CONVEX GEOMETRY AND GROUP CHOICE

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ABSTRACT

A geometrical approach to group choice theory is developed. Main binary relation spaces of weak preference, strict preference and indifference are described. Convexity in a binary relation space is studied in connection with the Pareto principle. A method of constructing admissible group decisions is suggested.

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1. Introduction

A general framework of a geometrical approach to the problem of aggregation of individual preferences in group choice theory is described in this paper. This approach is based on a study of geometrical structures of binary relation spaces. One of these stuctures - distance - has already been used for this purpose (see, for example, [1, 2, 5, 6, 11, 15]). The suggested approach is based on the notion of convexity in binary relation spaces; this notion turns out to be very closely related to the classical Pareto principle in group choice theory. In contrast to classical axiomatic methods a geometrical approach considers an iterated procedure of constructing more and more "narrow" sets of "admissible" preferences but does not define them by properties or formulas. Sets of admissible preferences - they are called "radicals" in the paper - are defined by simple geometrical properties and contain preferences which are close to Pareto nondominacy binary relation.

2. Preferences

A binary relation R on a set S is a subset of a direct product SxS. Both notations $(x,y) \in R$ and xRy will be used below; $\bar{R}$ denotes a complement of R in SxS and the converse R^{-1} is defined by $xR^{-1}y \leftrightarrow yRx$. Binary relations are classified here by their basic properties:

1. Reflexivity: $\forall x \in S$, xRx;
2. Antireflexivity: $\forall x \in S$, ~(xRx);
3. Symmetry: $\forall x, y \in S$, xRy $\rightarrow$ yRx;
4. Antisymmetry: $\forall x, y \in S$, (xRy & yRx) $\rightarrow$ x = y;
5. Asymmetry: $\forall x, y \in S$, xRy $\rightarrow$ ~(yRx);

6. Completeness: $\forall x, y \in S, (x \neq y) \rightarrow (xRy \vee yRx)$;

7. Transitivity: $\forall x, y, z \in S, (xRy \ \& \ yRz) \rightarrow (xRz)_L$;

8. Negative transitivity: $\forall x, y, z \in S, xRz \rightarrow (xRy \vee yRz)$.

Various combinations of these properties separate classes of binary relations. Firstly, the following classes of weak preferences are defined:

1. Weak Preferences (WP). These are reflexive complete binary relations.
2. Quasi-transitive Preferences (QT) are weak preferences with negative transitivity property (see [21]).
3. Orderings (O) are transitive weak preferences (complete preorders in [9]).
4. Linear Orderings (L) are antisymmetric orderings (complete orders in [9]). Obviously, we have the following inclusions:

$$\underline{L} \subset \underline{O} \subset \underline{QT} \subset \underline{WP} \tag{2.1}$$

We recall definitions of strict preferences and indifference relations for weak preferences [20]. Let R be a weak preference. A strict preference P for R is defined by

$$xPy \leftrightarrow [xRy \ \& \ \sim(yRx)] \tag{2.2}$$

An indifference relation I for R is defined by

$$xIy \leftrightarrow (xRy \ \& \ yRx) \tag{2.3}$$

Now, the corresponding classes of strict preferences are defined:

1. Strict Preferences (SP). These are symmetric binary relations.
2. Strict Partial Orderings (P) are transitive strict preferences.

3. Quasi-series (QS) are strict preferences with negative transitivity property (weak orders in [9]). If P is a quasi-series then there is a partition $S = S_1 \cup S_2 \cup \ldots \cup S_k$ such that xPy iff $x \in S_i$, $y \in S_j$ for $i < j$ (see [15]).

4. Strict Linear Orderings (SL) are complete strict orderings.

We have the following inclusions

$$\underline{SL} \subset \underline{QS} \subset \underline{P} \subset \underline{SP} \tag{2.4}$$

Finally, we define the corresponding classes of indifference relations:

1. Indifference Relations (I). These are reflexive symmetric relations.

2. Transitive-orientated Relations (TO) are indifference relations for quasi-transitive preferences. If I is a transitive-orientated relation then there is a strict partial ordering P such that the triple (P, I, P^{-1}) is a partition of a direct product SxS ([10]).

3. Equivalences (E) are transitive indifference relations.

4. Diagonal Relation (D) is a diagonal in SxS.

We have the following inclusions

$$\underline{D} \subset \underline{E} \subset \underline{TO} \subset \underline{I} \tag{2.5}$$

It follows from (2.2) and (2.3) that

$$P \cap I = \phi,\ P \cup I = R, \tag{2.6}$$

$$P = \overline{R}^{-1},\ R = \overline{P}^{-1},\ \text{and} \tag{2.7}$$

$$I = R \cap R^{-1} = \overline{P \cup P}^{-1}. \tag{2.8}$$

where R is a weak preference. Let R be any binary relation on S. Then

$\alpha : R \rightarrow \bar{R}^{-1}$ is a one-to-one mapping of the set of all binary relations onto itself. This is a mapping of period two, i.e., α^2 is an identity. By, (2.7), α establishes one-to-one correspondence between $\underline{WP}$ and $\underline{SP}$, $\underline{QT}$ and $\underline{P}$, $\underline{O}$ and $\underline{QS}$, $\underline{L}$ and $\underline{SL}$, respectively. Let us consider mappings $\beta : R \rightarrow R \cap R^{-1}$ and $\beta' : R \rightarrow \overline{R \cup R}^{-1}$. Then β maps $\underline{WP}$, $\underline{QT}$, $\underline{O}$ and $\underline{L}$ onto $\underline{I}$, $\underline{TO}$, $\underline{E}$ and $\underline{D}$, respectively. In the same way, β' maps $\underline{SP}$, $\underline{P}$, $\underline{QS}$ and $\underline{SL}$ onto $\underline{I}$, $\underline{TO}$, $\underline{E}$ and $\underline{D}$, respectively.

