PASSIVITY CRITERION FOR LTI N-PORTS

by

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ABSTRACT

Existing proofs of the passivity criterion for linear, time-invariant, distributed N-ports are either incorrect or too involved, requiring the use of advanced mathematics such as distribution theory. This paper presents a simple but completely rigorous proof using only basic real and complex analysis. For the sake of completeness we have included simple proofs of the classic Paley-Wiener Theorem and the Poisson formula for the half plane. Finally, we give a passivity criterion applicable to N-ports described by general coordinates, from which passivity criteria for any specific representation (e.g. impedance, admittance, hybrid, transmission, scattering, etc.) can be trivially derived.

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I. Introduction

In 1954 Raisbeck [1] proposed a general definition of passivity which would apply to distributed as well as lumped networks and gave an informal proof that a linear time-invariant (LTI) N-port is passive if and only if its impedance matrix is positive real. In 1958 Youla, Castriota, and Carlin published their classic paper on linear passive network theory [2] which included a formal proof of this passivity criterion, but the proof is fairly involved. Wohlers and Beltrami [3], [4] and Zemanian [5] gave simpler formal proofs using the theory of distributions, and several attempts were made to formalize Raisbeck's original argument; some (incorrect) proofs were even incorporated into textbooks (see [6]; [7], [8], [9]).

The primary purpose of this paper is to present a formal proof of the passivity criterion which is straightforward, intuitive, and makes the minimum appeal to advanced mathematics; in particular, no distribution theory is used. Much of the advanced mathematics used is condensed into a single theorem which characterizes LTI causal bounded operators in the frequency domain. We have called this theorem the Bochner-Paley-Wiener theorem since it is an easy consequence of their results, and give a proof in the appendix.

Our second purpose is to discuss some of the intricacies of the problem. We show that a reasonable assumption implies the less natural one ("solvability") that the set of all admissible v+i is dense in L_2^N. We examine the difference between passive devices which satisfy $\int_{-\infty}^{T} v^*i\,dt \geq 0$ for all T and devices for which $\int_{-\infty}^{\infty} v^*i\,dt \geq 0$ which we call weakly passive (the distinction is due to Wohlers; some authors have used weak passivity as their definition of passivity). We show that an

N-port N is passive if and only if it is weakly passive and has a causal scattering operator (which is not equivalent to its having a causal impedance operator even when N has both representations) and give a weaker criterion for N to be weakly passive. For example, consider a 1-port N with $v(t) = i(t) + \frac{1}{2} i(t+1)$. This N has an impedance $Z(j\omega) = 1 + \frac{1}{2} e^{j\omega}$ which has the analytic extension $Z(s) = 1 + \frac{1}{2} e^{s}$ in the whole complex plane. Parseval's relation shows that if i is admissible and $i \in L_2$, then $v \in L_2$ and $\int_{-\infty}^{\infty} vi\,dt \geq 0$, that is, N is weakly passive. $Z(s)$, though quite analytic, is far from positive real since $Z(\pi j+1) = 1 - \frac{1}{2} e < 0$, so this N is an example showing that proofs of the criterion assuming only weak passivity are incorrect.

Finally we give a passivity criterion for a device described by general coordinates from which specialized criteria in terms of any particular representation (e.g. admittance, hybrid, transmission, etc.) can be trivially derived.

We will use the following somewhat standard notation: $\bar{w}$ is the conjugate and w^* the conjugate transpose of $w \in C^N$, $|w| = (w^*w)^{1/2}$; L_2^N ($L_2^N(\mathbb{R})$) is the set of (measurable) C^N (R^N) valued functions of a real variable $f(t)$ with $\int_{-\infty}^{\infty} f^*(t)f(t)dt = \|f\|^2 < \infty$ (Lebesgue integral; functions which differ on a set of measure zero identified); L_2 is L_2^1. For $f \in L_2^N$, $\hat{f}(j\omega)$ is its Fourier tranform (= $\underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} f(t)e^{-j\omega t}\,dt$); if $f(t)$ is C^N-valued and $T \in \mathbb{R}$, f_T will denote the function which agrees with $f(t)$ for $t \leq T$ and which vanishes for $t \geq T$; L_{2e}^N ("Extended L_2^N") is the set of all f with $f_T \in L_2^N$ for all $T \in \mathbb{R}$; $\mu(E)$ will denote the Lebesgue measure of the (measurable) $E \in \mathbb{R}$, RHP will denote the open right half plane $\{z \in C | \mathrm{Re} z > 0\}$.

We will say a function $F(j\omega)$ defined only up to sets of measure zero has the analytic extension $F(s)$ in the RHP if $F(s)$ is analytic in the RHP and $\lim_{\sigma \to 0+} F(\sigma+j\omega)$ exists and equals $F(j\omega)$ for almost all $\omega \in \mathbb{R}$. We will routinely drop the qualifier "almost" from "almost all", trusting that the reader familiar with measure theory will be able to supply it where necessary.

An $\mathcal{N}$ -admissable signal or signal pair will mean a real valued signal or signal pair in L_{2e}^N which may appear across $\mathcal{N}$.[†]

II. Definition of Passivity and Statement of the Criterion

Following Youla et al. [2] we say $\mathcal{N}$ is passive if for all $\mathcal{N}$ -admissable port current-voltage pairs (i,v)

$$\text{for all } T \in \mathbb{R}, \int_{-\infty}^{T} v^*(t)i(t)dt \geq 0 \tag{2.1}$$

The use of the scattering variables[††] (a,b), where $a(t) = \frac{1}{2} v(t) + \frac{1}{2} i(t)$ and $b(t) = \frac{1}{2} v(t) - \frac{1}{2} i(t)$ is central to our argument, so we reformulate (2.1) as

$$\text{for all } T \in \mathbb{R}, \int_{-\infty}^{T} (a^*a-b^*b)dt \geq 0 \tag{2.2}$$

For reasons we will discuss later, we impose the following additional "solvability" assumption:

$$\text{The set of } \mathcal{N}\text{-admissable } a\text{'s is dense in } L_2^N(R) \tag{2.3}$$

[†]With only minor modification the entire theory may be formulated for complex signals, but we see no advantage.

