m} := \tilde{P}^{-1} = \mathcal{D}_{p\mathcal{R}} \mathcal{N}_{p\mathcal{R}}^{-1} \in (\tilde{\mathbb{C}}_z^{\mathbb{Z}})^{m\times m}$. Since $\tilde{P} \in \tilde{\ell}_1^{m\times m}$, hence $\det \mathcal{D}_{p\mathcal{R}}(z) \neq 0$, $\forall z \in D(1)^c$; thus the term in (8.18), $\gamma_{ij}(\cdot)\hat{n}_j(\cdot)$, is analytic in $D(1)^c$ if and only if $\pi_{ij}(\cdot)\hat{n}_j(\cdot)$ is analytic in $D(1)^c$. □

9. Concluding Discussion

In view of the need for a general theory to cover sampled-data systems obtained by sampling continuous-time linear time-invariant distributed systems, we have developed in Section 2 the algebra $b(\rho_0)$ which described a large class of such discrete-time systems. In contrast to the continuous-time distributed case which is plagued by difficult fine points of analysis, the discrete-time case can be treated by more straightforward methods: in particular, for any $\tilde{g} \in \tilde{b}(\rho_0)$, there is some $\rho \geq \rho_0$ such that $\tilde{g}$ is analytic and bounded in $|z| \geq \rho$, moreover $\tilde{g}(z)$ has a well-defined limit as $|z| \to \infty$. Such nice behavior at infinity is usually absent in transfer functions of continuous-time distributed systems (consider $\hat{g}(s) = e^{-sT}$). Consequently, this paper is essentially self-contained.

The model of system description in this paper, with transfer functions in $\tilde{b}(\rho_0)$, is far more general than the model with rational transfer functions (as demonstrated by the examples in Section 2.6): indeed, the algebra $\tilde{b}(\rho_0)$ includes, as a subalgebra, all the proper rational functions in z.

By generalizing in Section 3 the concept of matrix fraction representation to systems with $\tilde{b}(\rho_0)$-matrix transfer functions, we studied the dynamic interpretations of poles and transmission zeros for MIMO systems in Section 4. As in the rational case, each pole of $\tilde{b}(\rho_0)$-transfer functions can be activated individually by some appropriately chosen input signals (see Thm. 4.4). In contrast to the transmission zeros of the rational transfer function case, a transmission zero of a $\tilde{b}(\rho_0)$-transfer function cannot completely block out the corresponding exponential input-signal, but it can make the output asymptotically "small" compared to the blocked exponential; hence transmission zeros of $\tilde{b}(\rho_0)$-transfer functions still pose the same kind of nuisance on the tracking problem as those in the rational transfer function case (see e.g. [Des 4], [McF 1], [Dav 1]). Note that transmisssion zeros of $\tilde{b}(\rho_0)$-transfer functions also impose other limitations on the design of feedback systems, parallel to the rational case: Consider a feedback system with unity feedback, some zeros of the closed-loop characteristic function approach the open-loop transmission zeros under high gain; hence high loop gains lead to instability when there are $D(1)^c$-zeros in the plant transfer function (see discussions in [McF 1], [Dav 1] about similar behavior for continuous-time rational case). In addition, for a rather general feedback system defined as in Fig. 9-1, if $z \in D(1)^c$ is a transmission zero of the plant transfer function $\tilde{P}$, then under reasonable assumptions, for <u>any</u> controller transfer function $\tilde{C}$ and <u>any</u> feedback transfer function $\tilde{F}$ such that the closed-loop system is ℓ_p-stable $\forall p \in [1,\infty]$ (as defined in Section 5), the closed-loop transfer function $\tilde{H}_{yu}$ from input $\tilde{u}$ to output $\tilde{y}$ will have a transmission zero at z; thus the $D(1)^c$-transmission zeros of $\tilde{P}$ impose some fundamental limitations on the <u>achievable closed-loop transfer function</u> $\tilde{H}_{yu}$ (see [Che 1]). However, even though these transmission zeros cannot be removed by appropriate compensation, sometimes they can be relocated by judicious redesign of the actuators and/or sensors of the physical system [McF 1].

As for the analysis of interconnected systems using the notion of characteristic functions as described in Section 5, it is stressed that this method of analysis can be applied to any interconnection, as long as the well-posedness condition (i.e. $\lim_{|z|\to\infty} \det \mathcal{D}(z) \neq 0$) is satisfied.

Considering MIMO feedback systems, we studied in Section 6 the problem of closed-loop stability and in Section 7 the problem of designing a robust controller to achieve stabilization, tracking, and disturbance rejection. However, we have yet to investigate the possibility of designing controllers with proper rational transfer functions that can satisfy the same or relaxed specifications. We stress that if the design procedure 7.1 is applied to systems with rational transfer functions, then the controller is guaranteed to be proper, and arbitrary "dominant" closed-loop eigenvalue assignment is achieved.

When the given plant is square and stable, we have in Section 8 a procedure to design a feedback system so that the transfer function from the reference input to the plant output is decoupled (or, equally practicable, assigned a specific structure to satisfy other specifications), with arbitrary pole and zero assignment outside $D(\rho_0)$ (subject to, of course, the $D(1)^c$-zeros of the plant).

Hence, by combining the results of Sections 7 and 8, we conclude that, given any plant $\tilde{P} \in \tilde{b}(\rho_0)^{n_o \times n_i}$ that satisfies certain reasonable assumptions, we can design a feedback system with an inner loop to stabilize the plant (as in Section 7), and an outer loop to bring the overall system to satisfy certain specifications, e.g. decoupling (as in Section 8). We are of course aware that there are many important issues in control system design that are not addressed by the above methods.

Finally, we should point out that most of the results in this paper also apply to the continuous-time and lumped cases, by observing the similar algebraic structures of the different cases.

Appendix A: Proofs of Properties of $\ell_1(\rho_0)$, $\ell_{1-}(\rho_0)$

Proof of (2.2.1). $\ell_1(\rho_0)$ is defined by (2.4): it follows that $\ell_1(\rho_0)$ is a normed space over the field $\mathbb{C}$ with the usual definitions of addition for sequences, multiplication by scalars in $\mathbb{C}$, and a norm $\|\cdot\|_{\rho_0}: \ell_1(\rho_0) \to \mathbb{R}_+$ defined by

$$\|g\|_{\rho_0} := \sum_{k=0}^{\infty} |g(k)\rho_0^{-k}| \quad . \tag{A.1}$$

By definition, $g = (g(k))_{k=0}^{\infty} \in (\ell_1(\rho_0), \|\cdot\|_{\rho_0})$ if and only if $\gamma = (\gamma(k))_{k=0}^{\infty} \in (\ell_1, |\cdot|_1)$, where $\gamma(k) := g(k)\rho_0^{-k}$ $\forall k \in \mathbb{N}$, and $|\gamma|_1 := \|\gamma\|_{\rho_0=1} = \sum_{k=0}^{\infty} |\gamma(k)|$ is the usual norm defined on ℓ_1. This defines an isomorphism of $\ell_1(\rho_0)$ onto ℓ_1 with $\|g\|_{\rho_0} = |\gamma|_1$. Hence $(\ell_1(\rho_0), \|\cdot\|_{\rho_0})$ is a Banach space, since $(\ell_1, |\cdot|_1)$ is a Banach space [Die 2, Thm. 13.11.4 (using the counting measure)]. $\ell_1(\rho_0)$ also forms a commutative ring, with a "multiplication" in $\ell_1(\rho_0)$ defined as the convolution, namely,

$$f*g := \left(\sum_{j=0}^{k} f(k-j)g(j)\right)_{k=0}^{\infty} \quad \text{for } f, g \in \ell_1(\rho_0) \; . \tag{A.2}$$

Furthermore, the convolution satisfies the inequality

$$\begin{aligned} \|f*g\|_{\rho_0} &= \sum_{k=0}^{\infty} \left| \left(\sum_{j=0}^{k} f(k-j)g(j) \right) \rho_0^{-k} \right| \\ &\leq \sum_{k=0}^{\infty} \sum_{j=0}^{k} |f(k-j)\rho_0^{-(k-j)}| \, |g(j)\rho_0^{-j}| \\ &= \|f\|_{\rho_0} \|g\|_{\rho_0} \end{aligned} \tag{A.3}$$

where the last equality follows from [Apo 1, Thm. 8.4.6]. Note that $\delta_0 := (1,0,0,\ldots)$ is the neutral element of $\ell_1(\rho_0)$ under convolution:

indeed

$$\delta_0 * g = g * \delta_0 = g \quad \forall g \in \ell_1(\rho_0) \ , \tag{A.4}$$

and $\|\delta_0\|_{\rho_0} = 1$. Hence $\ell_1(\rho_0)$ is a commutative convolution (A.5) complex Banach algebra with unit [Rud 2, pp.227-228]. □

<u>Proof of (2.2.2)</u>. Follows immediately from (2.4) and the equivalence $\rho_1 < \rho_0 \Leftrightarrow \rho_1^{-k} > \rho_0^{-k}$, $\forall k \in \mathbb{N}^*$. □

<u>Proof of (2.2.3)</u>. Given any two nonzero elements $f = (f(m))_{m=0}^{\infty}$ and $g = (g(n))_{n=0}^{\infty}$ in $\mathbb{C}^{\mathbb{N}}$, let m_0 and n_0 be the least indices corresponding to a nonzero component of f and g respectively, then $h := f * g$ is nonzero because $h(m_0+n_0) = f(m_0)\cdot g(n_0) \neq 0$. □

<u>Proof of (2.2.4)</u>. (i) By assumption, $g \in \ell_1(\rho_0)$, then for $|z| \geq \rho_0$

$$|\tilde{g}(z)| = \left|\sum_{k=0}^{\infty} g(k) z^{-k}\right| \leq \sum_{k=0}^{\infty} |g(k)|\,|z|^{-k} \leq \sum_{k=0}^{\infty} |g(k)|\rho_0^{-k} = \|g\|_{\rho_0}$$

i.e. in $|z| \geq \rho_0$, the series defining $\tilde{g}(z)$ converges <u>absolutely</u> and is bounded by $\|g\|_{\rho_0}$.

(ii) For any $\varepsilon > 0$, the series defining $\tilde{g}(z)$ converges uniformly in $|z| \geq \rho_0 + \varepsilon$, hence $\tilde{g}(z)$ represents an analytic function in $D(\rho_0+\varepsilon)^c$.

(iii) Consider the definition of $\tilde{g}(z)$, as $|z| \to \infty$, $\tilde{g}(z) \to g(0)$. □

In order to prove Property (2.2.6) we need the concept of complex homomorphism. Let Δ denote the set of all complex homomorphisms mapping the Banach algebra $\ell_1(\rho_0)$ into $\mathbb{C}$ [Rud 1, Ch. 9][Rud 2, Ch. 11].

The following lemma characterizes Δ:

Lemma A.1. For $\phi: \ell_1(\rho_0) \to \mathbb{C}$,

$$\phi \in \Delta \Leftrightarrow \text{Either (a) } \phi(f) = f(0), \quad \forall f \in \ell_1(\rho_0) \tag{A.6}$$

$$\text{or (b) } \exists z \in D(\rho_0)^c \text{ such that}$$

$$\phi(f) = \tilde{f}(z), \quad \forall f \in \ell_1(\rho_0) . \tag{A.7}$$

□

Proof: ($\Leftarrow$) By definition, $\phi: \ell_1(\rho_0) \to \mathbb{C}$ belongs to Δ if and only if $\phi(f*g) = \phi(f)\phi(g)$, $\phi(\alpha f+\beta g) = \alpha\phi(f) + \beta\phi(g)$, $\forall f,g \in \ell_1(\rho_0)$, $\forall \alpha,\beta \in \mathbb{C}$. By direct calculation, these requirements are satisfied for any ϕ specified by (A.6) or (A.7).

($\Rightarrow$) Let $\phi_0: f \mapsto f(0)$ be the complex homomorphism defined by (A.6). Let $\Delta_1 := \Delta \backslash \{\phi_0\}$. According to the definitions of δ_1 and δ_k,

$$\delta_k = (\delta_1 *)^k = \delta_1 * \delta_1 * \cdots * \delta_1 , \quad k \geq 1 .$$

Hence, for any $\phi \in \Delta_1$,

$$\phi(\delta_k) = [\phi(\delta_1)]^k \quad \forall k \in \mathbb{N} . \tag{A.8}$$

Now for any homomorphism ϕ, $|\phi(g)| \leq \|g\|$ [Rud 1, Thm. 9.21], in particular

$$|\phi(\delta_1)| \leq \|\delta_1\|_{\rho_0} = \rho_0^{-1} .$$

Hence, for any $\phi \in \Delta_1$, there exists $z \in \mathbb{C}$ with $|z| \geq \rho_0$ such that

$$z^{-1} = \phi(\delta_1) . \tag{A.9}$$

Now $\forall \phi \in \Delta_1$ and $\forall g \in \ell_1(\rho_0)$, we obtain successfully

$$\phi(g) = \phi\left(\sum_{k=0}^{\infty} g(k)\delta_k\right)$$

$$= \sum_{k=0}^{\infty} g(k)\phi(\delta_k) \quad \text{by linearity of homomorphism } \phi$$

$$= \sum_{k=0}^{\infty} g(k)[\phi(\delta_1)]^k \quad \text{by (A.8)} . \qquad \text{(A.10)}$$

By using (A.9) in (A.10), we conclude that there exists $z \in D(\rho_0)^c$ such that

$$\phi(g) = \sum_{k=0}^{\infty} g(k)z^{-k} = \tilde{g}(z) . \qquad \square$$

Proof of (2.2.6). (2.7) $\Leftrightarrow$ (2.8). This is obvious by noting that $g(0) = \lim_{|z|\to\infty} \tilde{g}(z)$.

(2.6) $\Leftrightarrow$ (2.8). By [Rud 2, Thm. 11.5(c)], $g \in \ell_1(\rho_0)$ is invertible in $\ell_1(\rho_0)$ iff $\phi(g) \neq 0$ $\forall\phi \in \Delta$. The result follows immediately in view of Lemma A.1. $\square$

Proof of (2.2.7). (2.10)$\Rightarrow$(2.11). By contraposition, suppose $\inf_{|z|\geq\rho_0} |(\tilde{f}(z),\tilde{g}(z))| = 0$. Then there exists a sequence $(z_k)_{k=0}^{\infty}$ in $D(\rho_0)^c$ such that

$$\lim_{k\to\infty}|(\tilde{f}(z_k),\tilde{g}(z_k))| = 0 .$$

Hence, $\forall u,v \in \ell_1(\rho_0)$, which are necessarily bounded in $D(\rho_0)^c$,

$$\lim_{k\to\infty}(\tilde{u}\tilde{f} + \tilde{v}\tilde{g})(z_k) = 0 .$$

Then (2.10) cannot hold.

(2.11)$\Leftrightarrow$(2.12). This is obvious by noting that $(f(0),g(0)) = \lim_{|z|\to\infty} (\tilde{f}(z),\tilde{g}(z))$.

