NUMERICAL PROPERTIES OF ALGORITHMS FOR THE TIMING ANALYSIS OF MOS VLSI CIRCUITS

by

Giovami De Micheli and Alberto Sangiovanni-Vincentelli

Memorandum No. UCB/ERL M81/25

11 May 1981

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

NUMERICAL PROPERTIES OF ALGORITHMS FOR THE TIMING ANALYSIS

OF MOS VLSI CIRCUITS

Giovami De Micheli and Alberto Sangiovanni-Vincentelli

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, CA 94720

Abstract

Displacement techniques used for the timing analysis of VLSI circuits are presented under a new perspective. Their numerical properties such as stability, accuracy, consistency and convergence are investigated.

Research sponsored by the Defense Advanced Research Projects Agency (DOD). Monitored by Naval Electronic Systems Command under Contract No. N00039-K-0251, and by Rotary International.

1. INTRODUCTION

When analog voltage levels are critical to circuit performance, or where tightly coupled feedback loops are present, standard circuit simulators such as SPICE [1] or ASTAP [2] can be used to analyze the circuit. However, when the size of the circuit becomes large, the cost and the memory requirements of conventional circuit simulators become prohibitive and new techniques have to be used. The timing simulator MOTIS [3] was developed to simulate large scale integrated circuits. The Program MOTIS was a revolutionary simulator in two main respects:

a) It limited severely the types of networks it dealt with (MOS devices with quasi-unidirectional circuit models and a grounded capacitor on every node)

b) It discarded both sparse Gauss elimination and conventional Newton-Raphson iteration as solution methods.

In MOTIS Backward Euler formula was used to discretize the time derivative operator and a nonlinear Gauss -Jacobi like relaxation technique [4] was adopted to decouple the node equations at the nonlinear equation level. The algorithms of the timing simulators MOTIS-C [5] and SPLICE [6] perfected this technique. In particular, SPLICE used a nonlinear "Gauss-Seidel like" technique with a selective trace algorithm to exploit the "latency" [7][8] of large digital circuits. All of these algorithms did not carry the iteration of the relaxation methods to convergence: only one sweep was taken. Because of this, the numerical properties such as stability of the integraton formulae used to discretize the derivative operator no longer hold. These methods have indeed to be considered as *new integration methods*. Hence a complete analysis of their numerical properties has to be carried out to characterize them.

In this paper we formalize these relaxation or displacement methods and propose a generalization of a method presented for the first time in [9]. Then we propose a model to study formally the stability, accuracy, consistency and convergence properties of the methods. Based on this model, we evaluate the various methods and show that the method proposed in [9] has better stability and accuracy properties.

2. TIMING ANALYSIS ALGORITHMS

MOS VLSI circuits are often modeled as electrical circuits containing linear and nonlinear resistors (controlled sources are considered to be resistive elements according to [10]) and capacitors. Furthermore, a capacitor is connected from each node of the circuit to ground to model the time delay of a signal propagating through the circuit. Since each node has a capacitor to ground, the node equations have the following form:

$$C(v)\dot{v} + f(v,u(t)) = 0 \qquad (2.1)$$

$$v(0) = v_0.$$

$v \in R^n$; $u: R \to R^m$; $C(\cdot): R^n \to R^{n \times n}$; $f(\cdot,\cdot): R^n \times R^m \to R^n$.

$f(v,u(t)) = [f_1(v,u(t)), f_2(v,u(t)), \ldots, f_n(v,u(t))]^T$

where v is the vector of node voltages, u is the vector of independent source waveforms, $C(v)$ is the nonlinear nodal capacitance matrix and $f_i(v,u(t))$ is the sum of the currents feeding the capacitors connected to node i. In this paper we shall assume that no floating capacitor (i.e., capacitors connected between two non ground nodes) is present in the circuit. Therefore $C(v)$ is a diagonal matrix. We assume also that $C(v)^{-1}$ exists for all v of interest. Therefore we can simplify (2.1) as follows:

$$\dot{v} + F(v,u(t)) = 0 \qquad (2.2)$$

$$v(0) = v_0.$$

where:

$$F(v,u(t)) = C(v)^{-1} f(v,u(t)). \tag{2.3}$$

Algorithms used in the timing analysis of MOS and VLSI circuits discretize the derivative operator by Backward Euler [3][6] or trapezoidal formula [5]. In this paper we shall focus on the Backward Euler formula

$$\dot{v}_{k+1} = (v_{k+1} - v_k)/h. \tag{2.4}$$

where $h = t_{k+1} - t_k$ and v_{k+1} and v_k are the computed voltages of the node voltors at time t_{k+1} and t_k respectively. The solution of the resulting nonlinear system of equations:

$$v_{k+1} - v_k + hF(v_{k+1}, u(k_{k+1})) = 0. \tag{2.5}$$

is then approximated by one sweep of a displacement technique.

Program MOTIS [3] uses a Gauss-Jacobi like technique which yields the following set of decoupled equations:

$$v_{k+1}^1 - v_k^1 + hF_1(v_{k+1}^1, v_k^2, \ldots, v_k^n, u_1(t_{k+1})) = 0$$
$$v_{k+1}^2 - v_k^2 + hF_2(v_k^1, v_{k+1}^2, \ldots, v_k^n, u_2(t_{k+1})) = 0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$v_{k+1}^n - v_k^n + hF_n(v_k^1, v_k^2, \ldots, v_{k+1}^n, u_n(t_{k+1})) = 0$$

The solution of the decoupled nonlinear equations (2.6) is then approximted by taking a single step of a "regula falsi" iteration [11].

The MOTIS-C and SPLICE programs use a Gauss-Seidel like technique. In SPLICE this technique yields:

$$v_{k+1}^i - v_k^i + hF_i(\tilde{v}_{k+1,i}, u(t_{k+1})) = 0; \; i = 1, 2, \ldots, n. \tag{2.7}$$

where:

$$\tilde{v}_{k+1,i} = [v_{k+1}^1, \ldots, v_{k+1}^i, v_k^{i+1}, \ldots, v_k^n]^T. \tag{2.8}$$

The solution of (2.6) is then approximated by using one step of the Newton-Raphson algorithm.

Another displacement technique for the solution of (2.1) has been proposed for a simple circuit in [12]. This algorithm is a symmetric displacement method reminiscent of the alternating-direction implicit method [11] and of a method proposed in [9]. The basic idea here is to "symmetrize" the Gauss-Seidel scheme with a method that takes two half steps of size h/2 each: one half step is taken in the usual "forward" (i.e., cower triangular) direction, the second half step in the backward (i.e., upper triangular) direction. Letting:

$$\bar{v}_{l,i} = \begin{cases} [v_{l_1}, \ldots, v_l^i, v_{l-1/2}^{i+1}, \ldots, v_{l-1/2}^n]^T & \text{if } 2l \text{ is odd} \\ [v_{l-1/2}^i, \ldots, v_{l-1/2}^{i-1}, v_l^i, \ldots, v^n]^T & \text{if } 2l \text{ is even.} \end{cases} \tag{2.9}$$

the forward step yields:

$$v_{k+1/2}^i - v_k^i + \frac{h}{4} F_i(\bar{v}_{k+1/2,i}, u(t_{k+1/2})) + \frac{h}{4} F_i(\bar{v}_{k+1/2,i-1}, u(t_{k+1/2})) = 0 \quad i = l, 2, \ldots, n. \tag{2.10}$$

and the backward step:

$$v_{k+1}^i - v_k^i + \frac{h}{4} F_i(\bar{v}_{k+1,i}, u(t_{k+1})) + \frac{h}{4} F_i(\bar{v}_{k+1,i+1}, u(t_{k+1})) = 0 \quad i = n, n-1, \ldots, 1. \tag{2.11}$$

The solution of the decoupled equations is then approximated by taking one step of the Newton-Raphson algorithm. Note that all these methods do not solve (2.5) since only one sweep of the displacement iteration is taken. Therefore the stability and accuracy properties of the integration method used to discretize the derivative operator no longer hold.