[††]Here we assume port normalization impedances of 1Ω ; in general $a_k = \frac{1}{2R_k} v_k + \frac{R_k}{2} i_k$, $b_k = \frac{1}{2R_k} v_k - \frac{R_k}{2} i_k$.

Theorem 2.1 (Youla et al.) An N-port N is LTI and passive if and only if

(1) N has a scattering matrix, i.e. the set of admissable (a,b) are precisely

$$\{(a,b) \mid a \in L_2^N(R) \text{ and } \hat{b}(j\omega) = S(j\omega)\hat{a}(j\omega)\}$$

and (2) $S(j\omega)$ has the analytic extension $S(s)$ in the RHP with $I-S^*(s)S(s)$ positive semidefinite there.

We note that $S(j\omega)$ is defined only up to sets of measure zero, so that statements involving $S(j\omega)$ are to be interpreted as true almost everywhere, whereas statements involving the analytic function $S(s)$ are true everywhere. Our assumption that i,v and therefore a,b are real implies that $S(-j\omega) = \overline{S(j\omega)}$; finally let us note that (2) implies that $I - S^*(j\omega)S(j\omega)$ is positive semidefinite for almost all $\omega \in \mathbb{R}$ since it is almost everywhere $\lim_{\sigma \to 0+} \left(I-S^*(\sigma+j\omega)S(\sigma+j\omega)\right)$

III. Proof of Necessity of (1) and (2)

Throughout this section let N be a passive N-port.

Lemma 3.1 Suppose (a,b) is an admissable signal pair such that $a(t) = 0$ for $t < T$. Then $b(t) = 0$ for $t < T$.

Proof Since N is passive $\int_{-\infty}^{T} (a^*(t)a(t)-b^*(t)b(t))dt \geq 0$, so under the hypothesis of lemma 3.1, $-\int_{-\infty}^{T} b^*(t)b(t)dt = -\int_{-\infty}^{T} |b(t)|^2 dt \geq 0$. Thus $b(t) = 0$ for $t < T$. ¤

This simple lemma has extremely profound consequences!

Corollary 3.1 To each admissable $a \in L_2^N$, there is a unique b such that (a,b) is admissable. Furthermore $b \in L_2^N$ and $\|b\| \leq \|a\|$.

Proof Suppose $a \in L_2^N$ with (a,b) and (a,b') admissable. Since N is linear, $(0,b-b')$ is admissable. By lemma 3.1, $b(t) - b'(t) = 0$ for $t < T$ and T arbitrary, so $b = b'$. By passivity, we have for all $T \in \mathbb{R}$

$$\|a\|^2 \geq \int_{-\infty}^{T} a^*a \, dt \geq \int_{-\infty}^{T} b^*b \, dt$$

which proves $b \in L_2^N$ and $\|b\| \leq \|a\|$. ∎

Thus we may define a linear operator S on the set of admissable a's in L_2^N into $L_2^N(R)$ by $S(a) = b$. The last conclusion of Corollary 3.1 is that S is a <u>bounded</u> operator (hence continuous), and by our solvability assumption the set of admissable a's in L_2^N are dense in $L_2^N(R)$, so we may extend S in a unique way to an operator defined on all of $L_2^N(R)$, with $\|b\| \leq \|a\|$ valid for all $a \in L_2^N(R)$. Let us use the same symbol S for this extended operator.

<u>Corollary 3.2</u> S is a <u>causal</u> operator, that is, if $a(t) = a'(t)$ for $t < T$, then $Sa(t) = Sa'(t)$ for $t < T$.

<u>Proof</u> If $a(t) = a'(t)$ for $t < T$, then $(a-a', Sa-Sa')$ satisfies the hypothesis of lemma 3.1. Thus $Sa(t) = Sa'(t)$ for $t < T$. ∎

Thus, $S : L_2^N(R) \to L_2^N$ is a linear time invariant bounded causal operator. By the Bochner-Paley-Wiener theorem (see Section VII) S has a representation as: $\widehat{Sa(j\omega)} = S(j\omega)\hat{a}(j\omega)$ where the N×N matrix $S(j\omega)$ has the bounded analytic extension $S(s)$ in the RHP. We have shown that <u>all</u> $a \in L_2^N(R)$ are admissable and that N has a scattering matrix $S(j\omega)$. It remains to show $I-S^*(s)S(s)$ is positive semidefinite in the RHP.

<u>Lemma 3.3</u> For each $c \in C^N$, $c^*(I-S^*(j\omega)S(j\omega))c \geq 0$ for (almost all) $\omega \in \mathbb{R}$.

Note that this is weaker than $I-S^*(j\omega)S(j\omega)$ being positive semidefinite for (almost all) $\omega \in \mathbb{R}$.[†]

<u>Proof</u> Suppose for some $c \in C^N$ $c^*(I-S^*(j\omega)S(j\omega))c < 0$ for ω in some set $\bar{\Delta}$ of positive measure. We may take a subset Δ of $\bar{\Delta}$ with

[†] Lemma 3.3 says $\forall c \in C^N \, \exists N_c \subseteq \mathbb{R}\{\mu(N_c) = 0$ and $\omega \notin N_c \Rightarrow c^*(I-S^*(j\omega)S(j\omega))c \geq 0\}$ whereas this statement is $\exists N \subseteq \mathbb{R} \, \forall c \in C^N\{\mu(N) = 0$ and $\omega \notin N \Rightarrow c^*\big(I-S^*(j\omega)S(j\omega)\big)\, c \geq 0\}$. We shall see later that the stronger statement is true.