Classes of binary relations defined above are represented on fig. 2.1 (see, also, [17] where a similar diagram is introduced). Vertical arrows represent natural embeddings defined by (2.1), (2.4) and (2.5). The following theorem is proven in [13].

Theorem 2.1. Diagram 2.1 is a commutative diagram of mappings.

3. Binary relation spaces

Further in this paper, S is supposed to be a fixed finite set.

Each class of binary relations defined above have some natural geometrical structures. We will use the term "space" instead of "class" assuming these structures. Actually, these geometrical structures may be defined in a more general case. We begin with the following

Definition 3.1 ([12]). A binary relation space (over S) is any non-empty subset of the set of all binary relations on S.

Any class from diagram 2.1 may be regarded as an example of a binary relation space. Binary relations constituting a space $\underline{R}$ will be called points of this space.

Most important geometrical structures on binary relation spaces are listed below.

1. Partial ordering. This relation on a binary relation space is induced by a natural inclusion relation for binary relations regarded as subsets of SxS:

$$\forall R', R'' \in \underline{R},\ R' \underset{\sim}{\alpha} R'' \leftrightarrow R' \subseteq R'' \tag{3.1}$$

2. Betweeness relation. A point R lies between points R' and R" in a space $\underline{R}$ iff

$$R' \cap R'' \subseteq R \subseteq R' \cup R''. \tag{3.2}$$

[R',R"] denotes a set of all points of a space $\underline{R}$ which lie between R' and R". Betweenness relation (3.2) on a space $\underline{R}$ is induced by a well-known betweenness relation on the lattice of all binary relations (for general case see [3,4,19]).

3. Convexity. A natural notion of convexity is based on betweenness relation (3.2). A set X in a space $\underline{R}$ is said to be a convex set iff

$$(\forall R', R'')\ (R',R'' \in X \ \&\ R \in [R',R'']) \rightarrow (R \in X).$$

A convex hull C(X) of a set X is defined as the smallest convex set containing X. Convex structures on binary relation spaces were studied in [12] and [19].

4. Distance. Let us define a distance on a space $\underline{R}$ as a function

$$d(R',R'') = |R' \Delta R''| \tag{3.3}$$

where $|A|$ denotes a cardinality of a set A and Δ is an operation of symmetric difference. It is easy to verify that d satisfies usual axioms of distance (see [15]). For spaces from diagram 2.1 a distance can be defined axiomatically. In all cases studied thus defined distance is that of (3.3) within to a constant factor (see [14] for $\underline{E}$, [5] for

$\underline{P}$ and [9] for $\underline{O}$). Definitions (3.2) and (3.3) are consistent, since $R \in [R_1,R_2]$ iff $d(R_1,R) + d(R,R_2) = d(R_1,R_2)$.

It should be noted that any binary relation space $\underline{R}$ can be regarded as embedded in the lattice B(S) of all binary relations on S. Then all geometrical structures defined above are induced by corresponding structures on the lattice B(S). On the other hand, a space $\underline{R}$, generally speaking, is not a sublattice in B(S) with respect to operations of union and intersection. This is the reason why many obvious geometrical properties of lattices are not fulfilled in binary relation spaces (lattice-theoretical study of the subject can be found in [19]).

Geometrical structures on binary relation spaces defined above are closely related to each other; some more complex structures could be defined based on these structures, the most important of which is a notion of a linear segment.

Definition 3.2. A linear segment in a space $\underline{R}$ between points R' and R" is a sequence $R_1,\ldots,R_k$ of distinct points in $\underline{R}$ such that:

1) $R_1 = R'$ and $R_k = R''$,
2) $R_i \in [R_m,R_\ell]$ for $m \leq i \leq \ell$, and
3) $R \in [R_i,R_{i+1}] \rightarrow R = R_i$ or $R = R_{i+1}$, for all $i < k$.

Theorem 3.1. ([12]). In any binary relation space $\underline{R}$ there is a linear segment between any two points.

Linear segments in binary relation spaces are a natural extension of the notion of segment in usual geometry. In particular, the following property is fulfilled:

$$d(R',R'') = \sum_{i=1}^{k-1} d(R_i,R_{i+1}),$$

where $R_1,\ldots,R_k$ is a linear segment between R' and R". However, it should be noted that binary relation spaces admit, generally speaking, a few different linear segments between given points. In spite of this fact, linear segments are very useful in the study of binary relation spaces.

