PROBABILISTIC DYNAMIC SECURITY ASSESSMENT OF POWER SYSTEMS PART I: BASIC MODEL

by

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ABSTRACT

A comprehensive framework for power system security assessment which incorporates probabilistic aspects of disturbances and system dynamic responses to disturbances is presented. Standard mathematical models for power system (steady-state) power flow analysis and transient stability (dynamic) analysis are used. The probability distribution of the time to insecurity is shown to be obtainable from the solution of a linear vector differential equation. The coefficients of the differential equation contain the transition rates of system structural changes and a set of transition probabilities defined in terms of the steady-state and the dynamic security regions. These regions are defined in the space of power injections. Upper and lower bounds on the time to insecurity distribution are obtained.

1. INTRODUCTION

The concept of power system security was introduced by Dy Liacco [1-3]. Security is considered to be an instantaneous, time-varying condition that is a function of the robustness of the system relative to imminent disturbances [4-5]. A working definition of security introduced by Dy Liacco [1-3,6] employs a deterministic framework in which the robustness of the system is tested, using the steady-state model, with respect to a set of selected disturbances, or contingencies. However, because of uncertainty involved in the prediction of "imminent disturbances," a probabilistic framework for security assessment should be used. Furthermore most of the major power system breakdowns are caused by problems associated with system dynamic responses [7], dynamic models of the system should be included in security assessment. In this two-part paper we introduce a comprehensive framework for probabilistic dynamic security assessment of power systems. We say a power system is steady-state secure with respectto a given configuration (system structure) and load if it can be operated in normal state with no overload in any component. We say a power system is dynamically secure with respect to a given configuration, fault and load if it is transiently stable. Steady-state and dynamic security regions are defined in the space of power injections (load demand and generation). The time to the first instance at which the injection leaves the security region is defined as the measure of system security. It is shown that the probability distributionto the time to insecurity can be obtained from the solution of a vector differential equation. Upper and lower bounds for the time-to-insecurity distribution are obtained. In this paper (Part I) the system configurations during faults, though considered in the computation of dynamic security regions, are neglected in the consideration of the evolution of changes in system configurations. In Part II the system configurations during faults are considered. It is shown that the results of this paper are obtained as a limiting case when the short circuit duration approaches zero.

Comprehensive survey papers on conventional deterministic steady-state security assessment are available [1-4,6,9]. A simple measure of system security without considering the transmission system is the spinning reserve. Probability methods have been introduced for spinning reserve calculations [17, 18 Chapter 8]. An extension of this approach was suggested in [19]. A rather general conceptual approach to probabilistic dynamic security assessment in terms of stochastic differential equations was suggested in [5].

Most notations used in the text are standard. We use $\utilde{J}$ and $\utilde{y}$ to denote random variables, whether it is a scalar random variable or a vector random variable should be clear by the value it takes, a scalar J or a vectory $\underline{y}$. The probability density of $\utilde{y}$ is denoted by $f_{\utilde{y}}(\underline{y})$. The ith component of the vector $\underline{y}$ is denoted by y_i. The little oh notation, $o(\Delta t)$, is used to represent a term such that $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} \to 0$.

2. OVERALL FRAMEWORK

2.1 Modeling Considerations

A (bulk) power system consists of generations and loads, interconnected together by transmission lines and transformers. Each generation subsystem supplies real and reactive powers to the system.† It has a synchronous generator driven by either a steam or a hydro-turbine. The

†The reactive power from a generator may be positive or negative.

synchronous machine has its associated control devices for excitation control and speed-governor-turbine control. There are dynamics assoicated with machines, control devices and boilers [8]. Each load is an aggregate of real and reactive power demand at a substation. There are dynamics associated with load demands. Except for the study of electromagnetic transients, which are not considered here, the transmission subsystem is considered to be in sinusoidal steady-state. There is a maximum power carrying capability of a line or a transformer set by its thermal limit. (In current practice the maximum power carrying capability of some lines are limited by stability consideration.) During operation a component may fail, causing an outage. There are also protective devices to switch off a component when an abnormality is detected. Therefore, the configuration of the system depends on the availability of its components. The dynamic responses of the system variables, e.g., voltages, machine fluxes, speeds, phase angles, exciter voltages, etc., of course are affected by system configuration.

The disturbances on the system may be classified into load disturbance and event disturbance [9]. Load disturbances are the small random fluctuations of load demands. Event disturbances may be further classified into exogenous disturbances and endogenous disturbances. An exogenous disturbance is defined to be a short-circuit or a component outage or addition caused by outside forces, for example, a short-circuit on a transmission line due to lightning, a generator outage due to equipment failure, a generator addition after repairment, a sudden large load change. An endogenous disturbance is defined to be a component outage caused by an event developed in the power system, for example, disconnecting a shorted line, disconnecting an overloaded line, a generator outage due to loss of synchronism, load shedding, etc. The consequence of an event disturbance is the change of system configuration. The probability of occurrence of an exogenous disturbance is time-varying and may depend on ambient conditions. The occurrence of an endogenous disturbance is governed by the state in which the system is in.

The time between changes in system configuration is much longer than the time between changes in system variables associated with physical components, e.g., voltages, machine angles. The former is basically discrete in nature whereas the latter is continuous. Therefore we suggest the use of a two-level model to view the system dynamics. The first level of the model describes the evolution of system configurations. The second level of the model describes the trajectories of the system variables associated with component dynamics. The two levels are coupled.

2.2 State Variables

The first level of our model describes the evolution of system configurations that are discrete. Let N(t) denote the number of transitions of system configurations in [0,t] and $J_{N(t)}$ denote the system configuration immediately after the N(t)-th transition has occurred. $J_{N(t)}$ for any t is a discrete variable and takes values i,j,k etc. The change in configuration is a result of random event disturbances. Thus $J_{\underset{\sim}{N}(t)}$ is indeed a random process. We assume that the random process $J_{\underset{\sim}{N}}(t)$ satisfies:

Assumption A1: $J_{\underset{\sim}{N}(t)}$ is a Markov process and the probability of two or more transitions in a small interval Δt is of the order of $o(\Delta t)$, i.e.,

$$\Pr\{\underset{\sim}{N}(t+\Delta t) - \underset{\sim}{N}(t) \geq 2\} = o(\Delta t), \forall t$$

The second level of our model describes the trajectories of system variables when the configuration is given. The moving force behind the variations in system variables is the power injection (power generation and load demand). Let $\underline{y}(t)$ denote the power injection at time t. Because of random load disturbances, $\underset{\sim}{y}(t)$ is a random process. We assume that the random process $\underset{\sim}{y}(t)$ satisfies:

Assumption A2: The random process $\{\underset{\sim}{y}(t), t \geq 0\}$ is separable and sample piecewise differentiable, i.e., with probability 1, every sample function $\underline{y}(t)$ is piecewise differentiable. Furthermore, $\{\underset{\sim}{y}(t)\}$ is independent of $\{\underset{\sim}{J}_{N(t)}\}$.

We define the state of the system to be $\underline{\sigma}(t) = (J_{N(t)}, \underline{y}(t))$.

2.3 Security Regions

Because of different time scales of interest conventionally studies of power system response are divided into transient stability analysis for large disturbance, steady-state (or small disturbance) stability analysis†, and steady-state (power flow) analysis. Transient stability is considered when there is an event disturbance, i.e., a change in system configuration. Small disturbance stability and power flows are considered when system configuration is fixed.

2.3.1 Steady-state security region

A power system in steady-state for a given system configuration j is described by the power flow equations

$$f_j(\underline{x}) = \underline{y} \tag{1}$$

†This is the standard IEEE terms [25]. Small disturbance stability is called dynamic stability by some authors.

The system has to be operated within the operating limits of its components. This is described by a set of inequality constraints, known as the <u>security constraints</u>

$$g_j(\underline{x}) \leq 0 \tag{2}$$

Moreover it is required that the operating point be small-disturbance stable, i.e., the eigenvalues of $\frac{\partial f_j}{\partial x}$ at the operating point lie in the open left half of the complex plane.

The <u>steady-state security region</u> $\Omega_{ss}(j)$ for the system configuration j is defined to be the set of $\underline{y}$ for which there exists a solution† to the power flow equation (1) and all the solutions satisfy the security constraints (2) and small disturbance stability requirements, i.e.,

$$\Omega_{ss}(j) \stackrel{\Delta}{=} \left\{ \underline{y} \mid \exists\ \underline{x} \text{ s.t. } f_j(\underline{x}) = \underline{y},\ g_j(\underline{x}) \leq 0 \text{ and the eigenvalues of } \frac{\partial f_j}{\partial \underline{x}}(\underline{x}) \text{ lie in the open left half of the complex plane} \right\} \tag{3}$$

<u>Remarks</u>

1) Because of the fluctuating load demand the so-called steady-state analysis of power system is only an approximation. We shall investigate the validity of this "quasi" steady-state approximation.

   Suppose indeed the system variables are changing dynamically††:

$$\underline{\dot{x}}^*(t) = f(\underline{x}^*(t)) - \underline{y}(t)$$

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† For ease of presentation, all the mappings under consideration are assumed to be one to one.