$0 < \mu(\Delta) < \infty$ and $\Delta \in [0,\infty)$ or $\Delta \in (-\infty,0]$ such that $c^*(I-S^*(j\omega)S(j\omega))c < -\varepsilon < 0$ for $\omega \in \Delta$. Then for $-\omega \in \Delta$, $\bar{c}^*(I-S^*(j\omega)S(j\omega))\bar{c} = \overline{c^*(I-S^*(-j\omega)S(-j\omega))c} < -\varepsilon$. Define $\hat{a}(j\omega)$ by

$$\hat{a}(j\omega) = \begin{cases} c & \omega \in \Delta \\ \bar{c} & -\omega \in \Delta \\ 0 & \text{elsewhere} \end{cases}$$

Then $\|\hat{a}(j\omega)\| = \sqrt{2\mu(\Delta)}\ |c| < \infty$, so $\hat{a} \in L_2^N$. Consequently $\hat{a}(j\omega)$ is the transform of an $a(t) \in L_2^N$, and since $\hat{a}(-j\omega) = \overline{\hat{a}(j\omega)}$, $a(t) \in L_2^N(R)$ and is thus admissable since all $a(t) \in L_2^N(R)$ are admissable.

Intuitively, $a(t)$ is a signal bandlimited to the set Δ where $c^*(I-S^*(j\omega)S(j\omega))c < 0$. If we apply this signal to N, the Parseval relation yields

$$\int_{-\infty}^{\infty} (a^*a-b^*b)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{a}(j\omega)^*(I-S^*(j\omega)S(j\omega))\hat{a}(j\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{\Delta} c^*(I-S^*(j\omega)S(j\omega))c\ d\omega$$

$$+ \frac{1}{2\pi}\int_{-\Delta} \bar{c}^*(I-S^*(j\omega)S(j\omega))\bar{c}\ d\omega < \frac{-\mu(\Delta)\varepsilon}{\pi} < 0$$

Since $\int_{-\infty}^{\infty} (a^*a-b^*b)dt = \lim_{T\to\infty} \int_{-\infty}^{T} (a^*a-b^*b)dt$, there is a T_0 with $\int_{-\infty}^{T_0} (a^*a-b^*b)dt < 0$, contradicting N's passivity. This establishes lemma 3.3. ¤

Theorem 3.1 $I-S^*(s)S(s)$ is positive semidefinite in the RHP.

Proof From our remarks after Corollary 3.1 we know $S(s)$ is bounded in the RHP. Hence Poisson's representation is valid ([10]; see Section VII for proof): For $s_0 = \sigma_0 + j\omega_0$, $\sigma_0 > 0$

$$S(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{(\omega-\omega_0)^2+\sigma_0^2}\ S(j\omega)d\omega$$

Let $c \in C^N$. Then

$$|S(s_0)c| = \left|\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{(\omega-\omega_0)^2+\sigma_0^2} S(j\omega)d\omega\right| \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{(\omega-\omega_0)^2+\sigma_0^2} |S(j\omega)c|d\omega$$

By lemma 3.3 $c^*(I-S^*(j\omega)S(j\omega))c = |c|^2 - |S(j\omega)c|^2 \geq 0$ for (almost all) $\omega \in \mathbb{R}$, so $|c| \leq |S(j\omega)c|$ for (almost all) $\omega \in \mathbb{R}$ and

$$|S(s_0)c| \leq |c| \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{(\omega-\omega_0)^2+\sigma_0^2} d\omega = |c|$$

Thus $|c|^2 - |S(s_0)c|^2 = c^*\left(I-S^*(s_0)S(s_0)\right)c \geq 0$ establishing Theorem 3.1 and the necessity of (1) and (2) in Theorem 2.1. ¤

IV. Sufficiency of (1) and (2)

We assume now (1) and (2), that is N has a scattering matrix $S(j\omega)$ which has the analytic extension $S(s)$ in the RHP, and that $I-S^*(s)S(s)$ is positive semidefinite in the RHP. (1) implies that N is solvable since the set of admissable a's is $L_2^N(R)$. (2) implies that $S(s)$ is bounded in the RHP, for if e_k is the kth standard basis vector $(0,\ldots,1,\ldots 0)^*$, $e_k^*(I-S^*(s)S(s))e_k = 1 - \sum_{j=1}^{N} |S_{ij}(s)|^2 \geq 0$, so that $|S_{ij}(s)| \leq 1$ for $s \in$ RHP. By the Bochner-Paley-Wiener theorem, S is the frequency domain representation of a LTI bounded causal operator $S : L_2^N(R) \to L_2^N(R)$ (see Section VII). It remains only to establish (2.2). If $a \in L_{2e}^N(R)$ then

$$\int_{-\infty}^{T} (a^*a-(Sa)^*Sa\ dt = \int_{-\infty}^{\infty} (a_T^*a_T-(Sa)_T^*(Sa)_T)dt$$

$$= \int_{-\infty}^{\infty} (a_T^*a_T-(Sa_T)_T^*(Sa_T)_T)dt$$

$$= \int_{-\infty}^{\infty} (a_T^*a_T-(Sa_T)^*(Sa_T))dt + \int_{T}^{\infty} (Sa_T)^*(Sa_T)dt$$

Since S is causal. Note the second integral exists since $a_T \in L_2^N(R)$ and $S : L_2^N(R) \to L_2^N(R)$.

$$\geq \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{a}_T^*(j\omega)(I-S^*(j\omega)S(j\omega))\hat{a}_T(j\omega)d\omega \geq 0$$

since $I-S^*(j\omega)S(j\omega)$ is positive semidefinite for (almost all) $\omega \in \mathbb{R}$. This proves N is passive and completes the proof of theorem 2.1. ¤

V. Discussion

In this section we examine the definition of passivity we have used and our proof of the passivity criterion.