(2.12)$\Rightarrow$(2.10). Consider the ideal in $\ell_1(\rho_0)$ generated by f and g

$$I := \{h \mid h = u*f + v*g,\ u,v \in \ell_1(\rho_0)\} .$$

Either (i) $I = \ell_1(\rho_0)$, or (ii) $I \subsetneq \ell_1(\rho_0)$. If (i) holds, $\delta_0 \in I$ and we obtain (2.10). Otherwise, (ii) I is a proper ideal of $\ell_1(\rho_0)$, then by [Rud 2, Thm. 11.3(a)] I is contained in some maximal ideal $M \subsetneq \ell_1(\rho_0)$. By [Rud 2, Thm. 11.5(a)], there exists $\phi_M \in \Delta$ such that $M = \phi_M^{-1}(0)$. Hence $I \subset \phi_M^{-1}(0)$, i.e.

$$\phi_M(h) = 0 \quad \forall h \in I .$$

By Lemma A.1, <u>either</u>

(a) $\phi_M = \phi_0$: $f \mapsto f(0)$, then $f, g \in I$ implies

$$\phi_M(f) = f(0) = 0 \quad \text{and} \quad \phi_M(g) = g(0) = 0$$

and so $|(f(0),g(0))| = 0$ which contradicts (2.12)(i);

<u>or</u> (b) if $\phi_M \neq \phi_0$, then there exists $z' \in D(\rho_0)^c$ (where z' is specified by ϕ_M) such that

$$\phi_M(h) = \tilde{h}(z') \quad \forall h \in \ell_1(\rho_0) .$$

With this particular z', since $f, g \in I$,

$$\phi_M(f) = \tilde{f}(z') = 0 \quad \text{and} \quad \phi_M(g) = \tilde{g}(z') = 0$$

and so $|(\tilde{f}(z'),\tilde{g}(z'))| = 0$ which contradicts (2.12)(ii). Thus we conclude that $I = \ell_1(\rho_0)$ must hold. $\square$

<u>Proof of (2.3.2)</u>. (i) and (ii) hold because $\tilde{g} \in \tilde{\ell}_1(\rho_g)$ for some $\rho_g < \rho_0$.

(iii) Let $\alpha \in \mathbb{N}$ be the least index corresponding to a nonzero component of g. Then $g(\alpha) \neq 0$ and

$$\tilde{g}(z) = \sum_{k=\alpha}^{\infty} g(k)z^{-k} = g(\alpha)z^{-\alpha}\left[1 + \sum_{k=0}^{\infty} \frac{g(\alpha+1+k)}{g(\alpha)} z^{-(k+1)}\right] . \tag{A.11}$$

Since $g \in \ell_{1-}(\rho_0)$, hence $\left(\frac{g(\alpha+1+k)}{g(\alpha)}\right)_{k=0}^{\infty} \in \ell_{1-}(\rho_0)$; thus there exists $\rho \geq \rho_0$ such that

$$\left| \sum_{k=0}^{\infty} \frac{g(\alpha+1+k)}{g(\alpha)} z^{-(k+1)} \right| \leq |z|^{-1} \cdot \sum_{k=0}^{\infty} \left|\frac{g(\alpha+1+k)}{g(\alpha)}\right| |z|^{-k} < 1 \quad \forall |z| \geq \rho .$$

Hence by (A.11), $\tilde{g}(z) \neq 0 \;\; \forall |z| \geq \rho$; and so the zeros of $\tilde{g}(\cdot)$ in $D(\rho_0)^c$ are all inside the <u>compact</u> annulus $\{z | \rho_0 \leq |z| \leq \rho\}$. Since the zeros of $\tilde{g}(\cdot)$ are isolated in the region of analyticity, $\tilde{g}(\cdot)$ can have at most a finite number of zeros in the compact set $\{z | \rho_0 \leq |z| \leq \rho\}$, hence a finite number of zeros in $D(\rho_0)^c$. □

Before we prove Property (2.3.3), we consider next:

<u>Lemma A.2</u>. Let $0 \leq \rho_1 < \rho_0$, and let $f: D(\rho_1)^c \to \mathbb{R}$ be continuous at every point of $S := \{z \,\big|\, |z| = \rho_0\}$. If $f(z) > 0 \;\; \forall |z| = \rho_0$, then $\exists \rho_2 \in \,]\rho_1, \rho_0[$ such that

$$f(z) > 0 \quad \forall |z| \in [\rho_2, \rho_0] .$$ □

<u>Proof</u>. For the sake of contradiction, suppose that given any $\rho_2 \in \,]\rho_1, \rho_0[$, $\exists |z| \in [\rho_2, \rho_0]$ such that $f(z) \leq 0$. Hence we can construct a sequence $(z_k)_{k=1}^{\infty}$ with $|z_k| \in \,]\rho_1, \rho_0]$ such that $|z_k| \to \rho_0$ and $f(z_k) \leq 0$, $k = 1,2,\ldots$. By compactness of the closed ball $\overline{D(\rho_0)}$, $(z_k)_{k=1}^{\infty}$ must have a convergent subsequence specified by some index set K, i.e. $\exists K \subset \mathbb{N}$ such that $z_k \xrightarrow{K} z^*$ for some

$z^* \in \overline{D(\rho_0)}$. Furthermore, $|z^*| = \lim_{\substack{k\to\infty \\ k\in K}} |z_k| = \rho_0 \Rightarrow z^* \in S$. Hence $f(z_k) \leq 0$, $k \in K \Rightarrow f(z^*) \leq 0$, because f is continuous at $z^* \in S$; and this contradicts the hypothesis. □

Proof of (2.3.3). (2.16)$\Leftrightarrow$(2.17). This is obvious by noting that $g(0) = \lim_{|z|\to\infty} \tilde{g}(z)$.

(2.15)$\Rightarrow$(2.16). $g \in \ell_{1-}(\rho_0)$ implies that $g \in \ell_1(\rho_1)$ for some $\rho_1 < \rho_0$. It has an inverse in $\ell_{1-}(\rho_0)$ implies that for some $\rho_2 < \rho_0$, $\exists g^{-1} \in \ell_1(\rho_2)$ with $g * g^{-1} = g^{-1} * g = \delta_0$. Hence, $g \in \ell_1(\rho_3)$ has an inverse in $\ell_1(\rho_3)$, where $\rho_3 := \max(\rho_1, \rho_2) < \rho_0$. By Inversion Theorem (2.2.6), $\inf_{|z|\geq\rho_3} |\tilde{g}(z)| > 0$. Hence,

$$\inf_{|z|\geq\rho_0} |\tilde{g}(z)| \geq \inf_{|z|\geq\rho_3} |\tilde{g}(z)| > 0 .$$

(2.15)$\Leftarrow$(2.16). $g \in \ell_{1-}(\rho_0)$ implies that $\exists \rho_1 < \rho_0$ such that $g \in \ell_1(\rho_1)$. Hence the map $z \mapsto |\tilde{g}(z)|$ is defined for $|z| \geq \rho_1$ and is continuous at every point of $S := \{z \mid |z| = \rho_0\}$. By Lemma A.2, $|\tilde{g}(z)| > 0$ on S implies that $\exists \rho_2 \in]\rho_1, \rho_0[$ such that $|\tilde{g}(z)| > 0$ $\forall |z| \in [\rho_2, \rho_0]$. Hence $\inf_{|z|\geq\rho_2} |\tilde{g}(z)| > 0$. By Inversion Theorem (2.2.6), g has an inverse in $\ell_1(\rho_2)$, hence in $\ell_{1-}(\rho_0)$. □

The next theorem is needed in some subsequent proofs.

Theorem A.1: Decomposition Theorem. Let $\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$, and let $p \in D(\rho_0)^c$. Under these conditions, $\exists \rho_1, \rho_2$ satisfying $0 \leq \rho_2 < \rho_1 < \rho_0$ and such that

$$\frac{\tilde{g}(z)}{z-p} = \frac{\tilde{g}(p)}{z-p} + \tilde{\xi}(z) \quad \forall z \in D(\rho_2)^c \setminus \{p\} \qquad \text{(A.15)}$$

where both $\tilde{\xi}$ and $z \mapsto z\tilde{\xi}(z)$ belong to $\tilde{\ell}_{1-}(\rho_1)$. □

Proof. With $\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$, $\exists \rho_2 < \rho_0$ such that $\tilde{g} \in \tilde{\ell}_1(\rho_2)$ and hence $\tilde{g}$ is analytic in $|z| > \rho_2$ and bounded in $|z| \geq \rho_2$. Since $\tilde{g}$ is analytic at p with $|p| \geq \rho_0 > \rho_2$, and since $\tilde{g}(z) - \tilde{g}(p)$ has a zero at p, we see that

$$\tilde{\zeta}(z) := \frac{\tilde{g}(z)-\tilde{g}(p)}{z-p}$$

is analytic at p, hence it is analytic for $|z| > \rho_2$; note that $\tilde{\zeta}(z) \to 0$ as $|z| \to \infty$. Hence, by Laurent expansion, we obtain

$$\tilde{\zeta}(z) = \sum_{k=0}^{\infty} \zeta_k z^{-k} \quad \text{with} \quad \zeta(0) := \lim_{|z|\to\infty} \tilde{\zeta}(z) = 0$$

which converges absolutely for $|z| > \rho_2$. Hence, $\forall \rho_1 \in]\rho_2, \rho_0]$, $\sum_{k=0}^{\infty} |\zeta(k)| \rho_1^{-k} < \infty$, i.e. $\tilde{\zeta} \in \tilde{\ell}_{1-}(\rho_1)$ for any $\rho_1 \in]\rho_2, \rho_0[$. Since $\zeta(0) = 0$, we see that $z \mapsto z\tilde{\zeta}(z) \in \tilde{\ell}_{1-}(\rho_1)$ for the same ρ_1. □

Remark. (i) The decomposition theorem (A.15) expresses $\tilde{g}(z)/(z-p)$ as the sum of the principal part $\tilde{g}(p)/(z-p)$ of the Laurent expansion at p of $\tilde{g}(z)/(z-p)$, and of the remainder term $\tilde{\zeta} \in \tilde{\ell}_{1-}(\rho_1)$. Through repeated use of the decomposition theorem, for any $\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$, any $p \in D(\rho_0)^c$, and any $m \in \mathbb{N}^*$, the same conclusion holds for $\tilde{g}(z)/(z-p)^m$, giving

$$\frac{\tilde{g}(z)}{(z-p)^m} = \sum_{k=1}^{m} \frac{r_k}{(z-p)^k} + \tilde{\zeta}(z) \qquad \text{(A.16)}$$

where $\tilde{\zeta} \in \tilde{\ell}_{1-}(\rho_1)$ for some $\rho_1 < \rho_0$, and $r_k \in \mathbb{C}$, $k = 1,2,\ldots,m$. Repeated application of this last result (A.16) proves the following: For any $\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$, $\nu \in \mathbb{N}^*$, and any $p_\alpha \in D(\rho_0)^c$, $m_\alpha \in \mathbb{N}^*$, $\alpha = 1,2,\ldots,\nu$, $\tilde{g}(z)/\prod_{\alpha=1}^{\nu}(z-p_\alpha)^{m_\alpha}$ can be expressed as the sum of the principal parts of the Laurent expansions of $\tilde{g}(z)/\prod_{\alpha=1}^{\nu}(z-p_\alpha)^{m_\alpha}$ at the p_α's, and of a remainder term $\tilde{\zeta}$, i.e.

$$\frac{\tilde{g}(z)}{\prod_{\alpha=1}^{\nu}(z-p_\alpha)^{m_\alpha}} = \sum_{\alpha=1}^{\nu}\left(\sum_{k=1}^{m_\alpha}\frac{r_{\alpha k}}{(z-p_\alpha)^k}\right)+\tilde{\zeta}(z)\ , \tag{A.17}$$

where $\tilde{\zeta}\in\tilde{\ell}_{1-}(\rho_1)$ for some $\rho_1<\rho_0$, and $r_{\alpha k}\in\mathbb{C}$, $\alpha=1,2,\ldots,\nu$, $k=1,2,\ldots,m_\alpha$.

(ii) If, in (A.15), $\tilde{g}(p)=0$, then $\tilde{f}:=\tilde{g}(z)/(z-p)\in\tilde{\ell}_{1-}(\rho_1)$. Here $f(0)=\lim_{|z|\to\infty}\tilde{f}(z)=0$. We can easily verify that, $\forall a\in D(\rho_0)$,

$$\tilde{f}(z)(z-a) = \tilde{g}(z)(z-a)/(z-p)\in\tilde{\ell}_{1-}(\rho_1)\subset\tilde{\ell}_{1-}(\rho_0)\ . \tag{A.18}$$

Thus $\tilde{g}(z)$ can be expressed as a product of factors in $\tilde{\ell}_{1-}(\rho_0)$ given by

$$\tilde{g}(z) = [(z-p)/(z-a)][\tilde{g}(z)(z-a)/(z-p)]\ . \tag{A.19}$$

□

<u>Corollary A.1a</u>. Let $\tilde{g}\in\tilde{\ell}_{1-}(\rho_0)$ and let $p\in D(\rho_0)^c$.

(i) If $a\in\mathbb{C}$ such that $a\neq p$, then $\exists 0\le\rho_1<\rho_0$ such that

$$\tilde{g}(z)\frac{(z-a)}{(z-p)} = \tilde{g}(p)\frac{(p-a)}{(z-p)}+\tilde{\zeta}(z)\quad \forall z\in D(\rho_1)^c\backslash\{p\} \tag{A.20}$$

where $\tilde{\zeta}\in\tilde{\ell}_{1-}(\rho_0)$.

(ii) If $a\in D(\rho_0)$ and $\tilde{g}$ has an m^{th}-order zero at p, then

$$\frac{(z-p)}{(z-a)}\in\tilde{\ell}^{\infty}_{1-}(\rho_0) \tag{A.21}$$

and $\tilde{g}(z)\frac{(z-a)}{(z-p)}\in\tilde{\ell}_{1-}(\rho_0)$ and has an $(m-1)^{th}$ order zero at p. □

<u>Proof</u>. (i) By Theorem A.1, multiplying equation (A.15) by $(z-a)$, we see that $\exists\rho_1',\rho_2$ satisfying $0\le\rho_2<\rho_1'<\rho_0$ such that

$$\tilde{g}(z)\frac{(z-a)}{(z-p)} = \tilde{g}(p)\frac{(z-a)}{(z-p)} + \tilde{\zeta}_1(z)(z-a) \quad \forall z \in D(\rho_2)^c \setminus \{p\} \tag{A.22}$$

where $\tilde{\zeta}_1 \in \tilde{\ell}_{1-}(\rho_1')$. Now since $\frac{z-a}{z-p} = 1 + \frac{p-a}{z-p}$, we have, $\forall z \in D(\rho_2)^c \setminus \{p\}$

$$\tilde{g}(z)\frac{(z-a)}{(z-p)} = \tilde{g}(p)\frac{(p-a)}{(z-p)} + \tilde{\zeta}_1(z)(z-a) + \tilde{g}(p) \ . \tag{A.23}$$

Here, we define the last two terms of (A.23) as

$$\begin{aligned}\tilde{\zeta}(z) &:= \tilde{\zeta}_1(z)(z-a) + \tilde{g}(p) \\ &= \tilde{\zeta}_1(z)(z-a) + \tilde{g}(z) - \tilde{\zeta}_1(z)(z-p) \quad \text{using (A.22)} \\ &= \tilde{\zeta}_1(z)(p-a) + \tilde{g}(z) \in \tilde{\ell}_{1-}(\rho_0) \ .\end{aligned} \tag{A.24}$$

The proof is complete by defining any $\rho_1 \in [\rho_2, \rho_0[$.

(ii) This follows immediately from (i) and the fact that $a \in D(\rho_0)$. □

<u>Corollary A.1b</u>. Let $\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$ have zeros $z_\alpha \in D(\rho_0)^c$ of multiplicities m_α, respectively for $\alpha = 1,2,\ldots,\nu$. Let $m := \sum_{\alpha=1}^{\nu} m_\alpha$. Let $a_\beta \in D(\rho_0)$, $\beta = 1,2,\ldots,m$. Under these conditions

(i)
$$\tilde{g} = \tilde{b}\tilde{c} \tag{A.25}$$

where $\tilde{b} \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$, $\tilde{c} \in \tilde{\ell}_{1-}(\rho_0)$ are defined for some $\rho_1 < \rho_0$ by

$$\tilde{b}(z) := \prod_{\alpha=1}^{\nu} (z-z_\alpha)^{m_\alpha} \Big/ \prod_{\beta=1}^{m} (z-a_\beta) \tag{A.26}$$

$$\tilde{c}(z) := \tilde{g}(z) \prod_{\beta=1}^{m} (z-a_\beta) \Big/ \prod_{\alpha=1}^{\nu} (z-z_\alpha)^{m_\alpha}, \quad |z| \geq \rho_1 \tag{A.27}$$

and
$$\tilde{c}(z_\alpha) \neq 0 \ , \quad \alpha = 1,2,\ldots,\nu \ . \tag{A.28}$$

(ii) If, in addition, $\tilde{g} \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$ and z_α, $\alpha = 1,2,\ldots,\nu$, are the only zeros of $\tilde{g}$ in $D(\rho_0)^c$, then $\tilde{c}$ is an invertible element of $\tilde{\ell}_{1-}(\rho_0)$. □

Proof. (i) This follows immediately by repeated application of Corollary A.1a(ii).

