TRAFFIC MODELLING IN LARGE SCALE TELEPHONE NETWORKS

by

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ABSTRACT

The problem studied here concerns modelling of call losses due to the trunk blocking phenomenon in telephone networks. Using classical assumptions such as Poissonian distribution for inputs (call attempts) and exponential distribution for service time (call duration), a Markovian representation of telephone process provides a very accurate model for studying this process. However, such a "microscopic" representation cannot be used for real networks because of the rapidly increasing number of states to be considered. The Markovian model is useful for finding analytical formulas of trunk blocking probabilities and also for comparing some approximate models we try to build in this paper.

The first part of this study is devoted to the elaboration of a dynamic analytical model of traffic where each variable represents the average state of the corresponding trunk. In the second part, we introduce the concept of "over-variant" and "under-variant" processes. In some cases, representing traffic both by its mean and variance leads to better estimation of blocking probabilities. Finally, we try to generalize the "equivalent trunk" theory introduced by R. I. Wilkinson [1].

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I. INTRODUCTION

A telephone network consists of a set of trunks each with a finite number of servers, linking a set of nodes representing switching centers. Calls arrive in this network at some origin nodes. It is generally assumed that calls entering the system are Poisson distributed. The arriving calls are routed through the network link by link, according to a routing policy (load sharing, overflow rerouting), until a free connection can be established between an origin-destination pair of nodes. Successful calls remain in the system for a random conversation time (holding time) which is assumed exponential [7]. Calls that fail to find this connection are lost to the network and do not reenter the system (lost calls cleared [5]).

Analysis of telephone processes is aimed at building models to estimate the average number of calls which find a free connection for each origin-destination pair. In real networks, this analysis does not provide analytical solutions. Markovian models often lead to untractable numerical problems mainly due to the large number of possible states. Therefore, various approximate models and methods are generally used. The most common approximation is to assume the independence of different flows in the network and then to simplify the whole analysis with a link by link study [6] [11]. However, the resulting problem in each link is not necessarily simple to model. Even with the simplifying assumption of Poisson distributed input flows, the flow of calls in the network does not keep the Poisson Property due to routing and transit operations. The study of such distributions has been made for particular cases and generally appears to be very complicated [4]. Methods for finding equivalent processes have also been proposed [2] [3]. The accuracy and computation burden associated with each of these methods [10] may determine their applicability for engineering purposes.

The work presented in this report divides into two main parts. In the first (2nd and 3rd section) we introduce the notion of traffic (offered, carried, lost traffic) usually associated with telephone process. In the case of one trunk, transient state is described by means of a dynamic model of traffic. This model then is extended to a network. Numerical comparisons with Markovian model show that the approximate model provides a useful tool to represent traffic dynamics in a network.

In the second part (4th section) we analyze the second moment of telephone processes. A traffic model using both the average and the variance of telephone process provides a better approximation. The main result presented in this part is the equivalent trunk method for under-variant traffic as a complement to the equivalent random theory proposed by R. I. Wilkinson [1].

II. Study of Telephone Process on One Trunk:

II.1) General assumptions

- arrivals

We assume that call attempts can be represented by a Poissonian distribution with a constant rate λ; Probability of k call attempts during a time interval θ:

$$\frac{(\lambda\theta)^k}{k!} e^{-\lambda\theta} \tag{2.1}$$

For small time interval dt the probability of more than one arrival is of the order dt^2 and the probability of one arrival is:

$$\lambda\, dt \tag{2.2}$$

- <u>Service time</u>

We assume that service time (including both conversation time and switching times) obeys an exponential distribution with an average duration T. The probability that the duration of a call is included between t and t + dt is:

$$1/T\ e^{-t/T} \tag{2.3}$$

If i circuits of a trunk are busy at time t (i.e. i calls are present at time t), then the probability that one of these circuits be released in a small interval dt is:

$$\frac{idt}{T} \tag{2.4}$$

- <u>Lost calls cleared</u>

If N is the trunk capacity (i.e. number of circuits), then no more than N simultaneous calls can be carried by this trunk. If a call arrives when the trunk is busy (state N) it is immediately eliminated (no waiting time). Lost calls are considered cleared once and for all (see Fig. 2.1).

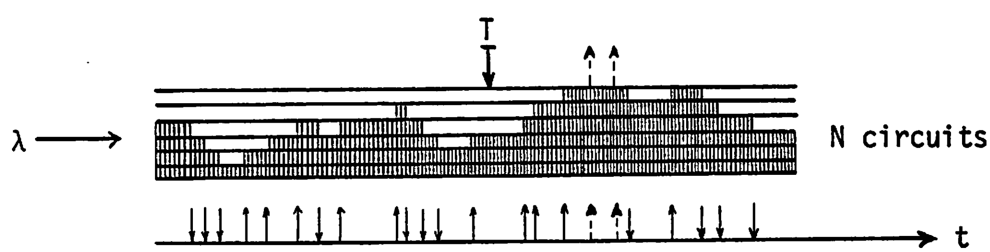
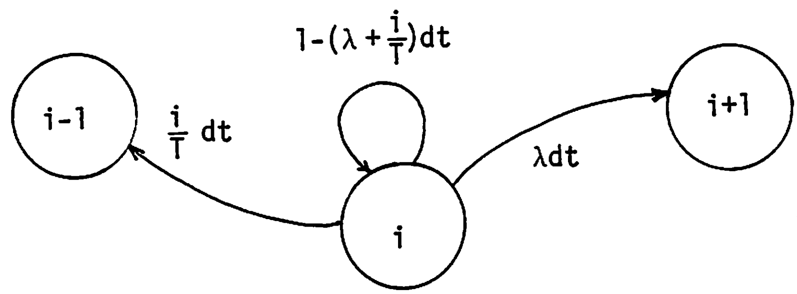


Figure 2.1

II.2) <u>Markovian representation</u>:

Since at each instant t either one departure or one arrival occur in the trunk with probabilities independent of t, telephone process is a Markovian birth-death process with the following transition diagram:



Let P(t) the probability vector, each component of which represents the probability of having i circuits busy and $\dot{P}$ the associated derivative. Then, the dynamics of the telephone process can be represented by the following equation:

$$\dot{P} = QP \tag{2.5}$$

where Q is the rate matrix. In the case of one trunk this (N+1) × (N+1) matrix has the following structure (derived from the transition diagram):

$$Q : \begin{bmatrix} -\lambda & 1/T & & & 0 \\ \lambda & -\lambda-1/T & 2/T & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda & -\lambda-\frac{N+1}{T} & N/T \\ & 0 & & & \\ & & & \lambda & -N/T \end{bmatrix}$$

2.3 <u>Steady State Solution</u>

We first consider the steady state equilibrium behavior of the telephone process.

Let us denote $\lim_{t\to\infty} P_i(t)$ by P_i, the solution of the system of N + 1 equations:

$$\sum_{j=0}^{N} Q(i,j)P_j = 0 \qquad i = 0,\ldots,N \tag{2.6}$$

<u>Blocking Probability</u>:

In order to get an analytical solution for (2.6), it is necessary to solve the recurrence:

$$\lambda P_{i-1} - (\lambda + \frac{i}{T})P_i + (\frac{i+1}{T})P_{i+1} = 0, \; i = 0,\ldots,N-1 \quad (P_i = 0, \; i < 0)$$

$$P_{N-1} - \frac{N}{T} P_N = 0 \tag{2.7}$$

The solution of which is:

$$P_i = \frac{y^i}{i!} P_0 \text{ with } y = \lambda.T \tag{2.8}$$

Then, with normalization condition one gets:

$$P_i = \frac{\frac{Y^i}{i!}}{\sum_{j=0}^{N} \frac{Y^j}{j!}} \tag{2.9}$$

The blocking probability of the trunk which in fact is P_N, is well known as the Erlang's lost calls cleared formula:

$$P_N = E(N,Y) = \frac{Y^N/N!}{\sum_{j=0}^{N} \frac{Y^j}{j!}} \tag{2.10}$$

The computationof E(N,Y) is simplified by the use of the following recursive formula:

$$\sum_{j=0}^{N} \frac{Y^j}{j!} = \underbrace{\sum_{j=0}^{N-1} \frac{Y^j}{j!}}_{\substack{\text{denominator of} \\ E(N-1,Y)}} + \frac{Y}{N} \underbrace{\frac{Y^{N-1}}{(N-1)!}}_{\substack{\text{numerator of} \\ E(N-1,Y)}}$$

$$E(N,Y) = \frac{\frac{Y}{N} E(N-1,Y)}{1 + \frac{Y}{N} E(N-1,Y)} \tag{2.11}$$

Taking initial value E(0,Y) = 1, this formula is convenient for computing E(N,Y) even for large values of N.

<u>mean state</u>:

Let X denote the average number of circuits busy in the trunk:

$$X = \sum_{i=0}^{N} i \, P_i$$

but

$$\sum_{i=0}^{N} i \frac{Y^i}{i!} = \sum_{i=0}^{N} Y \frac{Y^{i-1}}{(i-1)!} = Y\{\sum_{i=0}^{N} \frac{Y^i}{i!} - \frac{Y^N}{N!}\}$$

therefore

$$\frac{\sum_{i=0}^{N} \frac{Y^i}{i!} - \frac{Y^N}{N!}}{\sum_{i=0}^{N} \frac{Y^i}{i!}} = Y\left\{1 - \frac{\frac{Y^N}{N!}}{\sum_{i=0}^{N} \frac{Y^i}{i!}}\right\}$$

$$X = Y\{1-E(N,Y)\} \tag{2.12}$$

X is also called the carried traffic.