Let us first consider the energy integral (2.2). Several authors use the alternate integral

$$\int_{-\infty}^{\infty} (a^*a-b^*b)dt \geq 0 \tag{5.1}$$

where $a,b \in L_2^N(R)$ instead of the extended spaces $L_{2e}^N(R)$. Let us call an N-port N weakly passive if it satisfies (5.1) and (2.3) (solvability); Wohlers [3,4] points out that weak passivity has the advantage of being independent of causality. We can prove a theorem analogous to (2.1) for weakly passive N-ports:

Theorem 5.1 An N-port N is LTI and weakly passive if and only if

(1) N has a scattering matrix $S(j\omega)$

(2) $I-S^*(j\omega)S(j\omega)$ is positive semidefinite for (almost all) $\omega \in \mathbb{R}$

The difference between this and (2.1) is that $S(j\omega)$ need not have an analytic extension into the RHP, and when it does $I-S^*(s)S(s)$ need not be positive semidefinite there (cf. example in Section I).

Proof Corollary 3.2 is easily checked for a LTI weakly passive N. Bochner's theorem applies directly and we conclude N has a scattering matrix $S(j\omega)$. If $I-S^*(j\omega)S(j\omega)$ were not positive semidefinite in some set $\bar{\Delta}$ of positive measure, we can construct a (measurable) $\hat{a}(j\omega)$ supported on $\Delta \cup -\Delta$ with $\hat{a}(j\omega)^*\hat{a}(j\omega) = 1$ and $\hat{a}(j\omega)^*(I-S^*(j\omega)S(j\omega))a(j\omega) < -\varepsilon < 0$ for $\omega \in \Delta \cup -\Delta$ where $\mu(\Delta) < \infty$ and $\Delta \subseteq [0,\infty)$ or $(-\infty,0]$, and $\hat{a}(-j\omega) = \overline{\hat{a}(j\omega)}$ as in lemma 3.3. Then $\hat{a}(j\omega) \in L_2^N$ and corresponds $a(t) \in L_2^N(R)$ for which

$$\int_{-\infty}^{\infty} (a^*a-b^*b)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{a}^*(j\omega)(I-S^*(j\omega)S(j\omega))\,\hat{a}(j\omega)d\omega$$
$$\leq \frac{-\varepsilon\mu(\Delta)}{\pi} < 0$$

which contradicts weak passivity (5.1).

The converse is easily proved, for suppose N has a scattering matrix $S(j\omega)$ with $I-S^*(j\omega)S(j\omega)$ positive semidefinite for $\omega \in \mathbf{R}$. Then N is clearly LTI and $S(j\omega)$ is bounded so if $a \in L_2^N(R)$, $\hat{b}(j\omega) = S(j\omega)\hat{a}(j\omega) \in L_2^N$, so $b \in L_2^N$. Furthermore

$$\int_{-\infty}^{\infty} (a^*a-b^*b)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{a}^*(j\omega)(I-S^*(j\omega)S(j\omega))\hat{a}(j\omega)d\omega \geq 0$$

So that N is <u>weakly passive</u>. ◻

An example of a <u>weakly passive</u> but not <u>passive</u> 1-port is N given by

$$b(t) = \int_{-\infty}^{\infty} \frac{\sin^2\pi\tau}{\pi^2\tau^2}\, a(t-\tau)d\tau \tag{5.2}$$

for which $S(j\omega) = \begin{cases} 1-|\omega| & |\omega| \leq 1 \\ 0 & |\omega| > 1 \end{cases}$

Note that $S(j\omega)$ has no analytic extension into the RHP and that S is not a causal operator. Another example is a -1 H inductor, which has

$$S(j\omega) = \frac{1+j\omega}{1-j\omega}$$

Here S is <u>not</u> a causal operator, even though its impedance operator $Z = \frac{-d}{dt}$ is!

The relation between <u>weak passivity</u> and <u>passivity</u> is simple:

<u>Theorem 5.2</u> An N-port N is <u>passive</u> if and only if it is <u>weakly passive</u> and its <u>scattering operator</u> S <u>is causal</u>.

<u>Proof</u> If N is <u>passive</u>, then it is clearly <u>weakly passive</u> and we have seen in corollary 3.2 that its S is causal. Conversely, if N is <u>weakly passive</u> then its S is a bounded operator and if it is causal

then $(S(a_T))_T = (Sa)_T$. (Noting $a_T \in \text{Dom } S$ since S is bounded).
Following the argument in Section IV, if $a \in L_{2e}^N$ then $a_T \in L_2^N$ and

$$\int_{-\infty}^{T} (a^*a-b^*b)dt = \int_{-\infty}^{\infty} (a_T^*a_T-(Sa_T)^*(Sa_T))dt + \int_{T}^{\infty} (Sa_T)^*(Sa_T)dt \geq 0$$

Thus N is passive. ¤

One final remark concerning weak passivity is in order. Any proof of the passivity criterion theorem 3.1 which has as hypothesis only weak passivity without the auxiliary assumption that S is causal in incorrect. Mere analyticity of Z or S is not enough, though boundedness of S is (see Section VII Bochner-Paley-Wiener theorem; cf [7], [8], [9]). Nor is the assumption that Z is causal adequate, as the -1 H inductor shows.†

†It is interesting to note that Raisbeck's original definition of passivity is what we call weak passivity together with the additional assumption that Z is causal, so that his criterion is not quite right.