(ii) If $\tilde{g} \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$, then

$$\lim_{|z|\to\infty} \tilde{c}(z) = \lim_{|z|\to\infty} \tilde{g}(z)/\tilde{b}(z) = g(0) \neq 0 \ . \tag{A.29}$$

Hence $\tilde{c} \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$. From (A.27), the only possible zeros of $\tilde{c}$ in $D(\rho_0)^c$ are the zeros of $\tilde{g}$. Now that z_α, $\alpha = 1,2,\ldots,\nu$, are the only zeros of $\tilde{g}$ in $D(\rho_0)^c$, then in view of (A.28),

$$\tilde{c}(z) \neq 0 \quad \forall z \in D(\rho_0)^c \ . \tag{A.30}$$

Hence by Property (2.3.3), $\tilde{c}$ is an invertible element of $\tilde{\ell}_{1-}(\rho_0)$. □

Corollary A.1c. Let $\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$. Let $\nu \in \mathbb{N}^*$ and $p_\alpha \in D(\rho_0)^c$, $m_\alpha \in \mathbb{N}^*$, $\alpha = 1,2,\ldots,\nu$. Let $m := \sum_{\alpha=1}^{\nu} m_\alpha$, and $a_\beta \in D(\rho_0)$, $\beta = 1,2,\ldots,m$. Under these conditions $\exists \rho_1 \in [0,\rho_0[$ such that

$$\tilde{g}(z) \prod_{\beta=1}^{m} (z-a_\beta) / \prod_{\alpha=1}^{\nu} (z-p_\alpha)^{m_\alpha} = \sum_{\alpha=1}^{\nu} \sum_{k=0}^{m_\alpha - 1} r_{\alpha k}/(z-p_\alpha)^{m_\alpha - k} + \tilde{g}_p(z) \qquad \forall |z| \geq \rho_1 \tag{A.31}$$

where

(i) $\tilde{g}_p \in \tilde{\ell}_{1-}(\rho_0)$

(ii) for $\alpha = 1,2,\ldots,\nu$, and $k = 0,1,\ldots,m_\alpha - 1$, $r_{\alpha k} \in \mathbb{C}$ is given by

$$r_{\alpha k} = \frac{1}{k!} \frac{d^k}{dz^k}\Big[\tilde{g}(z) \prod_{\beta=1}^{m} (z-a_\beta) / \prod_{\substack{i=1 \\ i\neq\alpha}}^{\nu} (z-p_i)^{m_i}\Big]\Bigg|_{z=p_\alpha} \tag{A.32}$$

(iii) for $\alpha = 1,2,\ldots,\nu$

$$\tilde{g}(p_\alpha) \neq 0 \Rightarrow r_{\alpha 0} \neq 0 \ . \tag{A.33}$$

□

Proof: This is achieved through multiple applications of Corollary A.1a. □

Proof of (2.3.4): $\tilde{\ell}_{1-}(\rho_0)$ is a Euclidean ring

Since $\tilde{\ell}_{1-}(\rho_0)$ is an integral domain (entire ring), it suffices to prove [Sig 1, p.132] [Her 1, p.143] that the gauge $\gamma: \tilde{\ell}_{1-}(\rho_0)\backslash\{0\} \to \mathbb{N}$ defined in (2.18) satisfies

(i) $\gamma(\tilde{f}\tilde{g}) \geq \gamma(\tilde{f}) \quad \forall f,g \in \tilde{\ell}_{1-}(\rho_0)\backslash\{0\}$ (A.35)

(ii) a Euclidean algorithm exists: $\forall \tilde{f},\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$, $\tilde{f} \neq 0$, $\exists \tilde{q},\tilde{r} \in \tilde{\ell}_{1-}(\rho_0)$ such that

$$\tilde{g} = \tilde{q}\tilde{f} + \tilde{r} \tag{A.36}$$

with either $0 \leq \gamma(\tilde{r}) < \gamma(\tilde{f})$ or $\tilde{r} = 0$.

Observe that when $\tilde{g} \neq 0$, $\text{ord}(\tilde{g})$ is finite, and the last term of (2.18) is finite due to Property (2.3.2)(iii); hence the gauge γ in (2.18) is well-defined. Before we carry out the proof, we study the following with the gauge γ defined as in (2.18).

Fact A.1. For any nonzero $\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$, we can decompose

$$\tilde{g} = \tilde{g}_u\tilde{g}_s \tag{A.37}$$

such that

(i) $\tilde{g}_u \in \mathbb{C}[z^{-1}]$ with $\bar{\partial}[\tilde{g}_u] = \gamma(\tilde{g})$ and (A.38)

every zero z_0 of $\tilde{g}_u$ satisfies $|z_0^{-1}| \leq \rho_0$ (A.39)

(ii) $\tilde{g}_s \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$ is an invertible element of $\tilde{\ell}_{1-}(\rho_0)$. (A.40)

Proof of Fact A.1. Define $\tilde{g}_0 \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$ by

$$\tilde{g}(z) =: z^{-\text{ord}(\tilde{g})} \cdot \tilde{g}_0(z) \ . \tag{A.41}$$

Then the list of zeros of $\tilde{g}$ (including multiplicities) and that of $\tilde{g}_0$ are identical. Let $n_g \in \mathbb{C}[z]$ be a polynomial whose zeros are exactly all those of $\tilde{g}$ in $D(\rho_0)^c$, counting multiplicities. Then, by definition of n_g and by (2.18)

$$\gamma(\tilde{g}) = \text{ord}(\tilde{g}) + \partial[n_g] \; . \tag{A.42}$$

Note that n_g has no zero at $z = 0$, hence $\frac{n_g(z)}{z^{\partial[n_g]}}$ is a polynomial in z^{-1} with z^{-1}-degree equal to $\partial[n_g]$. Define $\tilde{g}_s \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$ by

$$\tilde{g}_0(z) =: \frac{n_g(z)}{z^{\partial[n_g]}} \cdot \tilde{g}_s(z) \; . \tag{A.43}$$

Equivalently, $\tilde{g}_s(z) := \tilde{g}_0(z)\frac{z^{\partial[n_g]}}{n_g(z)}$, and by Corollary A.1b, $\tilde{g}_s \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$ is an invertible element of $\tilde{\ell}_{1-}(\rho_0)$.

Define $\tilde{g}_u \in \mathbb{C}[z^{-1}]$, a polynomial in z^{-1}, by

$$\tilde{g}_u(z) := z^{-\text{ord}(\tilde{g})} \cdot \frac{n_g(z)}{z^{\partial[n_g]}} \; . \tag{A.44}$$

Then combining (A.41) and (A.43), we obtain

$$\tilde{g} = \tilde{g}_u \cdot \tilde{g}_s$$

and

$$\bar{\partial}[\tilde{g}_u] = \text{ord}(\tilde{g}) + \partial[n_g] = \gamma(\tilde{g}) \quad \text{by (A.42)} \; . \tag{A.45}$$

□

<u>Procedure A.1</u>: <u>Euclidean Algorithm for $\tilde{\ell}_{1-}(\rho_0)$</u>

<u>Given</u> $\tilde{f}, \tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$, $\tilde{f} \neq 0$.

<u>Step 1</u>. Decompose $\tilde{f}, \tilde{g}$ into

$$\tilde{f} = \tilde{f}_u \cdot \tilde{f}_s \; , \quad \tilde{g} = \tilde{g}_u \cdot \tilde{g}_s \tag{A.46}$$

as in Fact A.1.

<u>Step 2</u>. Use the Euclidean algorithm in $\mathbb{C}[z^{-1}]$ and find $\tilde{q}_t, \tilde{r}_t \in \mathbb{C}[z^{-1}]$ such that

$$\tilde{g}_u = \tilde{q}_t \tilde{f}_u + \tilde{r}_t \tag{A.47}$$

where either $\partial[\tilde{r}_t] < \partial[\tilde{f}_u]$ or $\tilde{r}_t = 0$.

<u>Step 3</u>. Define $\tilde{q}, \tilde{r} \in \tilde{\ell}_{1-}(\rho_0)$ by

$$\tilde{q} := \tilde{q}_t \frac{\tilde{g}_s}{\tilde{f}_s} \tag{A.48}$$

and

$$\tilde{r} := \tilde{r}_t \cdot \tilde{g}_s \ . \tag{A.49}$$

Such $\tilde{q}, \tilde{r}$ satisfy

$$\tilde{g} = \tilde{q}\tilde{f} + \tilde{r} \tag{A.50}$$

with either $0 \leq \gamma(\tilde{r}) < \gamma(\tilde{f})$ or $\tilde{r} = 0$. □

We now continue the proof of (2.3.4):

(i) $\forall$nonzero $\tilde{f}, \tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$,

$$\mathrm{ord}(\tilde{f}\tilde{g}) = \mathrm{ord}(\tilde{f}) + \mathrm{ord}(\tilde{g}) \geq 0 \ , \tag{A.51}$$

and, by counting zeros of the analytic functions $\tilde{f}$ and $\tilde{g}$,

$$\partial[n_{fg}] = \partial[n_f \cdot n_g] = \partial[n_f] + \partial[n_g] \ . \tag{A.52}$$

Hence

$$\begin{aligned} \gamma(\tilde{f}\tilde{g}) &= \mathrm{ord}(\tilde{f}\tilde{g}) + \partial[n_{fg}] \\ &= \mathrm{ord}(\tilde{f}) + \mathrm{ord}(\tilde{g}) + \partial[n_f] + \partial[n_g] \\ &= \gamma(\tilde{f}) + \gamma(\tilde{g}) \geq \gamma(\tilde{f}) \ . \end{aligned} \tag{A.53}$$

(ii) Next, we prove that the Euclidean algorithm gives the desired result: Step 1 follows from Fact A.1, and Step 2 is self-explanatory. For Step 3, since $\tilde{g}_s, \tilde{f}_s$ are invertible elements of $\tilde{\ell}_{1-}(\rho_0)$ and $\tilde{q}_t, \tilde{r}_t \in \mathbb{C}[z^{-1}] \subset \tilde{\ell}_{1-}(\rho_0)$, hence $\tilde{q}, \tilde{r}$ defined in

(A.48), (A.49) belong to $\tilde{\ell}_{1-}(\rho_0)$. Multiplying (A.47) by $\tilde{g}_s$, we get

$$\tilde{g}_u \cdot \tilde{g}_s = (\tilde{q}_t \cdot \frac{\tilde{g}_s}{\tilde{f}_s}) \cdot \tilde{f}_u \tilde{f}_s + \tilde{r}_t \tilde{g}_s \tag{A.54}$$

which gives (A.50). Finally,

$$\tilde{r} = 0 \quad \text{if} \quad \tilde{r}_t = 0 \text{ or } \tilde{g}_s = 0 \;;$$

otherwise, by (A.47)

$$0 \leq \gamma(\tilde{r}) \leq \bar{\partial}[\tilde{r}_t] < \bar{\partial}[\tilde{f}_u] = \gamma(\tilde{f}) \;. \tag{A.55}$$

□

Proof of (2.3.6). (2.19)⇔(2.20). (2.19) holds if and only if δ_0 is a gcd(f,g), which holds if and only if (2.20) holds [McL 1, Thm. 25, p.154].

(2.21)⇔(2.22). This is obvious by observing that $(f(0),g(0)) = \lim_{|z|\to\infty} (\tilde{f}(z),\tilde{g}(z))$.

(2.20)⇒(2.21). By definition, $\exists u, v \in \ell_{1-}(\rho_0)$ such that $u*f + v*g = \delta_0$. With $f, g, u, v \in \ell_{1-}(\rho_0)$, there exists $\rho_1 < \rho_0$ such that $f, g, u, v \in \ell_1(\rho_1)$. Hence, f and g are coprime in $\ell_1(\rho_1)$. By Property (2.2.7), $\inf_{|z|\geq\rho_1} |(\tilde{f}(z),\tilde{g}(z))| > 0$, and thus

$$\inf_{|z|\geq\rho_0} |(\tilde{f}(z),\tilde{g}(z))| \geq \inf_{|z|\geq\rho_1} |(\tilde{f}(z),\tilde{g}(z))| > 0 \;.$$

(2.20)⇐(2.21). With $f, g \in \ell_{1-}(\rho_0)$, there exists $\rho_1 < \rho_0$ such that $f, g \in \ell_1(\rho_1)$. Hence the map $z \mapsto |(\tilde{f}(z),\tilde{g}(z))|$ is defined for $|z| \geq \rho_1$ and is continuous on $S := \{z \,\big|\, |z| = \rho_0\}$. By Lemma A.2, $|(\tilde{f}(z),\tilde{g}(z))| > 0$ on S implies that $\exists \rho_2 \in]\rho_1,\rho_0[$ such that $|(\tilde{f}(z),\tilde{g}(z))| > 0 \;\; \forall |z| \in [\rho_2,\rho_0]$. Hence

$\inf_{|z|\geq\rho_2} |(\tilde{f}(z),\tilde{g}(z))| > 0$. By Property (2.2.7), f, g are coprime in $\ell_1(\rho_2)$, i.e. $\exists u, v \in \ell_1(\rho_2) \subset \ell_{1-}(\rho_0)$ such that

$$u*f + v*g = \delta_0 \, . \qquad \square$$

Appendix B: Proofs of Theorems and Lemmas

Proof of Lemma 2.1. By definition, $\tilde{g} \in \tilde{b}(\rho_0)$ implies that $\exists \tilde{\tilde{n}} \in \tilde{\ell}_{1-}(\rho_0)$ and $\tilde{\tilde{d}} \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$ such that $\tilde{g} = \tilde{\tilde{n}}/\tilde{\tilde{d}}$. If $(\tilde{\tilde{n}},\tilde{\tilde{d}})$ are ρ_0-coprime, then they give a ρ_0-representation of $\tilde{g}$. Otherwise, since $\tilde{\ell}_{1-}(\rho_0)$ is a Euclidean ring, we can find a $\gcd(\tilde{\tilde{n}},\tilde{\tilde{d}})$ in $\tilde{\ell}_{1-}(\rho_0)$: call it $\tilde{c}$. Define $\tilde{d} := \tilde{\tilde{d}}/\tilde{c}$ and $\tilde{n} := \tilde{\tilde{n}}/\tilde{c}$. By definition of the greatest common divisor $\tilde{c}$, $\tilde{d}$ and $\tilde{n}$ belong to $\tilde{\ell}_{1-}(\rho_0)$ and are ρ_0-coprime. Furthermore, both $\tilde{d}$, $\tilde{c} \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$ because $\tilde{\tilde{d}} \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$. □

Proof of Lemma 2.2. $\tilde{n} \in \tilde{\ell}_{1-}(\rho_0)$ and $\tilde{d} \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$ imply that $\tilde{n}$, $\tilde{d}$ are both analytic in $D(\rho_0)^c$. Since $\tilde{n}$, $\tilde{d}$ are ρ_0-coprime, $\tilde{n}$ and $\tilde{d}$ have no common zeros in $D(\rho_0)^c$. Hence statements (i) and (ii) of the lemma follow. □

Proof of Theorem 2.1. The steps in Procedure 2.1 are justified in the following:

Step 1. The ρ_0-representation exists by Lemma 2.1.