At the conclusion of this section we return to diagram 2.1 of binary relation spaces. Theorem 2.1 can be completed by the following

Theorem 3.2. The mapping α is a dual isomorphism of spaces $\underline{WP}$, $\underline{QT}$, $\underline{O}$, $\underline{L}$ and $\underline{SP}$, $\underline{P}$, $\underline{QS}$, $\underline{SL}$, respectively, with respect to main geometrical structures, i.e.

1) α inverts the partial ordering $\underset{\sim}{\alpha}$;

2) α preserves the betweenness relation;

3) α preserves distances;

4) α transfers convex sets into convex sets.

By this theorem, all spaces of weak preferences are dual isomorphic to correspondent spaces of strict preferences.

4. Pareto-convexity

Geometrical properties of, well-known in a group choice theory, Pareto principle will be studied in this section. The following version of this principle is employed in this paper (see [15] for motivation):

Pareto Principle. Let $R_1,\ldots,R_n$ be individual preference relations. Then a group preference R satisfies

$$\bigcap_i R_i \subseteq R \subseteq \bigcup_i R_i \qquad (4.1)$$

From a formal standpoint, (4.1) is an immediate generalization of betweenness relation (3.2). By using geometrical language, we read (4.1):

"point R lies between points $R_1,\ldots,R_n$". A point R satisfying (4.1) is also called a Pareto-point of a set $\{R_1,\ldots,R_n\}$.

A fixed binary relation space $\underline{R}$ is considered below. The set of all Pareto-points of a set $X \subseteq \underline{R}$ will be denoted by P(X).

Definition 4.1. A set X is said to be a Pareto-convex set iff

$R \in P(X) \rightarrow R \in X.$

The following theorem shows that a set of all Pareto-points of a set X can be regarded as a Pareto-convex hull of this set.

Theorem 4.1. ([12]). P(X) is the smallest Pareto-convex set containing X.

Connections between convexity and Pareto-convexity are studied in the rest of this section.

Theorem 4.2. ([12]). Any Pareto-convex set in $\underline{R}$ is a convex set in this space. In particular, $C(X) \subseteq P(X)$ for all X.

The following example shows that C(X) can be a proper subset of P(X).

Example 4.1. Let us consider a space $\underline{E}$ over a set $S = \{a,b,c\}$. Points of $\underline{E}$ can be regarded as partitions of S. Let us consider points $I_1 = \{a\} \cup \{b,c\}$, $I_2 = \{b\} \cup \{a,c\}$, $I_3 = \{c\} \cup \{a,b\}$. Let $X = \{I_1,I_2,I_3\}$. It is easy to verify that $I_4 = \{a,b,c\}$ belongs to P(X) and does not belong to C(X).

Hence, convexity and Pareto-convexity are different notions in the space $\underline{E}$. It turns out that the space $\underline{E}$ (and, possibly, the space $\underline{TO}$) is an exception among spaces from diagram 2.1. The following theorem is the main result of this paper.

Theorem 4.3. For all spaces of preference relations from diagram 2.1 a set is a Pareto-convex set iff it is a convex set.

There is an important class of binary relation spaces for which the previous theorem is true. We say that two distinct points R' and R" in the space R are adjacent if they constitute a linear segment in R. (Note, that in any space a linear segment is a sequence of adjacent points, lying between two given points and "joining" them).

Definition 4.2. A space R is said to be a complete space iff for any two adjacent points in this space the symmetric difference of these points (regarded as subsets in SxS) is a singleton.

Obviously, all preference spaces from the fourth "floor" of diagram 2.1 are complete ones. Completeness of spaces QT and P was established in [12]. All other spaces from diagram 2.1 are non-complete ones.

In [12] the following theorem is proven.

Theorem 4.4. Convexity and Pareto-convexity coincide in complete binary relation spaces.

Hence, theorem 4.4 provides the proof of theorem 4.3 for spaces of preference relations on the fourth and third "floors" of diagram 2.1.

Spaces I, L and SL have a structure which is very close to that of complete spaces (adjacent points in these spaces are distinguished on symmetric pairs (x,y) and (y,x)). By using results of [19], the statement of theorem 4.3 can be easily proven for these spaces (see, however, [8] where the result is proven for L and [18].

It suffices now to prove theorem 4.3 for the space O only. We begin with the following lemmas.

Lemma 4.1. (Generalized Szpilrajn's theorem [22]) Let $P \subseteq R$ where P is a reflexive partial ordering and R is an ordering. Then there is a linear ordering L such that

$$P \subseteq L \subseteq R.$$

<u>Proof</u>. Omitted. See [7].

We denote $U = S \times S$ - a universal relation with domain S; $P \cdot Q$ denotes a composition of binary relations P and Q.

<u>Lemma 4.2</u>. Let R_1 and R_2 be orderings such that $I_1 \neq I_2$. Then there is an ordering $R \in [R_1, R_2]$ such that $R \subset R_1$ or $R \subset R_2$.