We now turn to the second requirement for passivity, "solvability" (2.3), which may seem a bit technical. The obvious example of a device which satisfies (2.2) (has positive energy integrals) but not (2.3) (is not solvable) is the 1-port nullator characterized by $v = i = 0$. We will now show that all LTI N-ports which satisfy (2.2) but not (2.3) exhibit a generalization of this pathological behavior, and give a more natural assumption which implies (2.3).

Let us consider first a 1-port N which satisfies (2.2) but not necessarily (2.3). In this case, corollary 3.1 shows that S exists and is a bounded operator from its domain (which we assume here is not necessarily dense in $L_2(R)$) into L_2. Thus we may extend S to be defined on the closure M of its domain. M is a closed, translation invariant subspace of $L_2(R)$ which by a theorem of Bochner and Wiener [11] may be described by

$$M = \{a \in L_2(R) \mid a(j\omega) = 0 \text{ for (almost all) } |\omega| \in E\}$$

where $E \subseteq [0,\infty)$ is a set of positive measure if (2.3) is not satisfied, and may be taken to be the empty set if (2.3) is satisfied. Thus the admissable a's in L_2 are simply those whose spectrum vanishes on a certain set E of frequencies, that is, N acts as a frequency selective nullator.

We now make the observation that if a signal $a(t)$ which is not identically zero satisfies $a(t) = 0$ for $t < 0$ (let us call such a signal positively supported) then $\hat{a}(j\omega)$ vanishes for ω in a set of measure zero. This is easily seen from the fact that $\hat{a}(j\omega)$ has an analytic extension in the RHP which would vanish identically if $\hat{a}(j\omega)$ vanished on a set of positive measure, or from the well known version of the Paley-Wiener theorem which asserts

$$\int_{-\infty}^{\infty} \frac{|\ln|\hat{a}(j\omega)||}{1+\omega^2}\, d\omega < \infty \tag{5.3}$$

Thus N is a very strange device indeed, for the only positively supported admissable a(t) is 0. This precludes any testing of the device in the laboratory (a pathology shared by some non-causal devices). It is very natural, if not philosophically necessary, to assume this cannot happen. Specifically, if we make the assumption:

> If there is any nonzero admissable a, then there is a positively supported (nonzero admissable) one. (5.4)

then we may conclude that N is either a nullator or satisfies (2.3) and hence is passive.

The generalization to N-ports is straightforward even though the closed translation invariant subspaces are more complicated. In this case M may be described by

$$M = \{a \in L_2^N(R) \mid k_\alpha^* a(j\omega) = 0 \text{ for (almost all) } |\omega| \in E_\alpha, \text{ for all } \alpha \in A\} \tag{5.5}$$

where $\{(k_\alpha, E_\alpha) \mid \alpha \in A\}$ is a collection of pairs of complex N-vectors and subsets of $[0,\infty)$. For example, consider the 2-port N characterized by

$$\begin{aligned} M = \{a \in L_2^2(R) \mid & a_1(j\omega) + a_2(j\omega) = 0 \text{ for } |\omega| \in [0,1] \\ & \text{and } \hat{a}_1(j\omega) - \hat{a}_2(j\omega) = 0 \text{ for } |\omega| \in [2,3]\} \end{aligned}$$

and b = a when $a \in M$.

Thus $i_1 = i_2 = 0$ always, and v_1 and v_2 are constrained as follows: for signals bandlimited to [0,1], $v_1 = -v_2$; for signals bandlimited to [2,3], $v_1 = v_2$; for all other signals $v_1 = v_2 = 0$. Thus in the laboratory, N would appear (!) to be a 2-port nullator. This is the pathological behavior we eliminate with the following assumption:

There is a positively supported admissable a such that if $k^* a(t) = 0$ for all t, then $k = 0$. Informally, the infinite collection of vectors $\{a(t) | 0 \leq t < \infty\}$ spans $\mathbb{R}^N$[†] (5.6)

5.6 is implied by the existence of positively supported admissable $\{a_1, \ldots a_N\}$ with $\{a_1(t), \ldots a_N(t)\}$ spanning $\mathbb{R}^N$ for t in some set of positive measure. This is the generalization of (5.4) which implies N is solvable. For suppose one of the sets E_α in (5.5) has positive measure. Then since $k^* \hat{a}(j\omega)$ vanishes for ω in a set of positive measure, $k^* \hat{a}(j\omega) = 0$ for all ω and hence $k^* a(t) = 0$ for all t, so by (5.6) $k = 0$. Thus $M = L_2^N(R)$ and N is solvable.

The reader may have wondered why we have used the scattering representation as opposed to the more common impedance representation, used for example in Raisbeck's original informal argument. There are two reasons: certain passive devices such as open circuits do not have an impedance representation, and more important, for a passive device the scattering operator S is bounded whereas the impedance operator Z need not be. The recognition of the importance of the scattering representation for passive networks is of course due to Youla et al.

The boundedness of S is crucial to our proof. First it allows us to use the Bochner-Paley-Wiener theorem to show that a passive N has a scattering matrix $S(s)$. Distribution theory must be used to prove that N has an impedance matrix $Z(s)$ (assuming it has an impedance representation). Even assuming the existence of $Z(s)$, as Raisbeck and Kuo do, it may be unbounded.

[†] Technically, we must require that this remain true no matter how $a(t)$ is redefined for t in sets of measure zero.