Step 2. According to Property (2.3.2)(iii), $\tilde{\tilde{d}} \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$ can have at most a finite number of zeros in $D(\rho_0)^c$.

Step 3. By Corollary A.1b, $\tilde{c} \in \tilde{\ell}_{1-}(\rho_0)$ is invertible.

Step 4. Definition 2.3 is satisfied by $(\tilde{n},\tilde{d})$. In particular, since $\inf_{|z|\geq\rho_0} |\tilde{c}(z)| > 0$ and $|\tilde{c}(z)| \leq \|\tilde{c}\|_{\rho_0}$ $\forall z \in D(\rho_0)^c$,

$$\operatorname{rank}(\tilde{n}(z),\tilde{d}(z)) = \operatorname{rank}(\tilde{\tilde{n}}(z),\tilde{\tilde{d}}(z))\cdot\tilde{c}(z)^{-1} = 1 \quad \forall z \in D(\rho_0)^c .$$

Furthermore, by definition in Step 3, $\tilde{d} \in \mathcal{A}^{\infty}(\rho_0)$, $\lim_{|z|\to\infty} \tilde{d}(z) = 1$, and $Z[\tilde{d}] \subset D(\rho_0)^c$. □

Proof of Theorem 2.2. $(\Rightarrow)$ By assumption $\tilde{g} \in \tilde{b}(\rho_0)$. If $\tilde{g} \in \tilde{\ell}_{1-}(\rho_0)$, then (2.36) holds with $\tilde{r} \equiv 0$. Now suppose $\tilde{g} \notin \tilde{\ell}_{1-}(\rho_0)$. By Theorem 2.1, $\tilde{g}$ has a normalized ρ_0-representation $(\tilde{n},\tilde{d})$ such that $\tilde{n} \in \tilde{\ell}_{1-}(\rho_0)$ and $\tilde{d} \in \mathcal{R}^{\infty}(\rho_0)$. Since $\tilde{g} \notin \tilde{\ell}_{1-}(\rho_0)$, then $\tilde{d} \not\equiv 1$ and hence for some $\nu \in \mathbb{N}^*$, there are $p_\alpha \in D(\rho_0)^c$ and $m_\alpha \in \mathbb{N}^*$ where $\alpha = 1,2,\ldots,\nu$, and for $m := \sum_{\alpha=1}^{\nu} m_\alpha$, there are $a_\beta \in D(\rho_0)$, $\beta = 1,2,\ldots,m$, such that

$$\tilde{d}(z) = \prod_{\alpha=1}^{\nu} (z-p_\alpha)^{m_\alpha} / \prod_{\beta=1}^{m} (z-a_\beta)$$

with $\tilde{n}(p_\alpha) \neq 0$, $\alpha = 1,2,\ldots,\nu$.

Hence (2.36) with all its properties (2.37)-(2.39b) follows from Corollary A.1c, where $\tilde{g}$ is replaced by $\tilde{n}$ and where

$$\sum_{\alpha=1}^{\nu} \sum_{k=0}^{m_\alpha-1} r_{\alpha k}/(z-p_\alpha)^{m_\alpha-k} =: \tilde{r}(z) .$$

$(\Leftarrow)$ Proof by construction, following Procedure 2.2: Step 1 is self-explanatory. The pair $(\tilde{n},\tilde{d})$ generated by steps 2 and 3 satisfy the following:

(i) by (2.43), $\tilde{d} \in \mathcal{R}^{\infty}(\rho_0) \subset \tilde{\ell}^{\infty}_{1-}(\rho_0)$ with $\lim_{|z|\to\infty} \tilde{d}(z) = 1$ and $Z[\tilde{d}] \subset D(\rho_0)^c$; by (2.44), $\tilde{n} \in \tilde{\ell}_{1-}(\rho_0)$.

(ii) $\tilde{g} = n_r/d_r + \tilde{q}$ by (2.36) and (2.41)

$= (n_r+\tilde{q}d_r)/d_r$

$= \tilde{n}/\tilde{d}$ by (2.43) and (2.44)

(iii) Since (n_r,d_r) are coprime polynomials, then from (2.44), $\forall |z| \geq \rho_0$

$$d_r(z) = 0 \;\Rightarrow\; \tilde{n}(z) = n_r(z)/z^{\nu} \neq 0 .$$

Hence

$$\operatorname{rank}\begin{bmatrix}\tilde{n}(z)\\ \tilde{d}(z)\end{bmatrix} = \operatorname{rank}\begin{bmatrix}n_r(z)\\ d_r(z)\end{bmatrix} = 1 \quad \forall |z| \geq \rho_0 ,$$

so $$|(\tilde{n}(z),\tilde{d}(z))| > 0 \quad \forall |z| \geq \rho_0 ,$$

and by Remark 2.1, $(\tilde{n},\tilde{d})$ are ρ_0-coprime. Therefore $(\tilde{n},\tilde{d})$ is a normalized ρ_0-representation of $\tilde{g}$. □

Proof of Theorem 2.3. By Procedure 2.1 (normalization), we note that any ρ_0-representation $(\tilde{n},\tilde{d})$ is equal to the product of its normalized form with an invertible element of $\tilde{\ell}_{1-}(\rho_0)$. Hence without loss of generality, we assume that both $(\tilde{n},\tilde{d})$ and $(\tilde{\tilde{n}},\tilde{\tilde{d}})$ are normalized. By Lemma 2.2, $\tilde{g}$ has an m^{th} order pole at $p \in D(\rho_0)^c$ if and only if $\tilde{d}$ (ditto for $\tilde{\tilde{d}}$) has an m^{th} order zero at $p \in D(\rho_0)^c$. Let d_r be given by the coprime factorization in (2.46). By (2.39b), $\tilde{g}$ has an m^{th} order pole at $p \in D(\rho_0)^c$ if and only if d_r has an m^{th} order zero at $p \in D(\rho_0)^c$. Since $\tilde{d}$, $\tilde{\tilde{d}} \in \mathcal{A}^\infty(\rho_0)$ and $d_r \in \mathbb{C}[z]$ have zeros only in $D(\rho_0)^c$, they have zeros of the same order at the same locations in $D(\rho_0)^c$, and nowhere else; furthermore, $\lim_{|z|\to\infty} \tilde{d}(z) = \lim_{|z|\to\infty} \tilde{\tilde{d}}(z) = 1$, hence

$$\tilde{d} = d_r/n_h , \quad \tilde{\tilde{d}} = d_r/d_h$$

where n_h, $d_h \in \mathbb{C}[z]$ are monic polynomials of the same degree as d_r, and have zeros only in $D(\rho_0)$. Hence $\tilde{h} := n_h/d_h$ is an invertible element of $\tilde{\ell}_{1-}(\rho_0)$, and is rational. Then $\tilde{h} = n_h/d_h = \tilde{\tilde{d}}/\tilde{d} = \tilde{\tilde{n}}/\tilde{n}$. □

Proof of Theorem 2.4. $(\Rightarrow)$ By assumption, $\exists \tilde{h} \in \tilde{b}(\rho_0)$ such that $\tilde{g}\tilde{h} = \tilde{h}\tilde{g} \equiv 1$ in $D(\rho_0)^c$. Hence $\lim_{|z|\to\infty} (\tilde{g}\tilde{h})(z) = 1$. Since

$$\lim_{|z|\to\infty} |\tilde{h}(z)| = |h(0)| < \infty ,$$

then $$g(0) = \lim_{|z|\to\infty} |\tilde{g}(z)| = h(0)^{-1} \neq 0 \,.$$

($\Leftarrow$) Let $(\tilde{n},\tilde{d})$ be a ρ_0-representation of $\tilde{g}$. Then by assumption, $g(0) = \lim_{|z|\to\infty} \tilde{g}(z) = \lim_{|z|\to\infty} \tilde{n}(z)/\tilde{d}(z) = n(0)/d(0) \neq 0$. Furthermore, $d(0) = \lim_{|z|\to\infty} \tilde{d}(z) \neq 0$ because $\tilde{d} \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$. Hence $\lim_{|z|\to\infty} \tilde{n}(z) = n(0) = g(0)d(0) \neq 0$. Thus $\tilde{n} \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$ and $\tilde{g}^{-1} := \tilde{d}/\tilde{n}$ belongs to $\tilde{b}(\rho_0)$ and is the inverse of $\tilde{g}$ in $\tilde{b}(\rho_0)$. □

Proof of Lemma 3.1. This is immediate by Cramer's rule and Property (2.2.6) (respectively Property (2.3.3)). □

Proof of Lemma 3.2. This is immediate by Cramer's rule and Theorem 2.4. □

Proof of Lemma 3.3. This is immediate by applying Theorem 2.2 to every element of $\tilde{G}$, so that for $i = 1,2,\ldots,n_0$, $j = 1,2,\ldots,n_i$,

$$\tilde{g}_{ij} = \tilde{r}_{ij} + \tilde{q}_{ij}$$

where $\tilde{g}_{ij}$, $\tilde{r}_{ij}$, $\tilde{q}_{ij}$ satisfy Theorem 2.2. □

Proof of Lemma 3.4. $(N_r,\mathcal{D}_r)$ are ρ_0-r.c. if and only if I_{n_i} is a g.c.r.d. of $(N_r,\mathcal{D}_r)$, which holds if and only if (3.4) holds [McD 1, p.35]. This proves Lemma 3.4(r). The proof of Lemma 3.4(ℓ) is similar. □

Proof of Theorem 3.1.

Case 1. If $\tilde{G} \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_i}$, the theorem is immediately verified by choosing

$$N_r := \tilde{G} \;;\; \mathcal{D}_r := I_{n_i} \;;\; \mathcal{U}_r := 0 \;;\; \mathcal{V}_r := I_{n_i} \,,$$
$$N_\ell := \tilde{G} \;;\; \mathcal{D}_\ell := I_{n_o} \;;\; \mathcal{U}_\ell := 0 \;;\; \mathcal{V}_\ell := I_{n_o} \,.$$

Case 2. If $\tilde{G} \notin \tilde{\mathcal{L}}_{1-}(\rho_0)^{n_o \times n_i}$, we use Procedure 3.1 to find the eight matrices that satisfy the theorem.

Step 1 is self-explanatory.

Step 2. Since all elements r_{ij} of $\tilde{R}$ in (3.8) are elements of $\mathbb{C}_p(z)$ and have poles only in $D(\rho_0)^c$, they admit a rational ρ_0-representation (n_{ij}, d_{ij}) with $n_{ij} \in \mathcal{R}(\rho_0)$ and $d_{ij} \in \mathcal{R}^\infty(\rho_0)$ such that (n_{ij}, d_{ij}) are ρ_0-coprime, with respect to $\mathcal{R}(\rho_0)$. It is then possible to construct a least common multiple $d_j \in \mathcal{R}^\infty(\rho_0)$ of all denominators $d_{ij} \in \mathcal{R}^\infty(\rho_0)$ of column j [McL 1, Ch. IV, §10]. Hence $\hat{N}_r := [n_{ij}d_j/d_{ij}] \in \mathcal{R}(\rho_0)^{n_o \times n_i}$ and $\hat{D}_r := \operatorname{diag}(d_j)_{j=1}^{n_i} \in \mathcal{R}(\rho_0)^{n_i \times n_i}$ satisfy the conditions of Step 2.

Step 3. $\hat{M}$ is full rank because $\det \hat{D}_r \in \mathcal{R}^\infty(\rho_0)$ and thus $\det D_r$ is not the zero element of $\mathcal{R}(\rho_0)$. The rest is self-explanatory.

Step 4. Comment (i) relating to Step 4 holds as follows: Observe that all matrices in (3.10)-(3.12) have elements in $\mathcal{R}(\rho_0) \subset \tilde{\mathcal{L}}_{1-}(\rho_0)$ with $\det \hat{D}_r$, $\det \bar{R} \in \mathcal{R}^\infty(\rho_0)$. Moreover, from $\hat{M} = W^{-1}\begin{bmatrix} \bar{R} \\ \hline 0 \end{bmatrix}$,

$$\hat{D}_r = \bar{D}_r\bar{R} \ , \qquad \hat{N}_r = \bar{N}_r\bar{R}$$

hence $$\tilde{R} = \hat{N}_r\hat{D}_r^{-1} = \bar{N}_r\bar{D}_r^{-1} \quad \text{with} \quad \det \bar{D}_r \in \mathcal{R}^\infty(\rho_0)$$

and $$\bar{V}_r\bar{D}_r + \bar{U}_r\bar{N}_r \equiv I_{n_i} \ .$$

Hence $(\bar{N}_r, \bar{D}_r)$ is a ρ_0-r.r. of $\tilde{R}$, with $\bar{R}$ a g.c.r.d. of $\hat{N}_r$ and $\hat{D}_r$. From $\bar{W}\bar{W}^{-1} = I$, we also get

$$\tilde{R} = \bar{N}_r\bar{D}_r^{-1} = \bar{D}_\ell^{-1}\bar{N}_\ell \ ,$$

and $$\bar{N}_\ell\bar{U}_\ell + \bar{D}_\ell\bar{V}_\ell \equiv I_{n_o} \ .$$

Since $\bar{W}$ is invertible in $\mathcal{R}(\rho_0)^{(n_i+n_o)\times(n_i+n_o)}$, det $\bar{W}$ tends to a nonzero complex constant as $|z|\to\infty$. From (3.11), the matrix $[-\bar{N}_\ell \mid \bar{D}_\ell] = \bar{D}_\ell[-\tilde{R} \mid I_{n_o}]$, when evaluated at $z=\infty$, is a full rank matrix; hence det $\bar{D}_\ell \in \mathcal{R}(\rho_0)$ tends to a nonzero constant at infinity. Thus $(\bar{D}_\ell,\bar{N}_\ell)$ is a ρ_0-l.r. of $\tilde{R}$.

Step 5 is self-explanatory. Comment (ii) regarding Step 5 can be verified by using (3.13) and a little computation like the preceding comment. □

Proof of Corollary 3.1b. We present here the proof for Corollary 3.1b(r). The proof for Corollary 3.1b(ℓ) is similar and is omitted.

($\Rightarrow$) By assumption, (N_r,D_r) are ρ_0-r.c., hence by (3.4)

$$[V_r \mid U_r]\begin{bmatrix} D_r \\ \hline N_r \end{bmatrix}(z) = I_{n_i} \quad \forall z \in D(\rho_0)^c,$$

which implies

$$\operatorname{rank}\left([V_r \mid N_r]\begin{bmatrix} D_r \\ \hline N_r \end{bmatrix}(z)\right) = n_i \quad \forall z \in D(\rho_0)^c.$$

Hence, by Sylvester's inequality,

$$\operatorname{rank}\begin{bmatrix} D_r \\ \hline N_r \end{bmatrix}(z) \geq n_i \quad \forall z \in D(\rho_0)^c.$$

Equality holds because the matrix $[D_r^T \mid N_r^T]^T$ has only n_i columns.