In the sequel we will refer to the "time advancement" algorithms which use the Gauss-Jacobi, the Gauss-Seidel and modified symmetric Gauss-Seidel displacement step as Gauss-Jacobi, Gauss-Seidel and modified symmetric Gauss-

Seidel integration algorithms respectively. In the following section the numerical properties of these "time advancement" methods will be investigated.

3. NUMERICAL PROPERTIES OF TIMING ANALYSIS ALGORITHMS

The numerical properties of an integration method, such as stability, are studied on test problems [13][14], which are simple enough to allow a theoretical analysis but still so general that one can have insight about how the method behaves in general. For the commonly used multistep methods, the test problem consists of a linear time-invariant zero-input asymptotically stable differential equation. Unfortunately this simple test problem cannot be used to evaluate the displacement techniques introduced in section 2. In fact, each variable of the system of differential equations is treated differently according to the ordering in which equations are processed. Hence a more complex test problem is needed. The test problem we choose is a linear time-invariant zero-input asymptotically stable system of differential equations, i.e.:

$$\dot{x} = Ax \tag{3.1}$$

$$x(0) = x_0.$$

where $A \in R^{n \times n}$ and the set of eigenvalues (spectrum) of A, $\sigma(A)$, is in the open left half complex plane, i.e., $\sigma(A) \in C_0^-$. Let $A = L+D+U$, where L is strictly lower triangular, D is diagonal and U is strictly upper triangular. The displacement methods presented in section 2 applied to the test system (3.1) yield the following recursive relations:

a) Gauss-Jacobi integration algorithm:

$$[I-hD]x_{k+1} = [I+h(L+U)]x_k. \tag{3.2}$$

$$x_{k+1} = M_{GJ}(h)x_k. \tag{3.3}$$

where I is the identity matrix and

$$M_{GJ}(h) = [I-hD]^{-1}[I+h(L+U)]. \tag{3.4}$$

b) Gauss-Seidel integration algorithm:

$$[I-h(D+L)]x_{k+1} = [I+hU]x_k. \tag{3.5}$$

$$x_{k+1} = M_{GS}(h)x_k. \tag{3.6}$$

where

$$M_{GS}(h) = [I-h(D+L)]^{-1}[I+hU]. \tag{3.7}$$

c) Modified symmetric Gauss-Seidel integration algorithm:

Let:

$$A_L = L+1/2D \tag{3.8}$$

$$A_U = U+1/2D$$

Forward step:

$$[I-\frac{h}{4}(2L+D)]x_{k+1/2} = [I+\frac{h}{4}(D+2U)]x_k \tag{3.9}$$

$$[I-\frac{h}{2}A_L]x_{k+1/2} = [I+\frac{h}{2}A_U]x_k \tag{3.10}$$

$$x_{k+1/2} = [I-\frac{h}{2}A_L]^{-1}[I+\frac{h}{2}A_U]x_k. \tag{3.11}$$

Backward step:

$$[I-\frac{h}{4}(D+2U)]x_{k+1} = [I+\frac{h}{2}(2L+D)]x_{k+1/2} \tag{3.12}$$

$$x_{k+1} = [I-\frac{h}{2}A_U]^{-1}[I+\frac{h}{2}A_L]x_{k+1/2}. \tag{3.13}$$

Combining (3.11) and (3.13) we obtain:

$$x_{k+1} = M_S(h)x_k. \tag{3.14}$$

where

$$M_S(h) = [I-\frac{h}{2}A_U]^{-1}[I+\frac{h}{2}A_L][I-\frac{h}{2}A_L]^{-1}[I+\frac{h}{2}A_U] \tag{3.15}$$

The matrices $M_{GJ}(h)$, $M_{GS}(h)$ and $M_S(h)$ are called the companion matrices of the methods. If we denote with $M(h)$ the generic companion matrix of a method, we have:

$$x_k = [M(h)]^k x_0. \tag{3.16}$$

We define next the numerical properties of the integration algorithms described by (3.16) following the outlines of one-step integration methods applied to ordinary differential equations [13].