<u>Proof</u>. Let us define $R' = P_1 \cup (I_1 \cap R_2)$ and $R'' = P_2 \cup (I_2 \cap R_1)$. Then R' and R" are orderings. Indeed,

$$R' \cup (R')^{-1} = P_1 \cup (I_1 \cap R_2) \cup P_1^{-1} \cup (I_1^{-1} \cap R_2^{-1})$$

$$= P_1 \cup P_1^{-1} \cup [I_1 \cap (R_2 \cup R_2^{-1})] = P_1 \cup P_1^{-1} \cup I_1 = U$$

providing reflexivity and completeness of R'. Further, we have

$$R' \cdot R = [P_1 \cup (I_1 \cap R_2] \cdot [P_1 \cup (I_1 \cap R_2)] = (P_1 \cdot P_1)$$

$$\cup [(I_1 \cap R_2) \cdot P_1] \cup [P_1 \cdot (I_1 \cap R_2)] \cup [(I_1 \cap R_2) \cdot (I_1 \cap R_2)]$$

$$\subseteq P_1 \cup [P_1 \cap (R_2 \cdot P_1)] \cup [P_1 \cap (P_1 \cdot R_2)] \cup (I_1 \cap R_2) = R'.$$

since $I_1 \cdot P_1 = P_1 \cdot I_1 = P_1$ and $I_1 \cap R_2$ is a transitive binary relation. Hence, R' is an ordering. Similarly, R" is an ordering.

Let us prove that either $R' \subset R_1$ or $R'' \subset R_2$. We have

$$R' = P_1 \cup (I_1 \cap R_2) = (P_1 \cup I_1) \cap (P_1 \cup R_2)$$

$$= R_1 \cap (P_1 \cup R_2) \subseteq R_1.$$

In the same way, $R'' \subseteq R_2$. Let us suppose that $R' = R_1$ and $R'' = R_2$. Then

$$I_1 = R_1 \cap R_1^{-1} = R' \cap (R')^{-1} = [P_1 \cup (I_1 \cap R_2)] \cap [P_1^{-1} \cup (I_1^{-1} \cap R_2^{-1})]$$
$$= (P_1 \cap P_1^{-1}) \cup (I_1 \cap R_2 \cap P_1^{-1}) \cup (P_1 \cap I_1 \cap R_2^{-1}) \cup (I_1 \cap R_2 \cap R_2^{-1})$$
$$= I_1 \cap I_2.$$

Similarly, $I_2 = I_1 \cap I_2$, which imply $I_1 = I_2$. This contradiction shows that either $R' \subset R_1$ or $R'' \subset R_2$. Let us suppose, for instance, that $R' \subset R_1$. Then

$$R_1 \cap R_2 \subseteq (R_1 \cap P_1) \cup (R_1 \cap R_2) = R_1 \cap (P_1 \cup R_2)$$
$$= R' \subset R_1 \subseteq R_1 \cup R_2,$$

i.e. $R' \in [R_1,R_2]$ and one may take $R = R'$ to complete the proof. ¤

By R^a we denote a restriction of a binary relation R on a subset $S\backslash\{a\}$.

<u>Lemma 4.3</u>. Let $X = \{R_i\}$ be a convex set in $\underline{0}$ such that $\bigcup_i R_i = U$. Then $U \in X$.

<u>Proof</u>. By induction. The statement of the lemma is trivial if $k = |S| = 2$. Let $k > 2$ and a_1,a_2,a_3 be distinct elements in S. By inductive hypothesis, there are orderings R_{k_i} such that $R_{k_i}^{a_i} = U^{a_i}$ for $1 \leq i \leq 3$. Since each R_{k_i} is an ordering, we have either

$$R_{k_i} = U^{a_i} \cup \{(a_i,x) : x \in S\} \tag{4.2}$$

or

$$R_{k_i} = U^{a_i} \cup \{(x,a_i) : x \in S\} \tag{4.3}$$

for all i.

Let us suppose that $R_{k_i} \cup R_{k_j} \neq U$ for all i and j. Then all R_{k_i} are different orderings. Indeed, if, say, $R_{k_1} = R_{k_2}$ then

$R_{k_1} = R_{k_2} = U$, by definition of R_{k_i}, and $R_{k_1} \cup R_{k_2} = U$. Moreover, for $i \neq j$, R_{k_i} and R_{k_j} must have different representations (4.2) and (4.3), for in the opposite case $R_{k_i} \cup R_{k_j} = U$. It is impossible, since we have three different orderings R_{k_i}. This contradiction shows that $R_{k_i} \cup R_{k_j} = U$ for some i and j. Hence, $U \in [R_{k_i}, R_{k_j}]$, which implies $U \in X$. ⌑

The statement of theorem 4.3 follows from theorem 4.2 and the following lemma.

Lemma 4.4. Let $X = \{R_i\}$ be a convex set in $\underline{O}$ and

$$\bigcap_i R_i \subseteq R \subseteq \bigcup_i R_i .$$

Then $R \in X$.

Proof. 1) $\bigcup_i R_i = U$.