($\Leftarrow$) By Cramer's rule, $\tilde{G} := N_r D_r^{-1}$ belongs to $\tilde{b}(\rho_0)^{n_o\times n_i}$. Hence by Theorem 3.1, $\tilde{G}$ admits a ρ_0-r.r. $(\bar{N}_r,\bar{D}_r)$, i.e.

$$\tilde{G} = \bar{N}_r\bar{D}_r^{-1}$$

where $\bar{N}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_i}$, $\bar{\mathcal{D}}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$, $\det \bar{\mathcal{D}}_r \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$, and there exist $\bar{U}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o}$, $\bar{V}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ such that

$$\bar{U}_r \bar{N}_r + \bar{V}_r \bar{\mathcal{D}}_r \equiv I_{n_i} \ . \tag{B.20}$$

Define $R := \bar{\mathcal{D}}_r^{-1} \mathcal{D}_r$. Then $\mathcal{D}_r = \bar{\mathcal{D}}_r R$ and

$$N_r = N_r \mathcal{D}_r^{-1} \mathcal{D}_r = \bar{N}_r \bar{\mathcal{D}}_r^{-1} \mathcal{D}_r = \bar{N}_r R \ .$$

Using these identities in (B.20), we obtain

$$\bar{U}_r N_r + \bar{V}_r \mathcal{D}_r = R \ . \tag{B.21}$$

Hence $R \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$, and so

$$\det R = \det \mathcal{D}_r / \det \bar{\mathcal{D}}_r \in \tilde{\ell}_{1-}^{\infty}(\rho_0) \ . \tag{B.22}$$

By assumption,

$$n_i = \operatorname{rank} \begin{bmatrix} \mathcal{D}_r(z) \\ \hline N_r(z) \end{bmatrix} = \operatorname{rank} \left(\begin{bmatrix} \bar{\mathcal{D}}_r(z) \\ \hline \bar{N}_r(z) \end{bmatrix} R(z) \right) \quad \forall z \in D(\rho_0)^c \ .$$

Hence, by invoking Sylvester's inequality, $\operatorname{rank} R(z) = n_i \quad \forall z \in D(\rho_0)^c$,

i.e.
$$\det R(z) \neq 0 \quad \forall z \in D(\rho_0)^c \ . \tag{B.23}$$

Invoking Lemma 3.1, (B.22) and (B.23) together imply that R is an invertible element of $\tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$. Thus,

$$U_r := R^{-1} \bar{U}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o} \tag{B.24}$$

and
$$V_r := R^{-1} \bar{V}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i} \ . \tag{B.25}$$

Premultiply (B.21) by R^{-1} and using (B.24), (B.25), we obtain

$$\mathcal{U}_r N_r + \mathcal{V}_r \mathcal{D}_r \equiv I_{n_i} ,$$

i.e. $(N_r, \mathcal{D}_r)$ is ρ_0-r.c. □

Proof of Procedure 3.2(r). Steps 1, 2 and 3 are self-explanatory. To prove the comment in Step 4, we need to show that $\tilde{R} = \bar{N}_r \bar{\mathcal{D}}_r^{-1}$ with (a) $\bar{N}_r \in r(\rho_0)^{n_o \times n_i}$, $\bar{\mathcal{D}}_r \in r(\rho_0)^{n_i \times n_i}$; (b) $\det \bar{\mathcal{D}}_r \in r^{\infty}(\rho_0)$; (c) $(\bar{N}_r, \bar{\mathcal{D}}_r)$ are ρ_0-r.c. The fact that $\tilde{R} = \bar{N}_r \bar{\mathcal{D}}_r^{-1}$ is obvious from the procedure. Now consider:

(a) By Lemma 3.3, $\tilde{R}$ is strictly proper. Hence for $i = 1,2,\ldots,n_i$

$$\partial_{c_i}[N_r] < \partial_{c_i}[D_r] = \gamma_i = \partial_{c_i}[S] . \tag{B.26}$$

Since S is diagonal, it is column-reduced. Hence, using (B.26), $\bar{N}_r := N_r S^{-1} \in \mathbb{C}(z)^{n_o \times n_i}$ is strictly proper, and $\bar{\mathcal{D}}_r := D_r S^{-1} \in \mathbb{C}(z)^{n_i \times n_i}$ is proper. Furthermore, by construction, all poles of $\bar{N}_r$ and $\bar{\mathcal{D}}_r$ are at $z = 0$, hence $\bar{N}_r \in r(\rho_0)^{n_o \times n_i}$ and $\bar{\mathcal{D}}_r \in r(\rho_0)^{n_i \times n_i}$.

(b) By (3.21), $D_r \in \mathbb{C}[s]^{n_i \times n_i}$ is column-reduced, hence

$$\partial[\det D_r] = \sum_{i=1}^{n_i} \gamma_i = \partial[\det S] .$$

Therefore $\det \bar{\mathcal{D}}_r = \det D_r / \det S$ belongs to $r^{\infty}(\rho_0)$.

(c) Now (N_r, D_r) being r.c. implies

$$\operatorname{rank}\begin{bmatrix} D_r \\ \hdashline N_r \end{bmatrix}(z) = n_i \quad \forall z \in \mathbb{C} .$$

Hence

$$\operatorname{rank}\begin{bmatrix} \bar{\mathcal{D}}_r \\ \hdashline \bar{N}_r \end{bmatrix}(z) = \operatorname{rank}\left(\begin{bmatrix} D_r \\ \hdashline N_r \end{bmatrix} S^{-1}\right)(z) = n_i \quad \forall z \neq 0 . \tag{B.27}$$

By Corollary 3.1b(r), $(\bar{N}_r,\bar{D}_r)$ are ρ_0-r.c. This completes the proof of the comment that $(\bar{N}_r,\bar{D}_r)$ is a ρ_0-r.r. of $\tilde{R}$.

Similarly, to prove the concluding comment in Step 5, we have to show that $\tilde{G} = N_r D_r^{-1}$ with (a) $N_r \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_i}$, $D_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$; (b) $\det D_r \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$; (c) (N_r,D_r) are ρ_0-r.c. By (3.20)-(3.29),

$$\begin{aligned}\tilde{G} &= \tilde{R} + \tilde{Q} \\ &= (\bar{N}_r + \tilde{Q}\bar{D}_r)\bar{D}_r^{-1} \\ &= N_r D_r^{-1} .\end{aligned}$$

(a) Furthermore, by the closure properties of $\tilde{\ell}_{1-}(\rho_0)$ under addition and multiplication,

$$N_r \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_i} , \quad D_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i} .$$

(b) Also, $\det D_r = \det \bar{D}_r \in r^{\infty}(\rho_0) \subset \tilde{\ell}^{\infty}_{1-}(\rho_0)$.

(c) Since, by (B.27), $(\bar{N}_r,\bar{D}_r)$ are ρ_0-r.c., there exist

$$\bar{U}_r \in r(\rho_0)^{n_i \times n_o} \subset \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o} \text{ and } \bar{V}_r \in r(\rho_0)^{n_i \times n_i} \subset \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$$

such that

$$\bar{U}_r \bar{N}_r + \bar{V}_r \bar{D}_r \equiv I_{n_i} \quad \text{in} \quad D(\rho_0)^c .$$

Hence

$$\bar{U}_r(\bar{N}_r + \tilde{Q}\bar{D}_r) + (\bar{V}_r - \bar{U}_r\tilde{Q})D_r \equiv I_{n_i} \quad \text{in} \quad D(\rho_0)^c ,$$

i.e. (N_r,D_r) is ρ_0-r.c. because $U_r N_r + V_r D_r \equiv I_{n_i}$, with $U_r := \bar{U}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o}$ and $V_r := \bar{V}_r - \bar{U}_r\tilde{Q} \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$. □

<u>Proof of Theorem 3.2</u>. By assumption, there exist $\bar{U}_\ell$, $\bar{U}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o}$, $\bar{V}_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_o}$ and $\bar{V}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ such that

$$N_\ell \mathcal{D}_r = \mathcal{D}_\ell N_r \tag{B.28}$$

$$\bar{\mathcal{U}}_r N_r + \bar{\mathcal{V}}_r \mathcal{D}_r \equiv I_{n_i} \tag{B.29}$$

and
$$N_\ell \bar{\mathcal{U}}_\ell + \mathcal{D}_\ell \bar{\mathcal{V}}_\ell \equiv I_{n_o} . \tag{B.30}$$

Rewriting (B.28)-(B.30), we obtain

$$\left[\begin{array}{c|c} \bar{\mathcal{V}}_r & \bar{\mathcal{U}}_r \\ \hline -N_\ell & \mathcal{D}_\ell \end{array}\right] \left[\begin{array}{c|c} \mathcal{D}_r & -\bar{\mathcal{U}}_\ell \\ \hline N_r & \bar{\mathcal{V}}_\ell \end{array}\right] = \left[\begin{array}{c|c} I_{n_i} & X \\ \hline 0 & I_{n_o} \end{array}\right] \tag{B.31}$$

where $X \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o}$ due to the closure properties of $\tilde{\ell}_{1-}(\rho_0)$. Observe that the right-hand side of (B.31) has determinant unity, and is thus invertible in $\tilde{\ell}_{1-}(\rho_0)^{(n_i+n_o)\times(n_i+n_o)}$. Premultiplying (B.31) with the inverse $\left[\begin{array}{c|c} I & -X \\ \hline 0 & I \end{array}\right]$, we obtain (3.30) with

$$\mathcal{V}_r := \bar{\mathcal{V}}_r + XN_\ell , \qquad \mathcal{U}_r := \bar{\mathcal{U}}_r - X\mathcal{D}_\ell$$
$$\mathcal{V}_\ell := \bar{\mathcal{V}}_\ell \qquad , \qquad \mathcal{U}_\ell := \bar{\mathcal{U}}_\ell . \qquad \square$$

Proof of Theorem 3.3. We restrict the proof to the ρ_0-r.r. case: Define

$$R := (\mathcal{D}_r')^{-1}\mathcal{D}_r .$$

Since $\mathcal{D}_r, \mathcal{D}_r' \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ and $\det \mathcal{D}_r' \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$, it follows by Cramer's rule that $R \in \tilde{b}(\rho_0)^{n_i \times n_i}$. Furthermore,

$$\mathcal{D}_r = \mathcal{D}_r' R$$
$$N_r = N_r \mathcal{D}_r^{-1}\mathcal{D}_r = N_r'(\mathcal{D}_r')^{-1}\mathcal{D}_r = N_r' R$$

and thus (3.32) holds. By the ρ_0-r.c. property, there exist matrices $\mathcal{U}_r, \mathcal{V}_r, \mathcal{U}_r', \mathcal{V}_r'$ with elements in $\tilde{\ell}_{1-}(\rho_0)$ such that

$$\mathcal{U}_{\mathcal{R}}\mathcal{N}_{\mathcal{R}} + \mathcal{V}_{\mathcal{R}}\mathcal{D}_{\mathcal{R}} \equiv I_{n_i} \tag{B.32}$$

and

$$\mathcal{U}'_{\mathcal{R}}\mathcal{N}'_{\mathcal{R}} + \mathcal{V}'_{\mathcal{R}}\mathcal{D}'_{\mathcal{R}} \equiv I_{n_i} . \tag{B.33}$$

Postmultiplying (B.32) with $\mathcal{R}^{-1}$ and (B.33) with $\mathcal{R}$, we obtain

$$\mathcal{U}_{\mathcal{R}}\mathcal{N}'_{\mathcal{R}} + \mathcal{V}_{\mathcal{R}}\mathcal{D}'_{\mathcal{R}} \equiv \mathcal{R}^{-1} \tag{B.34}$$

$$\mathcal{U}'_{\mathcal{R}}\mathcal{N}_{\mathcal{R}} + \mathcal{V}'_{\mathcal{R}}\mathcal{D}_{\mathcal{R}} \equiv \mathcal{R} . \tag{B.35}$$

By the closure properties of $\tilde{\ell}_{1-}(\rho_0)$, $\mathcal{R}, \mathcal{R}^{-1} \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ and thus (3.31) follows. □

<u>Proof of Lemma 4.1</u>.

<u>Case 1</u>: only one pole at $z = 0$, i.e. $\nu = 1$, $p_1 = 0$.

$$\tilde{R}(z) = \sum_{i=1}^{m} Z_i z^{-i} .$$

By [Bro 1, Thm. 18-1] or [Kai 1, Lemma 6.5-7], the McMillan degree of $\tilde{R}$ is given by the rank of

$$H := \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_m \\ Z_2 & Z_3 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ Z_m & 0 & \cdots & 0 \end{bmatrix} . \tag{B.41}$$

<u>Case 2</u>: only one pole at $p \in \mathbb{C}$.

$$\tilde{R}(z) = \sum_{i=1}^{m} Z_i (z-p)^{-i} .$$

The change of variable $\lambda := z-p$, which defines $\tilde{R}' \in \mathbb{C}_p(\lambda)^{n_o \times n_i}$ by

$$\tilde{R}'(\lambda) := \tilde{R}(z) = \sum_{i=1}^{m} Z_i \lambda^{-i} , \qquad \text{(B.42)}$$

brings us back to Case 1. The conclusion follows by applying Case 1 to $\tilde{R}'(\lambda)$ to obtain a minimal realization (A,B,C) for $\tilde{R}'(\lambda)$, *thus* leading to a minimal realization $(A+pI,B,C)$ for $\tilde{R}(z)$ of dimension $r = \text{rank } H$, with H given in (B.41).

Case 3: General case with $\tilde{R}$ given by

$$\tilde{R}(z) = \sum_{\alpha=1}^{\nu} \sum_{i=1}^{m_\alpha} Z_{\alpha i}(z-p_\alpha)^{-i} .$$

For $\alpha = 1,2,\ldots,\nu$, define

$$\tilde{R}_\alpha(z) := \sum_{i=1}^{m_\alpha} Z_{\alpha i}(z-p_\alpha)^{-i} . \qquad \text{(B.46)}$$

and $r_\alpha := \text{rank } H_\alpha$ where H_α is defined as in (4.2). Then by Case 2, $\tilde{R}_\alpha$ has a minimal realization $(A_\alpha,B_\alpha,C_\alpha)$ with $A_\alpha \in \mathbb{C}^{r_\alpha \times r_\alpha}$. Letting

$$A := \text{diag}(A_1,A_2,\ldots,A_\nu) ,$$

$$B := [B_1^T | B_2^T | \cdots | B_\nu^T]^T , \qquad \text{(B.47)}$$

and

$$C := [C_1 | C_2 | \cdots | C_\nu] ,$$

the rank tests show that (A,B,C) is a minimal realization of $\tilde{R}$, and $A \in \mathbb{C}^{r\times r}$ with $r := \sum_{\alpha=1}^{\nu} r_\alpha$. Furthermore, by the block diagonal structure of A, the McMillan degree of p_α as a pole of $\tilde{R}$ is equal to the dimension of A_α, which is equal to $r_\alpha = \text{rank } H_\alpha$. □

Proof of Theorem 4.1. Parts (a) and (b) will only be proved for the ρ_0-r.r. case, the ρ_0-l.r. case is similar.

(a) ($\Leftarrow$) By assumption, there exist $\mathcal{U}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o}$ and $\mathcal{V}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ such that

$$\mathcal{U}_r \mathcal{N}_r + \mathcal{V}_r \mathcal{D}_r \equiv I_{n_i} .$$

Postmultiplying both sides by $\mathcal{D}_r^{-1}$, and noting $\tilde{G} = \mathcal{N}_r \mathcal{D}_r^{-1}$, we have

$$\mathcal{U}_r \tilde{G} + \mathcal{V}_r = \mathcal{D}_r^{-1} \tag{B.48}$$

where, for some $\rho_1 < \rho_0$, both sides of (B.48) are meromorphic in $D(\rho_1)^c$, and $\mathcal{U}_r$ and $\mathcal{V}_r$ are analytic and bounded in $D(\rho_1)^c$. Hence if $\det \mathcal{D}_r(p) = 0$, then $\mathcal{D}_r^{-1}$ is unbounded in any neighborhood about p. In view of (B.48), $\tilde{G}$ must have a pole at p.

($\Rightarrow$) $\tilde{G}$ has a pole at $p \in D(\rho_0)^c$ implies that it must be unbounded in any neighborhood about p. Since $\tilde{G} = \mathcal{N}_r \mathcal{D}_r^{-1}$ and $\mathcal{N}_r \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_i}$ is analytic and bounded in $D(\rho_1)^c$ for some $\rho_1 < \rho_0$, it follows that $\det \mathcal{D}_r(p) = 0$.