†† This may be considered as the differential equations describing the so-called long term dynamics.

and our "quasi" steady-state approximation is the power flow equations:

$$\underline{0} = f(\underline{x}(t)) - \underline{y}(t)$$

Consider the error $\Delta\underline{x} = \underline{x}^* - \underline{x}$, we have

$$\Delta\dot{\underline{x}} = f(\underline{x}^*(t)) - f(\underline{x}(t)) - \dot{\underline{x}}$$

$$= \left.\frac{\partial f}{\partial x}\right|_0 \Delta x - \left(\left.\frac{\partial f}{\partial x}\right|_0\right)^{-1} \dot{\underline{y}}(t)$$

Therefore for quasi steady-state approximation to be valid it is required that:

(i) all the eigenvalues of $\left.\frac{\partial f}{\partial x}\right|_0$ lie in the open left half of the complex plane

(ii) the rate of change of power injections $\dot{\underline{y}}$ is much smaller than the eigenvalues of $\left.\frac{\partial f}{\partial x}\right|_0$.

The first condition above is precisely the requirement that the system is small disturbance stable, which is the classical approximation of studying long term dynamics. The second condition requires that the change in power injections be relatively slow.

2) We [10] have formulated the steady-state security assessment problem as a mathematical problem of the existence of solutions to $f_j(\underline{x}) = \underline{y}$ within a region $\{\underline{x} | g_j(\underline{x}) \leq 0\}$. By the application of Leray-Schauder fixed point theorem and the analytic degree theory we have obtained a region which is a hyperbox in the space of injections

$$H(j) = \{\underline{y} | y_k^m(j) \leq y_k \leq y_k^M(j)\} \tag{4}$$

where the limits, $y_k^m(j)$ and $y_k^M(j)$, are explicitly defined in terms of the network parameters and the constraints on $\underline{x}$.

3) Previous attempts to obtain steady-state security regions using linearized power flow equations or rectangular-coordinate formulation have been reported in the literature with various degree of success [20-23].

2.3.2 Dynamic security region

Loss of transient stability is a severe breach of security. We will define dynamic security region after an event disturbance in terms of transient stability region. Two types of event disturbances will be considered, one, those cause component outages or additions, two, short circuit faults. We make the following assumption.

Assumption A3: Except for transitions from a short circuit fault situation, all transitions take place when the system has reached a steady-state.

2.3.2.1 Component outage or addition

Consider the situation where a disturbance in the form of a component outage or addition occurs at t and the system configuration changes from $J_0 = i$ to $J_1 = j$. Prior to the disturbance the system is assumed to be in steady-state and is governed by the power flow equations

$$f_i(\underline{x}_0) = \underline{y} \tag{5}$$

After the change, the system dynamics are governed by a set of differential equations:

$$\dot{\underline{x}}_1 = f_j(\underline{x}_1, \underline{y}) \tag{6}$$

We are concerned with the stability of the post-disturbance system (6). Assume that the equilibrium point of (6) is stable and let V be the region of asymptotic stability of (6) for the equilibrium point under consideration. This is the region in the post-disturbance system state space such that if the initial state $\underline{x}_1$ is in V then the trajectory of the post-disturbance system will be stable. The region V is dependent on the equilibrium point, hence the configuration j and the injection $\underline{y}$. We will denote this dependence explicitly by writing $V(j,\underline{y})$. The initial state $\underline{x}_1$ is related to the pre-disturbance state $\underline{x}_0$. Indeed $\underline{x}_0$ is part of $\underline{x}_1$. From the power flow map f_i in (5) we can obtain a region W in the space of injections $\underline{y}$ such that

$$W(i,j,\underline{y}^*) \triangleq \{\underline{y} | \exists\, \underline{x}_0 \text{ s.t. } f_i(\underline{x}_0) = \underline{y} \text{ and } \underline{x}_1(0) \in V(j,\underline{y}^*)\} \tag{7}$$

We define the dynamic security region for the system from i to j to be

$$\Omega_d(i,j) = \{\underline{y}^* | \underline{y}^* \in W(i,j,\underline{y}^*)\} \tag{8}$$

Note that under assumption A3 the dynamic security is considered only at the instance when a component outage or addition takes place.

2.3.2.2 Short-circuit fault

For transient stability analysis as a result of a short-circuit fault on the system, a power system can be considered as going through the changes in configuration from pre-fault system i, fault-on system k, to post-fault system j. The pre-fault system is considered to be in steady-state and is described by the power flow equations:

$$f_i(\underline{x}_0) = \underline{y} \tag{9}$$

The fault-on system is described by a set of differential equations from 0 to τ, where τ is the switching time,

$$\dot{\underline{x}}_1 = f_k(\underline{x}_1,\underline{y}) \tag{10}$$

The post-fault dynamics are

$$\dot{\underline{x}}_2 = f_j(\underline{x}_2,\underline{y}) \tag{11}$$

Let us point out that the change from i to k is the result of an exogenous disturbance and the change from k to j is the result of an endogenous disturbance. The former is probabilistic and the latter is deterministic.

Power system transient stability refers to the asymptotic stability of the post-fault system. Let $V(j,\underline{y})$ denote the region of asymptotic stability such that if the initial state $\underline{x}_2(0)$ is in $V(j,\underline{y})$ then the post-fault trajectory will be stable. The initial state $\underline{x}_2(0)$ of the post-fault system is the final state $\underline{x}_1(\tau)$ of the fault-on system. The fault-on system dynamics (10) can be considered as a mapping D which steers a point $\underline{x}_1(0)$ to $\underline{x}_1(\tau)$. The image U of the region V under the map D^{-1} (the inverse of D) is a region in the state-space of the fault-on system such that if $\underline{x}_1(0) \in U$ then the post-fault trajector will be stable, i.e., $U = D^{-1}(V)$ or

$$U(k,j,\underline{y},\tau) \triangleq \{\underline{x}_1(0) \mid D\underline{x}_1(0) \in V(j,\underline{y})\} \tag{12}$$

Note that the map D depends on the fault-on system configuration k and the switching time τ, hence U depends on k,j, $\underline{y}$, and τ.

The initial state $\underline{x}_1(0)$ is related to the pre-fault system state $\underline{x}_0$. Indeed $\underline{x}_0$ is part of $\underline{x}_1(0)$. From the power flow map f_i in (9) we can obtain a region W in the space of injections $\underline{y}$ such that

$$W(i,k,j,\underline{y}^*,\tau) \triangleq \{\underline{y} \mid \exists\, \underline{x}_0 \text{ s.t. } f_i(\underline{x}_0) = \underline{y} \text{ and } \underline{x}_1(0) \in U(k,j,\underline{y}^*,\tau)\} \tag{13}$$

We now make the following assumption.

Assumption A4: The system configuration during short-circuit is not considered in the first level system model.

Comment. The short-circuit duration is in the order of a few cycles (1 cycle = $\frac{1}{60}$ sec), whereas the time between component outages or additions may be minutes, hours or even longer. Assumption A4 removes deterministic transition due to endogenous disturbance, thus simplifies the model considerably. In Part II of this paper Assumption A4 will be relaxed. It is shown that the more general results obtained there reduce to our current results as $\tau \to 0$.

Knowing that j is the result of removing a short circuit on i, the configuration k can be deduced from the configurations i and j. Therefore we may write $W(i,j,\underline{y}^*,\tau)$ instead of (13). Under assumption A4, we define the dynamic security region for the system transition from i to j as a result of a short-circuit fault to be

$$\Omega_d(i,j,\tau) \stackrel{\Delta}{=} \{\underline{y}^* \mid \underline{y}^* \in W(i,j,\underline{y}^*,\tau)\} \tag{14}$$

Remarks

1) Under assumption A4 we do not consider the configuration k during the short-circuit in the first level of our model, however the effect of which on the second level of our model, namely, the dynamic security region, is considered.
2) In this case we view the short circuit fault the same as a component outage (i.e., the outage of the shorted component) except replacing $\Omega_d(i,j)$ in Eq. (8) by $\Omega_d(i,j,\tau)$ in Eq. (14). Alternatively we can view $\Omega_d(i,j) = \Omega_d(i,j,\tau)\big|_{\tau=0}$. For ease of presentation we will write the dynamic security region as $\Omega_d(i,j,\tau)$ with the understanding that if the transition from i to j involves no short circuit fault then $\tau=0$.

3) The switching time τ may depend on the transition, i.e., function of i and j.

4) A subset of the region of asymptotic stability may be obtained by Liapunov method [11-13] or energy function method [14]. The map D^{-1} is difficult to evaluate (solving differential equations). We suggest to avoid the direct evaluation of D^{-1} by a two-step approximation scheme. The first step is to approximate f_k by a simpler function, e.g., piecewise linear, and obtain a region inside U [15]. The second step is to make simplicial approximation of V and obtain a further reduced region inside U.

5) Note that under assumptions A3,A4, dynamic security is considered only at the instance when a fault occurs.

6) The regions $\Omega_{ss}(j)$, $\Omega_d(i,j,\tau)$ are time-invariant. They can be constructed off-line and stored for on-line security assessment.