Let X_{min} be a set of all minimal elements in X with respect to the partial ordering induced by inclusion (see section 3). By lemma 4.2, all orderings in X_{min} have the same indifference relation I. The equivalence relation I is a congruence relation for all orderings in X, by (4.4). Hence, without loosing generality, we may suppose that I is the identity relation on S. Then all elements in X_{min} are linear orderings, and $P = \bigcap_i R_i$ is a partial ordering. By lemma 4.1, there is a linear ordering L such that

$$P = \bigcap_i R_i \subseteq L \subseteq R.$$

Let $X_{min} = \{L_i\}$. Then $P = \bigcap_i L_i$, and we have

$$\bigcap_i L_i \subseteq L \tag{4.5}$$

which imply $\bigcup_i L_i \supseteq L$. Indeed, $\overline{L_i^{-1}} \cup D = L_i$ and $\overline{L^{-1}} \cup D = L$, where D is a diagonal relation. Then by (4.5),

$$\bigcup_i \overline{L_i^{-1}} \supseteq L^{-1}$$

which imply $\bigcup_i L_i \supseteq L$. Hence, L belongs to the set $P(X_{min})$ in the space L. But X_{min} is a convex set in L, since it contains all minimal elements of the convex set X. We have $P(X_{min}) = X_{min}$ in L, since theorem 4.3 is true in the space L. Hence, $L \in X_{min} \subseteq X$. By lemma 4.3, $U \in X$, which imply $R \in X$, since $L \subseteq R \subseteq U$ and X is a convex set.

2) $\bigcup_i R_i \neq U$.

By induction. The statement of lemma 4.4 is trivial if $k = |S| = 2$. Since $\bigcup_i R_i \neq U$, there is $(a,b) \notin \bigcup_i R_i$, i.e. $(a,b) \notin R_i$ for all i. (Note that $a \neq b$.) Moreover, we have $(b,a) \in \bigcap_i R_i$, since $(a,b) \notin R_i$ for all i. By inductive hypothesis , R^a and R^b coincide with some R_i^a and R_j^b, respectively. Let us prove, that $R \in [R_i, R_j]$.

Let $(x,y) \in R_i \cap R_j$, i.e. $(x,y) \in R_i$ and $(x,y) \in R_j$. The following cases are only possible ones since $(a,b) \in \bigcup_i R_i$:

1) $x \neq a$, $y \neq a$; then $(x,y) \in R_i^a = R^a \subset R$;

2) $x = a$, $y \neq b$; then $(x,y) \in R_j^b = R^b \subset R$;

3) $x \neq b$, $y = a$; then $(x,y) \in R_j^b = R^b \subset R$;

4) $x = b$, $y = a$; then $(x,y) = (b,a) \in \bigcap_i R_i \subseteq R$.

Hence, $(x,y) \in R$, which implies $R_i \cap R_j \subseteq R$.

Let now $(x,y) \in R$. We consider the same cases again, since $(a,b) \notin \bigcup_i R_i \supseteq RL$:

1) $x \neq a,\ y \neq a$; then $(x,y) \in R^a = R_i^a \subset R_i$;

2) $x = a,\ y \neq b$; then $(x,y) \in R^b = R_j^b \subset R_j$;

3) $x \neq b,\ y = a$; then $(x,y) \in R^b = R_j^b \subset R_j$;

4) $x = b,\ y = a$; then $(x,y) = (b,a) \in \bigcap_i R_i \subseteq R_i \cup R_j$.

Hence $R \subseteq R_i \cup R_j$.

Finally, $R \in X$ since $R \in [R_i, R_j]$ ¤

5. Geometrical principles of group choice

In this section we consider only binary relation spaces for which convexity and Pareto-convexity coincide. It may be, for example, any preference space from diagram 2.1 as well as any complete space. Then, from the geometrical standpoint, Pareto-points of a given set fill out a convex hull of this set. Each such point lies between the original points and can be regarded as an "admissible" group preference in the sense that it satisfies the Pareto principle. Unfortunately, convex hulls contain, as a rule, too many points and it is still a problem to define a set of admissible preferences which would have some kind of attractive geometrical properties, a relatively simple structure and would be more "narrow" then the Pareto one. A constructive method for this purpose is suggested below, based on a separation of a convex subset - radical - in every convex set. Geometrically, a radical lies "in the middle" of a given convex set and has some symmetry properties. But a radical itself may have a quite complex structure. Therefore, a sequence of embedded convex subsets is considered, such that each consequent set is a radical of the preceding one. Naturally, this sequence is stabilized and its least element is called a kernel of an initial set. One can consider a kernel as the smallest convex subset "lying in the middle" of a given set.

Radicals and kernels will be defined and studied in the next section. Here some general principles of geometrical approach are stated.

First of all, two peculiarities of the approach developed should be emphasized:

1) a set of admissible decisions is defined but not a unique decision,

2) a set of admissible decisions is defined by iterated algorithm of constructing more and more "narrow" nested convex subsets. These peculiarities cause some difficulties in the attempt to compare the geometrical approach with classical methods of group choice theory. Wherever it is possible, we will present such a comparison.

Principle 1. All individual preferences and admissible group preferences are considred as points in a given binary relation space $\underline{R}$. A set of admissible group decisions is defined for every subset in $\underline{R}$.

This principle can be regarded as analogous to the condition of unrestricted domain in group choice theory. In accordance to principle 1, a set of admissible preferences Y is some function F of subsets in $\underline{R}$:

$$F : X \to Y$$

where X and Y are subsets in $\underline{R}$ and X takes the values from all nonempty subsets in $\underline{R}$. A function F is considered in this paper as a generalized collective choice rule. Since F does not depend on the order of points of X, principle 1 contains anonymity condition too.