Definition 3.1. (Consistency)

An integration algorithm is consistent if its companion matrix can be expanded in power series as a function of the stepsize h as:

$$M(h) = I+hA+0(h^2). \tag{3.17}$$

Definition 3.2 (Stability)

An integration algorithm is stable if $\exists\, \delta>0$, $\exists\, N>0$ such that $\forall x_0 \in R^n$, $\exists\, \bar{k}>0$

$$||x_k||<N \qquad \forall k \geq \bar{k} \qquad \forall h \in [0,\delta), \tag{3.18}$$

where x_k is the sequence generated by the algorithm applied to the test problem according to (3.16)

Definition 3.3. (Convergence)

Let $x(t)$ be the exact solution of the test problem. An integration algorithm is convergent if the sequence of the computed solution converges uniformly to $x(t)$ as the stepsize h tends to zero.

Theorem 3.1.

Gauss-Jacobi, Gauss-Seidel and modified symmetric Gauss Seidel integration algorithms are consistent.

Proof

a) Let us consider Gauss-Jacobi integration algorithm first. To expand the companion matrix given by (3.4) in power series as a function of the stepsize h, we compute

$$\frac{d}{dh}M_{GJ}(h) = [I-hD]^{-1}D[I-hD]^{-1}[I+h(L+U)]+[I-hD]^{-1}(L+U) \tag{3.19}$$

and:

$$\frac{d}{dh}M_{GJ}(0) = D+L+U = A. \tag{3.20}$$

where $\frac{d}{dh}M_{GJ}(0)$ is the derivative of $M_{GJ}(h)$ evaluated at $h = 0$. It follows that

$$M_{GJ}(h) = I+hA+0(h^2). \tag{3.21}$$

b) The consistency of Gauss-Seidel integration algorithm follows, "mutatis mutandis," a similar argument.

c) For the modified symmetric Gauss-Seidel integration algorithm, we have:

$$\begin{aligned}\frac{d}{dh}M_S(h) = {} & [I-\frac{h}{2}A_U]^{-1}\frac{1}{2}A_U[I-\frac{h}{2}A_U]^{-1}[I+\frac{h}{2}A_L][I-\frac{h}{2}A_L]^{-1}[I+\frac{h}{2}A_U] \\ & +[I-\frac{h}{2}A_U]^{-1}\frac{1}{2}A_L[I-\frac{h}{2}A_L]^{-1}[I+\frac{h}{2}A_U] \\ & +[I+\frac{h}{2}A_U]^{-1}[I+\frac{h}{2}A_L][I-\frac{h}{2}A_L]^{-1}\frac{l}{2}A_L[I-\frac{h}{2}A_L]^{-1}[I+\frac{h}{2}A_U] \\ & +[I-\frac{h}{2}A_U]^{-1}[I+\frac{h}{2}A_L][I-\frac{h}{2}A_L]^{-1}\frac{1}{2}A_L.\end{aligned} \tag{3.22}$$

and

$$\frac{d}{dh}M_s(0) = \frac{1}{2}A_L+\frac{1}{2}A_U+\frac{1}{2}A_U+\frac{1}{2}A_L = A. \tag{3.23}$$

Hence:

$$M_s(h) = I+hA+0(h^2). \tag{3.24}$$

The definition of stability requires the boundness of the sequence at x_k for small values of the stepsize h. The following proposition relates the boundness of the sequence x_k with the spectrum of M(h).

Proposition 3.1 [15]

The sequence of vectors $\{x_k\}$ defined by (3.16) is bounded for a given vaue of the stepsize $\bar{h}$ if and only if the spectrum of $M(\bar{h})$ is contained in the unit ball B(0,1), i.e., $\sigma(M(\bar{h}))\subseteq B(0,1)$ and no multiple zero of the minimal polynomial of $M(h)$ has modulus equal to one.