(b) By Theorem 3.3, if $(\mathcal{N}_r, \mathcal{D}_r)$ and $(\mathcal{N}_r', \mathcal{D}_r')$ are two ρ_0-r.r.'s of $\tilde{G}$, then there exists $R \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ invertible in $\tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ such that $\mathcal{D}_r = \mathcal{D}_r' R$. Hence

$$\det \mathcal{D}_r = \det \mathcal{D}_r' \cdot \det R .$$

By Lemma 3.1, $\det R$ is an invertible element of $\tilde{\ell}_{1-}(\rho_0)$. Therefore $\det \mathcal{D}_r$ and $\det \mathcal{D}_r'$ are equal modulo an invertible element of $\tilde{\ell}_{1-}(\rho_0)$, which, by Property (2.3.3), has no zeros in $D(\rho_0)^c$. Hence, to prove Theorem 4.1(b), it suffices to show that it holds for a particular ρ_0-r.r. In particular, we choose the ρ_0-r.r. $(\mathcal{N}_r, \mathcal{D}_r)$ as given in

Procedure 3.2(r). Then, for $p \in D(\rho_0)^c$,

McMillan degree of p as a pole of $\tilde{G}$

$= $ McMillan degree of p as a pole of $\tilde{R}$ given in (3.20)

(by Remark 4.1(ii))

$= $ order of p as a zero of det $D_r \in \mathbb{C}[z]$

(by right-coprime polynomial matrix factorization)

$= $ order of p as a zero of det $\bar{\mathcal{D}}_r \in r(\rho_0)$

(by definition of $\bar{\mathcal{D}}_r$)

$= $ order of p as a zero of det $\mathcal{D}_r$

(since $\mathcal{D}_r = \bar{\mathcal{D}}_r$) .

(c) Consider $\tilde{r} := \det \mathcal{D}_r / \det \mathcal{D}_\ell$. Since $\det \mathcal{D}_r$ and $\det \mathcal{D}_\ell$ both belong to $\tilde{\ell}^{\infty}_{1-}(\rho_0)$, it follows that $\tilde{r}$ is an invertible element of $\tilde{b}(\rho_0) = [\tilde{\ell}_{1-}(\rho_0)]\cdot[\tilde{\ell}^{\infty}_{1-}(\rho_0)]^{-1}$. Furthermore, by Part (b), $\tilde{r}$ has neither zeros nor poles in $D(\rho_0)^c$. Hence $\tilde{r}$ and $\tilde{r}^{-1}$ belong to $\tilde{\ell}_{1-}(\rho_0)$. □

Proof of Lemma 4.2. We give here the proof for (4.13); the proof for (4.14) is similar. To prove that $(LE, R^{-1}\Psi_r)$ is a ρ_0-r.r. of $\tilde{G}$, we need to show that (i) $\tilde{G} = LE(R^{-1}\Psi_r)^{-1}$; (ii) $LE \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_i}$ and $R^{-1}\Psi_r \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$; (iii) $\det(R^{-1}\Psi_r) \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$; (iv) $(LE, R^{-1}\Psi_r)$ are ρ_0-r.c. Now condition (i) follows from (4.8), (4.10)-(4.12); condition (ii) holds because $L \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_o}$ and $R \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$ are unimodular; (iii) R being unimodular also implies that $\det(R^{-1}\Psi_r) = (\det R)^{-1} \cdot \prod_{i=1}^{r} \psi_i \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$; (iv) by construction of E and Ψ_r in (4.10) and (4.11),

$$\operatorname{rank}\begin{bmatrix} E \\ - \\ \Psi_{\ell} \end{bmatrix}(z) = n_i \quad \forall z \in D(\rho_0)^c ,$$

hence

$$\operatorname{rank}\begin{bmatrix} LE \\ - \\ R^{-1}\Psi_{\ell} \end{bmatrix}(z) = \operatorname{rank}\left[\begin{array}{c|c} L & 0 \\ \hline 0 & R^{-1} \end{array}\right]\begin{bmatrix} E \\ - \\ \Psi_{\ell} \end{bmatrix}(z) = n_i \quad \forall z \in D(\rho_0)^c ,$$

and thus $(LE, R^{-1}\Psi_{\ell})$ are ρ_0-r.c. in view of Corollary 3.1b(ℓ). □

<u>Proof of Theorem 4.3</u>. By Lemma 4.2, $(LE, R^{-1}\Psi_{\ell})$ is a ρ_0-r.r. Next, we note that $\det(R^{-1}\Psi_{\ell}) = (\det R)^{-1}\tilde{\chi}_G$; hence $\tilde{\chi}_G = \det(R^{-1}\Psi_{\ell})$ <u>modulo</u> an invertible element of $\tilde{\ell}_{1-}(\rho_0)$, which has neither zero nor pole in $D(\rho_0)^c$. The proof is complete by invoking Theorem 4.1. □

<u>Proof of Theorem 4.4</u>. Let $\tilde{G} = N_{\ell}\mathcal{D}_{\ell}^{-1}$ be a ρ_0-r.r.

($\Rightarrow$) Let $p \in D(\rho_0)^c$ be a pole of $\tilde{G}$. By Theorem 4.1, $\det \mathcal{D}_{\ell}(p) = 0$. Hence there is a <u>nonzero</u> $\xi \in \mathbb{C}^{n_i}$ such that

$$\mathcal{D}_{\ell}(p)\xi = \theta_{n_i} . \tag{B.51}$$

Choose the input $\tilde{e}(z) := \mathcal{D}_{\ell}(z)\xi\frac{z}{(z-p)}$. Then since $\mathcal{D}_{\ell} \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_i}$, we apply Theorem A.1 to the term $\mathcal{D}_{\ell}(z)\xi/(z-p)$ and get

$$\begin{aligned}\tilde{e}(z) &= \mathcal{D}_{\ell}(z)\xi\frac{z}{(z-p)} \\ &= \mathcal{D}_{\ell}(p)\xi\frac{z}{(z-p)} + z\tilde{\zeta}_e(z) \\ &= z\tilde{\zeta}_e(z) \qquad \text{by (B.51)}\end{aligned} \tag{B.52}$$

and hence $e \in \ell_{1-}(\rho_1)^{n_i}$ for some $\rho_1 \in]0,\rho_0[$. Next we calculate the output and apply Theorem A.1 to the term $N_{\ell}(z)\xi/(z-p)$:

$$\begin{aligned}\tilde{y}(z) &= N_r(z)\mathcal{D}_r(z)^{-1}\tilde{e}(z)\\ &= N_r(z)\xi\frac{z}{(z-p)}\\ &= N_r(p)\xi\frac{z}{(z-p)} + z\tilde{\zeta}_y(z)\end{aligned} \tag{B.53}$$

where, by defining $\tilde{h}(z) := z\tilde{\zeta}_y(z)$,

$$h \in \ell_{1-}(\rho_2)^{n_o}$$

for some $\rho_2 \in]0,\rho_0[$. Note that since $(N_r,\mathcal{D}_r)$ is ρ_0-r.c., then by Corollary 3.1b(r), $\mathrm{rank}[\mathcal{D}_r(p)^T \mid N_r(p)^T]^T = n_i$ and thus

$$\gamma := N_r(p)\xi \neq \theta_{n_o} .$$

($\Leftarrow$) By contradiction: If $p \in D(\rho_0)^c$ is not a pole of $\tilde{G}$, then for any input $e \in \ell_{1-}(\rho_1)^{n_i}$ $(\rho_1 < \rho_0)$, $\tilde{G}\tilde{e}$ is analytic at p and hence the output cannot contain a term of the form $\gamma \cdot p^k$. □

Proof of Lemma 4.3. Consider the McMillan form $M[\tilde{G}]$ of $\tilde{G}$ in (4.8)-(4.14). By Remark 4.4, since multiplication by unimodular matrices does not affect the rank of a matrix at any point in $D(\rho_0)^c$, hence $\forall z \in D(\rho_0)^c$

$$\begin{aligned}\mathrm{rank}[N_r(z)] &= \mathrm{rank}[LE](z) = \mathrm{rank}[E(z)]\\ &= \mathrm{rank}[ER](z) = \mathrm{rank}[N_\ell(z)] .\end{aligned}$$ □

Proof of Theorem 4.5. Let $\tilde{G} = \mathcal{D}_\ell^{-1}N_\ell$ be a ρ_0-l.r.

(a) Since $(\mathcal{D}_\ell,N_\ell)$ is ρ_0-l.c., there exist $V_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_o\times n_o}$ and $U_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_i\times n_o}$ such that

$$(\mathcal{D}_\ell V_\ell + N_\ell U_\ell)(z) = I_{n_o} \quad \forall z \in D(\rho_0)^c . \tag{B.64}$$

By (4.22), $\operatorname{rank}[N_\ell(z_0)] < n_i$. Hence there is a nonzero $\xi \in \mathbb{C}^{n_i}$ such that

$$N_\ell(z_0)\xi = \theta_{n_o} \; . \tag{B.65}$$

Now, since $N_\ell(z)\xi \in \tilde{\ell}_{1-}(\rho_0)^{n_o}$, by applying Theorem A.1 to the term $N_\ell(z)\xi/(z-z_0)$, we obtain

$$\begin{aligned} N_\ell(z)\xi\frac{z}{(z-z_0)} &= N_\ell(z_0)\xi\frac{z}{(z-z_0)} + z\tilde{\zeta}(z) \\ &= z\tilde{\zeta}(z) \quad \text{by (B.65)} \end{aligned} \tag{B.66}$$

and
$$z \mapsto z\tilde{\zeta}(z) \in \tilde{\ell}_{1-}(\rho_3)^{n_o} \quad \text{for some} \quad \rho_3 \in \,]0,\rho_0[\; . \tag{B.67}$$

Choose

$$\tilde{m}(z) := -U_\ell(z)\cdot z\tilde{\zeta}(z) \; . \tag{B.68}$$

By (B.67) and since $U_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o}$,

$$m \in \ell_{1-}(\rho_1)^{n_i} \quad \text{for some} \quad \rho_1 \in \,]0,\rho_0[\; .$$

Using the input defined by (4.24), we calculate

$$\begin{aligned} \tilde{y}(z) &= \tilde{G}(z)\tilde{e}(z) \\ &= D_\ell(z)^{-1}[z\tilde{\zeta}(z) - N_\ell(z)U_\ell(z)\cdot z\tilde{\zeta}(z)] && \text{by (B.66)} \\ &= D_\ell(z)^{-1}D_\ell(z)V_\ell(z)\cdot z\tilde{\zeta}(z) && \text{by (B.64)} \\ &= V_\ell(z)\cdot z\tilde{\zeta}(z) \; . \end{aligned} \tag{B.69}$$

So, by (B.67) and since $V_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_o}$

$$y \in \ell_{1-}(\rho_2)^{n_o} \quad \text{for some} \quad \rho_2 \in \,]0,\rho_0[\; .$$

(b) With the input e in (4.26), the output is given by

$$\tilde{y}(z) = \tilde{G}(z)\cdot\xi\frac{z}{(z-v)}$$

$$= \tilde{G}(v)\xi\frac{z}{(z-v)} + \mathcal{D}_\ell(z)^{-1}[\mathcal{N}_\ell(z) - \mathcal{D}_\ell(z)\mathcal{D}_\ell(v)^{-1}\mathcal{N}_\ell(v)]\xi\frac{z}{(z-v)} . \tag{B.70}$$

Since v is neither a zero nor a pole of $\tilde{G}$,

$$\tilde{G}(v)\xi \neq \theta_{n_o}$$

and the second term of (B.70) is analytic at $z = v$. (This follows because $v \in D(\rho_0)^c$ belongs to the domain of analyticity of $\mathcal{D}_\ell$, $\mathcal{N}_\ell$, and $\mathcal{D}_\ell^{-1}$.) □

Proof of Theorem 4.6. Let $G = \mathcal{D}_\ell^{-1}\mathcal{N}_\ell$ be a ρ_0-l.r. By (4.22), $\text{rank}[\mathcal{N}_\ell(z_0)] < n_o$, hence there exists a nonzero $\gamma \in \mathbb{C}^{n_o}$ such that

$$\gamma^*\mathcal{N}_\ell(z_0) = \theta^*_{n_i} . \tag{B.71}$$

Define

$$\eta^* := \gamma^*\mathcal{D}_\ell(z_0) . \tag{B.72}$$

Since $(\mathcal{D}_\ell,\mathcal{N}_\ell)$ is ρ_0-l.c., then by (B.71) and Corollary 3.1b(ℓ),

$$\eta \neq \theta_{n_o} .$$

Now given any $\xi \in \mathbb{C}^{n_i}$, choose $\pi \in \mathcal{A}(\rho_0)$ (the choice of $\pi(z)$ will be specified below, see (B.84)), and consider the input (4.28) with $m(z) := \pi(z)\xi$, i.e.

$$\tilde{e}(z) = [\frac{z}{(z-z_0)} + \pi(z)]\xi . \tag{B.73}$$

Hence

$$\eta^*\tilde{y}(z) = \gamma^*\mathcal{D}_\ell(z_0)\mathcal{D}_\ell(z)^{-1}\mathcal{N}_\ell(z)\xi[\frac{z}{(z-z_0)} + \pi(z)] . \tag{B.74}$$

Consider

$$\tilde{g} := \gamma^*\mathcal{D}_\ell(z_0)\mathcal{D}_\ell^{-1}\mathcal{N}_\ell\xi \in \tilde{b}(\rho_0) , \tag{B.75}$$

hence, by Theorem 2.1, $\tilde{g}$ admits a normalized ρ_0-representation

$$\tilde{g} = n/d \tag{B.76}$$

with $n \in \tilde{\ell}_{1-}(\rho_0)$ and $d \in \mathcal{R}^{\infty}(\rho_0)$. By (B.71), $\tilde{g}(z_0) = 0$. Hence by ρ_0-coprimeness

$$n(z_0) = 0 \ , \quad d(z_0) \neq 0 \ . \tag{B.77}$$

Applying Theorem A.1 to the term $n(z)/(z-z_0)$ and using (B.77), we obtain

$$n(z)\cdot\frac{z}{(z-z_0)} = \frac{n(z_0)}{(z-z_0)}\cdot z + z\tilde{\nu}(z) = z\tilde{\nu}(z) \tag{B.78}$$

where $\qquad z \mapsto z\tilde{\nu}(z) \in \tilde{\ell}_{1-}(\rho_1)$ for some $\rho_1 \in \,]0,\rho_0[$. (B.79)

Using (B.75), (B.76) and (B.78) in (B.74), we obtain

$$\eta * \tilde{y}(z) = z\tilde{\nu}(z)\cdot[1 + \frac{(z-z_0)}{z}\pi(z)]/d(z) \ . \tag{B.80}$$

We will show next that $\pi \in \mathcal{R}(\rho_0)$ can be chosen so that the second factor of (B.80) is a constant for all z. Once this is done, the conclusion (4.29) of the theorem follows.

Since $d \in \mathcal{R}^{\infty}(\rho_0)$, let

$$d(z) = a(z)/b(z) \tag{B.81}$$

where $a, b \in \mathbb{C}[z]$ are coprime polynomials such that b has no zeros in $D(\rho_0)^c$ and $\partial a = \partial b$. Since (B.77) holds, we can pick $\alpha \in \mathbb{C}$ such that

$$-1 + \alpha\cdot d(z_0) = 0 \ , \tag{B.82}$$

hence $(z-z_0)$ divides $[-b(z) + \alpha\cdot a(z)]$. Then with

$$p(z) := [-b(z) + \alpha\cdot a(z)]/(z-z_0) \ , \tag{B.83}$$

$p \in \mathbb{C}[z]$ and $\partial p < \partial b$. Now define

$$\pi(z) := zp(z)/b(z) \;, \tag{B.84}$$

hence $\pi \in \mathcal{R}(\rho_0)$. With this choice of π, the right-hand side of (B.80) reduces to $\alpha \cdot z\tilde{v}(z)$. □

Proof of Lemma 5.1. (5.14)$\Leftrightarrow$(5.15)$\Leftrightarrow$(5.16). Immediate by Theorem 4.1 because $(N_r, \mathcal{D})$ is a ρ_0-r.r. of $\tilde{G}_r$ and $(\mathcal{D}, N_\ell)$ is a ρ_0-l.r. of $\tilde{G}_\ell$.