The next principle is based on the notion of "congruence" in $\underline{R}$. One-to-one mapping $\delta : \underline{R} \to \underline{R}$ is said to be a transformation iff it preserves basic geometrical structures in $\underline{R}$. For example, if δ is a transformation, then $R \in [R', R'']$ iff $\delta(R) \in [\delta(R'), \delta(R'')]$. The set

T($\underline{R}$) of all transformation of $\underline{R}$ is a group with respect to natural operations of composition and inverse map. Two subsets in $\underline{R}$ are said to be <u>congruent</u> if and only if there is a transformation which transfers one of the given sets onto another. From the geometrical point of view congruent sets are regarded as identical. Now we are able to formulate

<u>Principle 2</u>. Let X' and X" be two congruent subsets in a space $\underline{R}$ and δ be any transformation such that $\delta(X') = X''$. Then correspondent sets Y' and Y" of admissible group preferences are congruent and $\delta(Y') = Y''$.

Let F be a generalized collective choice rule. Then we have, by principle 2, for any transformation δ

$$F(\delta(X)) = \delta(F(X)),$$

i.e. principle 2 states that generalized collective choice rules are commuted with transformations in spaces of binary relations.

Let us consider an important example of transformations. Let $\underline{R}$ be any preference space from diagram 2.1. Any permutation π of the set S induces a mapping $\pi^* : R \rightarrow R^{\pi}$ where $xR^{\pi}y$ iff $\pi(x)R\,\pi(y)$. It is easy to verify that π^* is a transformation of $\underline{R}$. These transformations form a subgroup $\Pi(\underline{R})$ of the group T($\underline{R}$). Now it is obvious that principle 2 contains a neutrality principle (a condition of object equality) (see [9], p. 138) for spaces in diagram 2.1.

<u>Principle 3</u>. $C(X') = C(X'')$ implies $F(X') = F(X'')$.

In accordance with this principle a generalized collective choice rule could be defined only for convex subsets in $\underline{R}$.

As it was already mentioned above, a set of Pareto-points P(X) of a given set X may be regarded as a set of admissible preferences.

By theorem 4.3, $P(X) = C(X)$ for all preference spaces in diagram 2.1, i.e. the rule $F = P$ satisfies principle 3. Principle 3 states that even for more "narrow" rules a group choice should be based on the set of Pareto-points of a given set, but not on this set itself. From this standpoint, sets of individual preferences, which have the same Pareto-set, are considered as equivalent.

In the form formulated principle 3 is a pure geometrical one; because of theorem 4.3, one can also consider this principle as a stronger variant of the classical Pareto condition.

6. Radicals and kernels

Only preference spaces from diagram 2.1 are considered in this section.

Let X be a convex set in a space $\underset{\sim}{R}$. The partial ordering in this space induces the partial ordering in X. Maximal and minimal elements of X with respect to this partial ordering are denoted $R_1^{max},\dots,R_s^{max}$ and $R_1^{min},\dots,R_t^{min}$, respectively. Evidently, X is a convex hull of the set of its maximal and minimal elements; X can be regarded as a union of all intervals $[R_i^{min},R_j^{max}] = X_{ij}$ such that $R_i^{min} \subseteq R_j^{max}$, because each point in X lies between some maximal and some minimal elements. We consider now $\underset{\sim}{R}$ as a subspace of the space $\underset{\sim}{B}$ of all binary relations on S. Then each X_{ij} is an intersection of $\underset{\sim}{R}$ with a hypercube $\tilde{X}_{ij}$ of all binary relations lying between R_i^{min} and R_j^{max}. Let M be a minimal hypercube in the space $\underset{\sim}{B}$ with vertices $\bigcup_i R_i^{min}$ and $\bigcap_j R_j^{max}$. Geometrically, this hypercube M may be regarded as lying "in the middle" between hypercubes $\tilde{X}_{ij}$. This general geometrical idea is a basis for the following

<u>Definition 6.1</u>. A <u>radical</u> $r(X)$ of a convex set in $\underset{\sim}{R}$ is the set of all points in $\underset{\sim}{R}$ lying between $\bigcup_i R_i^{min}$ and $\bigcap_j R_j^{max}$. For an arbitrary set X we set $r(X) = r(C(X))$.

Remark. Note that $\bigcup_i R_i^{min}$ and $\bigcap_j R_j^{max}$ do not, generally speaking belong to the space $\underline{R}$.

Theorem 6.1. A radical of any non-empty set is a non-empty convex subset of its convex hull in any preference space from diagram 2.1.

In the framework of the approach developed, r(X) is considered as a set of admissible group preferences which is, generally speaking, a more "narrow" one then P(X). But r(X) itself may be still too "large". Therefore we define a sequence of nested sets $r_k(X)$ by

$r_0(X) = P(X) = C(X)$,

$r_1(X) = r(X)$, and

$r_k(X) = r(r_{k-1}(X))$ for $k > 1$.

This sequence is stabilized because of finiteness of $\underline{R}$.

Definition 6.2. A set

$$k(X) = \bigcap_i r_i(X)$$

is said to be a kernel of X in the space $\underline{R}$.

From theorem 6.1 we obtain immediately

Theorem 6.2. A kernel k(X) of any non-empty set X is a non-empty convex subset of C(X) and r(k(X)) = k(X).