2.4 Security Assessment

We say that a system state $\underline{\sigma}(t) = (j,\underline{y})$ is secure at time t if

$$\begin{cases} \underline{y} \in \Omega_{ss}(j) & \text{if there is no transition at } t \\ \underline{y} \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) & \text{if there is a transition at } t \text{ from } i \text{ to } j \end{cases}$$

We shall use the notation $\underline{\sigma}(t) \in \Sigma$ to denote that $\underline{\sigma}(t)$ is secure at time t.

Suppose that presently $t = 0$ our system is secure, $\underline{\sigma}(0) \in \Sigma$. We define the time to insecurity T to be the first instance at which the system state leaves Σ, i.e., $\underline{\sigma}(t) \in \Sigma$ for $t \in [0,T)$ and $\underline{\sigma}(T) \notin \Sigma$. For convenience we will write $\underline{\sigma}[0,T) \subset \Sigma$ to represent $\underline{\sigma}(t) \in \Sigma$ for $t \in [0,T)$. The time to insecurity obviously is a natural measure of system security.

The disturbances are random in nature. Thus $\underset{\sim}{J}_{N(t)}$ and $\underset{\sim}{y}(t)$ are all random variables. The system state $\underset{\sim}{\sigma}(t) = (\underset{\sim}{J}_{N(t)}, \underset{\sim}{y}(t))$ is also a random variable. Consequently the time to insecurity $\underset{\sim}{T}$ is a random variable. Given that presently the system is secure, i.e., $\underset{\sim}{\sigma}(0) \in \sum$, let us define the probability that the system will remain secure up to t seconds ahead as F(t), i.e.,

$$F(t) \overset{\Delta}{=} \Pr\{ \underset{\sim}{T} > t \mid \underset{\sim}{\sigma}(0) \in \textstyle\sum \} \tag{16}$$

$$= \Pr\{\underset{\sim}{\sigma}[0,t] \subset \textstyle\sum \mid \underset{\sim}{\sigma}(0) \in \textstyle\sum \} \tag{17}$$

The function F(t) is a measure of system security in a probabilistic framework. We shall refer to F(t) as the probability distribution of the time to insecurity.†

In security assessment we assume that presently the system is secure $\underset{\sim}{\sigma}(0) \in \sum$. The information we have for the present state $\underset{\sim}{\sigma}(0)$ is given in terms of

(i) probability distribution of present power injection $f_{\underset{\sim}{y}}(\underline{y},t)\big|_{t=0}$.

(ii) probability distribution of configurations $\Pr\{\underset{\sim}{J}_{N(0)} = j\}$.

<u>Remarks</u>

1) From the on-line state estimator [6], we can obtain the distribution $f_{\underset{\sim}{y}}(\underline{y},t)\big|_{t=0}$. From the on-line network configurator, we can obtain $\Pr\{\underset{\sim}{J}_{N(0)} = j)$.

2) In what follows all of our probabilities are conditioning on the given initial-state probability distribution $f_{\underset{\sim}{y}}(\underline{y},t)\big|_{t=0}$, $\Pr\{\underset{\sim}{J}_{N(0)} = j\}$, and the fact that $\underset{\sim}{\sigma}(0) \in \sum$.

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†Conventionally $1-F(t) = \Pr\{T \leq t \mid \underset{\sim}{\sigma}(0) \in \sum\}$ is the probability distribution of $\underset{\sim}{T}$.

3. TIME TO INSECURITY

3.1 Model Simplifications

To facilitate the presentation of the basic ideas in the development of our results, we make for the time being the following simplifying assumptions which will be relaxed later.

Assumption SA1. The automatic relay reclosure is not considered.

Assumption SA2. A short-circuit fault and a component outage are not distinguished.

Under the assumptions A1-4 and SA1-2 the first level system model $\underset{\sim}{J}_{N(t)}$ is characterized by its transition rates. We define† the transition rate $\lambda_{ij}(t)$ as

$$\lambda_{ij}(t) \stackrel{\Delta}{=} \lim_{\Delta t \to 0} \frac{\Pr\left\{\underset{\sim}{J}_{N(t+\Delta t)} = j \,|\, \underset{\sim}{J}_{N(t)} = i\right\}}{\Delta t} \tag{19}$$

$$\lambda_{jj}(t) \stackrel{\Delta}{=} \sum_{\ell \neq j} \lambda_{j\ell}(t) \tag{20}$$

Remarks

1) The probability distribution of event disturbances is used to define the rates in (19). Note that faults are part of the event disturbances.
2) If the time to failure and the time to repair distributions of each component are exponential the transition rates (19) will be constants, independent of time.

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†We assume that all the limits in our definitions exist.

Consider the transition from i to j. At the second level system model, the following quantitites are of interest in security assessment. We define the <u>security transition probabilities</u> $\mu_{ij}(t)$ as

$$\mu_{ij}(t) \overset{\Delta}{=} \Pr\{\underset{\sim}{y}(t)\in\Omega_d(i,j,\tau) \cap \Omega_{ss}(j)\,|\,\underset{\sim}{y}(t)\in\Omega_{ss}(i)\} \quad i\neq j \tag{21}$$

$$\mu_{jj}(t) \overset{\Delta}{=} \lim_{\Delta t\to 0} \frac{\Pr\{\underset{\sim}{y}(t+\Delta t)\notin\Omega_{ss}(j)\,|\,\underset{\sim}{y}(t)\in\Omega_{ss}(j)\}}{\Delta t} \tag{22}$$

<u>Remarks</u>

1) The probability distribution of load disturbance enters into the definition of the security transition probability (21), (22).

2) We will discuss how to compute the security transition probabilities $\mu_{ij}(t)$ and $\mu_{jj}(t)$ from the probability distribution $f_{\underset{\sim}{y}}(\underline{y},t)$ in Section 4.

We now make an additional assumption A5, which is sort of like requiring the state $\underset{\sim}{\sigma}$ to have a Markov property with respect to the <u>set</u> $\sum$.

<u>Assumption A5</u>.

$$\Pr\left\{\underset{\sim}{J}_{N(t+\Delta t)}=j,\underset{\sim}{\sigma}[0,t+\Delta t]\subset\textstyle\sum\,|\,\underset{\sim}{J}_{N(t)}=i,\underset{\sim}{\sigma}[0,t]\subset\textstyle\sum\right\}$$
$$=\Pr\left\{\underset{\sim}{J}_{N(t+\Delta t)}=j,\underset{\sim}{\sigma}(t+\Delta t)\in\textstyle\sum\,|\,\underset{\sim}{J}_{N(t)}=i,\underset{\sim}{\sigma}(t)\in\textstyle\sum\right\}+o(\Delta t)$$

<u>Remark</u>. It is easy to see that if $\underset{\sim}{y}(t)$ is a constant $\underline{y}$, i.e., a deterministic quantity, then assumption A5 holds. On the other hand, Fact 1 below shows that, roughly speaking, if $\underset{\sim}{y}(t)$ takes any value of the set $\Omega_{ss}(i)$ equally likely, then assumption A5 also holds. The proof of Fact 1 is left in the Appendix.

Fact 1. Suppose that

(i) $\{\underset{\sim}{y}(t),\ t \geq 0\}$ is a Markov process.

(ii) For $i \neq j$,

$$\Pr\{\underset{\sim}{y}(t+\Delta t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) \mid \underset{\sim}{y}(t) \in \Omega_{ss}(i)\}$$

$$= \Pr\{\underset{\sim}{y}(t+\Delta t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) \mid \underset{\sim}{y}(t) \in \Omega_{ss}(i), \underset{\sim}{y}(t) = \underline{y}\} + \bar{0}(t)$$

for all $\underline{y} \in \Omega_{ss}(i)$,

where $\bar{0}(\Delta t) \to 0$ as $\Delta t \to 0$,

and

$$\Pr\{\underset{\sim}{y}(t+\Delta t) \in \Omega_{ss}(j) \mid \underset{\sim}{y}(t) \in \Omega_{ss}(j)\}$$

$$= \Pr\{\underset{\sim}{y}(t+\Delta t) \in \Omega_{ss}(j) \mid \underset{\sim}{y}(t) \in \Omega_{ss}(j), \underset{\sim}{y}(t) = \underline{y}\} + o(\Delta t)$$

for all $\underline{y} \in \Omega_{ss}(j)$

Then assumption A5 holds.

3.2 Probability Distribution of the Time to Insecurity

3.2.1 Reclosure and simultaneous disturbance neglected

The probability distribution of the time to insecurity F(t) by definition can be expressed as

$$F(t) = \Pr\{\underset{\sim}{\sigma}[0,t] \subset \Sigma \mid \underset{\sim}{\sigma}(0) \in \Sigma\} \tag{24}$$

$$= \sum_j \Pr\{\underset{\sim}{J}_{N(t)} = j,\ \underset{\sim}{\sigma}[0,t] \subset \Sigma \mid \underset{\sim}{\sigma}(0) \in \Sigma\} \tag{25}$$

Let us define

$$h_j(t) \triangleq \Pr\{\underset{\sim}{J}_{N(t)} = j,\ \underset{\sim}{\sigma}[0,t] \subset \Sigma \mid \underset{\sim}{\sigma}(0) \in \Sigma\} \tag{26}$$

We are going to show in Theorem 1 that the vector $\underline{h}(t) = (h_1(t), h_2(t)\ldots)^T$ satisfies a vector differential equation.