Remark

If we consider a trunk with a very large capacity such that $\frac{Y}{N} \to 0$, then

$$\lim_{\frac{Y}{N}\to 0} \sum_{i=0}^{N} \frac{Y^i}{i!} = e^Y$$

$$P_i = \frac{Y^i}{i!} e^{-Y} \tag{2.13}$$

In this case, the probability of i circuits busy is defined by a Poisson distribution. This means that the Poisson characteristic of traffic is preserved when blocking probability is very small.

$$X = \sum_i i\ P_i = Y \tag{2.13}$$

Definition

The offered traffic Y is the traffic which would be carried by a trunk having infinite capacity.

$$\text{carried traffic} = \text{offered traffic} - \text{lost traffic} \tag{2.12}$$

4. Transient State

We propose an extension of the previous results to study the transient behavior of the process. Let X(t) be the carried traffic at time t and let $\dot{X}$ be the associated derivative:

$$X(t) = \sum_{i=0}^{N} i \; P_i(t) \tag{2.14}$$

$$\dot{X}(t) = \sum_{i=0}^{N} i \; \dot{P}_i(t) \tag{2.15}$$

$$\dot{X} = \sum_{i=0}^{N} i\{\sum_{j=0}^{N} Q_{ij} \; P_j(t)\} \tag{2.15}$$

$$= \lambda\{\sum_{i=0}^{N-1} P_i(t)\} - \frac{1}{T}\{\sum_{i=0}^{N} i \; P_i(t)\}$$

$$\dot{X} = - \frac{X}{T} + \lambda\{1-P_N(t)\} \tag{2.16}$$

The differential equation (2.16) is useful in representing dynamics of calls in a trunk. However, the calculation of $P_N(t)$ requires the computation of all values $P_i(t)$ which is equivalent to solving the Markovian model (2.5). In fact model (2.16) would be very useful if a relation between $X(t)$ and $P_N(t)$ could be stated. Unfortunately such a relation, even in steady state, seems very difficult to obtain. In order to remove coupling between model (2.16) and Markovian model (2.5) we propose two approximations for estimating $P_N(t)$.

II.4.1. Offered Traffic Approximation

Let us consider $Y(t)$ the traffic carried by an infinite capacity trunk (i.e. offered traffic at time t) and $\dot{Y}$ the associated derivative:

$$\dot{Y}(t) = \sum_{i=0}^{\infty} i \; \dot{P}_i(t)$$

$$\dot{Y} = - \frac{Y(t)}{T} + \lambda \tag{2.17}$$

The approximation consists in using $Y(t)$ and Erlang's formula (only true in steady state) to estimate $P_N(t)$; let $P_N^*(t)$ be the estimator of $P_N(t)$:

$$P_N^*(t) = E(N,Y(t)) \tag{2.18}$$

and then,

$$\dot{X} = -\frac{X}{T} + \lambda(1-P_N^*).$$

Remark:

$$P_N^*(\infty) = P_N(\infty) = E(N,\lambda T)$$

Numerical Result

On figures 2.2 and 2.3 are plotted responses of $P_N(t)$ (using model (2.5)) and $P_N^*(t)$ (using (2.17) and (2.18)).

Initial Data Are:

$$P_0(0) = 1 \quad P_i(0) = 0 \quad \forall i \neq 0$$

$$Y(0) = 0$$

Capacity: $N = 20$

The transient part of P_N is always faster than that of $P_N^*(t)$; this difference increases when λT increases.

II.4.2. Approximation by Erlang Inverse

We call Erlang Inverse the function which would give the blocking probability of a trunk depending on the carried traffic X. No analytical solution has been found for such a function. But since P_N, for a fixed N, is a monotically increasing function of X it is easy to find the unique value P_N corresponding to X. In the appendix I we propose a very simple algorithm for solving this problem.

The approximation we propose, consists of computing by Erlang's formula the blocking probability $P_N(t)$ corresponding to X(t) (even if X(t) is a transient state). Let $\tilde{P}_N(t)$ be the estimator of $P_N(t)$. $\tilde{P}_N(t)$ is solution of the following equation:

$$\begin{cases} X(t) = \tilde{Y}(t)\{1-E(N,\tilde{Y}(t))\} \\ \tilde{P}_N(t) = E(N,\tilde{Y}(t)) \end{cases} \tag{2.19}$$

Remark:

When X(t) is such that $\tilde{Y}(t) = \lambda T$, then $\dot{X} = 0$, and the steady state exact solution (2.12) is well verified.

Numerical Results

With the same data we used in the previous approximation we plotted on Figures 2.4 and 2.5 the responses of $P_N(t)$ obtained by the Markovian model (2.5) and the responses of $\tilde{P}_N(t)$ obtained by (2.19) and (2.16).

The results show that transients of P_N and $\tilde{P}_N$ are very similar; $\tilde{P}_N$ seems to be a better estimator than P_N^*. The transient of $\tilde{P}_N(t)$ is always faster than the transient of P_N, but in contrast with P_N^*, the difference does not increase with λT.

On figures 2.6 and 2.7 a comparison is made between model (2.16) using estimator $\tilde{P}_N$ and a Monte-Carlo simulation. These results prove the accuracy obtained by model (2.16) and estimator $\tilde{P}_N$.

III. Study of Telephone Process on a Network

Although accurate results are obtained with estimator $\tilde{P}_N$, the improvement introduced by the dynamic model (2.16) and (2.19) compared to the Markovian model (2.5) is not obvious because of the burden to compute $\tilde{P}_N$.

In fact, we shall see in this section that an extension of model (2.16), representing the average state (or traffic) in a network, is profitable since a "microscopic" representation such as a Markovian model can no longer be used because of the rapidly increasing number of states to be considered.

III.1 Studied Network

The simplest network which can be studied consists of departure nodes, destination nodes and only one transit node (fig. 3.1). The process is