(5.16)$\Rightarrow$(5.17). Since $(N_r, \mathcal{D})$ is ρ_0-r.c., there exist matrices U_r, V_r with elements in $\tilde{\ell}_{1-}(\rho_0)$ such that

$$U_r N_r + V_r \mathcal{D} = I \;. \tag{B.90}$$

Postmultiply (B.90) by $\tilde{G}_\ell := \mathcal{D}^{-1} N_\ell$,

$$U_r \tilde{G} + V_r N_\ell = \tilde{G}_\ell \;. \tag{B.91}$$

Since all elements of U_r, V_r and N_ℓ are bounded in $D(\rho_1)^c$ for some $\rho_1 < \rho_0$, $\tilde{G}_\ell$ has a pole at $p \in D(\rho_0)^c$ implies that $\tilde{G}$ has a pole at p.

(5.16)$\Leftarrow$(5.17). Since $\tilde{G} = N_r \tilde{G}_\ell$ and all elements of N_r are bounded in $D(\rho_1)^c$ for some $\rho_1 < \rho_0$, $\tilde{G}$ has a pole at $p \in D(\rho_0)^c$ implies that $\tilde{G}_\ell$ has a pole at p. □

Proof of Theorem 5.1. ($\Leftarrow$) Since u has support $\{0\}$, then $\tilde{u}(z) \equiv u_0$ $\forall z \in \mathbb{C}$. Since $y(k)$ is $O(k^{m-1}\sigma^k)$ for large k, hence $\tilde{y} = \tilde{G}\tilde{u}$ must have a pole of order m at some p, where $|p| = \sigma \geq \rho_0$. Thus $\tilde{G}$ must have a pole at p of order at least m. The conclusion follows by Lemma 5.1.

($\Rightarrow$) Assume $\tilde{\chi}(p) = 0$, with $|p| = \sigma \geq \rho_0$. By Lemma 5.1, p is a pole of $\tilde{G}$. Let m be the order of p as a pole of $\tilde{G}$. Then there exists some $u_0 \in \mathbb{C}^{n_i}$ such that $\tilde{G}u_0$ has a pole at p of order m. Choose an input sequence $u := u_0 \cdot \delta_0 \in (\mathbb{C}^{\mathbb{N}})^{n_i}$. Then the output sequence y satisfies $\tilde{y} = \tilde{G}u_0$, which has a pole of order m at p. Compute the Laurent expansion of $\tilde{y}$ at p

$$\tilde{y}(z) = \frac{\xi_m}{(z-p)^m} + \frac{\xi_{m-1}}{(z-p)^{m-1}} + \cdots + \xi_0 + \xi_{-1}(z-p) + \cdots .$$

Hence, for large k, $y(k)$ includes a term $\xi_m \cdot \binom{k-1}{m-1} p^{(k-m)}$, which is $O(k^{m-1}\sigma^k)$. □

Proof of Theorem 5.2. ($\Rightarrow$) By contraposition: If there is $p \in D(\rho)^c$ such that $\tilde{\chi}(p) = 0$. Then by Lemma 5.1, $\tilde{G}$ has a pole at $p \in D(\rho)^c$. Hence $\tilde{G}$ is not bounded in $D(\rho)^c$ and cannot belong to $\tilde{\ell}_1(\rho)^{n_o \times n_i}$.

($\Leftarrow$) Since $\tilde{\chi}(z) \neq 0$ for all $z \in D(\rho)^c$ and $\tilde{\chi} \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$ (i.e. $\chi(0) \neq 0$), hence $\tilde{\chi} := \det \mathcal{D}$ is an invertible element of $\tilde{\ell}_1(\rho)$ by Property (2.2.6). Furthermore, since N_r, $\mathcal{D}$, N_ℓ all have elements in $\tilde{\ell}_{1-}(\rho_0) \subset \tilde{\ell}_1(\rho)$, it follows by Cramer's rule that $\tilde{G} := N_r \mathcal{D}^{-1} N_\ell$ belongs to $\tilde{\ell}_1(\rho)^{n_o \times n_i}$. □

Proof of Theorem 6.1. (i) follows from Lemma 5.1 and (6.9).

(ii) Since $\tilde{P} \in \tilde{b}(\rho_0)^{n_o \times n_i}$, thus by Theorem 3.1

$$\tilde{P} \text{ has a } \rho_0\text{-r.r. } (N_{pr}, \mathcal{D}_{pr}) . \tag{B.100}$$

Hence by applying Theorem 4.1(c) to (6.12) and (B.100), there exists $\tilde{r} \in \tilde{\ell}_{1-}(\rho_0)$ invertible in $\tilde{\ell}_{1-}(\rho_0)$ such that

$$\det \mathcal{D}_{pr} = \tilde{r} \cdot \det \mathcal{D}_{p\ell} . \tag{B.101}$$

Recall the terms defined in (6.1)-(6.14), and consider the following matrices in $\tilde{\ell}_{1-}(\rho_0)^{(n_i+n_o)\times(n_i+n_o)}$:

$$N := \begin{array}{c|c} & \begin{array}{cc} n_o & n_i \end{array} \\ \begin{array}{c} n_o \\ n_i \end{array} & \left[\begin{array}{c|c} 0 & N_{pr} \\ \hline -N_{cr} & 0 \end{array}\right] \end{array} \qquad \mathcal{D} := \begin{array}{c|c} & \begin{array}{cc} n_o & n_i \end{array} \\ \begin{array}{c} n_o \\ n_i \end{array} & \left[\begin{array}{c|c} \mathcal{D}_{cr} & 0 \\ \hline 0 & \mathcal{D}_{pr} \end{array}\right] \end{array} . \tag{B.102}$$

Then using Corollary 3.1b(r) $(N,\mathcal{D})$ is a ρ_0-r.r. of $\tilde{G} \in \tilde{b}(\rho_0)^{(n_i+n_o)\times(n_i+n_o)}$. Moreover, $(N,\mathcal{D})$ being ρ_0-r.c. implies that $(\mathcal{D},\mathcal{D}+N)$ is ρ_0-r.c. By (B.102)

$$\mathcal{D}+N = \left[\begin{array}{c|c} \mathcal{D}_{cr} & N_{pr} \\ \hline -N_{cr} & \mathcal{D}_{pr} \end{array}\right] = \left[\begin{array}{c|c} I & \tilde{P} \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|c} I+\tilde{P}\tilde{C} & 0 \\ \hline -\tilde{C} & I \end{array}\right] \left[\begin{array}{c|c} \mathcal{D}_{cr} & 0 \\ \hline 0 & \mathcal{D}_{pr} \end{array}\right]$$

and thus

$$\det[\mathcal{D}+N] = \det[I+\tilde{P}\tilde{C}]\cdot\det \mathcal{D}_{pr}\cdot\det \mathcal{D}_{cr} . \tag{B.103}$$

Hence using (6.7), $\det[\mathcal{D}+N] \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$, and so $(\mathcal{D},\mathcal{D}+N)$ is a ρ_0-r.r. of

$$\tilde{H}_{eu} = (I+\tilde{G})^{-1} = \mathcal{D}(\mathcal{D}+N)^{-1} \in \tilde{b}(\rho_0)^{(n_i+n_o)\times(n_i+n_o)} . \tag{B.104}$$

Similarly, $(J^{-1}N,\mathcal{D}+N)$ is a ρ_0-r.r. of

$$\tilde{H}_{yu} = J^{-1}\tilde{G}(I+\tilde{G})^{-1} = J^{-1}N(\mathcal{D}+N)^{-1} \in \tilde{b}(\rho_0)^{(n_i+n_o)\times(n_i+n_o)} \tag{B.105}$$

Now by (B.103), using (6.12), (6.13) and (B.101)

$$\begin{aligned} \det[\mathcal{D}+N] &= \det[I+\mathcal{D}_{p\ell}^{-1}N_{p\ell}N_{cr}\mathcal{D}_{cr}^{-1}]\cdot\tilde{r}\cdot\det \mathcal{D}_{p\ell}\cdot\det \mathcal{D}_{cr} \\ &= \tilde{r}\cdot\det[\mathcal{D}_{p\ell}\mathcal{D}_{cr}+N_{p\ell}N_{cr}] \\ &= \tilde{r}\cdot\tilde{\chi} \quad \text{by (6.14)} . \end{aligned} \tag{B.106}$$

Observe that $\tilde{r}$ is bounded and bounded away from zero in $D(\rho_1)^c$, for some $\rho_1 < \rho_0$. The conclusion follows by applying Theorem 4.1 to (B.104) and (B.105), using (B.106). □

Proof of Lemma 7.1. (7.5)$\Leftrightarrow$(7.6). This is immediate by (7.3).

(7.4)$\Leftarrow$(7.6). This is immediate by the second equation in (7.6).

(7.4)$\Rightarrow$(7.5). By (7.3), $(X^p, Y^p) := (U_\ell \mathcal{D}, V_\ell \mathcal{D})$ is a particular solution of (7.4). Hence (X,Y) is a solution of (7.4) if and only if

$$(X^h, Y^h) := (X - U_\ell \mathcal{D}, Y - V_\ell \mathcal{D}) \tag{B.110}$$

is a solution of the homogeneous equation

$$N_\ell X^h + \mathcal{D}_\ell Y^h = 0_{n_o \times n_o} . \tag{B.111}$$

It remains to prove that (X^h, Y^h) in (B.110) is equal to $(-\mathcal{D}_r N, N_r N)$ for some $N \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o}$. Define

$$N := -\mathcal{D}_r^{-1} X^h \in \tilde{b}(\rho_0)^{n_i \times n_o} . \tag{B.112}$$

Then by (B.111), using (7.2),

$$Y^h = -\mathcal{D}_\ell^{-1} N_\ell X^h = -N_r \mathcal{D}_r^{-1} X^h = N_r N . \tag{B.113}$$

By (7.3), $V_r \mathcal{D}_r + U_r N_r = I$; hence postmultiplying by N

$$-V_r X^h + U_r Y^h = N \tag{B.114}$$

and so $N \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o}$ by the closure properties of $\tilde{\ell}_{1-}(\rho_0)$. Thus by (B.110), (B.112) and (B.113), we obtain as required

$$X = U_\ell \mathcal{D} - \mathcal{D}_r N , \qquad Y = V_\ell \mathcal{D} + N_r N .$$

The proof of (7.7) proceeds as follows: ($\Leftarrow$) By Lemma 3.4(r), there exist $U \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_i}$, $V \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_o}$ such that $UN + V\mathcal{D} = I$. Premultiplying (7.6) by $[U \,|\, V]$, we get

$$I = (-UV_r + VN_\ell)X + (UU_r + VD_\ell)Y \ .$$

The matrices in parentheses have elements in $\tilde{\ell}_{1-}(\rho_0)$, due to the closure properties of $\tilde{\ell}_{1-}(\rho_0)$. Hence (X,Y) are ρ_0-r.c. by Lemma 3.4(r).

(7.7)($\Rightarrow$) follows by interchanging (N,D) and $(-X,Y)$ in the above argument, and by using (7.5) instead of (7.6).

(b) Since $N_\ell = D_\ell \tilde{G}$ and $\det D_\ell \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$

(i.e. $\lim_{|z|\to\infty} \det \mathcal{D}_\ell(z) \neq 0$),

$$\lim_{|z|\to\infty} N_\ell(z) = 0_{n_o \times n_i} \quad \text{by (7.8)} . \tag{B.115}$$

Now by (7.6), $\mathcal{D} = N_\ell X + \mathcal{D}_\ell Y \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_o}$. Hence using (B.115),

$$\lim_{|z|\to\infty} \det \mathcal{D}(z) = \lim_{|z|\to\infty} \det \mathcal{D}_\ell(z) \cdot \lim_{|z|\to\infty} \det Y(z) . \tag{B.116}$$

Conclusion (7.9) follows because $\lim_{|z|\to\infty} \det \mathcal{D}_\ell(z) \neq 0$. □

Lemma B.1. Let $\rho \in]0,1[$ and $g, u \in \ell_1(\rho)$. Then $y := g*u$ satisfies $y_k = o(\rho^k)$ as $k \to \infty$.

Proof. Since $g, u \in \ell_1(\rho)$, $\bar{g} := (g(k)\rho^{-k})_{k=0}^{\infty}$ and $\bar{u} := (u(k)\rho^{-k})_{k=0}^{\infty}$ belong to ℓ_1; hence $\bar{y} := \bar{g}*\bar{u} \in \ell_1$. By simple calculations, $\bar{y}$ and y are related by $\bar{y} = (y(k)\rho^{-k})_{k=0}^{\infty}$. Consequently, since $\bar{y} \in \ell_1$, $\bar{y}(k) = y(k)\rho^{-k} \to 0$ as $k \to \infty$; hence $y(k) = o(\rho^k)$ as $k \to \infty$. □

Proof of Theorem 7.1. (i) We first verify Procedure 7.1 step by step:

Steps 1 and 2 are self-explanatory.

Step 3. To show that $(\mathcal{D}_\ell, N_\ell)$ is a ρ_0-l.r. of $\tilde{F}$, note that $\mathcal{D}_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_o}$, $N_\ell \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_i}$ and $\det \mathcal{D}_\ell \in \tilde{\ell}_{1-}^{\infty}(\rho_0)$; $\tilde{F} = \mathcal{D}_\ell^{-1} N_\ell$. Furthermore, in view of (7.22), $\operatorname{rank}[\mathcal{D}_\ell(z) \mid N_\ell(z)] = n_o$, $\forall z \in D(\rho_0)^c$; hence $(\mathcal{D}_\ell, N_\ell)$ is ρ_0-l.c.

Step 4 is self-explanatory, in view of Lemma 7.1.

Step 5. To show that $(N_{cr}, \mathcal{D}_{cr})$ is a ρ_0-r.r. of $\tilde{C}$, it is immediate by definition that $\tilde{C} = N_{cr} \mathcal{D}_{cr}^{-1}$, where $N_{cr} \in \tilde{\ell}_{1-}(\rho_0)^{n_i \times n_o}$ and $\mathcal{D}_{cr} \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_o}$. Moreover, by (7.32), (7.33) and Lemma 7.1(a)

$$\mathcal{D} = N_\ell X + \mathcal{D}_\ell Y$$
$$= N_{p\ell} X + \frac{\phi}{d}\mathcal{D}_{p\ell} Y \ ; \qquad \text{(B.117)}$$

hence by (7.11) and Lemma 7.1(b), $\det \mathcal{D}_{cr} = \det(\frac{\phi}{d}Y) \in \tilde{\ell}^{\infty}_{1-}(\rho_0)$. Lastly, it remains to show that $(N_{cr}, \mathcal{D}_{cr})$ are ρ_0-r.c.: by (B.117) and (7.16),

$$\mathcal{D}(z) = N_{p\ell}(z) X(z) \quad \forall z \in Z[\phi_u] \cup Z[\phi_w] \ ; \qquad \text{(B.118)}$$

by (7.25), $\det \mathcal{D}(z) \neq 0 \ , \quad \forall z \in Z[\phi_u] \cup Z[\phi_w] \subset D(1)^c$;

hence by (B.118),

$$\operatorname{rank}[X(z)] = n_o \quad \forall z \in Z[\phi_u] \cup Z[\phi_w] \ ; \qquad \text{(B.119)}$$

since (X, Y) are ρ_0-r.c. by Lemma 7.1, then using (B.119)

$$\operatorname{rank}\begin{bmatrix} X \\ -- \\ Y\frac{\phi}{d} \end{bmatrix}(z) = n_o \quad \forall z \in D(\rho_0)^c \ ; \qquad \text{(B.120)}$$

hence according to Corollary 3.1b(r), $(N_{cr}, \mathcal{D}_{cr})$ is ρ_0-r.c.