Theorem 1. Under assumptions A1-5 and SA1-2, the time to insecurity distribution

$$F(t) = \sum_j h_j(t) \tag{27}$$

can be obtained from the solution of the following vector differential equation

$$\frac{d}{dt}\,\underline{h}(t) = \Lambda(t)\underline{h}(t) \tag{28}$$

where the ji-th element of the matrix $\Lambda(t)$ is

$$\Lambda_{ji}(t) = \lambda_{ij}(t)\mu_{ij}(t) \qquad i \neq j \tag{29}$$

$$\Lambda_{jj}(t) = -\lambda_{jj}(t) - \mu_{jj}(t) \tag{30}$$

The initial conditions of $\underline{h}(0)$ are given by

$$h_j(0) = \frac{\Pr\{\underset{\sim}{J}_{N(0)} = j\}\Pr\{\underset{\sim}{y}(0) \in \Omega_{ss}(j)\,|\,\underset{\sim}{y}(0)\}}{\sum_j \Pr\{\underset{\sim}{J}_{N(0)} = j\}\Pr\{\underset{\sim}{y}(0) \in \Omega_{ss}(j)\,|\,\underset{\sim}{y}(0)\}} \tag{31}$$

Remark. Equation (28) is analogous to the Kolmogorov differential equation for finite-state continuous time Markov chains. We shall therefore call the elements of the matrix $\Lambda(t)$ in (29)(30) the security transition rates.

Proof of Theorem 1

Consider $h_j(t+\Delta t)$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j,\ \underset{\sim}{\sigma}[0,t+\Delta t] \subset \textstyle\sum \,|\, \underset{\sim}{\sigma}(0) \in \textstyle\sum\}$$

$$= \sum_{i \neq j} \Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j, \underset{\sim}{\sigma}[0,t+\Delta t] \subset \textstyle\sum \,|\, \underset{\sim}{J}_{N(t)} = i, \underset{\sim}{\sigma}[0,t] \subset \textstyle\sum\} h_i(t)$$

$$+ \Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j, \underset{\sim}{\sigma}[0,t+\Delta t] \subset \textstyle\sum \,|\, \underset{\sim}{J}_{N(t)} = j, \underset{\sim}{\sigma}[0,t] \subset \textstyle\sum\} h_j(t) \tag{32}$$

Because of assumptions A1 and A5

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j, \underset{\sim}{\sigma}[0,t+\Delta t] \subset \textstyle\sum \mid \underset{\sim}{J}_{N(t)} = i, \underset{\sim}{\sigma}[0,t] \subset \textstyle\sum\}$$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j, \underset{\sim}{\sigma}[0,t+\Delta t] \subset \textstyle\sum \mid \underset{\sim}{J}_{N(t)} = i, \underset{\sim}{\sigma}[0,t] \subset \textstyle\sum, \text{ one transition in } [t,t+\Delta t]\} + o(\Delta t)$$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j, \underset{\sim}{\sigma}(t+\Delta t) \in \textstyle\sum \mid \underset{\sim}{J}_{N(t)} = i, \underset{\sim}{\sigma}(t) \in \textstyle\sum, \text{ one transition in } [t,t+\Delta t]\} + o(\Delta t) \qquad (33)$$

By the definition of $\sum$, this is equal to

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j, \underset{\sim}{y}(t+\Delta t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) \mid \underset{\sim}{J}_{N(t)} = i, \underset{\sim}{y}(t) \in \Omega_{ss}(i)\} + o(\Delta t) \qquad (34)$$

Since $\{\underset{\sim}{J}_{N(t)}\}$ and $\{\underset{\sim}{y}(t)\}$ are independent, the first term of (34) can be expressed as a product

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j \mid \underset{\sim}{J}_{N(t)} = i\}$$

$$\Pr\{\underset{\sim}{y}(t+\Delta t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) \mid \underset{\sim}{y}(t) \in \Omega_{ss}(i)\} + o(\Delta t) \qquad (35)$$

The first term of (35) is $\lambda_{ij}(t)\Delta t + o(\Delta t)$ by definition (19). Thus the first term in (32) becomes

$$\sum_{i \neq j} \lambda_{ij}(t)\Delta t \Pr\{\underset{\sim}{y}(t+\Delta t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) \mid \underset{\sim}{y}(t) \in \Omega_{ss}(i)\}\, h_i(t) + o(\Delta t) \qquad (36)$$

Now consider the second term in (32). By conditioning on the number of transitions in $[t,t+\Delta t]$ we get

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j, \underset{\sim}{\sigma}[0,t+\Delta t] \subset \textstyle\sum \mid \underset{\sim}{J}_{N(t)} = j, \underset{\sim}{\sigma}[0,t] \subset \textstyle\sum\} \qquad (37)$$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j, \underset{\sim}{\sigma}[0,t+\Delta t] \subset \textstyle\sum \mid \underset{\sim}{J}_{N(t)} = j, \underset{\sim}{\sigma}[0,t] \subset \textstyle\sum, \text{ no transition in } [t,t+\Delta t]\} + o(\Delta t) \qquad (38)$$

Note that we have used the facts that (i) if there is exactly one transition in $[t,t+\Delta t]$ the probability of the term in (37) vanishes (ii) the probability of having two or more transitions in $[t,t+\Delta t]$, by assumption A1, is $o(\Delta t)$. By assumption A5 and the definition of secure state (38) becomes

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j, \underset{\sim}{y}(t+\Delta t) \in \Omega_{ss}(j) \mid \underset{\sim}{J}_{N(t)} = j, \underset{\sim}{y}(t) \in \Omega_{ss}(j)\} + o(\Delta t) \tag{39}$$

Since $\{\underset{\sim}{J}_{N(t)}\}$ and $\{\underset{\sim}{y}(t)\}$ are independent, Eq. (39) becomes

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j \mid \underset{\sim}{J}_{N(t)} = j\}\Pr\{\underset{\sim}{y}(t+\Delta t) \in \Omega_{ss}(j) \mid \underset{\sim}{y}(t) \in \Omega_{ss}(j)\} + o(\Delta t) \tag{40}$$

Note that by using (20) and (22), we get

$$\begin{aligned}
&\Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j \mid \underset{\sim}{J}_{N(t)} = j\} \\
&= 1 - \sum_{\ell \neq j} \Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = \ell \mid \underset{\sim}{J}_{N(t)} = j\} \\
&= 1 - \sum_{\ell \neq j} \lambda_{j\ell}(t)\Delta t + o(\Delta t) \\
&= 1 - \lambda_{jj}(t)\Delta t + o(\Delta t)
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
&\Pr\{\underset{\sim}{y}(t+\Delta t) \in \Omega_{ss}(j) \mid \underset{\sim}{y}(t) \in \Omega_{ss}(j)\} \\
&= 1 - \mu_{jj}(t)\Delta t + o(\Delta t)
\end{aligned} \tag{42}$$

Therefore the second term in (32) becomes

$$\begin{aligned}
&\{(1-\lambda_{jj}(t)\Delta t)(1-\mu_{jj}(t)\Delta t + o(\Delta t)) + o(\Delta t)\}\ h_j(t) \\
&= h_j(t) - \{\lambda_{jj}(t) + \mu_{jj}(t)\}\Delta t\ h_j(t) + o(\Delta t)
\end{aligned} \tag{43}$$

Substituting (36) and (43) into (32) and rearranging terms, we get

$$\frac{d}{dt} h_j(t) = \lim_{\Delta t \to 0} \frac{h_j(t+\Delta t) - h_j(t)}{\Delta t}$$

$$= \sum_{i \neq j} \lambda_{ij}(t)\mu^*_{ij}(t)h_i(t) - \{\lambda_{jj}(t) + \mu_{jj}(t)\}\, h_j(t) \qquad (44)$$

where

$$\mu^*_{ij}(t) \triangleq \lim_{\Delta t \to 0} \Pr\{\underset{\sim}{y}(t+\Delta t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) | \underset{\sim}{y}(t) \in \Omega_{ss}(i)\} \qquad (45)$$

We claim that $\mu^*_{ij}(t)$ in (45) is equal to $\mu_{ij}(t)$ in (21) except on a set with measure zero. First we are going to show that under assumption A2,

$$\mu^*_{ij}(t) = \Pr\{\underset{\sim}{y}(t^+) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) | \underset{\sim}{y}(t) \in \Omega_{ss}(i)\} \qquad (46)$$

By sequential continuity of probability measure, Eq. (45) becomes

$$\mu^*_{ij}(t) = \Pr\{ \lim_{\Delta t \to 0} \underset{\sim}{y}(t+\Delta t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) | \underset{\sim}{y}(t) \in \Omega_{ss}(i)\} \qquad (47)$$

We claim that for a fixed t $\{\omega | \lim_{\Delta t \to 0} \underset{\sim}{y}(t+\Delta t) \neq \underset{\sim}{y}(t^+)\}$ where ω denotes sample point is a set with measure zero. Let $\Delta t_n \geq 0$, $\Delta t_n \to 0$, and define $E_n \triangleq \{\omega | \underset{\sim}{y}(t+\Delta t_n) \neq \underset{\sim}{y}(t^+)\}$. Note that any point belongs to $\limsup E_n \triangleq \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n$ iff it belongs to infinitely many terms in the sequence $\{E_n, n \geq 1\}$. By assumption A2 $\limsup E_n$ is a set with measure zero. The fact that $\{\omega | \lim_{\Delta t_n \to 0} \underset{\sim}{y}(t+\Delta t) \neq \underset{\sim}{y}(t^+)\} \subset \limsup E_n$ implies that $\{\omega | \lim_{\Delta t_n \to 0} \underset{\sim}{y}(t+\Delta t) \neq \underset{\sim}{y}(t^+)\}$ is a set with measure zero. By the separability assumption this implies $\{\omega | \lim_{\Delta t \to 0} \underset{\sim}{y}(t+\Delta t) \neq \underset{\sim}{y}(t^+)\}$ is a set with measure zero. So $\lim_{\Delta t \to 0} \underset{\sim}{y}(t+\Delta t) = \underset{\sim}{y}(t^+)$ except on a set with measure zero and the proof of the claim (46) is complete.