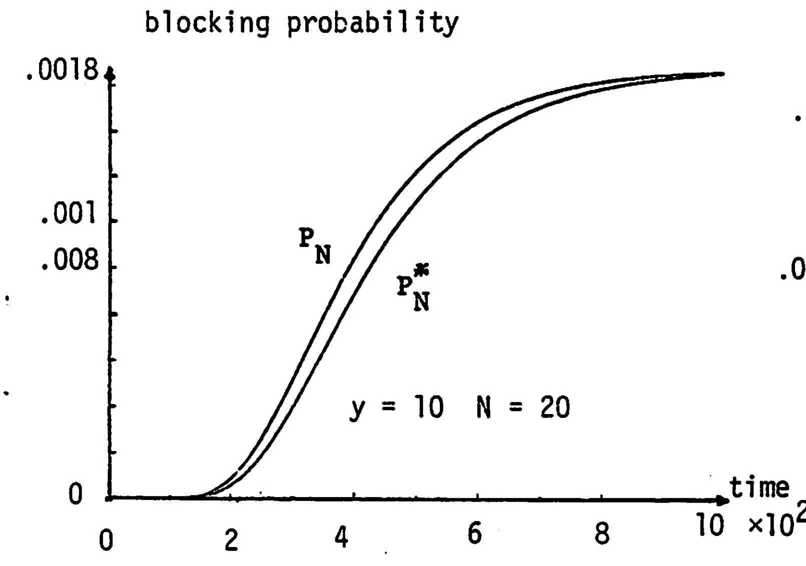


Fig. 2.2

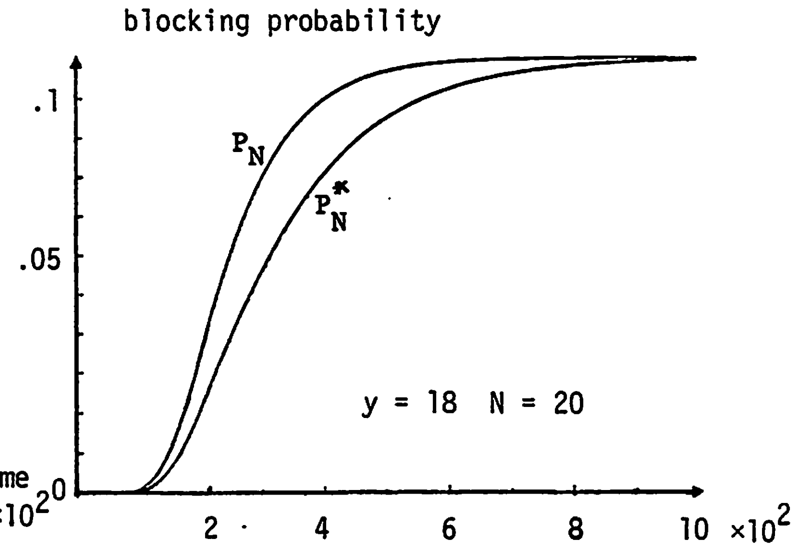


Fig. 2.3

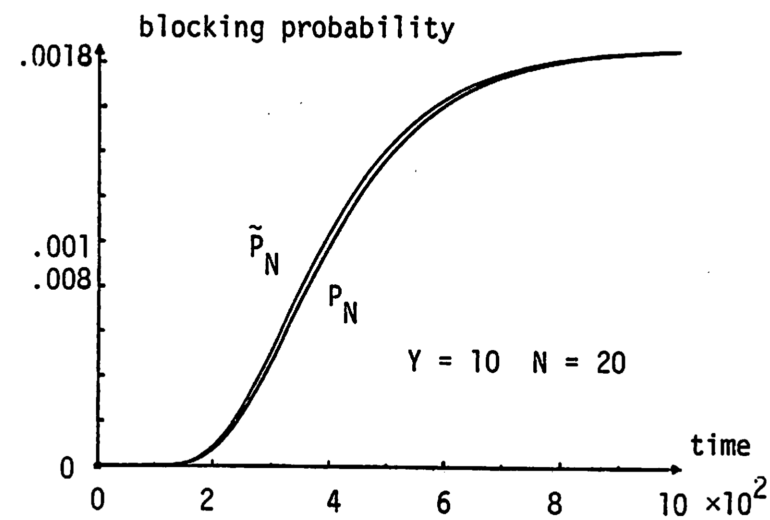


Fig. 2.4

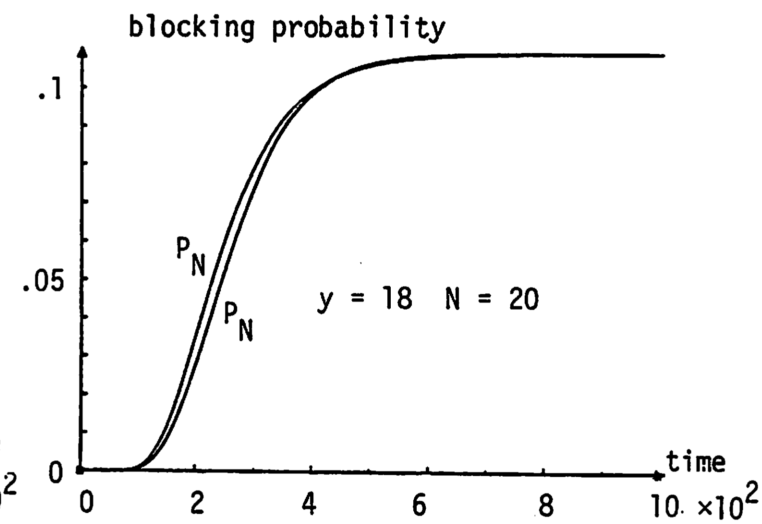


Fig. 2.5

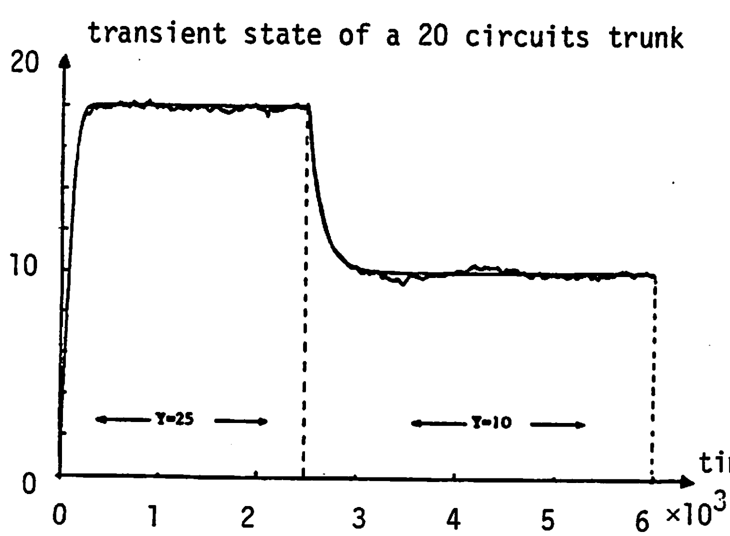


Fig. 2.6

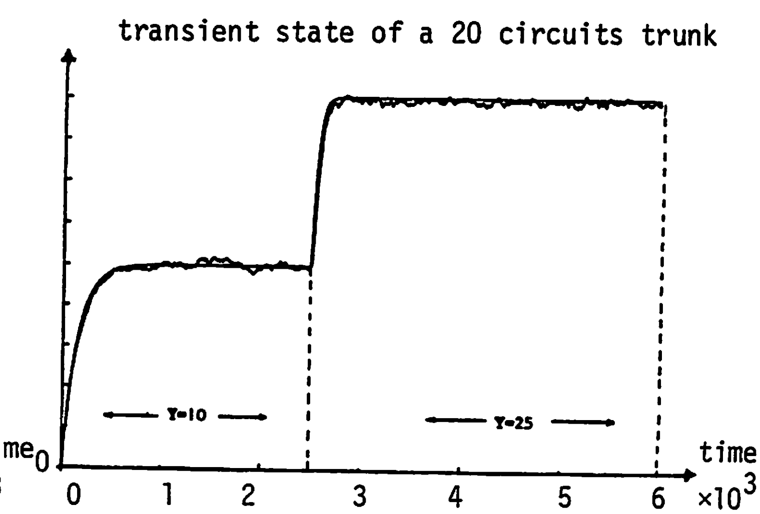


Fig. 2.7

—— : $\dot{x} = -\frac{x}{T} + \lambda\{1-\tilde{P}_N(t)\}$

: average of 300 random simulations

defined with the same assumptions as in paragraph II.1:

- Trunks linking i to k and k to j have finite capacity N_{ik} and N_{kj}.
- Conversation time exponentially distributed with average T.
- Poisson distribution for call attempts; λ_{ij} being the arrival rate of each flow (i,j).
- Lost calls cleared.

III.2 Markovian Model

The state of network is completely determined by the state of every path connecting the set of nodes I to the set of nodes J. The associated probability vector P has to contain all combinations of possible states and therefore the dimension of this vector becomes enormous. Thus, it is easy to understand why such a "microscopic" model cannot be used to represent the telephone process in a network. In fact, we use this model only to compare approximate models of traffic.

Transition Diagram

Let us denote by $\{\rho_{11},\rho_{12},\rho_{13},\ldots,\rho_{21},\rho_{ij},\ldots,\rho_{IJ}\}$ the state of the network where the component ρ_{ij} represents the number of busy circuits between i and j. As in the case of a single trunk, we consider only the adjacent states; the corresponding transition diagram is given in fig. 3.2 where:

- $$a_{ij} = \begin{cases} 0 \text{ if } \sum_j \rho_{ij} = N_{ik} \text{ or } \sum_i \rho_{ij} = N_{kj} \\ \lambda_{ij}\, dt \text{ otherwise} \end{cases}$$

- $$d_{ij} = \begin{cases} 0 \text{ if } \rho_{ij} = 0 \\ \frac{\rho_{ij}}{T}\, dt \text{ otherwise} \end{cases}$$

Let us denote by $P_{N_{ij}}(t)$, the blocking probability of path (i,j), and by $x_{ij}(t)$ the active number of calls from i to j:

$$P_{N_{ij}}(t) = \text{Probability}\{x_{ik}(t) = N_{ik} \text{ or } X_{kj}(t) = N_{kj}\} \tag{3.1}$$

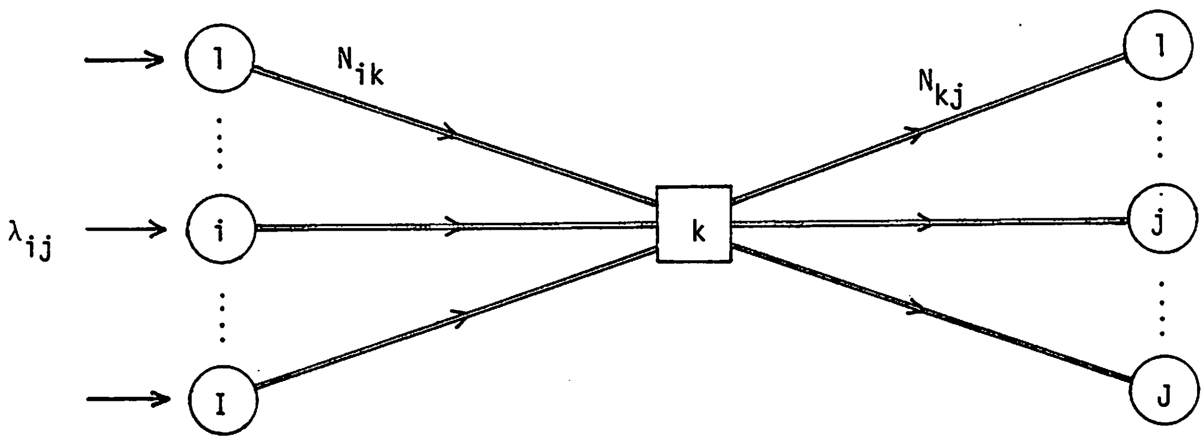


Fig. 3.1. One transit network

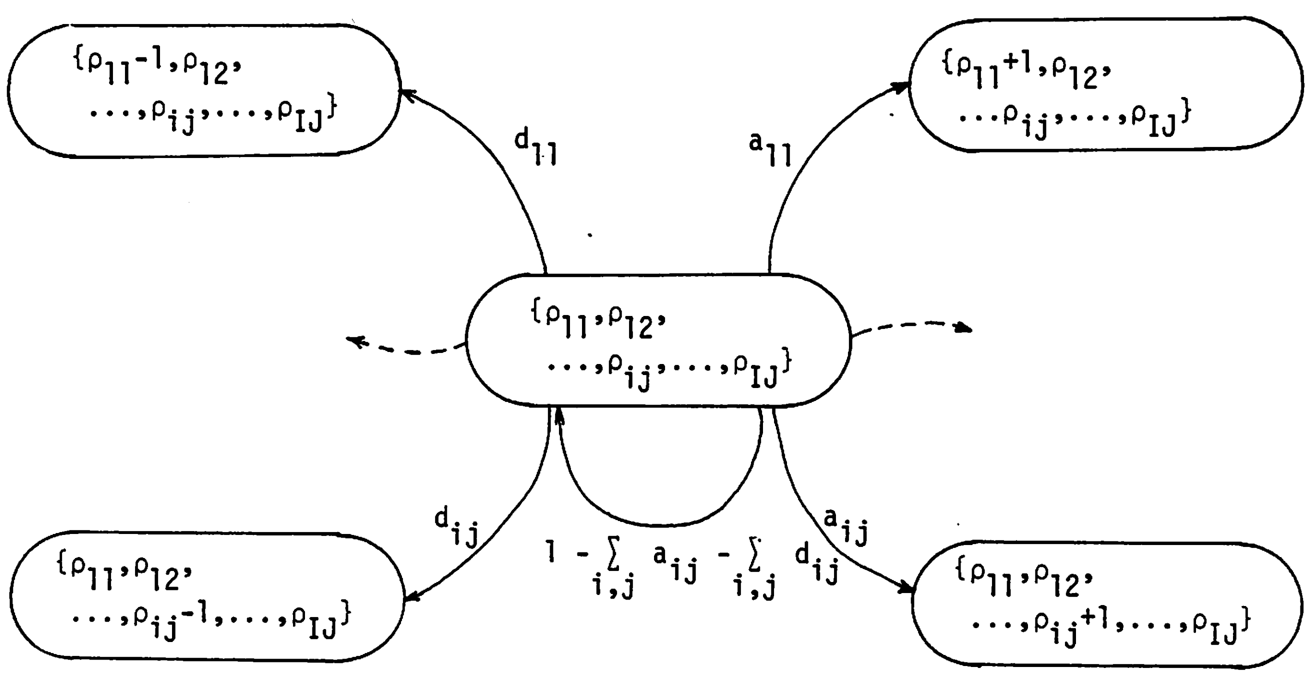


Fig. 3.2. Transition diagram

$$\text{with}\begin{cases} x_{ik}(t) = \sum_j x_{ij}(t) \\ x_{kj}(t) = \sum_i x_{ij}(t) \end{cases}$$

It can be verified [12] that the steady state probability satisfies:

$$P\{x_{11} = \rho_{11},\ldots,x_{ij} = \rho_{ij},\ldots,x_{IJ} = \rho_{IJ}\} = \underset{i,j}{\pi} \frac{Y_{ij}^{\rho_{ij}}}{\rho_{ij}!} P(0) \tag{3.1}$$

It can be verified that dynamic equation (2.16) transforms, in this case, to the following:

$$\dot{X}_{ij} = \frac{X_{ij}(t)}{T} + \lambda_{ij}\{1-P_{N_{ij}}(t)\} \tag{3.2}$$

$X_{ij}(t)$ defines the average number of circuits busy between i and j at time t and $\dot{X}_{ij}$ is its derivative.