In the sequel we restrict our analysis to the case in which the stepsize is constant. From Proposition 3.1 it is immediate to derive the following theorem:

Theorem 3.2

An integration algorithm is stable if and only if $\exists\delta>0$ such that $\forall h\in[0,\delta)$ the spectrum of $M(h)$ is contained in the unit ball $B(0,1)$ and no multiple zero of the minimal polynomial of $M(h)$ has modulus equal to one.

Theorem 3.3

Gauss-Jacobi, Gauss-Seidel and modified symmetric Gauss-Seidel integration algorithms are stable.

Proof.

From the consistency of the above mentioned algorithms we have

$$M(h) = I+hA+0(h^2). \tag{3.25}$$

By the spectral mapping theorem [15]

$$\sigma(M(h)) = \{\xi_i|_i = 1+h\lambda_i+0(h^2);\ \lambda_i\in\sigma(A);\quad i = 1,2,\ldots,\sigma\}. \tag{3.26}$$

From (3.26) we have:

$$|\xi_i| = |1+h\lambda_i+0(h^2)|,\quad i = 1,2,\ldots,\sigma, \tag{3.27}$$

and

$$|\xi_i|^2 = [1+hRe(\lambda_i)]^2+[hIm(\lambda_i)]^2+0(h^2). \tag{3.28}$$

Since $M(0) = I$, its eigenvalues are all 1, and 1 is a simple zero of the minimal polynomial of the identity matrix. Therefore from Theorem 3.2 it is sufficient to show that:

$$\sigma(M(h))\subset B(0,1)\qquad \forall h\in(0,\delta). \tag{3.29}$$

i.e. from (3.28)

$$|\xi_i|2<1\quad \forall h\in(0,\delta),\quad i = 1,2,\ldots,\sigma \tag{3.30}$$

From (3.30), we have:

$$2Re(\lambda_i)+h(Re^2(\lambda_i)+Im^2(\lambda_i))+O(h)<0\quad i = 1,2,\ldots,\sigma \tag{3.31}$$

$$2Re(\lambda_i)+O(h)<0 \quad i = 1,2,\ldots,\sigma \tag{3.32}$$

Since by assumption $Re(\lambda_i)<0$, i = 1,2,...., σ, $\exists\delta>0$, such that $\forall h\in(0,\delta)$,

$$\sigma(M(h))\subset B(0,1). \tag{3.33}$$

Corollary 3.1

Gauss Jacobi - Gauss Seidel and modified symmetric Gauss Seidel integration algorithms are convergent.

Proof.

Follows from Theorems 3.1, 3,3 and the classical convergence theorem.

For computationl efficiency, it would be highly desirable that the stepsize be limited only by accuracy considerations as in the case of the implicit backward differentiation formulas [13]. In the case of classical multistep methods, the concept of A-stability [14] and stiff-stability [13] have been introduced to test the "unconditional" stability of multistep methods. For the "time-advancement" techniques introduced in this paper, it would make sense to define a similar concept. Unfortunately, general results of "unconditional" stability are not available for the test problem previously defined, but only for a subclass, the subclass characterized by a symmetric A matrix.

Definition 3.4. ($\mathcal{A}$-stability)

An integration method is $\mathcal{A}$-stable if $\exists$ N > 0 such that $\forall x_0\in R^n$, $\exists\bar{k}$

$$||x_k||<N \quad \forall k\geq\bar{k} \quad \forall h\in[0,\infty). \tag{3.34}$$

where $\{x_k\}$ is the sequence generated by the method applied to the test problem (3.1) with A symmetric.

Theorem 3.4

The modified symmetrixc Gauss Seidel method is $\mathcal{A}$ stable.

Proof.