The coefficients of the d.e. (44) with $\mu^*_{ij}(t)$ given by (46) and the d.e. (28) with $\mu_{ij}(t)$ given by (21) are equal except on a set of t with measure zero. Moreover the off-diagonal coefficients of these d.e.

are bounded. Hence [24] their solutions are identical for $t \geq 0$.

Now we shall derive the initial conditions (31). By definition (26)

$$h_j(0) = \Pr\{J_{\tilde{N}(0)} = j, \underset{\sim}{\sigma}(0) \in \textstyle\sum \mid \underset{\sim}{\sigma}(0) \in \sum\}$$

$$= \frac{\Pr\{J_{\tilde{N}(0)} = j, \underset{\sim}{\sigma}(0) \in \sum\}}{\Pr\{\underset{\sim}{\sigma}(0) \in \sum\}}$$

$$= \frac{\Pr\{J_{\tilde{N}(0)} = j\} \Pr\{\underset{\sim}{y}(0) \in \Omega_{ss}(j) \mid \underset{\sim}{y}(0)\}}{\sum_j \Pr\{J_{\tilde{N}(0)} = j\} \Pr\{\underset{\sim}{y}(0) \in \Omega_{ss}(j) \mid \underset{\sim}{y}(0)\}} \tag{54}$$

which is (31). ¤

3.2.2 Simultaneous disturbance considered

If we relax assumption SA2 we have to consider two possible transitions from a configuration i to a configuration j having less components. One case is that the transition is the result of component outages and the other is that the transition is the result of the removal of a short-circuit fault. Let $\underset{\sim}{d}(i,j)$ be the indicator of which type of disturbance occurs in the transition from i to j:

$$d(i,j) = \begin{cases} 0 & \text{if component outage has occurred} \\ s & \text{if short-circuit fault has occurred} \end{cases} \tag{55}$$

Let us define two types of transition probabilities and security transition probabilities:

$$\lambda^0_{ij}(t) \triangleq \lim_{\Delta t \to 0} \frac{\Pr\{J_{\tilde{N}(t+\Delta t)} = j, \underset{\sim}{d}(i,j) = 0 \mid J_{\tilde{N}(t)} = i\}}{\Delta t}, \quad i \neq j \tag{56}$$

$$\lambda^s_{ij}(t) \triangleq \lim_{\Delta t \to 0} \frac{\Pr\{J_{\tilde{N}(t+\Delta t)} = j, \underset{\sim}{d}(i,j) = s \mid J_{\tilde{N}(t)} = i\}}{\Delta t}, \quad i \neq j \tag{57}$$

$$\lambda'_{jj}(t) \triangleq \sum_{\ell \neq j} \{\lambda^{o}_{j\ell}(t) + \lambda^{s}_{j\ell}(t)\} \tag{58}$$

$$\mu^{o}_{ij}(t) \triangleq \Pr\{\underset{\sim}{y}(t) \in \Omega_d(i,j) \cap \Omega_{ss}(j) \,|\, \underset{\sim}{y}(t) \in \Omega_{ss}(i), \underset{\sim}{d}(i,j) = 0\},\ i \neq j \tag{59}$$

$$\mu^{s}_{ij}(t) \triangleq \Pr\{\underset{\sim}{y}(t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) \,|\, \underset{\sim}{y}(t) \in \Omega_{ss}(i), \underset{\sim}{d}(i,j) = s\},\ i \neq j \tag{60}$$

$$\mu_{jj}(t) \triangleq \lim_{\Delta t \to 0} \frac{\Pr\{\underset{\sim}{y}(t+\Delta t) \notin \Omega_{ss}(j) \,|\, \underset{\sim}{y}(t) \in \Omega_{ss}(j)\}}{\Delta t} \tag{61}$$

Corollary 1. Under assumptions A1-5 and SA1, the results of Theorem 1 holds with the elements of the matrix $\Lambda(t)$ replaced by

$$\Lambda_{ji}(t) = \lambda^{o}_{ij}(t)\mu^{o}_{ij}(t) + \lambda^{s}_{ij}(t)\mu^{s}_{ij}(t) \tag{62}$$

$$\Lambda_{jj}(t) = -\lambda'_{jj}(t) - \mu_{jj}(t) \tag{63}$$

The same proof for Theorem 1 holds, mutatis mutandis for Corollary 1.

3.2.3 Automatic reclosure considered

Most of the short-circuit faults on transmission lines are temporary. They may be cleared by opening and immediately re-closing the circuit breakers protecting the lines. If the automatic reclosure is successful in clearing the fault, the system returns to its original configuration. On the other hand if the short-circuit fault persists the circuit breakers will remain open and the system will be in a different configuration. Consider the situation where without reclosure the result of the short-circuit fault would be the transition of system configuration from i to j. Let $\pi_{ij}(t)$ be the probability that in this case the reclosure will be successful in clearing the fault.

$$\pi_{ij}(t) \stackrel{\Delta}{=} \Pr\left\{\begin{matrix}\text{reclosure clears the}\\ \text{fault successfully}\end{matrix}\middle| \begin{matrix}\text{transition at time t from i to j}\\ \text{(if reclosure unsuccessful)}\end{matrix}\right\} \quad (64)$$

For a successful reclosure the system configuration is back to where it was before the fault but the system variables undergo dynamics. Let us define $\Omega_d(i,j,\tau,r)$ to be the dynamic security region for breakers reclose r seconds after their opening, where i is the pre-fault configuration and the breakers openned τ seconds after a fault, resulting in configuration j. Let us define the following security transition probabilities due to reclosure

$$\mu_{ij}^{sc}(t) \stackrel{\Delta}{=} \Pr\{\underset{\sim}{y}(t) \in \Omega_d(i,j,\tau,r) \cap \Omega_{ss}(j) \,|\, \underset{\sim}{y}(t) \in \Omega_{ss}(i), \underset{\sim}{d}(i,j) = s\} \quad i \neq j$$

<u>Corollary 2</u>. Under assumptions A1-5, the results of Theorem 1 holds with the elements of the matrix $\Lambda(t)$ replaced by

$$\Lambda_{ji}(t) = (1 - \pi_{ij}(t))\lambda_{ij}^{s}(t)\mu_{ij}^{s}(t) + \lambda_{ij}^{o}(t)\mu_{ij}^{o}(t) \quad (65)$$

$$\Lambda_{jj}(t) = -\lambda_{jj}'(t) - \mu_{jj}(t) + \sum_{\ell \neq j} \pi_{j\ell}(t)\lambda_{j\ell}^{s}(t)\mu_{j\ell}^{sc}(t) \quad (66)$$

The proof of Corollary 2 is left in the Appendix.

3.3 <u>Upper and Lower Bounds on F(t)</u>

In this section we present an upper bound and a lower bound for the time to insecurity distribution F(t). For ease of presentation the proof of the result, Theorem 2, is given under the simplifying assumptions SA1-2. Obviously using the modifications (65)(66) assumptions SA1 and SA2 can be relaxed.

<u>Theorem 2</u>.

$$e^{-\int_0^t \max_j r_j(\tau)d\tau} \leq F(t) \leq e^{-\int_0^t \min_j r_j(\tau)d\tau} \quad (67)$$

where

$$r_i(t) \overset{\Delta}{=} \mu_{ii}(t) + \sum_{j\neq i} \lambda_{ij}(t)\ \{1-\mu_{ij}(t)\} \tag{68}$$

Remark. The function $r_i(t)$ has a physical interpretation. Substituting (21) (22) into (68), we have

$$r_i(t) = \lim_{\Delta t \to 0} \frac{\Pr\{\underset{\sim}{y}(t+\Delta t) \notin \Omega_{ss}(i) | \underset{\sim}{y}(t) \in \Omega_{ss}(i)\}}{\Delta t}$$

$$+ \sum_{j\neq i} \lambda_{ij}(t)\ \Pr\{\underset{\sim}{y}(t) \notin \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) | \underset{\sim}{y}(t) \cap \Omega_{ss}(i)\} \tag{69}$$

The first term is the "rate" from secure region into insecure region due to steady-state security violation and the second term is the "rate" from secure region into insecure region due to dynamic security violation. Thus $r_i(t)$ can be interpreted as the "rate from secure region into insecure region" at configuration i.