Notions introduced will now be illustrated by examples of spaces from diagram 2.1. By theorem 3.1 it is sufficient to consider only weak preference spaces $\underline{L}$, $\underline{O}$, $\underline{QT}$ and $\underline{WP}$.

The space $\underline{QT}$ from this standpoint was studied in [12]. Each convex set X in $\underline{QT}$ has a unique maximal element. It implies immediately

$$k(X) = r(X) = \{R \in \underline{QT} \mid R \in [\bigcup_i R_i^{min}, R^{max}]\} \quad (6.1)$$

Since $\bigcup_i R_i^{min}$ belongs to QT, kernels in QT have a very simple structure: they are intervals [R',R"] such that $R' \subseteq R"$, R', R" $\in$ QT and any such interval is a kernel of some convex set (for example, of itself). The same description is true (trivially) in the space WP.

The space L provides an example, in some sense, opposed to that of QT. The partial ordering is trivial in L, because linear orderings are not comparable by inclusion. Hence,

$$k(X) = r(X) = X$$

for any convex subset in the space L. Therefore, the approach developed in this paper permits to separate the only class of admissible preferences, namely, the class of Pareto-points. A weakness of geometrical approach in the case of L is related to the exclusively "homogeneous" (from a geometrical point of view) structure of convex subsets in this space. In this sense, the problem in question is analogous to the search for a "middle vertex" in a regular polygon.

Even simple examples demonstrate the non-triviality of introduced notions in the case of the space O. In particular, theorem 6.1 is non-trivial just in this case. We consider here only an important non-geometrical property of radicals in O. Let X be a convex hull of the set $\{R_1,\dots,R_m\}$ of individual preferences (orderings) in the space O. We define R_Π as a Pareto non-dominancy relation for $\{R_1,\dots,R_m\}$, i.e.

$$xR_\Pi y \leftrightarrow \sim[(\forall i)yR_i x \;\&\; (\exists i)yP_i x]$$

(see [20], p. 52). Note, that, generally speaking, R_Π does not belong to the space O. It may be proven that R_Π lies between $\bigcup_i R_i^{min}$ and

$\bigcap_j R_j^{max}$ in the space $\underset{\sim}{B}$. Hence, if R_Π belongs to $\underset{\sim}{0}$, then this relation belongs to the radical $r(X)$ of the set $X = \{R_1,\ldots,R_m\}$. Based upon this fact, points of the radical in the space 0 may be regarded as orderings similar to the Pareto non-dominancy relation. It could be considered as a non-geometrical argument in favor of study of radicals in the space $\underset{\sim}{0}$.

We complete this section with the proof of theorem 6.1. As it was mentioned above it suffices to prove it only for the space $\underset{\sim}{0}$.

Proof of theorem 6.1. Let us consider a relation $I = \bigcap_i I_i^{max}$. It is easy to verify that I is a congruence relation for $\bigcap_i R_i^{max}$. By Szpilrajn's theorem [22] there is an ordering R such that I is an indifference relation for R and $R \supseteq \bigcap_i R_i^{max}$. Obviously,

$$(\bigcap_i R_i^{max}) \cap (\bigcup_i R_i^{min}) \subseteq R$$

Let us prove that

$$R \subseteq (\bigcap_i R_i^{max}) \cup (\bigcup_i R_i^{min}).$$

In the opposite case there is $(a,b) \in R$ such that

$$(a,b) \notin (\bigcap_i R_i^{max}) \cup (\bigcup_i R_i^{min})$$

or, equivalently,

$$(a,b) \notin \bigcap_i R_i^{max} \text{ and } (a,b) \notin \bigcup R_i^{min} . \tag{6.2}$$

Then $(a,b) \in P$. Indeed, if $(a,b) \in I$, then

$$(a,b) \in \bigcap_i I_i^{max} \subseteq \bigcap_i R_i^{max},$$

which contradicts to (6.2). Hence, $(a,b) \in P$. We have $P \subseteq \bigcup_i P_i^{max}$, since $R \supseteq \bigcap_i R_i^{max}$. Hence, $(a,b) \in P_i^{max}$ for some i. On the other hand $(a,b) \notin \bigcup_i R_i^{min}$ implies $(b,a) \in \bigcap_i R_i^{min} \subseteq \bigcap_i R_i^{max}$, which, in its turn, implies $(b,a) \in P_i^{max}$ and $(b,a) \in R_i^{max}$. This contradiction completes the proof. ¤

7. Conclusion

In this section some general remarks and problems related to the geometrical approach to group choice are listed.

1) Diagram 2.1 can be extended by inclusion of different spaces of preferences. For example, the space of all semiorders [14] and related spaces of weak preferences and indifference relations are naturally included in the framework of diagram 2.1 (in this connection see [17]).

2) Basic geometrical structures of binary relation spaces are defined in this paper internally whereby a structure of their points (binary relations). In [19] a lattice-theoretical approach to this problem was suggested, which involves only "external" definitions of geometrical structures. This approach gives an equivalent result in case of complete binary relation spaces. It is an interesting problem to find an "external" geometrical description of such spaces as, for example, incomplete spaces $\underline{O}$ and $\underline{E}$.