In steady state we get:

$$X_{ij} = Y_{ij}\{1-P_{N_{ij}}\},\ Y_{ij} = \lambda_{ij}T \tag{3.3}$$

Then, total traffic carried by trunk ik is:

$$X_{ij} = \sum_{j=1}^{J} X_{ij} \tag{3.4}$$

and total traffic carried by trunk kj is:

$$X_{kj} = \sum_{i=1}^{I} X_{ij} \tag{3.5}$$

The problem in equation (3.3) is to compute $P_{N_{ij}}$. If the network is sufficiently large to assume that the processes on ik and kj trunks are independent, then we can write:

$$P_{N_{ij}} = P_{N_{ik}} + P_{N_{kj}} - P_{N_{ik}}P_{N_{kj}} \tag{3.6}$$

$P_{N_{ik}}$: blocking probability of trunk ik

$P_{N_{kj}}$: blocking probability of trunk kj

Using relation (3.6) we present two different ways for calculating the blocking probabilities $P_{N_{ik}}$ and $P_{N_{kj}}$.

III.3 Erlang Fluid Approximation Model

This model, currently used by telephone engineers, allows traffic calculation as if the telephone process was a fluid flowing from a trunk to the next one. Assuming that the traffic obeys a Poisson distribution, blocking probabilities are computed by Erlang's formula. Let Y_{ik} be the total traffic offered to trunk ik.

$$Y_{ik} = \sum_{j=1}^{J} Y_{ij} \tag{3.7}$$

If the process on trunk ik is considered independent of the remaining trunks, then, Erlang formula (2.10) gives the blocking probability of trunk ik:

$$P^*_{N_{ik}} = E(N_{ik}, Y_{ik}) \tag{3.8}$$

Let Y_{kj} be the total traffic offered to trunk kj:

$$Y_{kj} = \sum_{i=1}^{I} Y_{ij}\{1-P^*_{N_{ik}}\} \tag{3.9}$$

If we assume that Y_{kj} has a Poisson distribution, blocking probability of trunk kj is calculated by Erlang formula:

$$P^*_{N_{kj}} = E(N_{kj}, Y_{kj}) \tag{3.10}$$

Then, using (3.6), traffic carried by each flow (i,j) is:

$$X_{ij} = Y_{ij}(1-P^*_{N_{ik}})(1-P^*_{N_{kj}}) \tag{3.11}$$

III.4 Erlang Inverse Approximation

With the independence assumption (3.6), the dynamic model (3.2) can be rewritten:

$$\dot{X}_{ij} = \frac{X_{ij}(t)}{T} + \lambda_{ij}(1-P_{N_{ik}}(t))(1-P_{N_{kj}}(t)) \tag{3.12}$$

Using the estimator $\tilde{P}_N$ we obtain:

$$\begin{cases} \dot{X}_{ik} = -\dfrac{X_{ik}}{T} + \sum\limits_{j} \lambda_{ij}(1-\tilde{P}_{N_{ik}}(t))(1-\tilde{P}_{N_{kj}}(t)) & i = 1,\dots,I \\ \dot{X}_{kj} = -\dfrac{X_{kj}}{T} + \sum\limits_{i} \lambda_{ij}(1-\tilde{P}_{N_{ik}}(t))(1-\tilde{P}_{N_{kj}}(t)) & j = 1,\dots,J \end{cases} \tag{3.13}$$

$$\text{with}\quad \begin{cases} \tilde{P}_{N_{ik}}(t) = E(N_{ik},\tilde{Y}_{ik}(t)) \\ X_{ik}(t) = \tilde{Y}_{ik}(t)\{1-E(N_{ik},\tilde{Y}_{ik})\} \\ \tilde{P}_{N_{kj}}(t) = E(N_{kj},\tilde{Y}_{kj}(t)) \\ X_{kj}(t) = Y_{kj}(t)\{1-E(N_{kj},\tilde{Y}_{kj})\} \end{cases}$$

In steady state (3.13)

$$\begin{cases} X_{ik} = \sum_{j=1}^{J} Y_{ij}(1-\tilde{P}_{N_{ik}})(1-\tilde{P}_{N_{kj}}) & i = 1,\ldots,I \\ X_{kj} = \sum_{I=1}^{I} Y_{ij}(1-\tilde{P}_{N_{ik}})(1-\tilde{P}_{N_{kj}}) & j = 1,\ldots,J \end{cases} \tag{3.14}$$

This system of nonlinear equations can be solved using a relaxation algorithm, see appendix II.

III.5 Numerical Results

We consider a small network constituted of 3 departure nodes, 3 destination nodes and one transit node. We use the following data:

- capacity:

$$N_{ik} : \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad N_{kj} : [3\ 4\ 6]$$

- offered traffic matrix:

$$Y_{ij} : \begin{bmatrix} 0.5 & 0.5 & 0.8 \\ 0.6 & 0.8 & 1.2 \\ 1.3 & 1.4 & 2.7 \end{bmatrix}$$

In order to compare the two models presented previously, the following table gives the steady state probabilities and carried traffic obtained by (3.11), (3.14) and the exact values computed by the Markovian model.

		Steady state solution of Markovian model $P_{N_{ij}}$	Erlang Inverse $\tilde{P}_{N_{ij}}$	Fluid approximation model $P^*_{N_{ij}}$
blocking probabilities of paths (i,j)	P_{11}	0.4489	0.4588	0.4838
	P_{12}	0.3926	0.4002	0.4320
	P_{13}	0.3817	0.3924	0.4236
	P_{21}	0.2895	0.2998	0.3169
	P_{22}	0.2097	0.2240	0.2479
	P_{23}	0.1964	0.2139	0.2368
	P_{31}	0.3225	0.3369	0.3658
	P_{32}	0.2511	0.2652	0.3021
	P_{33}	0.2339	0.2556	0.2918
traffic carried by trunks ik	X_{IK}	1.074	1.057	1.003
	X_{2K}	2.023	1.984	1.928
	X_{3K}	3.998	3.901	3.714
traffic carried by trunks kj	X_{KI}	1.583	1.553	1.493
	X_{K2}	1.984	1.949	1.863
	X_{k3}	3.527	3.439	3.298

Table 1

The results presented in this table show clearly that the approximate models (3.11) and (3.14) always overestimate losses compared to the exact solution given by the Markovian model. The error is about 2 percent. However, model (3.14) always gives a better approximation than model (3.11).

IV. Over-Variant and Under-Variant Telephone Processes

Let us consider a telephone process described by probabilities P_i (probability of i busy circuits) with an average X and a variance V:

$$X = \sum_{i=0}^{\infty} i \, P_i$$

$$V = \sum_{i=0}^{\infty} i^2 P_i - X^2$$

- if $\frac{V}{X} = 1$ (see 2.13), the offered traffic is assumed Poissonian. Therefore all the results of section II can be applied.
- if $\frac{V}{X} > 1$ (over-variant) or $\frac{V}{X} < 1$ (under-variant) previous results cannot be applied.

In this paragraph we investigate when over-variance and under-variance are created and how to calculate carried traffic, lost traffic in such a case.

IV.1 Over-Variant Process

IV.1.1 Overflow Rerouting

Overflow rerouting is the most commonly used routing policy in telephone networks. Let us consider traffic at rate λ arriving at two trunks. One of these trunks, with capacity N_1, is called the first-choice trunk, the other one is called the second-choice trunk (capacity N_2). The routing consists in first trying the first choice trunk. If all circuits are busy in this trunk (state N_1), calls are switched to the second-choice trunk (fig. 4.1). Overflow traffic is of the over-variant type.

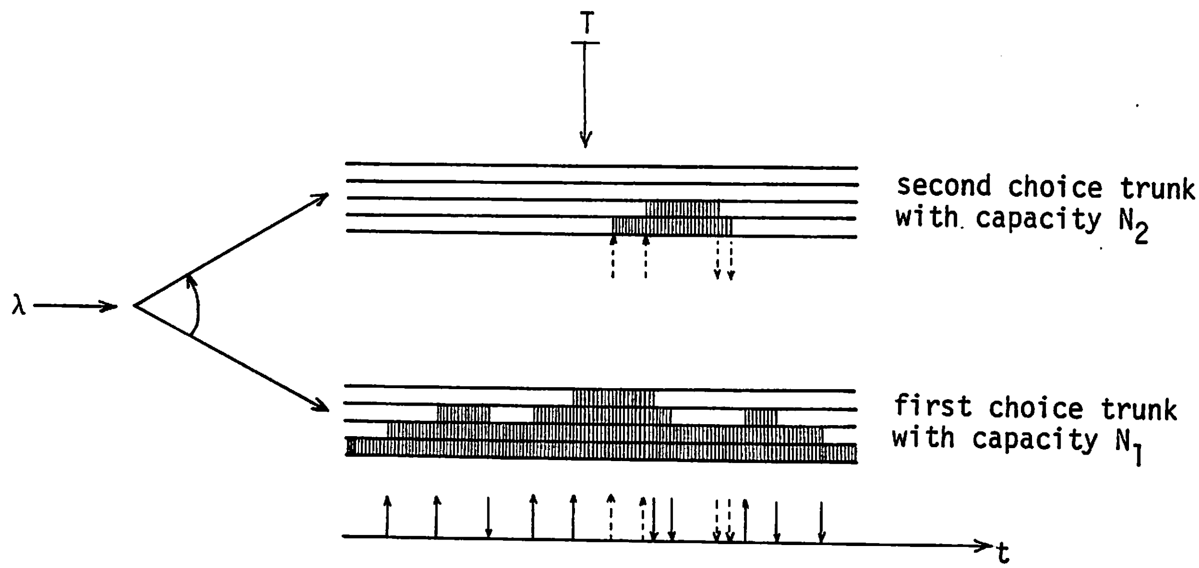


Fig. 4.1. Calls overflowing from the first choice trunk to the second choice trunk.