We can have $Zi \notin L_2^N$, even if $i \in L_2^N$, so that Parseval's relation must be used with care. Furthermore Poisson's representation is not valid. For example, if N is a (quite passive) 1 H inductor, $\frac{\sigma_0 j\omega}{(\omega-\omega_0)^2+\sigma_0^2}$ is not even integrable (e.g. [1] line 16, [9] line 5). In the sufficiency proof we considered a_T, admissable since all of $L_2^N(R)$ was known to be admissable; this too was a consequence of the boundedness of S . The same argument fails for Z, since its domain may be a proper subset of $L_2^N(R)$. With the inductor above, i_T need not be admissable since i_T is generally not differentiable. This is only a partial list, but we can say that arguments using Z instead of S cannot be made formal without considerable trouble.

VI. Passivity Criterion with General Coordinates

In this section we consider the use of variables other than the scattering variables. Specifically, we consider the variables ξ and η related to v and i by

$$\begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} \tag{6.1}$$

where $\Omega = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a real invertible 2N x 2N matrix. We shall say a LTI N-port N has an Ω-representation if for each N-admissable $\xi(t)$ there is a unique N-admissable $\eta(t)$, in other words there is an (LTI) operator Λ with $\xi = \Lambda\eta$. We assume neither that the domain of Λ includes $L_2^N(R)$ nor that Λ is bounded. For example an inductor has an $\Omega = I_2$ representation with $\Lambda\eta = L\dot{\eta}$; we call this the impedance representation and Λ the impedance operator. By suitable choice of Ω, this general framework includes the scattering, impedance, admittance, hybrid, and transmission representations. We will recast theorem 2.1 into a form applicable to N-ports having some general Ω-representation; in the particular case of the impedance representation this will be the original Raisbeck proposition.

Theorem 6.1 N is LTI passive and has an Ω-representation if and only if it is solvable and $\hat{\xi}(j\omega) = \hat{\Lambda}(j\omega)\hat{\eta}(j\omega)$ where $\hat{\Lambda}(j\omega)$ has the meromorphic extension $\hat{\Lambda}(s)$ in the RHP and $D(s) + D(s)^*$ is positive semidefinite† in the RHP, where $D(s) = [a\hat{\Lambda}(s)+b]^*[c\hat{\Lambda}(s)+d]$.

Proof Suppose first N is passive and has an Ω-representation $\xi = \Lambda\eta$. By theorem 2.1 we know N has a scattering matrix $S(j\omega)$ with analytic extension $S(s)$, $I-S(s)^*S(s)$ positive semidefinite in the RHP. This and (6.1) imply $\hat{\xi}(j\omega) = \hat{\Lambda}(j\omega)\hat{\eta}(j\omega)$ where $\hat{\Lambda}(j\omega)$ has the meromorphic extension in the RHP

$$\hat{\Lambda}(s) = [(\alpha-\beta)S(s)+\alpha+\beta][(\gamma-\delta)S(s)+\gamma+\delta]^{-1}$$

Furthermore $S(s) = [(a-c)\hat{\Lambda}(s)+b-d][(a+c)\hat{\Lambda}(s)+b+d]^{-1}$, so $[(s+c)\hat{\Lambda}(s)+b+d]^*[I-S(s)^*S(s)][(a+c)\hat{\Lambda}(s)+b+d] = D(s) + D(s)^*$ is positive semidefinite in the RHP.

Conversely suppose N is solvable with $D(s) + D(s)^*$ positive semidefinite in the RHP. Solvability implies $((a+d)\hat{\Lambda}(s)+b+d)$ is invertible except on a set E of isolated points (i.e. is invertible as a meromorphic matrix). Moreover N has a scattering matrix $S(j\omega)$ with meromorphic extension $S(s)$ in the RHP given by

$$S(s) = [(a-c)\hat{\Lambda}(s)+b-d][(a+c)\hat{\Lambda}(s)+b+d]^{-1}.$$

For $s \in$ RHP, $s \notin E$,

$$[(a+c)\hat{\Lambda}(s)+b+d]^{-1*}[D(s)+D(s)^*][(a+c)\hat{\Lambda}(s)+b+d]^{-1} = I - S(s)^*S(s)$$

is positive semidefinite; thus $S(s)$ is bounded there and consequently analytic in the RHP. By theorem 2.1 N is passive. ◻

Corollary 6.2 Assuming N is LTI, solvable, and has an impedance (Z), admittance (Y), or hybrid (H) representation,

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†We mean positive semidefinite where defined, i.e. except at the RHP poles of $\hat{\Lambda}(s)$.

(a) (Raisbeck proposition) N is passive if and only if $Z(s) + Z(s)^*$ is positive semidefinite in the RHP. In this case we can show $Z(s)$ is in fact analytic.

(b) N is passive if and only if $Y(s) + Y(s)^*$ is positive semidefinite in the RHP. $Y(s)$ is in fact analytic.

(c) N is passive if and only if

$$\left[\begin{array}{c|c} H_{11}+H_{11}^* & H_{12}+H_{21}^* \\ \hline H_{21}+H_{12}^* & H_{22}+H_{22}^* \end{array}\right]$$

is positive semidefinite in the RHP, where $a = c = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right]$, $b = d = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I \end{array}\right]$ and $\hat{\Lambda} = \left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right]$.

VII. Appendix: Proofs of Mathematical Theorems

(A) Bochner-Paley- Wiener theorem:

S is a LTI bounded causal operator : $L_2^N \to L_2^N$ if and only if $\widehat{Sa}(j\omega) = S(j\omega)\hat{a}(j\omega)$ where $S(j\omega)$ has a bounded analytic extension $S(s)$ in the RHP.