Proof of Theorem 2

We first claim that $h_j(t) \geq 0$ for $t \geq 0$. From (31), $h_j(0) \geq 0$ for all j. Let t' be the first instance a component $h_i(t') = 0$. Then

$$\frac{d}{dt} h_i(t') = \sum_{j\neq i} \lambda_{ji}(t)\mu_{ji}(t)h_j(t) > 0$$

The claim is thus proved.

Combining (27) and (28), we get

$$\frac{d}{dt} F(t) = \sum_j \sum_i \Lambda_{ji}(t)h_i(t)$$

$$= \sum_i \sum_j \Lambda_{ji}(t)h_i(t) \tag{70}$$

Since $h_i(t) \geq 0$, we have

$$\min_i \sum_j \Lambda_{ji}(t)\, F(t) \le \frac{dF(t)}{dt} \le \max_i \sum_j \Lambda_{ji}(t) F(t) \tag{71}$$

Consider $\max_i \sum_j \Lambda_{ji}(t)$, substituting (29), (30), (20) into it, we get

$$\begin{aligned}
&\max_i \sum_j \Lambda_{ji}(t) \\
&= \max_i \{-\lambda_{ii}(t) - \mu_{ii}(t) + \sum_{j\neq i} \lambda_{ij}(t)\mu_{ij}(t)\} \\
&= -\min_i \{\mu_{ii}(t) + \sum_{j\neq i} \lambda_{ij}(t)(1 - \mu_{ij}(t))\} \\
&= -\min_i r_i(t)
\end{aligned} \tag{72}$$

The last equation is obtained by definition (68).

Substituting (72) into (71) and upon integration, we get the right half of (67). Similarly for the left half. ¤

Remark. From (31), we see $F(0) = 1$. Clearly $r_j(t) \ge 0$. Equation (67) implies $0 \le F(t) \le 1$. Furthermore Eqs. (71) (72) implies $\frac{dF(t)}{dt} \le 0$. This checks with the fact that F(t) is equal to one minus a probability distribution.

4. COMPUTATION OF SECURITY TRANSITION PROBABILITIES

In this section we take up the problem of computing the security transition probabilities $\mu_{ij}(t)$ and $\mu_{jj}(t)$ defined in (21) and (22), respectively, from the probability distribution of $\underset{\sim}{y}(t)$.

Let the mean function of $\underset{\sim}{y}(t)$ be denoted by $\underline{m}(t)$, i.e.,

$$\underline{m}(t) \stackrel{\Delta}{=} E\, \underset{\sim}{y}(t) \tag{73}$$

By assumption A2, $\underline{m}(t)$ is piecewise differentiable. Let

$$\underset{\sim}{z}(t) \stackrel{\Delta}{=} \dot{\underset{\sim}{y}}(t) - \dot{\underline{m}}(t) \tag{74}$$

which is sample piecewise continuous by assumption A2. Let $f_{\underset{\sim}{y},\underset{\sim}{z}}(\underline{y},\underline{z},t)$ denote the joint probability density function of $\underset{\sim}{y}(t)$ and $\underset{\sim}{z}(t)$.

Theorem 3. Under assumptions A2, A5 and assuming the surface of $\Omega_{ss}(j)$ is piecewise smooth and orientable [16, p. 351] we have

$$\mu_{ij}(t) = \frac{\int_{\underline{y} \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(i)} f_{\underset{\sim}{y}}(\underline{y},t)d\underline{y}}{\int_{\underline{y} \in \Omega_{ss}(i)} f_{\underset{\sim}{y}}(\underline{y},t)d\underline{y}} \tag{75}$$

$$\mu_{jj}(t) = \frac{\int_{\underline{z}} \int_{\Omega^+_{ss}(j)} f_{\underset{\sim}{y},\underset{\sim}{z}}(\underline{y},\underline{z},t)(\underline{\dot{m}}(t)+\underline{z})\cdot\vec{n}d\underline{s}d\underline{z}}{\int_{\underline{y} \in \Omega_{ss}(j)} f_{\underset{\sim}{y}}(\underline{y},t)d\underline{y}} \tag{76}$$

where $\vec{n}$ denote the unit outward normal vector of the boundary (surface) of $\Omega_{ss}(j)$ and $\int_{\Omega^+_{ss}(j)} f_{\underset{\sim}{y},\underset{\sim}{z}}(\underline{y},\underline{z},t)(\underline{\dot{m}}(t)+\underline{z})\cdot\vec{n}\,\underline{ds}$ denote the surface integral of $f_{\underset{\sim}{y},\underset{\sim}{z}}(\underline{y},\underline{z},t)(\underline{\dot{m}}(t)+\underline{z})$ over that part of the boundary of $\Omega_{ss}(j)$ on which $(\underline{\dot{m}}(t)+\underline{z})\cdot\vec{n} \geq 0$.

Proof of Theorem 3.

Equation (75) is direct from the definite (21). Consider

$$\mu_{jj}(t) = \lim_{\Delta t \to 0} \frac{\Pr\{\underset{\sim}{y}(t+\Delta t) \notin \Omega_{ss}(j) | \underset{\sim}{y}(t) \in \Omega_{ss}(j)\}}{\Delta t} \tag{77}$$

From the definition (74) clearly

$$\underset{\sim}{y}(t) = \underset{\sim}{a} + \int_0^t \underset{\sim}{z}(\tau)\, d\tau + \underline{m}(t) \tag{78}$$

where $\underset{\sim}{a} = \underset{\sim}{y}(0) - \underline{m}(0)$ has zero mean. Since $\{\underset{\sim}{z}(t)\}$ is sample piecewise continuous, with probability 1 for all sample functions $\underline{y}(t)$ we have

$$\underline{y}(t+\Delta t) = \underline{y}(t) + [\underline{z}(t)+\dot{\underline{m}}(t)]\Delta t + o(\Delta t) \tag{79}$$

Let $\bar{\Omega}_{ss}(j)$ denote the complement of $\Omega_{ss}(j)$. We can express

$$\Pr\{\underset{\sim}{y}(t+\Delta t) \notin \Omega_{ss}(j),\ \underset{\sim}{y}(t) \in \Omega_{ss}(j)\}$$

$$= \Pr\{\underset{\sim}{y}(t) + [\underset{\sim}{z}(t)+\dot{\underline{m}}(t)]\Delta t + o(\Delta t) \in \bar{\Omega}_{ss}(j),\ \underset{\sim}{y}(t) \in \Omega_{ss}(j)\}$$

$$= \int_{\underline{z}} \Pr\{\underset{\sim}{y}(t) \in \bar{\Omega}_{ss}(j) - [\underset{\sim}{z}(t)+\dot{\underline{m}}(t)]\Delta t - o(\Delta t), \underset{\sim}{y}(t) \in \Omega_{ss}(j) \mid \underset{\sim}{z}(t) = \underline{z}\}$$

$$\Pr\{\underset{\sim}{z}(t) = \underline{z}\}d\underline{z} \tag{80}$$

But

$$\Pr\{\underset{\sim}{y}(t) \in \bar{\Omega}_{ss}(j) - [\underset{\sim}{z}(t)+\dot{\underline{m}}(t)]\Delta t - o(\Delta t), \underset{\sim}{y}(t) \in \Omega_{ss}(j) \mid \underset{\sim}{z}(t) = \underline{z}\}$$

$$= \int_{\underline{y} \in \{\bar{\Omega}_{ss}(j) - [\underline{z}+\dot{\underline{m}}(t)]\Delta t\} \cap \Omega_{ss}(j)} f_{\underset{\sim}{y}|\underset{\sim}{z}}(\underline{y},t)d\underline{y} + o(\Delta t) \tag{81}$$

The integral in (81) can be considered as an analogy of calculating the total mass inside R := $\{\bar{\Omega}_{ss}(j) - [\underline{z}+\dot{\underline{m}}(t)]\Delta t\} \cap \Omega_{ss}(j)$. We decompose R into small boxes with base ds on $\Omega^{+}_{ss}(j)$ and height $\vec{n}\cdot[\underline{z}+\dot{\underline{m}}(t)]\Delta t$. The mass inside these small boxes equals $f_{\underset{\sim}{y}|\underset{\sim}{z}}(\underline{y},t)\vec{n}\cdot[\underline{z}+\dot{\underline{m}}(t)]\Delta t ds$ (Fig. 1). When we sum over $\Omega^{+}_{ss}(j)$ gives

$$= \int_{\Omega^{+}_{ss}(j)} f_{\underset{\sim}{y}|\underset{\sim}{z}}(\underline{y},t)n \cdot [\underline{z}+\dot{\underline{m}}(t)]\Delta t \ \ ds + o(\Delta t) \tag{82}$$

Substituting (82) (80) into (77), we have

$$\mu_{jj}(t) = \frac{\lim\limits_{\Delta t \to 0} \frac{1}{\Delta t} \Pr\{\underset{\sim}{y}(t+\Delta t) \notin \Omega_{ss}(j), \underset{\sim}{y}(t) \in \Omega_{ss}(j)\}}{\Pr\{\underset{\sim}{y}(t) \in \Omega_{ss}(j)\}}$$

$$= \frac{\lim\limits_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{\underline{z}}\int_{\Omega^{+}_{ss}(j)} f_{\underset{\sim}{y}|\underset{\sim}{z}}(\underline{y},t)\vec{n}\cdot[\underline{z}+\dot{\underline{m}}(t)]\Delta t \ ds \cdot \Pr\{z(t) = \underline{z}\}d\underline{z} + o(\Delta t)\right]}{\Pr\{\underset{\sim}{y}(t) \in \Omega_{ss}(j)\}}$$

(83)

which reduces to (76).