Throughout the procedure, all matrices concerned have elements corresponding to sequences in $\mathbb{R}^{\mathbb{N}}$: this property is preserved in C.

(ii) By (6.14), (7.35) and (B.117), the characteristic function of the feedback system S is

$$\tilde{\chi} = \det \mathcal{D} \ ; \qquad \text{(B.121)}$$

hence by (7.25), $Z[\tilde{\chi}; D(\rho_0)^c] = \Lambda$ and condition (b) of Problem (SP) is satisfied. Furthermore by definition of Λ, $\tilde{\chi}(z) \neq 0 \quad \forall z \in D(1)^c$; hence condition (a) is also satisfied in view of Remark 6.2. To show that condition (c) of problem (SP) holds, we calculate first the transfer functions for the maps $u_s \mapsto e_s$ and $w_p \mapsto e_s$, respectively,

$$\begin{aligned}\tilde{H}_{e_s u_s} &= [I + \tilde{P}\tilde{C}]^{-1} \\ &= \mathcal{D}_{cr}[\mathcal{D}_{p\ell}\mathcal{D}_{cr} + N_{p\ell}N_{cr}]^{-1}\mathcal{D}_{p\ell} \\ &= \frac{\phi}{d}\mathcal{Y}\mathcal{D}^{-1}\mathcal{D}_{p\ell} \ ,\end{aligned} \tag{B.122}$$

and

$$\begin{aligned}\tilde{H}_{e_s w_p} &= -[I + \tilde{P}\tilde{C}]^{-1}\tilde{P} \\ &= -\mathcal{D}_{cr}[\mathcal{D}_{p\ell}\mathcal{D}_{cr} + N_{p\ell}N_{cr}]^{-1}N_{p\ell} \\ &= -\frac{\phi}{d}\mathcal{Y}\mathcal{D}^{-1}N_{p\ell} \ .\end{aligned} \tag{B.123}$$

Since the list Λ is finite, there exists some $\rho \in]\rho_0,1[$ such that $\det \mathcal{D}(z) \neq 0 \ \ \forall z \in D(\rho)^c$; hence by applying Corollary 5.2b to $\mathcal{D}^{-1} \in \tilde{b}(\rho_0)^{n_o \times n_o}$, we have $\mathcal{D}^{-1} \in \tilde{\ell}_1(\rho)^{n_o \times n_o}$; consequently

$$\mathcal{Y}\mathcal{D}^{-1}\mathcal{D}_{p\ell} \in \tilde{\ell}_1(\rho)^{n_o \times n_o} \ , \qquad -\mathcal{Y}\mathcal{D}^{-1}N_{p\ell} \in \tilde{\ell}_1(\rho)^{n_o \times n_i} \ . \tag{B.124}$$

Also, by construction of ϕ and d,

$$\frac{\phi}{d}\tilde{u}_s = \frac{\phi}{d} \cdot \frac{v_u}{\phi_u} \in \tilde{\ell}_{1-}(\rho_0)^{n_o} \subset \tilde{\ell}_1(\rho)^{n_o} \tag{B.125}$$

and

$$\frac{\phi}{d}\tilde{w}_p = \frac{\phi}{d} \cdot \frac{v_w}{\phi_w} \in \tilde{\ell}_{1-}(\rho_0)^{n_i} \subset \tilde{\ell}_1(\rho)^{n_i} \ . \tag{B.126}$$

Now, for arbitrary v_u and v_w satisfying (7.13) and (7.14)

$$\begin{aligned}\tilde{e}_s &= \tilde{H}_{e_s u_s}\tilde{u}_s + \tilde{H}_{e_s w_p}\tilde{w}_p \\ &= [\mathcal{Y}\mathcal{D}^{-1}\mathcal{D}_{p\ell}]\cdot[\tfrac{\phi}{d}\tilde{u}_s] + [-\mathcal{Y}\mathcal{D}^{-1}N_{p\ell}]\cdot[\tfrac{\phi}{d}w_p] \ .\end{aligned} \tag{B.127}$$

Applying Lemma B.1 to (B.127), using (B.124), (B.125) and (B.126), we have

$$e_s(k) = o(\rho^k) \quad \text{as} \quad k \to \infty \ . \tag{B.128}$$

To check condition (d), consider any perturbed plant $\tilde{\tilde{P}} \in \tilde{b}(\rho_0)^{n_o \times n_i}$ satisfying (7.10), and for which the feedback system S with controller

$\tilde{C}$ and plant $\tilde{\tilde{P}}$ has transfer functions $\tilde{\tilde{H}}_{eu}$, $\tilde{\tilde{H}}_{yu}$ that are ℓ_p-stable, $\forall p \in [1,\infty]$. By Theorem 3.1 $\tilde{\tilde{P}}$ admits a ρ_0-l.r. $(\bar{\mathcal{D}}_{p\ell},\bar{\mathcal{N}}_{p\ell})$ and the characteristic function $\tilde{\chi}$ becomes

$$\tilde{\tilde{\chi}} = \det \bar{\mathcal{D}} \tag{B.129}$$

where

$$\bar{\mathcal{D}} := \bar{\mathcal{D}}_{p\ell}\mathcal{D}_{cr} + \bar{\mathcal{N}}_{p\ell}\mathcal{N}_{cr} \in \tilde{\ell}_{1-}(\rho_0)^{n_o \times n_o} .$$

By Corollary 5.2a, the ℓ_p-stability of the perturbed feedback system implies that $\tilde{\tilde{\chi}}(z) \neq 0$ $\forall z \in D(1)^c$ and, since $\tilde{\tilde{\chi}} \in \tilde{\ell}_{1-}(\rho_0)$ and $D(1)$ is compact, there is a $\bar{\rho} \in [\rho_0,1[$ such that $\det \bar{\mathcal{D}}(z) = \tilde{\tilde{\chi}}(z) \neq 0$, $\forall z \in D(\bar{\rho})^c$. Calculating as above, we obtain that

$$\tilde{\tilde{H}}_{e_s u_s} = \frac{\phi}{d}\mathcal{Y}\bar{\mathcal{D}}^{-1}\bar{\mathcal{D}}_{p\ell} \quad \text{and} \quad \tilde{\tilde{H}}_{e_s w_p} = -\frac{\phi}{d}\mathcal{Y}\bar{\mathcal{D}}^{-1}\bar{\mathcal{N}}_{p\ell} \tag{B.134}$$

where $\mathcal{Y}\bar{\mathcal{D}}^{-1}\mathcal{D}_{p\ell} \in \tilde{\ell}_1(\bar{\rho})^{n_o \times n_o}$, $\quad -\mathcal{Y}\bar{\mathcal{D}}^{-1}\bar{\mathcal{N}}_{p\ell} \in \tilde{\ell}_1(\bar{\rho})^{n_o \times n_i}$. (B.135)

The arguments of (B.125)-(B.128) can be repeated here and thus condition (d) of problem (SP) is satisfied. □

Proof of Theorem 8.1. The proof of this theorem with the general $\mathbb{A}$, $\mathbb{B}$ notation can be found in [Des 5]. Just to demonstrate this theorem in terms of the system descriptions we are now concerned with, we give the proof for equivalence (8.15):

(⇒) (8.13) and $\tilde{Q} \in \tilde{\ell}_{1s}^{m\times m}$ imply that all elements of $\tilde{H}_{yu}$ (see (8.9a)) are in $\tilde{\ell}_{1s}^{m\times m}$. Since $\tilde{P}$ is in the radical $\tilde{b}_s(\rho_0)^{m\times m}$, $I - \tilde{P}\tilde{Q}$ has an inverse in $\tilde{b}(\rho_0)^{m\times m}$; and since $\tilde{Q}$ is in $\tilde{\ell}_{1s}^{m\times m}$, (8.9b) shows that $\tilde{C}$ is in the radical $\tilde{b}_s(\rho_0)^{m\times m}$.

(⇐) This is immediate since $\tilde{Q}$ is a submatrix of $\tilde{H}_{yu}$. □

Proof of Theorem 8.3. It suffices to show that $\tilde{H}_{y_s u_s}$ and $\tilde{Q}$ satisfy Theorem 8.2. We note that since $\tilde{P} \in \tilde{\ell}_{1s}^{m\times m}$, then in (8.21)

$$\mathrm{ord}_{c_j}[\tilde{P}^{-1}] \leq 0 \ , \quad j = 1,2,\ldots,m \ .$$

Hence in (8.19), $\tilde{H}_{y_s u_s} \in \mathcal{A}(1)^{m\times m} \cap \mathbb{R}(s)^{m\times m} \subset \tilde{\ell}_1^{m\times m}$. Furthermore, $\tilde{Q} := \tilde{P}^{-1}\tilde{H}_{y_s u_s} \in (\tilde{\mathbb{R}}^{\mathbb{N}})^{m\times m}$. Since <u>all poles of</u> $\tilde{P}^{-1}$ in $D(\rho_0)^c$ are <u>zeros of</u> $\tilde{P}$, and these poles are cancelled by $\hat{n}_j$, $j = 1,2,\ldots,m$ in (8.18), hence $\tilde{Q}$ is analytic in $D(1)^c$. By Remark 2.2(ii),

$$\tilde{Q} \in \tilde{\ell}_1^{m\times m} \cap (\tilde{\mathbb{R}}^{\mathbb{N}})^{m\times m} \ . \qquad \square$$

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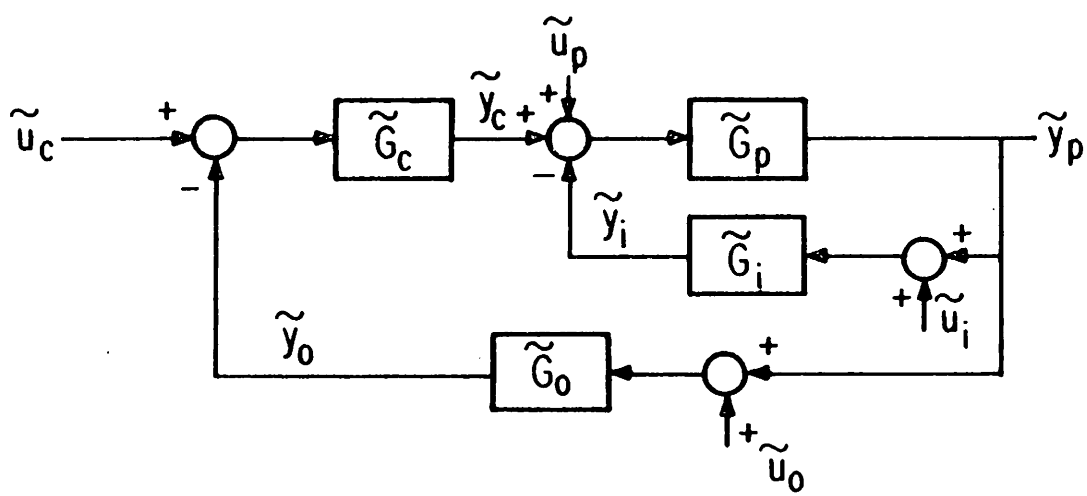

Fig. 5-1. Example of an interconnected system.

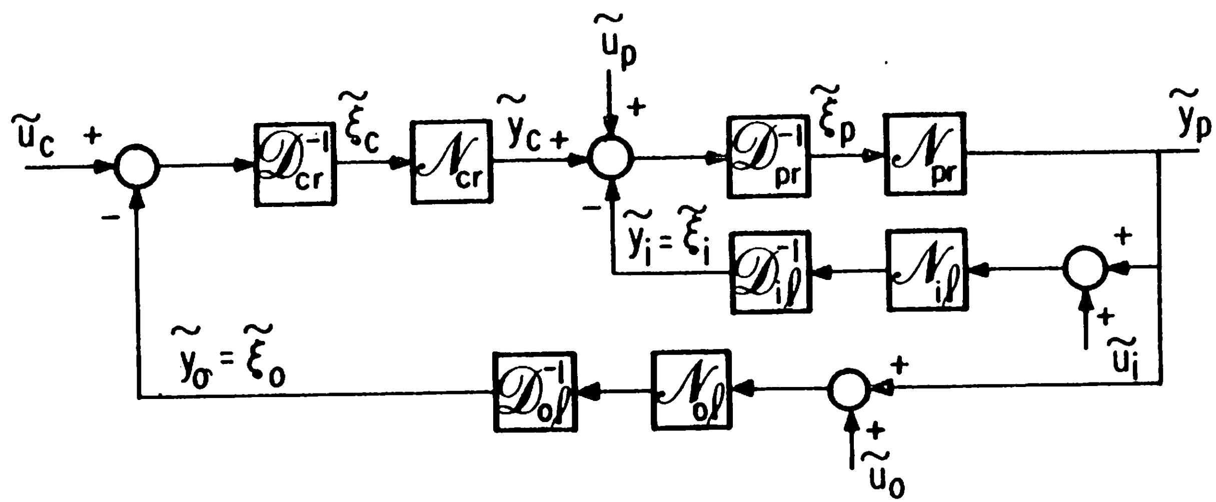

Fig. 5-2. Interconnected system of Fig. 5-1 with coprime factorization for each subsystem.

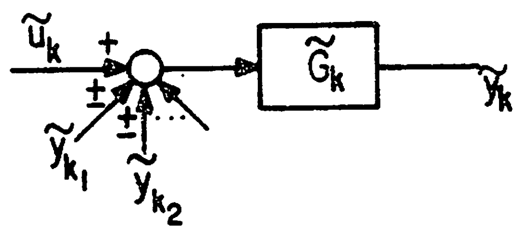

Fig. 5-3. Model for each individual system.

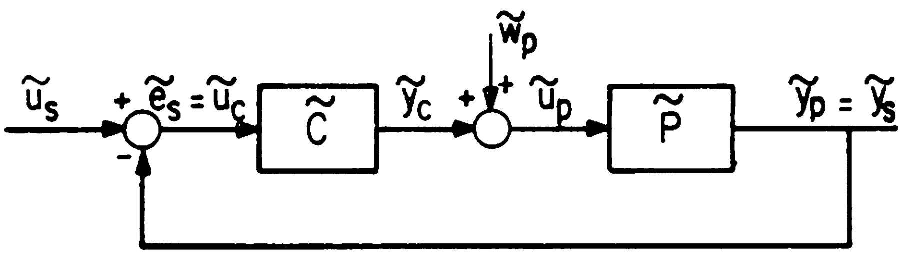

Fig. 6-1. Feedback system S with plant $\tilde{P}$ and controller $\tilde{C}$.

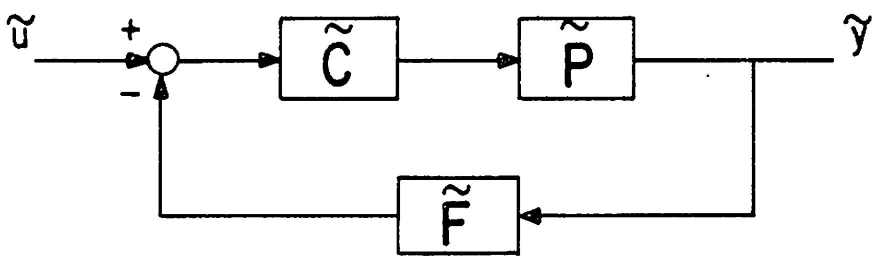

Fig. 9-1. General feedback system.