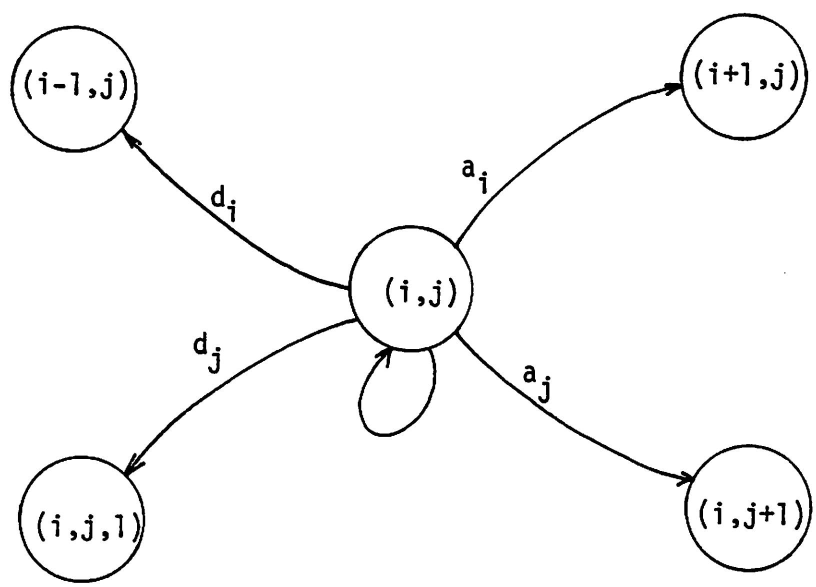


Fig. 4.2. Transition diagram associated with overflow rerouting.

IV.1.2 Transition Diagram

Let i denote the state of the first choice trunk and j the state of the second-choice trunk. The process is identified by the set of possible states (i,j) ($i = 0,\ldots,N_1$; $j = 0,\ldots,N_2$) and by the probabilities P_{ij}(= Probability{$x_1 = i$ and $x_2 = j$}). For a small time interval dt, we only consider transitions to the adjacent stated of (i,j); The transition diagram is given in figure 4.2 where:

$$d_i = \frac{i}{T}\,dt,\quad d_j = \frac{j}{T}\,dt$$

$$a_i = \begin{cases}\lambda\,dt & \text{if } i < N_1\\ 0 & \text{if } i = N_1\end{cases},\quad a_j = \begin{cases}0 & \text{if } i < N_1 \text{ or } j = N_2\\ \lambda\,dt & \text{if } i = N_1 \text{ and } j < N_2\end{cases}$$

IV.1.3 Equilibrium Equations

These equations describe the steady state solution of Markov model $\dot{P} = QP$. Let us denote L_{ij} the equation corresponding to the component $\dot{P}_{ij}$ of vector $\dot{P}$. We get a set of $(N_1+1)\,(N_2+1)$ equations $L_{ij} = 0$ as follows:

$$\begin{cases} L_{ij} : \lambda P_{i-1\,j} - (\lambda + \frac{i+j}{T})P_{ij} + \frac{i+1}{T} P_{i+1j} + \frac{j+1}{T} P_{ij+1} \\ (\forall i = 0,\ldots,N_1-1;\ \forall j = 0,\ldots,N_2-1) \\ L_{N_1j} : \lambda P_{N_1j-1} + \lambda P_{N_1-1\,j} - (\lambda + \frac{N_1+j}{T})P_{N_1j} + \frac{j+1}{T} P_{N_1j+1} \\ L_{N_1N_2} : \lambda P_{N_1N_2-1} + \lambda P_{N_1-1\,N_2} - \frac{N_1+N_2}{T} P_{N_1N_2} \end{cases} \tag{4.1}$$

IV.1.4 Recurrences on Probabilities

Define P_{i*} and P_{*j} as follows:

$$P_{i*} = \text{Probability}\{x_1 = i,\ \forall x_2\} = \sum_j P_{ij}$$
$$P_{*j} = \text{Probability}\{x_2 = j,\ \forall x_1\} = \sum_i P_{ij} \tag{4.2}$$

By summing the equations L_{ij} one gets:

$$\begin{cases} \sum_j L_{0j} \rightarrow -YP_{0*} + P_{1*} = 0 \\ \sum_j L_{1j} \rightarrow YP_{0*} - (Y+1)P_{1*} + 2P_{2*} = 0 \\ \vdots \\ \sum_j L_{N_1 j} \rightarrow YP_{N_1-1*} - N_1 P_{N_1*} = 0 \end{cases} \qquad Y = \lambda T$$

$$\Rightarrow P_{i*} = \frac{Y}{i} P_{i-1*} \quad i = 0,\ldots,N_1 \tag{4.3}$$

$$\begin{cases} \sum_i L_{i0} \rightarrow -Y_{P_{N_1 0}} + P_{*1} = 0 \\ \sum_i L_{i1} \rightarrow -\frac{YP}{2}{}_{N_1 1} + P_{*2} = 0 \\ \vdots \\ \sum_i L_{ij} \rightarrow -\frac{YP}{j}{}_{N_1 j} + P_{*j+1} = 0 \end{cases}$$

$$\Rightarrow P_{*j} = \frac{Y}{j} P_{N_1 j-1} \quad j = 0,\ldots,N_2 \tag{4.4}$$

Of course, the solution of recurrence (4.3) gives the Erlang's formula:

$$P_{N_1*} = E(N_1, Y) \tag{4.5}$$

Hence, all the results obtained in section II apply to the process of the first-choice trunk.

However relation (4.4) does not allow the calculation of P_{*N_2}. We shall see later, how traffic characteristics of the second choice-trunk can be computed.

Summing the equations L_{ij}, in another way, we get:

$$\begin{cases} L_{00} & \rightarrow -YP_{00} + (P_{10}+P_{01}) = 0 \\ (L_{10}+L_{01}) & \rightarrow YP_{00} - (Y+1)(P_{10}+P_{01}) + 2(P_{20}+P_{11}+P_{02}) = 0 \\ (L_{02}+L_{20}+L_{11}) & \rightarrow Y(P_{01}+P_{10}) - (Y+2)(P_{20}+P_{02}+P_{11}) + 3(P_{03}+P_{30}+P_{21}+P_{12}) = 0 \\ \vdots & \\ L_{N_1N_2} & \rightarrow Y(P_{N_1-1N_2}+P_{N_1N_2-1}) - (N_1+N_2)P_{N_1N_2} = 0 \end{cases}$$

By successive substitution in the above, we get:

$$\sum_{i,j\in S_k(i,j)} P_{ij} = \frac{Y^k}{k!} P_{00} \tag{4.6}$$

$$S_k(i,j) : \{i,j/i+j = k;\ 0 \leq i \leq N_1,\ 0 \leq j < N_2\}$$

and

$$P_{N_1N_2} = \frac{Y^{(N_1+N_2)}}{(N_1+N_2)!} P_{00}$$

but:

$$\sum_{k=0}^{N_1+N_2} \sum_{i,j\in S_k(i,j)} P_{ij} = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} P_{ij} = 1$$

thus we get:

$$P_{00} = \frac{1}{\sum_{k=0}^{N_1+N_2} \frac{Y^K}{k!}}$$

and therefore:

$$P_{N_1N_2} = \frac{\frac{Y^{N_1+N_2}}{(N_1+N_2)!}}{\sum_{k=0}^{N_1+N_2} \frac{Y^k}{k!}} = E(N_1+N_2,Y) \tag{4.7}$$

(4.7) proves the following intuitive result: The probability all circuits in the first and second choice trunk are busy is equal to the blocking probability of a trunk having a capacity (N_1+N_2).

IV.1.5 Characteristics of Traffic Offered to the Second Choice Trunk

By definition the traffic offered to the second choice trunk is the traffic which would be carried by this trunk having an infinite capacity. Let Y' and V' be the average and the variance of this traffic. Then,

• average $$Y' = \sum_{j=0}^{\infty} jP_{*j} \qquad (4.8)$$

$$(4.8) + (4.4) \Rightarrow Y' = \sum_{j=0}^{\infty} j \frac{Y}{j} P_{N_1 j-1} = YP_{N_1 *} = YE(N_1,Y) \qquad (4.9)$$

• variance $$V' = \sum_{j=0}^{\infty} j^2 P_{*j} - Y'^2 \qquad (4.10)$$

but

$$\sum_{j=0}^{\infty} j^2 P_{*j} = Y \sum_{j=1}^{\infty} jP_{N_1 j-1} = Y\{\sum_{j=1}^{\infty} jP_{N_1 j} + P_{N_1 *}\}$$

$$= Y \sum_{j=1}^{\infty} jP_{N_1 j} + Y' \qquad (4.11)$$

In order to calculate $\sum_j j\, P_{N_1 j}$ it would be useful to find a recurrence between $\sum_j jP_{ij}$. Let us sum the equations L_{ij} in the following way:

$$\begin{cases} \sum_{j=0}^{\infty} jL_{0j} \rightarrow \sum_{j=0}^{\infty} jP_{1j} = (Y+1) \sum_{j=0}^{\infty} jP_{0j} \\ \sum_{j=0}^{\infty} jL_{ij} \rightarrow \sum_{j=0}^{\infty} jP_{i+1j} = \frac{Y+i+1}{i+1} \sum_{j=0}^{\infty} jP_{ij} - \frac{Y}{i+1} \sum_{j=0}^{\infty} jP_{i-1j} \\ i = 0,\ldots,N_1-1 \\ \sum_{j=0}^{\infty} jL_{N_1 j} \rightarrow \sum_{j=0}^{\infty} jP_{N_1 j} = \frac{Y}{N_1+1} \sum_{j=0}^{\infty} jP_{N_1-1 j} + \frac{Y}{N_1+1} P_{N_1 *} \end{cases}$$

Let us denote:

$$A_i = \sum_{j=0}^{\infty} jP_{ij} \text{ and } B_i = \sum_{k=0}^{i} \frac{Y^k}{k!}$$

we find:

$$\left.\begin{aligned} A_1 &= B_1\, A_0 \\ A_2 &= B_2\, A_0 \\ &\vdots \\ A_{N_1} &= B_{N_1}\, A_0 \end{aligned}\right\} \tag{4.12}$$

and

$$A_{N_1} = \frac{Y}{N_1+1}\{A_{N_1-1}+P_{N_1}*\} \tag{4.13}$$

$$(4.12) + (4.13) \Rightarrow S_0 = \frac{\frac{Y}{N_1+1}\, P_{N_1}*}{B_{N_1} - \frac{Y}{N_1+1}\, B_{N_1-1}} = \frac{YP_{N_1}*}{B_{N_1}\{N_1+1-Y+YP_{N_1}*\}}$$

therefore we get:

$$\sum_{j=0}^{\infty} j\, P_{N_1 j} = A_{N_1} = B_{N_1} S_0 = \frac{YP_{N_1}*}{N_1+1-Y+Y\, P_{N_1}*} \tag{4.14}$$

Finally from (4.10) and (4.11) and (4.14) we obtain the well known formula of variance:

$$V' = Y'\{1 - Y + \frac{Y}{N_1+1+Y'-Y}\} \tag{4.15}$$

R. I. Wilkinson [1] demonstrated this formula using factorial moment generating function. The plotting of V' and Y' for different values of N_1 and Y shows clearly that V' is greater or equal to Y'. Hence, offered traffic to second choice trunk is an over-variant process ($\frac{V'}{Y'}$ is called peakedness factor).