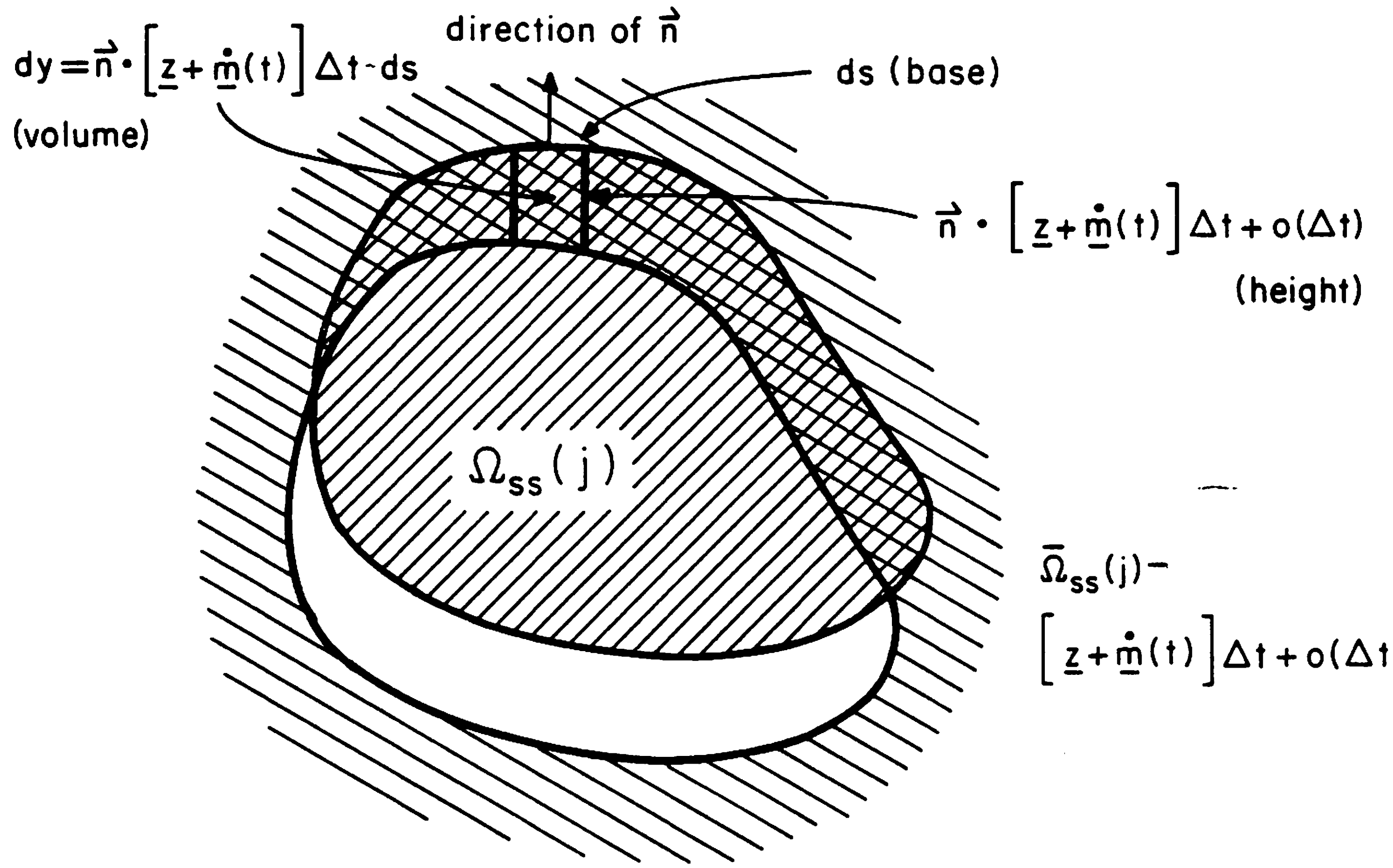

Fig. 1. The change from volume integral to surface integral.

5. CONCLUSIONS

We have presented a comprehensive framework for probabilistic dynamic security assessment in which the conventional models for steady-state and dynamic analyses of power systems can be incorporated. We have derived a vector differential equation whose solution gives us the probability distribution of the time to insecurity. The coefficients of the differential equation are computed in terms of the steady-state and dynamic security regions. We have indicated how these regions may be computed. In practice we expect only to be able to compute subsets of these regions. We have begun research in this area. When subsets of the security regions are used the time to insecurity derived in this paper should be interpreted instead as the time to leave security.

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APPENDIX

1. Proof of Fact 1.

Case 1) For $i \neq j$

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j,\ \underset{\sim}{\sigma}(t+\Delta t) \in \Sigma \mid \underset{\sim}{J}_{N(t)} = i, \underset{\sim}{\sigma}(t) \in \Sigma\} \tag{100}$$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j, \underset{\sim}{y}(t+\Delta t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) \mid \underset{\sim}{J}_{N(t)} = i, \underset{\sim}{y}(t) \in \Omega_{ss}(i)\}$$

$$= [\lambda_{ij}(t)\Delta t + o(\Delta t)]\ \Pr\{\underset{\sim}{y}(t+\Delta t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) \mid \underset{\sim}{y}(t) \in \Omega_{ss}(i)\} \tag{101}$$

The last equality follows from the fact that $\{\underset{\sim}{J}_{N(t)}\}$ and $\{\underset{\sim}{y}(t)\}$ are independent and the definition of $\lambda_{ij}(t)$. Substituting condition (ii) of Fact 1 into (101) we obtain

$$= [\lambda_{ij}(t)\Delta t + o(\Delta t)] \Pr\{\underset{\sim}{y}(t+\Delta t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) \mid \underset{\sim}{y}(t) \in \Omega_{ss}(i), \underset{\sim}{y}(t) = \underline{y}\} + o(\Delta t)$$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j, \underset{\sim}{y}(t+\Delta t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) \mid \underset{\sim}{J}_{N(t)} = i, \underset{\sim}{y}(t) \in \Omega_{ss}(i), \underset{\sim}{y}(t) = \underline{y}\} + o(\Delta t) \tag{102}$$

The last equality follows again from the fact that $\{\underset{\sim}{J}_{N(t)}\}$ and $\{\underset{\sim}{y}(t)\}$ are independent and the definition of $\lambda_{ij}(t)$. On the other hand,

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j, \underset{\sim}{\sigma}[t,t+\Delta t] \subset \Sigma \mid \underset{\sim}{J}_{N(t)} = i, \underset{\sim}{\sigma}[0,t] \subset \Sigma\} \tag{103}$$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j,\ \underset{\sim}{\sigma}(t+\Delta t) \in \Sigma \mid \underset{\sim}{J}_{N(t)} = i,\ \underset{\sim}{\sigma}(t) \in \Sigma,\ \underset{\sim}{\sigma}[0,t) \subset \Sigma\}$$

$$= \int_{\underline{y} \in \Omega_{ss}(i)} \Pr\{\underset{\sim}{J}_{N(t+\Delta t)} = j,\ \underset{\sim}{y}(t+\Delta t) \in \Omega_d(i,j,\tau) \cap \Omega_{ss}(j) \mid \underset{\sim}{J}_{N(t)} = i,\ \underset{\sim}{y}(t) \in \Omega_{ss}(i),\ \underset{\sim}{y}(t) = \underline{y},\ \underset{\sim}{\sigma}[0,t] \subset \Sigma\}$$
$$\cdot \Pr\{\underset{\sim}{y}(t) = \underline{y} \mid \underset{\sim}{J}_{N(t)} = i,\ \underset{\sim}{y}(t) \in \Omega_{ss}(i),\ \sigma[0,t) \subset \Sigma\} d\underline{y} \tag{104}$$

Since $\{y(t)\}$ is Markov, (104) becomes

$$= \int_{\underline{y}\in\Omega_{ss}(i)} \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j,\ \underset{\sim}{y}(t+\Delta t)\in\Omega_d(i,j,\tau)\cap\Omega_{ss}(j) \mid \underset{\sim}{J}_{N(t)}=i,\ \underset{\sim}{y}(t)\in\Omega_{ss}(i),\ \underset{\sim}{y}(t)=\underline{y}\}$$

$$\cdot \Pr\{\underset{\sim}{y}(t)=\underline{y} \mid \underset{\sim}{J}_{N(t)}=i,\ \underset{\sim}{y}(t)\in\Omega_{ss}(i),\ \underset{\sim}{\sigma}[0,t)\subset\textstyle\sum\}d\underline{y} \tag{105}$$

Since (100) is equal to (102), substituting (100) into (105), we obtain

$$= \int_{\underline{y}\in\Omega_{ss}(i)} \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j,\ \underset{\sim}{\sigma}(t+\Delta t)\in\textstyle\sum \mid \underset{\sim}{J}_{N(t)}=i,\ \underset{\sim}{\sigma}(t)\in\textstyle\sum\}$$

$$\cdot \Pr\{\underset{\sim}{y}(t)=\underline{y} \mid \underset{\sim}{J}_{N(t)}=i,\ \underset{\sim}{y}(t)\in\Omega_{ss}(i),\ \underset{\sim}{\sigma}[0,t)\in\textstyle\sum\}\,d\underline{y} + o(\Delta t) \tag{106}$$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j,\ \underset{\sim}{\sigma}(t+\Delta t)\in\textstyle\sum \mid \underset{\sim}{J}_{N(t)}=i,\ \underset{\sim}{\sigma}(t)\in\textstyle\sum\} + o(\Delta t) \tag{107}$$

Thus we have shown that assumption A5 holds under conditions (i) and (ii) for $i\neq j$.