Since A is symmetric and $\sigma(A)\in C_0^-$, A is a negative definite matrix. For $h = 0$, $M_S(0) = I$, the eigenvalues of $M_S(0)$ are all 1 and 1 is a simple zero of the minimal polynomial. Hence we need only to see where the eigenvalues of $M_S(h)$ lie when $h\in(0,\infty)$. Let us apply to $M_S(h)$ a similarity transformation:

$$\bar{M}_S(h) = [I-\frac{h}{2}A_U]M_S(h)[I-\frac{h}{2}A_U]^{-1}. \tag{3.35}$$

and factorize $\bar{M}_s$ as:

$$\bar{M}_S(h) = P(h)Q(h). \tag{3.36}$$

where

$$P(h) = [I+\frac{h}{2}A_L][I-\frac{h}{2}A_L]^{-1} \tag{3.37}$$

$$Q(h) = [I+\frac{h}{2}A_U][I-\frac{h}{2}A_U]^{-1}. \tag{3.38}$$

Now:

$$||P(h)||_2^2 = \max_{x\neq 0}\frac{<[I+\frac{h}{2}A_L][I-\frac{h}{2}A_L]^{-1}x,[I+\frac{h}{2}A_L][I-\frac{h}{2}A_L]x>}{<x,x>} \tag{3.39}$$

Let:

$$y = [I-\frac{h}{2}A_L]^{-1}x \quad . \tag{3.40}$$

Then:

$$||P(h)||_2^2 = \max_{y\neq 0}\frac{<[I+\frac{h}{2}a_L]y,[I+\frac{h}{2}A_L]y>}{<[I-\frac{h}{2}A_L]y,[I-\frac{h}{2}A_L]y>} \tag{3.41}$$

$$= \max_{y\neq 0}\frac{<y,y>+\frac{h}{2}<y,Ay>+\frac{h^2}{4}<y,A_UA_Ly>}{<y,y>-\frac{h}{2}<y,Ay>+\frac{h^2}{4}<y,A_UA_Ly>} \tag{3.42}$$

Since $\forall y$, $<y,A_UA_Ly > 0$, and A is negative definite

$$||P(h)||_2^2<1\forall(h)\in(0,\infty). \tag{3.41}$$

Hence:

$$||P(h)||_2<1 \quad \forall h \in (0,\infty). \tag{3.42}$$

It can be proved in a similar way that

$$||Q(h)||_2<1 \quad \forall h \in (0,\infty). \tag{3.43}$$

Hence:

$$||\bar{M}_S(h)|| \leq ||P(h)|| \ \ ||Q(h)||<1 \quad \forall h \in (0,\infty). \tag{3.44}$$

and:

$$\sigma(\bar{M}_S(h)) = \sigma(M_S(h)) \in B(0,1), \forall h \in (0,\infty).$$

Remark

Note that we cannot prove any $\mathcal{A}$ stability result for the Gauss-Jacobi and the Gauss-Seidel integration methods. In our practical experiments, we have seen that when applied to circuit problems, the modified symmetric Gauss-Seidel method is indeed "more stable" than the other two methods.

Now we are going to discuss the accuracy of the integration methods presented in this paper. Once more, we are going to define accuracy in terms of the test problem (3.1).

Definition 3.5

Let $x(t_k)$ be the exact value of the solution of the test problem at time t_k. Let x_k be the computed solution at time t_k assuming $x_{k-1} = x(t_{k-1})$ i.e., that no error has been made in computing the previous time point-value of x. Letting h $= t_k - t_{k-1}$, the local truncation error is defined to be

$$\varepsilon = ||x(t_k) - x_k|| \tag{3.46}$$

If $\varepsilon = O(h^{r+1})$, r is said to be the order of the integraton method [13].

Theorem 3.5

Gauss-Jacobi and Gauss-Seidel integration methods are first order integration algorithms.

Proof:

From (3.46) we have:

$$\varepsilon = ||x(t_k)-x_k|| \tag{3.47}$$

$$= ||(e^{hA}-M)x_{k-1}||. \tag{3.48}$$

By expanding e^{hA} in power series of h and by Theorem 3.1,

$$\varepsilon = ||\{I+hA+O(h^2)-I-hA-O(h^2)\}x_{k-1}|| \tag{3.49}$$

$$= 0(h^2)$$

Theorem 3.6

The modified symmetric Gauss-Seidel algorithm is a second-order integration algorithm.