IV.1.6 Characteristics of Traffic Carried by the Second Choice Trunk

Let X_2 and V_2 the average and the variance of the traffic carried by the second choice trunk. Then, average

$$X_2 = \sum_{j=0}^{N_2} j\ P_{*j} = Y \sum_{j=0}^{N_2} P_{N_1 J-1} \Rightarrow X_2 = Y\{P_{N_1 *} - P_{N_1 N_2}\} \qquad (4.16)$$

with

$$P_{N_1 *} = E(N_1, Y)$$

and

$$P_{N_1 N_2} = E(N_1 + N_2, Y)$$

<u>variance</u>

$$V_2 = \sum_{j=0}^{N_2} j^2 P_{*j} = Y \sum_{j=1}^{N_2} j\ P_{N_1 j-1} - X_2^2$$

Using the same demonstration than for V' we get:

$$V_2 = X_2\{1 - X_2 + \frac{1}{N_1 + 1 - Y + Y'}\} - YN_2\ P_{N_1 N_2} \qquad (4.17)$$

<u>Remark</u>:

Generally the following approximation is made

$$P_{N_1 N_2} = P_{N_1 *}\ P_{*N_2} \Rightarrow \tilde{P}_{*N_2} = \frac{E(N_1 + N_2, Y)}{E(N_1, Y)} \qquad (4.18)$$

Using approximation (4.18) carried traffic can be calculated as follows:

$$\tilde{X}_2 = Y'(1 - \tilde{P}_{*N_2}) \qquad (4.19)$$

This is an exact formula for the traffic carried by the second choice trunk:

$$(4.16),\ (4.19) \Rightarrow \tilde{X}_2 = X_2$$

In the same way, it is usual to find the same approximation for the variance of the carried traffic:

$$\tilde{V}_2 = V'(1 - \tilde{P}_{*N_2}) \qquad (4.20)$$

$$(4.15),\ (4.17) \rightarrow \tilde{V}_2 \neq V_2$$

Depending on Y' and N_2, the difference between V_2' and V_2 may be large. In fact, there is no reason for (4.20) to be a good approximation.

IV.1.7 Equivalent Trunk Theory for Overvariant Traffic

Let us consider a system composed of several first choice trunks overflowing to a common second choice trunk (fig. 4.3).

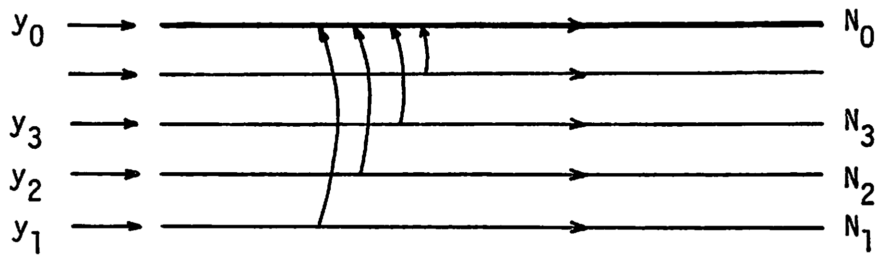


Fig. 4.3. Traffics overflowing to a common second choice trunk.

The study of traffic carried by trunk 0 is much more complicated than in the previous example; in fact no exact formula has been found for the carried traffic on trunk 0. The equivalent trunk theory elaborated by R. I. Wilkinson provides a powerful tool since the system of figure 4.3 can be reduced to the simplest case represented on figure 4.1, and then all the results obtained previously can be applied.

Let Y_i' and V_i' be respectively the average and the variance of the traffic overflowing from trunk i,

$$(4.9),\ (4.15) \rightarrow \begin{cases} Y_i' = Y_i\ E(N_i, Y_i) \\ V_i' = Y_i'\{1 - Y_i' + \dfrac{1}{N_i+1+Y_i'-Y_i}\} \end{cases}$$

If the blocking probability of trunk 0 is small then the overflow traffics can be assumed independent and then the resulting total traffic offered to trunk 0 has the following characteristics:

$$\begin{cases} Y' = \sum_{i=1}^{I} Y_i' + Y_0 \\ V' = \sum_{i=1}^{I} V_i' + Y_0 \end{cases} \qquad (4.21)$$

This over-variant traffic ($V' > Y'$) can be viewed as a traffic overflowing from a unique trunk. Let N^* and Y^* be the capacity and the offered traffic of this equivalent trunk. These two variables have to satisfy the two nonlinear equations of overflow traffic:

$$\begin{cases} Y' = Y^* E(N^*,Y^*) \\ V' = Y'\{1 - Y' + \frac{1}{N^*+1+Y'-Y^*}\} \end{cases} \tag{4.22}$$

(In Appendix III we present an algorithm for solving (4.22)). Traffic carried by trunk 0 is computed by (4.16):

$$X_0 = Y^*\{E(N^*,Y^*)-E(N^*+N_0,Y^*)\} \tag{4.23}$$

The contribution of traffic i is:

$$X_0^i = Y_i' \frac{X_0}{Y'} \tag{4.24}$$

IV.2 Under-Variant Process

Let us consider a trunk with capacity N and with Poisson offered traffic $Y = \lambda T$. We demonstrated in paragraph II.3 that carried traffic has a Poisson distribution when $\lambda T \ll N$. When blocking probability cannot be neglected, this property doesn't remain, and it is interesting to know the impact of blocking probability upon variance of carried traffic.

IV.2.1 Variance of Carried Traffic

The probability of state i is:

$$P_i = \frac{\frac{Y^i}{i!}}{\sum_{j=0}^{N} \frac{Y^j}{j!}}$$

and

$$P_i = \frac{Y}{i} P_{i-1} \tag{4.25}$$

By definition the variance of this process is:

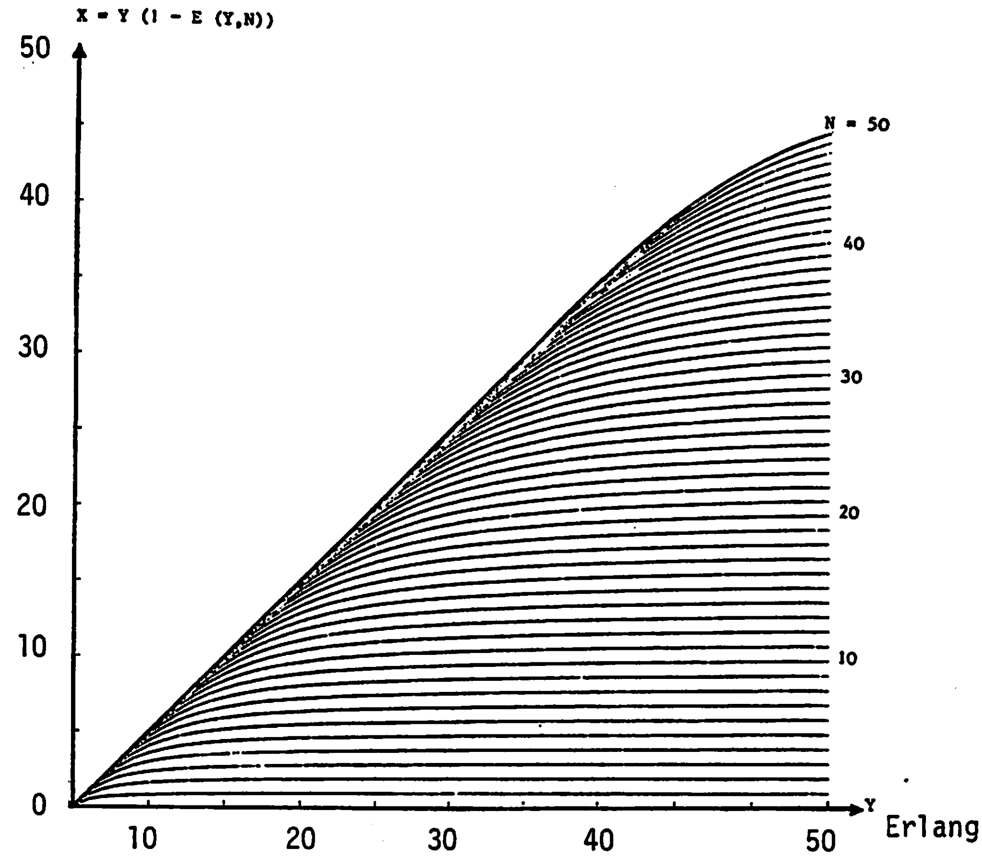

X = Y (1 - E (Y,N))
50
40
30
20
10
0
N = 50
40
30
20
10
10
20
30
40
50
Y
Erlang

Fig. 4.4

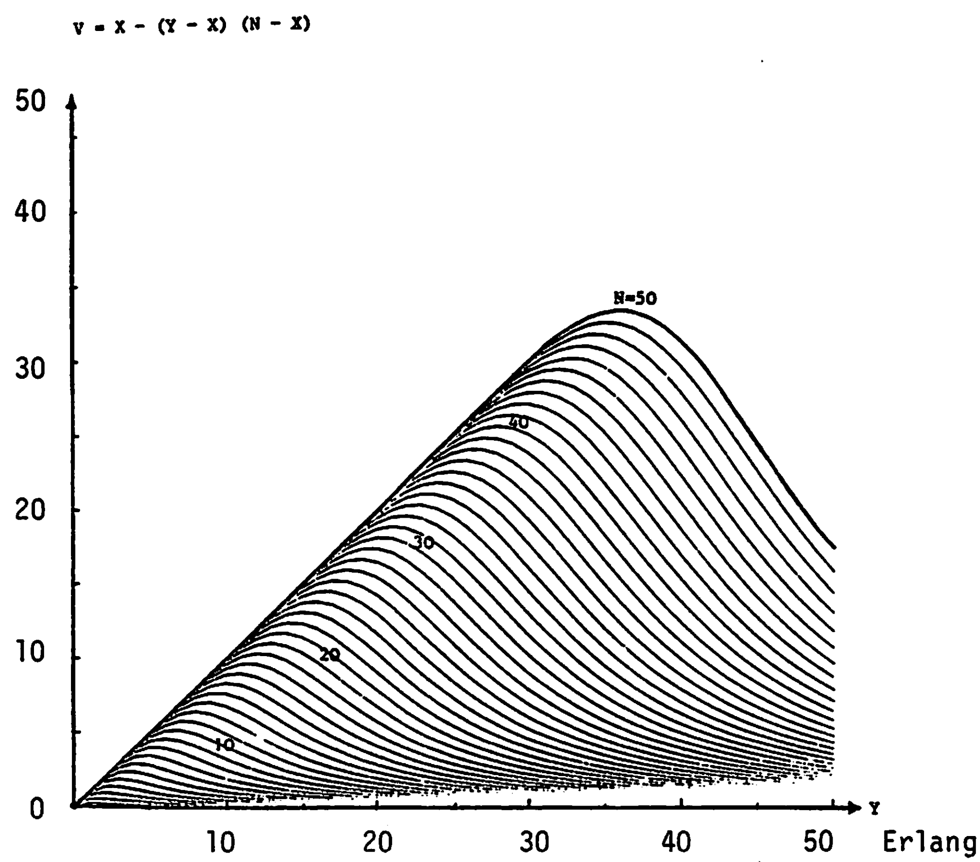

V = X - (Y - X) (N - X)
50
40
30
20
10
0
N=50
40
30
20
10
10
20
30
40
50
Y
Erlang

Fig. 4.5

$$V = \sum_{i=0}^{N} i^2 \, P_i - X^2 \qquad (4.26)$$

X being the average of the process:

$$X = Y(1-P_N)$$

$$(4.25), (4.26) \rightarrow V = Y\{\sum_{i=0}^{N} i^2 \frac{P_{i-1}}{i}\} - X^2$$

$$= Y\{\sum_{i=0}^{N} i \, P_{i-1}\} - X^2$$

$$= Y\{\sum_{i=0}^{N-1} i \, P_i + \sum_{i=0}^{N-1} P_i\} - X^2$$

but $\sum_{i=0}^{N} i \, P_i = X \Rightarrow \sum_{i=0}^{N-1} i \, P_i = X - NP_N$, then $V = Y\{X-NP_N+1-P_N\} - X^2$.

Finally, rearranging this expression, we get:

$$V = X - (Y-X)(N-X) \qquad (4.27)$$

Since $X \leq N$ and $X \leq Y$ it is obvious that $V \leq X$.

$\frac{V}{X}$ can be called "saturation factor"

Hence, the traffic carried by a trunk, can be an undervariant process. X and V are plotted on figures 4.4 and 4.5 for different values of Y and N; these figures show that when Y is approximately greater than 60% then V is smaller than X.

IV.2.2 Two Trunks in Series

Let us consider two trunks ik and kj in series, and let Y be the offered traffic at node i (fig. 4.6).

It is obvious that calls are rejected only by the trunk having the smallest capacity. In fact this system is equivalent to a single trunk from the point of view of call losses.(fig. 4.7).

The traffic carried between nodes i and j is:

$$X_{ij} = Y\{1-E(N_{ij},Y)\} \qquad (4.28)$$

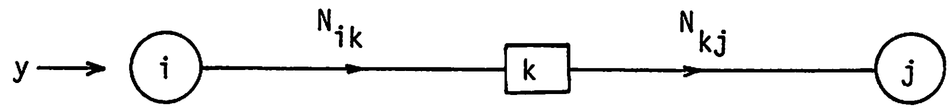
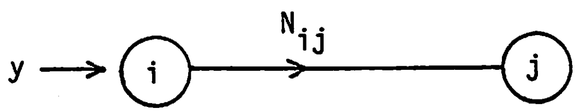


Fig. 4.6. Two trunks in series.



$N_{ij} = \min\{N_{ik}, N_{kj}\}$

Fig. 4.7. Equivalent trunk.

Remark

The Erlang fluid approximation presented in paragraph III.3 implies:

$$\tilde{X}_{ij} = Y(1-\tilde{P}_{ik})(1-\tilde{P}_{kj}) \tag{4.29}$$

$$\text{with} \begin{cases} \tilde{P}_{ik} = E(N_{ik}, Y) \\ \tilde{P}_{kj} = E(N_{kj}, Y_{kj}), \; Y_{kj} = Y(1-\tilde{P}_{ik}) \end{cases}$$

$(4.29),\ (4.28) \rightarrow \tilde{X}_{ij} < X_{ij}$

IV.2.3 <u>The Equivalent Trunk Theory for Under-Variant Traffic</u>

The model we propose here is an extension of Erlang fluid approximation model (we suppose traffic flows through the network like a fluid). The only difference with model of paragraph III.3 is the introduction of variance to compute blocking probabilities.

We make the same assumption as for over-variant traffics: the addition of several traffics into a node is equivalent to an unique traffic having an average and a variance equal to the sum of individual averages and variances.

Let us consider the following network:

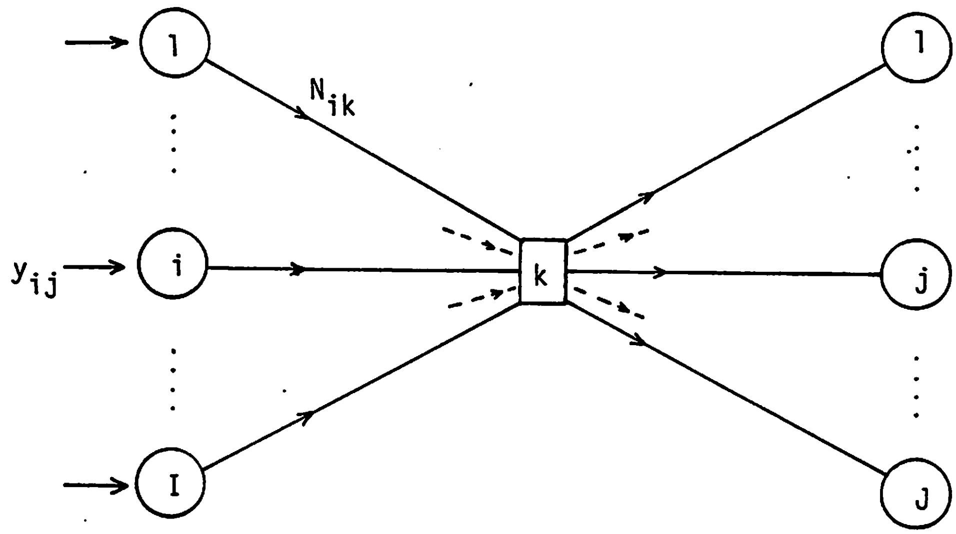


Fig. 4.8

Let Y_{ij}^k and V_{ij}^k be the average and variance of traffic carried by trunks ik:

$$\begin{cases} Y_{ij}^k = Y_{ij}\{1-E(N_{ik},\sum_j Y_{ij})\} \\ V_{ij}^k = Y_{ij}^k - (Y_{ij}-Y_{ij}^k)(N_{ik}-Y_{ij}^k) \end{cases} \quad \begin{matrix} i = 1,\dots,I \\ j = 1,\dots,J \end{matrix} \tag{4.30}$$

All these traffics add at node k; we denote by Y_j the average and V_j the variance of the resulting traffic offered to trunks j.

$$\begin{cases} Y_j = \sum_i Y_{ij}^k \\ V_j = \sum_i V_{ij}^k \end{cases} \quad j = 1,\dots,J \tag{4.31}$$

If $V_j < X_j$ the traffic offered to trunk kj is under variant; then we can assume that this traffic has the characteristics of the traffic which would be carried by a unique trunk. This equivalent trunk with capacity N_j^* and offered traffic Y_j^* has to satisfy the following system of equations (given by (2.12) and (4.27)):

$$\begin{cases} Y_j = Y^*\{1-E(N_j^*,Y_j^*)\} \\ V_j = Y_j - (Y_j^*-Y_j)(N_j^*-Y_j) \end{cases} \quad j = 1,\dots,J \tag{4.32}$$

The network of figure 4.8 decomposes into J simple networks. Each of them is composed of two trunks in series:

From IV.2.2, we can easily calculate the traffic carried by trunks kj:

$$X_{kj} = Y_j^*\{1-E(\min(N_j^*,N_{kj}),Y_j^*)\} \qquad (4.33)$$

$$\forall j = 1,\ldots,J$$

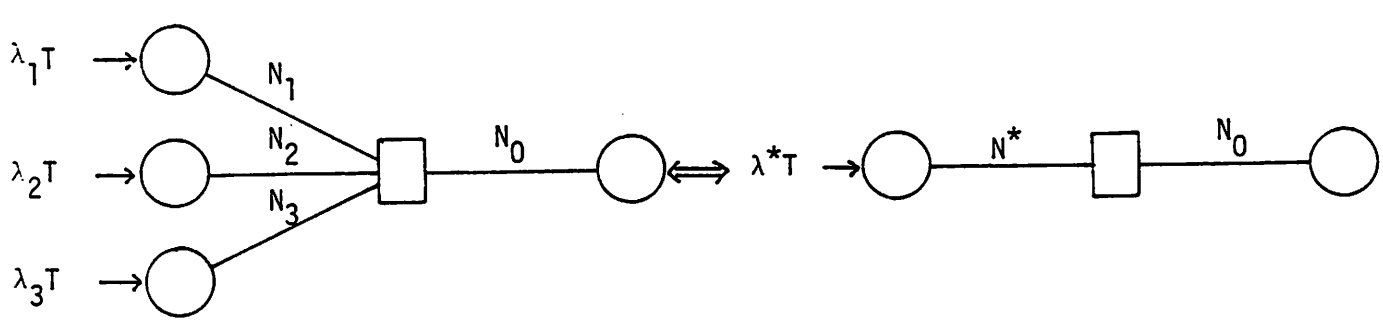
In fact if $N^* < N_{kj}$ then $X_{kj} = Y_j$ meaning that trunk kj is never blocked.