Case 2) For $i=j$

Under the condition (ii) we have,

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j,\ \underset{\sim}{\sigma}(t+\Delta t)\in\textstyle\sum \mid \underset{\sim}{J}_{N(t)}=j,\ \underset{\sim}{\sigma}(t)\in\textstyle\sum\} \tag{108}$$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j \mid \underset{\sim}{J}_{N(t)}=j\}\Pr\{\underset{\sim}{y}(t+\Delta t)\in\Omega_{ss}(j) \mid \underset{\sim}{y}(t)\in\Omega_{ss}(j)\}$$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j \mid \underset{\sim}{J}_{N(t)}=j\}\Pr\{\underset{\sim}{y}(t+\Delta t)\in\Omega_{ss}(j) \mid \underset{\sim}{y}(t)\in\Omega_{ss}(j),\ \underset{\sim}{y}(t)=\underline{y}\}+o(\Delta t)$$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j,\ \underset{\sim}{y}(t+\Delta t)\in\Omega_{ss}(j) \mid \underset{\sim}{J}_{N(t)}=j,\ \underset{\sim}{y}(t)\in\Omega_{ss}(j),\underset{\sim}{y}(t)=\underline{y}\}+o(\Delta t) \tag{109}$$

where we have again used the fact that $\{\underset{\sim}{J}_{N(t)}\}$ and $\{\underset{\sim}{y}(t)\}$ are independent. On the other hand,

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j,\ \underset{\sim}{\sigma}[t,t+\Delta t]\subset\Sigma\,|\,\underset{\sim}{J}_{N(t)}=i,\ \underset{\sim}{\sigma}[0,t]\subset\Sigma\} \tag{110}$$

$$= \int_{\underline{y}\in\Omega_{ss}(j)} \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j,\ \underset{\sim}{y}(t+\Delta t)\in\Omega_{ss}(j)\,|\,\underset{\sim}{J}_{N(t)}=j,\ \underset{\sim}{y}(t)\in\Omega_{ss}(j), \underset{\sim}{y}(t)=\underline{y}\}$$

$$\cdot\Pr\{\underset{\sim}{y}(t)=\underline{y}\,|\,\underset{\sim}{J}_{N(t)}=j, \underset{\sim}{y}(t)\in\Omega_{ss}(j),\ \underset{\sim}{\sigma}[0,t)\subset\Sigma\}\,d\underline{y} \tag{111}$$

Substituting (108) into (111) we reach the conclusion that assumption A5 holds under conditions (i) and (ii) also for $i=j$. ¤

2. <u>Proof of Corollary 2:</u>

Proceed the same as in the proof of Theorem 1. Let us consider

$$h_j(t+\Delta t) = \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j,\ \underset{\sim}{\sigma}[0,t+\Delta t]\subset\Sigma\,|\,\underset{\sim}{\sigma}(0)\in\Sigma\}$$

$$= \sum_{i\neq j} \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j,\ \underset{\sim}{\sigma}[0,t+\Delta t]\subset\Sigma\,|\,\underset{\sim}{J}_{N(t)}=i,$$

$$\underset{\sim}{\sigma}[0,t]\subset\Sigma\}h_i(t)$$

$$+ \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j, \underset{\sim}{\sigma}[0,t+\Delta t]\subset\Sigma\,|\,\underset{\sim}{J}_{N(t)}=j,$$

$$\underset{\sim}{\sigma}[0,t]\subset\Sigma\}h_j(t) \tag{112}$$

By assumption A1, $j\neq i$

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j, \underset{\sim}{\sigma}[0,t+\Delta t]\subset\Sigma\,|\,\underset{\sim}{J}_{N(t)}=i, \underset{\sim}{\sigma}[0,t]\subset\Sigma\}$$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j, \underset{\sim}{\sigma}[0,t+\Delta t]\subset\Sigma\,|\,\underset{\sim}{J}_{N(t)}=i, \underset{\sim}{\sigma}[0,t]\subset\Sigma\ ,$$

$$\text{one transition in } [t,\ t+\Delta t] + o(\Delta t) \tag{113}$$

Equation (113) can be further written as

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j, \underset{\sim}{\sigma}[0,t+\Delta t]\subset\Sigma,\ d(i,j)=0\,|\,\underset{\sim}{J}_{N(t)}=i, \underset{\sim}{\sigma}[0,t]\subset\Sigma,$$

$$\text{one transition in } [t,t+\Delta t]\}$$

$$+ \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j, \underset{\sim}{\sigma}[0,t+\Delta t]\subset\Sigma,\ d(i,j)=s\,|\,\underset{\sim}{J}_{N(t)}=i,$$

$$\underset{\sim}{\sigma}[0,t]\subset\Sigma\ ,\ \text{one transition in } [t,t+\Delta t]\} \tag{114}$$

Since $i \neq j$ and there is one transition in $[t,t+\Delta t]$ implies the second term in (114) corresponding to unsuccessful reclosure. Thus by the same reasoning in the proof of Theorem 1, (114) becomes

$$[\lambda_{ij}^{o}(t)\mu_{ij}^{o}(t) + (1 - \pi_{ij}(t))\ \lambda_{ij}^{s}(t)\mu_{ij}^{s}(t)]\Delta t + o(\Delta t) \tag{115}$$

Now consider the second term in (112). By conditioning on the number of transitions in $[t,t+\Delta t]$ we get

$$\Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j, \underset{\sim}{\sigma}[0,t+\Delta t] \subset \textstyle\sum \mid \underset{\sim}{J}_{N(t)}=j, \underset{\sim}{\sigma}[0,t] \subset \textstyle\sum\}$$

$$= \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j, \underset{\sim}{\sigma}[0,t+t] \subset \textstyle\sum \mid \underset{\sim}{J}_{N(t)}=j, \underset{\sim}{\sigma}[0,t] \subset \textstyle\sum,$$

$$\text{no transition in } [t,t+\Delta t]\}$$

$$+ \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j, \underset{\sim}{\sigma}[0,t+\Delta t] \subset \textstyle\sum \mid \underset{\sim}{J}_{N(t)}=j, \underset{\sim}{\sigma}[0,t] \subset \textstyle\sum,$$

$$\text{one transition in } [t,t+\Delta t]\} + o(\Delta t) \tag{116}$$

The first term of (116) can be reduced to

$$[1 - \lambda_{jj}'(t) - \mu_{jj}(t)]\Delta t \tag{117}$$

The second term of (116) corresponding to successful reclosure, can be written as

$$\sum_{\ell \neq j} \Pr\{\underset{\sim}{J}_{N(t+\Delta t)}=j, \text{ transition from } j \text{ to } \ell \text{ (if reclosure}$$

$$\text{unsuccessful) in } [t,t+\Delta t], \underset{\sim}{\sigma}[0,t+\Delta t] \subset \textstyle\sum \mid \underset{\sim}{J}_{N(t)}=j,$$

$$\sigma[0,t] \subset \textstyle\sum \text{ , one transition in } [t,t+\Delta t]\}$$

$$= \sum_{\ell \neq j} \lambda_{j\ell}^{s}(t)\Delta t\mu_{j\ell}^{sc}(t)\pi_{j\ell}(t) + o(\Delta t) \tag{118}$$

Substitute (118), (117), (115) into (112), upon taking $\Delta t \to 0$ we have

$$\frac{d}{dt} h_j(t) = \sum_{i \neq j} [(1 - \pi_{ij}(t))\lambda^s_{ij}(t)\mu^s_{ij}(t) + \lambda^o_{ij}(t)\mu^o_{ij}(t)]h_i(t)$$

$$+ [-\lambda'_{jj}(t) - \mu_{jj}(t) + \sum_{\ell \neq j} \lambda^s_{j\ell}(t)\mu^{sc}_{j\ell}(t)\pi_{j\ell}(t)]h_j(t) \tag{119}$$