Proof.

Since matrices $[I+\frac{h}{2}A_L]$ and $[I+\frac{h}{2}A_U]^{-1}$ commute, then:

$$M_S = [I-\frac{h}{2}A_U]^{-1}[I-\frac{h}{2}A_L]^{-1}[I+\frac{h}{2}A_L][I+\frac{h}{2}A_U] \tag{3.50}$$

$$= [I-\frac{h}{2}A+\frac{h^2}{4}A_LA_U]^{-1}[I+\frac{h}{2}+\frac{h^2}{4}A_LA_U] \tag{3.51}$$

$$= I+hA+\frac{h^2}{2}A+O(h^3). \tag{3.52}$$

Hence:

$$\varepsilon = ||(e^{hA}-M_S)x_{k-1}|| = O(h^3). \tag{3.53}$$

In circuit analysis, another important criterion for evaluating the accuracy of an integration method, is what we define "waveform accuracy." In general, the computed solution of a system of differential equations is the superposition of a principal solution and parasitic solutions [13]. Parasitic solutions are generated by the numerical approximations and, in particular for the algorithm we are dealing with in this paper, by the displacement technique used.

Proposition 3.2

Oscillatory parasitic components are present in the computers solution if the spectrum of the companion matrix M(h) contains complex conjugate eigenvalues.

If the original system to be analyzed does not contain an oscillatory component, the presence of such a component in the computed solution can be misleading in the evaluation of the performances of the system [16]. Therefore we introduce a subclass of the test problem, characterized by $\sigma(A) \in R_0^-$; i.e., the set of test problems whih does not have an oscillatory component in the solution, and we look for bounds an the oscillatory components of the computed solutions.

Theorem 3.7

Let $\sigma(A) \in R_0^-$. The imaginary part of the eigenvalues of the companion matrix of Gauss-Jacobi, Gauss-Seidel, and modified symmetrix Gauss-Seidel integration methods is bounded by a i quadratic function of the stepsize h

$$i.e. \ \max_i |Im(\xi_i)| = 0(h^2) \tag{3.54}$$

Proof.

From Theorem (3.1)

$$M(h) = I + hA + O(h^2) \tag{3.55}$$

Hence

$$(M(h)) = \{\xi_i \mid \xi_i\} = 1 + \lambda_i h + O(h^2); \quad \lambda_i \in \sigma(A) \tag{3.56}$$

and

$$Im(\xi_i) = O(h^2) \quad \forall i \,. \tag{3.57}$$

Remark

The theorem essentially says that by choosing an appropriately small stepsize h, the parasitic oscillatory solutions can be made negligible with respect to the principal solution.

If we restrict the class of the test problems to the subclass characterized by a symmetric A matrix, then we can prove a much stranger result for the modified symmetric Gauss-Seidel integration method.

Theorem 3.8

If A is a real symmetric matrix, the spectrum of the companion matrix of the modified symmetric Gauss-Seidel integration method is real, i.e., no oscillatory parasitic components are present in the computed solution.

Proof

Let us factorize matrix M_s as in (3.51)

$$M_s = PQ \tag{3.58}$$

$$P = [I-\frac{h}{2}A+h^24A_LA_U]^{-1} \tag{3.59}$$

$$Q = [I+\frac{h}{2}A+\frac{h^2}{4}A_La_U] \tag{3.60}$$

Since A_LA_U is a positive semidefinite symmetric matrix, -A is symmetric and positive definite follows that P is symmetric positive definite matrix. Matrix Q is the sum of symmetric matrices, hence symmetric. Since

$$P = \sum_{i=1}^{\infty}\lambda_i R_i. \tag{3.61}$$

where λ_i are the eigenvalues and R_i are the residues of matrix P, then

$$p^{1/2} = \sum_{i=1}^{\sigma}\sqrt{\lambda_i}R_i \tag{3.62}$$

$p^{1/2}$ is a symmetric matrix, since the residues R_i are symmetric matrices.1 Let us consider now the similarity transformation:

$$\mathbf{M}_s = p^{-1/2}M_sp^{1/2} \tag{3.63}$$

$$= p^{1/2}QP^{1/2}. \tag{3.64}$$

Matrix $\mathbf{M}_s$ is symmetric and therefore has real eigenvalues. Then by similarity also M_s has real eigenvalues.