This theorem characterizes a very important class of operators and is well known. It is usually proved using distribution theory, where the boundedness conditions may be dropped. But the fact that the scattering operator of a passive device is bounded allows us to use this (weaker) version which is easily proved without distribution theory. It is a simple consequence of two classic theorems: the Bochner theorem [11] which states that $\Pi : L_2 \to L_2$ is LTI bounded if and only if $\widehat{\Pi a}(j\omega) = T(j\omega)\hat{a}(j\omega)$ where $T(j\omega)$ is essentially bounded; and the Paley-Wiener theorem [12] which states that $a \in L_2$ and is positively supported ($a(t)=0, t<0$) if and only if $\hat{a}(j\omega)$ has an analytic extension $\hat{a}(s)$ in the RHP such that for some k and all $\sigma \geq 0$,

$$\int_{-\infty}^{\infty} |a(\sigma+j\omega)|^2 \, d\omega \leq k \tag{7.1}$$

we will also use a corollary of the Paley-Wiener theorem due to Titchmarsh. For completeness we give a sketch of the

Proof of Paley-Wiener theorem

Suppose first $a \in L_2$ and is positively supported. It is then well known that $\hat{a}(s) = \int_0^{\infty} e^{-st} a(t)dt$ defines an analytic function for $\operatorname{Re} s > 0$ with $\lim_{\sigma\to 0^+} \hat{a}(\sigma+j\omega) = \hat{a}(j\omega)$ for almost all $\omega \in \mathbb{R}$. Furthermore for $\sigma \geq 0$

$$\|a\|^2 \geq \int_{-\infty}^{\infty} e^{-2\sigma t}|a(t)|^2 dt = \int_{-\infty}^{\infty} |e^{-\sigma t}a(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{a}(\sigma+j\omega)|^2 d\omega$$

using the Parseval relation. Thus (7.1) holds for $k = 2\pi\|a\|^2$. Conversely suppose $\hat{a}(j\omega)$ has the analytic extension $\hat{a}(s)$ in the RHP satisfying (7.1). In particular for $\sigma = 0$ we have $\hat{a}(j\omega) \in L_2$ and so is the Fourier transform of an $a(t) \in L_2$. Since $\hat{a}(s)$ has a domain of analyticity including the RHP, we conclude $e^{-\sigma t}a(t) \in L_2$ for all $\sigma \geq 0$ and $\hat{a}(\sigma+j\omega) = \widehat{e^{-\sigma t}a(t)}(j\omega)$ (we have used the fact that the extension $\hat{a}(s)$ is unique). If $a(t)$ were not positively supported, then for some $-\delta < 0$ $\int_{-\infty}^{-\delta} |a(t)|^2 dt = \varepsilon > 0$.
For $\sigma \geq 0$, $\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{a}(\sigma+j\omega)|^2 d\omega = \|e^{-\sigma t}a(t)\|^2 = \int_{-\infty}^{\infty} e^{-2\sigma t}|a(t)|^2 dt$

$$\geq \int_{-\infty}^{\infty} e^{-2\sigma t}|a(t)|^2 dt \geq e^{2\sigma\delta}\varepsilon$$

which contradicts (7.1) for $\sigma > \frac{1}{2\delta}\ln\frac{k}{2\pi\varepsilon}$. ¤

Proof of Bochner-Paley-Wiener theorem

We will prove the theorem for $N = 1$; the generalization to $N > 1$ is immediate. Suppose first S is defined by $\widehat{Sa}(j\omega) = S(j\omega)\hat{a}(j\omega)$, $S(j\omega)$ having a bounded analytic extension $S(s)$ in the RHP. It is obvious that S is linear and time invariant. $S(j\omega)$ is essentially bounded since it is almost everywhere $\lim_{\sigma\to 0^+} S(\sigma+j\omega)$. Thus for $a \in L_2$, $\hat{a} \in L_2$ and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\widehat{Sa}(j\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{a}(j\omega)|^2|S(j\omega)|^2 d\omega$$

$$\leq M^2\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{a}(j\omega)|^2 d\omega = M^2\|a\|^2$$

where $|S(s)| \leq M$ for $s \in$ RHP. So $Sa \in L_2$ and $\|Sa\| \leq M\|a\|$ so that S is bounded (this was the easy half of the Bochner theorem). It remains to show S is causal. Suppose $a(t)$ is positively supported. By the Paley-Wiener theorem $\hat{a}(j\omega)$ has the analytic extension $\hat{a}(s)$ satisfying (7.1).

Since $\widehat{Sa}(j\omega) = S(j\omega)\hat{a}(j\omega)$, $\widehat{Sa}(j\omega)$ has the analytic extension $S(s)\hat{a}(s)$ in the RHP. For $\sigma \geq 0$

$$\int_{-\infty}^{\infty} |S(\sigma+j\omega)\hat{a}(\sigma+j\omega)|^2 d\omega \leq M^2 \int_{-\infty}^{\infty} |\hat{a}(\sigma+j\omega)|^2 d\omega \leq M^2 2\pi \|a\|^2$$

By the Paley-Wiener theorem we conclude $S(j\omega)\hat{a}(j\omega)$ is the Fourier transform of a positively supported element of L_2, that is, $Sa(t) = 0$, $t < 0$. Thus S is causal.