3) An interesting problem is a study of "morphisms" of binary relation spaces, in particular, a full description of transformations in these spaces. For example, the space L admits a wider group of transformations then that generated by permutations. We have seen already that transformations generated by permutations have a natural interpretation in terms in neutrality principle. What kind of general group choice principles correspond to the full group of transformations?

4) The following peculiarity of the approach developed should be noted. In all cases, when k(X) = C(X), i.e. when a kernel coincides with a convex hull of a given set, it could be stated that X is a "bad"

set from the geometrical choice point of view. It means that C(X) is too homogeneous set to select a "middle" of it: all points of C(X) have equal rights to be chosen as group preferences. A good example of this situation is an interval $X = [R',R"]$ where $R' \subseteq R"$ and both points belong to a space considered.

5) Is it true that a kernel is always an interval $[R',R"]$ with $R' \subseteq R"$? The only non-trivial case is the space $\underline{O}$ (and $\underline{QS}$). Many interesting properties of radicals in $\underline{O}$ may be proven which provide a basis for their effective construction. The author intends to study these problems in future papers.

6) It should be noted that the approach developed in the paper does not take into account a "multiplicity" of points; namely, if two or more individuals have the same preference, they are represented by a single point in a preference space. More general approach should involve "weights" of points and is not considered in this paper. Nevertheless, note that any Pareto principle has the same "shortcoming".

7) Metric structures - which have not been involved in our study - would provide more advantages in a geometrical construction of group preferences. For example, "means" and "medians" may be used to separate more "narrow" sets of admissible preferences from kernels (see, for example, [2]).

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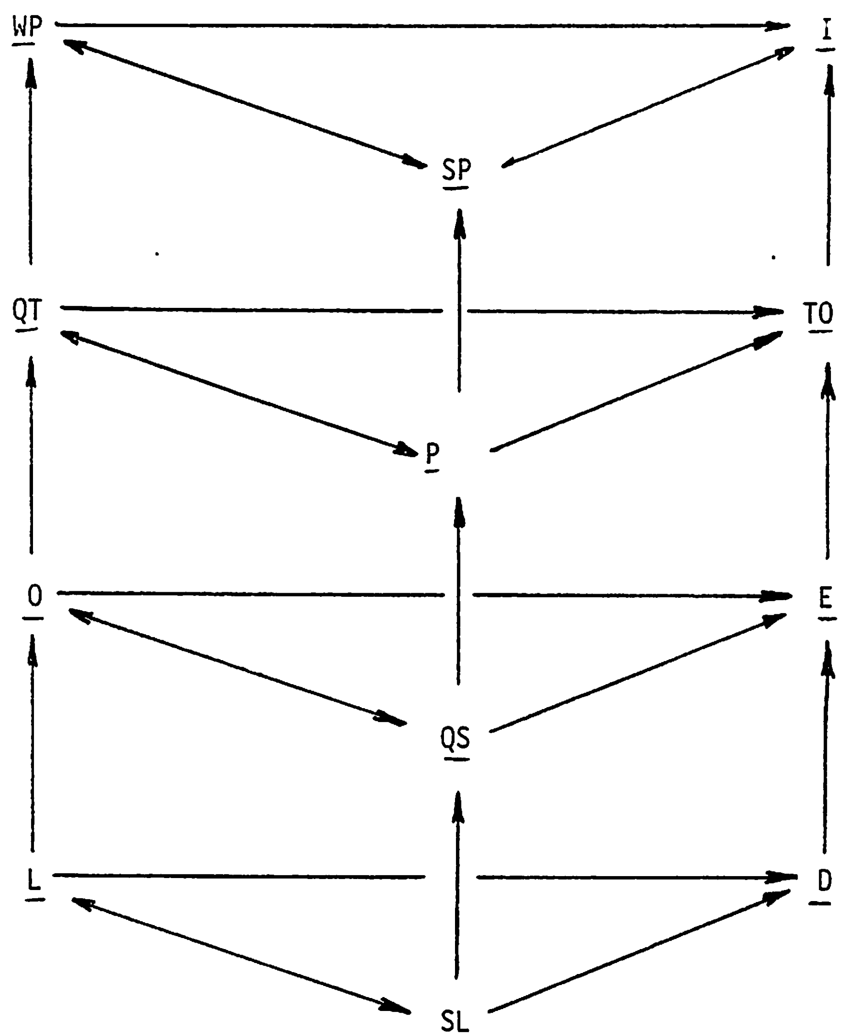

Fig. 2.1

Figure Caption

Figure 2.1. Diagram of Binary relation spaces.

List of Symbols

$\underline{WP}$, $\underline{QT}$, $\underline{O}$, $\underline{L}$, $\underline{SP}$, $\underline{P}$, $\underline{QS}$, $\underline{SL}$, $\underline{I}$, $\underline{TO}$, $\underline{E}$, $\underline{D}$, $\underline{R}$, $\underline{B}$	should be bold script
α, β, δ, π	small Greek letters
Δ, Σ, Π	capital Greek letters
$\forall$, $\sim$, $\&$, $\in$, $\vee$	logical symbols
$\rightarrow$	arrows
ϕ	symbol for empty set
$\underset{\sim}{\alpha}$	alpha with tilde underneath
$\cap$, $\cup$, $\subseteq$, $\supseteq$	set-theoretical symbols