Let X_{ij}^k be the contribution of traffic Y_{ij} to the total traffic carried by trunk kj; in order to compute X_{ij}^k we use the classical assumption that the carried traffic of one flow is proportional to the offered traffic of this flow:

$$X_{ij}^k = \frac{Y_{ij}^k}{Y_j} X_{kj} \qquad (4.34)$$

IV.2.4 Numerical Results

In order to compare the equivalent trunk method with Erlang fluid approximation and with the exact solution given by stationary probabilities of the Markovian model, we study the following network constituted of 3 source nodes i one transit node k and one destination node j:



The results obtained by the three models are presented in the following table:

		EXACT SOLUTION	EQUIVALENT TRAFFIC THEORY	FIRST MOMENT APPROXIMATION
$\lambda_1 T=25$ $N_1=20$	m	68.16	67.74	66.47
$\lambda_2 T=35$ $N_2=30$	x_1	15.52	14.87	14.59
$\lambda_3 T=45$ $N_3=40$	x_2	22.83	22.55	22.13
$\lambda^* T=95$ $N^*=87$ $N_0=70$	x_3	29.81	30.31	29.74
$\lambda_1 T=20$ $N_1=15$	m	44.27	44.25	42.05
$\lambda_2 T=21$ $N_2=20$	x_1	13.08	12.93	12.29
$\lambda_3 T=19$ $N_3=18$	x_2	16.48	16.54	15.71
$\lambda^* T=53$ $N^*=50$ $N_0=48$	x_3	14.71	14.78	14.04

These results as well as other results obtained with different data show that the equivalent trunk theory improves the accuracy of fluid model usually used to describe large telephone networks.

IV.3 Generalization of the Equivalent Trunk Theory

The equivalent trunk theory for overflow traffic developed by R. I. Wilkinson and the extension to under-variant traffic presented previously provides a general method to compute blocking probabilities when non-Poisson traffic is offered to a trunk.

Let V be the variance and Y be the average of traffic offered to a trunk having capacity N. Depending on whether $V < Y$ or $V > Y$ or $V = Y$ there are three ways to compute X, the traffic carried by this trunk.

a) $V = Y$

$X = Y\{1-E(N,Y)\}$

b) $V > Y$

find (N^*,Y^*) such that:
$$\begin{cases} Y = Y^*E(N^*,Y^*) \\ V = Y\{1 - Y + \dfrac{Y^*}{N+1+Y-Y^*}\} \end{cases}$$

$$\Rightarrow X = Y^*\{E(N^*,Y^*)-E(N+N^*,Y^*)\}$$

c) $V < Y$

find N^*, Y^*
$$\begin{cases} Y = Y^*\{1-E(N^*,Y^*)\} \\ V = Y - (Y^*Y)\ (N^*-Y) \end{cases}$$

$$X = Y^*\{1-E(\min(N^*,N),Y^*)\}$$

V. Conclusion

In this report we discussed different ways of estimating call losses in a telephone network. Basic assumptions such as Poisson arrivals, exponential holding time and lost calls cleared were made. Some classical results were rederived and new theoretical and practical results were presented.

It is obvious that a "microscopic" representation of the telephone process such as a Markovian model, cannot be used when considering large networks, because of the rapid increase of elementary states of the system with respect to the number of trunks and the capacity of these trunks.

However, in the simplest case of one and two trunks, we have shown that exact formulas of blocking probability and average and variance of offered (or carried) traffic can be obtained. The purpose of modelling was to extend these theoretical results to larger networks by doing some judicious approximations and to validate them by numerical comparisons.

In the second and third paragraph the basic idea was to use the Erlang formula for estimating the transient blocking probability. The plots of $\tilde{P}_N(t)$ and $P_N(t)$ shows that $\tilde{P}_N(t)$ is a good estimator for the blocking probability $P_N(t)$. The computation of $\tilde{P}_N$ is not simpler than that of $P_N(t)$ in the case of one trunk. However an improvement is obtained when considering several interconnected trunks (model (3.13)). In such a case model (3.13) gives better results than the classical Erlang fluid approximation (model (3.11)).

In the last section the idea of equivalent trunk, developed earlier by R. I. Wilkinson, was extended to a network constituted of trunks interconnected by a transit node. The concepts of over variant traffic and under variant traffic allows the generalization of equivalent trunk theory. When analyzing large networks exact blocking probabilities (or lost traffic) are extremely difficult to obtain. Then, the most convenient approximation consists to assume that the telephone process flows through the network like a fluid. The errors introduced by the Erlang fluid approximation are due to the fact that this model assumes that the Poisson distribution of the input process is not modified by the network. Representing traffic both by its variance and its average allows to take into account peakedness factor (over variant traffic) and the corresponding equivalent trunk then increases the accuracy of blocking probability estimation.

References

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Appendix I

Given the carried traffic X and the capacity N of a trunk, the following algorithm finds the offered traffic Y* such that:

$$X = Y^*\{1-E(N,Y^*)\}$$

Step 1: $i = 0 \quad Y^i = 0 \quad$ select $\varepsilon > 0$

Step 2: compute $E(N,Y^i)$; $X^i = Y^i\{1-E(N,Y^i)\}$ and $\frac{dX^i}{dY^i} = 1 - E(N,Y^i)\{N+1-X^i\}$

Step 3:

$$\text{if } \frac{\|X^i-X\|}{X} \begin{cases} \bullet \leq \varepsilon \Rightarrow Y^* = Y^i \\ \qquad \tilde{P} = E(N,Y^i) \\ \qquad \text{stop} \\ \bullet > \varepsilon \text{ go to step 4} \end{cases}$$

Step 4:

$$Y^{i+1} = Y^i + \frac{X-X^i}{\frac{dX^i}{dY^i}}$$

$i = i + 1$ go to step 2.

Appendix II

Steady state solution of model (3.13) is computed by a relaxation algorithm

$$X_{ik} = \sum_j Y_{ij}(1-\tilde{P}_{N_{ik}})(1-\tilde{P}_{N_{kj}})$$

$$X_{kj} = \sum_i Y_{ij}(1-\tilde{P}_{N_{ik}})(1-\tilde{P}_{N_{kj}})$$

Step 1:

initialize $\beta^0_{kj} = 1 \quad \forall j$

$\ell = 1$ go to Step 2

Step 2:

$$\alpha^{\ell}_{ik} = 1 - E(N_{ik}, \sum_j Y_{ij}\beta^{\ell-1}_{kj})$$

$$\beta^{\ell}_{kj} = 1 - E(N_{kj}, \sum_i Y_{ij}\alpha^{\ell}_{ik})$$

go to step 3

Step 3:

- if $\|\alpha^{\ell}-\alpha^{\ell-1}\|$ and $\|\beta^{\ell}-\beta^{\ell-1}\| \leq \varepsilon$

  $\tilde{P}_{N_{ik}} = 1 - \alpha^{\ell}_{ik}$

  $\tilde{P}_{N_{kj}} = 1 - \beta^{\ell}_{kj}$
- otherwise $\ell = \ell + 1$ go to Step 2.

Appendix III

Given an over-variant traffic having a variance V' and an average Y', the following algorithm finds the offered traffic Y* and the capacity N* of the first choice equivalent trunk such that:

$$\begin{cases} Y' = Y^*E(N^*,Y^*) \\ V' = Y'\{1 - Y' + \frac{1}{N^*+1+Y'-Y^*}\} \end{cases}$$

Step 1: $i = 0 \quad j = 0$, step length $\alpha > 1$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, maximum capacity $\bar{N}$

$Y^0 = 2Y'$, $N^0 = I(Y')$ $\quad I(\cdot)$ = integral part

Step 2: Compute

$$Y'^i = Y^i E(N^j, Y^i)$$

and

$$\frac{dY'^i}{dY^i} = E(N^j,Y^i)\{N^j+1+Y'^i-Y^i\}$$

- if $\frac{\|Y'^i - Y'\|}{Y'} \leq \varepsilon_1$ go to Step 4.
- otherwise next step

Step 3:

$$Y^{i+1} = Y^i + \frac{Y'-Y'^i}{\frac{dY'^i}{dY^i}}$$

$i = i + 1$ go to Step 2.

Step 4:

$$V'^j = Y'\{1 - Y' + \frac{Y^i}{N^j+1-Y^i+Y'}\}$$

- if $\frac{V'^j - V'}{V'} \leq \varepsilon_2 \qquad \begin{matrix} Y^* = Y^i \\ N^* = N^j \end{matrix}$

In order to get better precision, it is preferable to test adjacent value of N^j : $N^j - 1$ or $N^j + 1$

- otherwise adjust step length α

$$* \ \frac{\| V'^{j} - V' \|}{V'} < \frac{\| V'^{j-1} - V' \|}{V'} \rightarrow \alpha^{j+1} = 1.2\alpha^{j}$$

$$* \ \frac{\| V'^{j} - V' \|}{V'} > \frac{\| V'^{j-1} - V' \|}{V'} \rightarrow \alpha^{j+1} = 0.8\alpha^{j}$$

go to next step

Step 5:

$$N^{j+1} = N^{j} + I(\alpha^{j+1} \frac{V'^{j} - V'}{V'})$$

- if $N^{j+1} < 1 \rightarrow N^{j+1} = 1$

- if $N^{j+1} > \bar{N} \rightarrow N^{j+1} = \bar{N}$

$j = j + 1$ GO TO STEP 2