4. CONCLUSIONS

We have investigated the numerical properties of certain displacement techniques used for the timing analysis of VLSI, MOS circuits. from stability and accuracy viewpoint, the modified symmetric Gauss-Seidel integration algorithm outperforming the other two methods: the Gauss-Jacobi method used in MOTIS and Gauss-Seidel method used in MOTIS-C and SPLICE. The algorityhms have been discussed for circuits containing no floating capacitors. When floating capacitors are present, the algorithms have to be modified to deal with the additional coupling between equations introduced by the capacitors. The analysis of the modified algorithms is complex and is carried out in [17], where experimental results are also presented and discussed. We believe that these methods will replace the traditional circuit simulator techniques based on sparsity techniques, Newton-Raphson methods and stiffy stable integration formulae, for the analysis of digital very large scale integrated circuits.

5. REFERENCES

[1] W. Nagel, "SPICE2, A computer program to simulate semiconductor circuits," University of California, Berkeley, ERL Memo No. ERL-M520, May 1975.

[2] "Advanced statistical analysis program (ASTAP)", Program reference manual, Pub. No. SH20-1118-0, IBM Corp. Data Proc. Div., White Plains, N.Y. 10604.

[3] B. R. Chawla H. K. Gummel P. Kozah, "MOTIS-an M.O.S. timing simulator," Trans. IEEE vol. CAS/22, pp. 901-909, Dec. 75.

[4] M. Ortega W. C. Reinboldt, "Iterative Solutions of Non-linear Equations in Several Variables" Academic Press 70.

[5] S> . P. Fan, M. Y. Hsueh, A. R. Newton, D. O. Pederson, "MOTIS-C A new circuit simulator for M. O. E. L. S. I. circuits," Proc. IEEE ISCAS Apr. 77.

[6] A. R. Newton, "The Simulation of Large Scale Integrated Circuits," Memorandum UCB/ERL M78/52, July 78.

[7] A. R. Newton, "Timing, Logic and Mixed-mode simulation for large M.O.S. integrated circuits," NATO ASI notes Sagesta, Urbino, Italy, 1980.

[8] N. B. G. Rabbat, A. L. Sangiovanni-Vincentelli and H. Y. Hsieh, "A multilevel Newton Algorithm with macro modeling and latency for the analysis of large scale nonlinear circuits in the time domain," IEEE Circuits and Systems, vol. CAS 26, No. 29, pp. 733-741, Sept. 1979.

[9] W. Kahan, private communication.

[10] C. A. Desoer and E. S. Kuh, "Basi Circuit Theory," McGraw Hill, 1969.

[11] E. Isaacson, H. B. Keller, "Analysis of Numerical Methods," Wiley 1966.

[12] G. De Micheli, A. R. Newton, A. L. Sangiovanni-Vincentelli, "New algorithms for timing analysis of large circuits," International Symposium on Circuits

and Systems, Houston, April 11980.

[13] C. W. Gear, "Numerical Initial Value Problems for Ordinary Differential Equations," Prentice-Hall, 1971.

[14] G. Dahlquist and A. Byorch, "Numerical Methods," Prentice Hall, 1974.

[15] C. A. Desoer, "Notes for a second course on linear ysystems," D. Van Nostrand, 1970.

[16] A R. Newton, "The Timing Analysis of Floating Capacitors for Timing Simulation," Proc. 13th Asilomar Conference on Circuits Systems and Computers" Asilomar, CA Nov. 1979.

[17] G. De Micheli, A. R. Newton and A. L. Sangiovanni-Vincentelli, "Symmetric Displacement Algorithms for the Timing Analysis of VLSI MOS circuits," submitted for publication.