Suppose now S is LTI, bounded and causal: $L_2 \to L_2$. By the (harder half of the) Bochner theorem, $\widehat{Sa}(j\omega) = S(j\omega)\hat{a}(j\omega)$ where $|S(j\omega)| \leq M$ for (almost all) $\omega \in \mathbb{R}$. We must show $S(j\omega)$ has a bounded analytic extension into the RHP. Let $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$. Consider $a(t) = \begin{cases} e^{-\bar{s}_0 t} & t \geq 0 \\ 0 & t<0 \end{cases}$ $\hat{a}(j\omega) = \dfrac{1}{j\omega+\bar{s}_0}$ so $\widehat{Sa}(j\omega) = \dfrac{S(j\omega)}{j\omega+\bar{s}_0}$. Since $a(t)$ is positively supported and S is causal, Sa is positively supported and so by the Paley-Wiener theorem $\widehat{Sa}(j\omega)$ has the analytic extension $\widehat{Sa}(s)$ in the RHP. Thus $S(j\omega)$ has the analytic extension $\widehat{Sa}(s)(s+\bar{s}_0)$ in the RHP, which must be independent of s_0 since it is unique. By the Titchmarsh theorem [13]

$$\widehat{Sa}(s_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\widehat{Sa}(j\omega)d\omega}{s_0-j\omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S(j\omega)d\omega}{(s_0-j\omega)(j\omega+\bar{s}_0)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S(j\omega)d\omega}{\sigma_0^2+(\omega-\omega_0)^2}$$

Hence $|\widehat{Sa}(s_0)| \leq \dfrac{M}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{d\omega}{\sigma_0^2+(\omega-\omega_0)^2} = \frac{M}{2\sigma_0}$

So $|S(s_0)| = |\widehat{Sa}(s_0)(s_0+\bar{s}_0)| = 2\sigma_0|\widehat{Sa}(s_0)| \leq M$, that is, $S(s)$ is bounded in the RHP. ◻

(B) Poisson's formula for the half plane [10]. Suppose $S(j\omega)$ has a bounded analytic extension $S(s)$ in the RHP. Let $s_0 = \sigma_0 + j\omega_0$, $\sigma_0 > 0$. Then

$$S(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0 S(j\omega)d\omega}{(\omega-\omega_0)^2+\sigma_0^2}$$

Again we prove this only for N = 1.

<u>Proof</u> Consider the contours $\Gamma_{\varepsilon,R}$ oriented as shown in Fig. 1.

$(\frac{1}{z-s_0} - \frac{1}{z+\bar{s}_0})S(z)$ is meromorphic in the RHP, its only pole there at s_0 with residue $S(s_0)$. By Cauchy's theorem for $\varepsilon < \sigma_0$ and $R > |s_0|$,

$$\frac{1}{2\pi j}\int_{\Gamma_{\varepsilon,R}} (\frac{1}{z-s_0} - \frac{1}{z+\bar{s}_0})S(z)dz = \frac{1}{\pi}\int_{\Gamma_{\varepsilon,R}} \frac{\sigma_0 S(z)dz}{(z-s_0)(z+\bar{s}_0)} = S(s_0)$$

Letting $\varepsilon \to 0$ we conclude by Lebesque's bounded convergence theorem

$$\frac{1}{\pi}\int_{-R}^{R} \frac{\sigma_0 S(j\omega)d\omega}{(\omega-\omega_0)^2+\sigma_0^2} + \frac{1}{\pi j}\int_{\gamma_R} \frac{\sigma_0 S(z)dz}{(z-s_0)(z+\bar{s}_0)} = S(s_0)$$

where γ_R is the semicircle of radius R centered at the origin and oriented positively. But

$$\left| \frac{1}{\pi j}\int_{\gamma_R} \frac{\sigma_0 S(z)dz}{(z-s_0)(z+\bar{s}_0)} \right| \leq \frac{\sigma_0 MR}{(R-|S_0|)^2}$$

where $|S(z)| \leq M$ in the RHP. Finally since $\frac{\sigma_0 S(j\omega)}{(\omega-\omega_0)^2+\sigma^2}$ is integrable

$$\lim_{R\to\infty} \frac{1}{\pi}\int_{-R}^{R} \frac{\sigma_0 S(j\omega)d\omega}{(\omega-\omega_0)^2+\sigma_0^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0 S(j\omega)d\omega}{(\omega-\omega_0)^2+\sigma_0^2} = S(s_0) \qquad \square$$

Note in particular that Poisson's formula is not valid if $|S(z)|$ grows as fast as $|z|$ in the RHP.

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Figure Caption

Fig. 1. The contour $\Gamma_{\varepsilon,R}$ used for proving Poisson's formula.

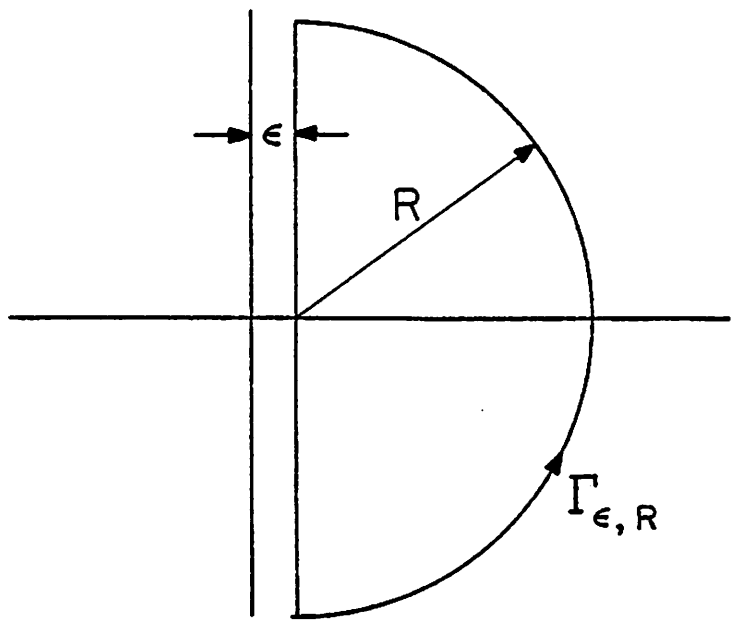

figure 1