WHEN IS A DEVICE PASSIVE?

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ABSTRACT

Passivity is perhaps the most basic concept in circuit theory. Unfortunately, the existing definitions of passivity are too restrictive and often contradict one another. In this paper, a new passivity definition is proposed which is applicable to all n-port and (n+1)-terminal devices -- including time-varying, nonlinear, and distributed circuit elements. This definition generalizes and reconciles several recent conflicting definitions.

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1. INTRODUCTION

Several recent publications, see for example [1,2], have made it clear that there is still not universal agreement on how the term "passivity" should be defined. Because passivity is a long-standing and often-used concept, it is undesirable that the word be used with several different meanings. The purpose of this paper is to propose a definition which, in effect, reconciles the differences among the existing competing definitions. As a means to this end, it is necessary to look very carefully at what is meant by a model of a device. In fact, there are three concepts to be explored: (a) passivity of a state-space model; (b) passivity of an input-output model; (c) passivity of a device. The existing literature treats issues (a) and (b). Our aim here is to tie together (a) and (b), and thence to look at (c). The bulk of this paper is an extension of several results of the state-space theory, to a situation where a state-space model need not exist.

Superficially, it would appear that one could define passivity via the inequality

$$\int_{t_0}^{t} v(t)^T i(t)dt \geq 0 \qquad (1.1)$$

The difficulty with this approach is that it does not account for initial stored energy. One must require that (1.1) only hold when the circuit is started in a "relaxed state" [3]; alternatively, one must modify (1.1) to allow explicitly for stored energy terms, as in [4]. For the classes of circuits treated in [3] and [4] these two definitions are adequate and indeed equivalent. In general, though, it is surprisingly difficult to state precisely what is meant by the terms "relaxed state" and "stored energy," so that the definitions become ambiguous [8].

A resolution of these problems has been presented in [2], but only for the case where a state-space model can be written down. To allow for the widest possible class of circuits, we need to work with what is generally called an input-output model. This allows us to bypass the concept of "state," and specify a device purely in terms of its admissible signals (voltages and currents). A new complication, though , is that the input-output model implicitly assumes fixed initial conditions. That is, it describes only one _frame_ of the device, for the given or implied initial conditions, rather than the whole device. (In contrast, a state-space model automatically includes a specification of the effect of initial conditions, via the parameter called initial state.) One effect is that the input-output definition of passivity in [5] is not equivalent to that in [4], although they look the same. Even the stability properties of a "passive" circuit depend crucially on which definition of passivity one uses [6].

The approach in [1] and [6] _does_ allow for the effect of initial conditions, while in other ways retaining the spirit of the input-output approach as in [5]. However the central results of [1,6] depend in an essential way on the existence of a state space. To our knowledge, there is no treatment in the literature of device models with the generality allowed by an input-output approach, retaining at the same time the ability to describe initial condition effects. The present paper provides such an approach. The main contributions are

(i) A framework (_Section 2_) which allows one to discuss initial condition effects in an input-output setting;

(ii) The introduction of the concept of _attainability_ (_Section 3_), which is closely allied to the state-space notion of reachability; and the extension (Theorem 3.5) to an input-output setting of an important property of stored energy;

(iii) The reconciliation (_Section 4_) of several apparently conflicting definitions of passivity.

Because we do not necessarily assume the existence of a state-space, we need an extremely general definition of a "device." All that is required is a description of the set of admissible signals at the device's interface to the external world. In particular, the device might be an n-port as in Fig. 1a, or an (n+1)-terminal device as in Fig. 1b. In those cases, the "signal" could be a vector of port or terminal voltages and currents. Another possibility is a "distributed port" where voltage and current are a function of space coordinates, so that the signal at each time is an infinite-dimensional vector. _Time-varying_ circuits are allowed. It is not assumed that to each voltage there exists a _unique_ current, nor vice versa. Incidentally, we do not assume _causality_, for an important reason: in an electrical n-port, it is often not clear which of the signal components (voltages and currents) should be called "inputs," and which should be called "outputs." Circuit analysts commonly make an arbitrary choice. To avoid such complications, we have chosen simply to abolish the distinction between inputs and outputs.

2. DEVICE MODELS

In this section, we shall briefly discuss passivity for the case where a state-space model is available, and then show how some of the state-space properties can be carried over to a situation where only an input-output description is available. Since our main interest lies in looking at the input-output case, only an outline of the details for the state-space case will be given; for a more rigorous treatment, the interested reader should consult [2].

For reasons that will later become clear, we need to consider devices with possibly variable parameters. Thus, with a device $\mathcal{D}$ we associate a state-space model S(p) -- or, more precisely, the collection of all S(p) as p takes all values in some set $\mathcal{P}$. That is, for each $p \in \mathcal{P}$, S(p) is a state-space description (satisfying the usual axioms -- see for example [2]), with state-space Σ, input space $\mathcal{U}$, and output space $\mathcal{Y}$. To keep the notation simple, we suppose that Σ, $\mathcal{U}$ and $\mathcal{Y}$ do not depend on p. For much of what follows, there is actually no need to distinguish between inputs and outputs. Indeed, for many electrical circuits the designation of some port voltages and currents as "inputs" and others as "outputs" creates an asymmetry which is somewhat artificial. It makes sense, then, to define a <u>signal space</u> $\mathcal{S}$ containing all signals of interest. Most commonly, we will find that $\mathcal{S}$ can be decomposed into the form $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$, because the "interesting" signals of an n-port tend to occur in pairs (e.g., the voltage and current at each port). The obvious choice is $\mathcal{S} = \mathcal{U} \times \mathcal{Y}$; but, as the following examples indicate, there are sometimes better choices.

(2.1) <u>Example</u>. For the nonlinear two-port described by

$$v_1 = f_1(i_1, v_2)$$
$$i_2 = f_2(i_1, v_2)$$

we have a null state-space. If we want our mathematical model to describe, say, voltages and currents which are continuous functions of time, then $\mathcal{U} = \mathcal{Y} = C_2[0,\infty)$, where $C_n[0,\infty)$ is the space of continuous n-vector functions of time. The variables i_1 and v_2 are inputs, and v_1 and i_2 are outputs. But since the distinction between inputs and outputs is irrelevant to our present purposes, we can simply lump all four variables together as a vector $s \in \mathcal{S}$, where $\mathcal{S} = C_4[0,\infty)$. Some ways of writing s(t) are as $s(t) = [v_1(t)\ v_2(t)\ i_1(t)\ i_2(t)]^T$, or $s(t) = [v_1(t)\ i_1(t)\ v_2(t)\ i_2(t)]^T$, or

$$s(t) = \begin{bmatrix} v_1(t) & i_1(t) \\ v_2(t) & i_2(t) \end{bmatrix}, \text{ or } s(t) = \left(\begin{bmatrix} v_1(t) \\ i_1(t) \end{bmatrix}, \begin{bmatrix} v_2(t) \\ i_2(t) \end{bmatrix} \right)$$

or several other possibilities. Which of these is chosen is purely a matter of notational convention, and does not affect the basic theory. ¤

(2.2) Example. For the element described by

$$\frac{dv(t)}{dt} = f\left(\frac{di(t)}{dt}\right)$$

A simple state-space description is

$$\dot{x}_1 = u$$
$$\dot{x}_2 = f(u)$$

with output equation

$$y = \begin{bmatrix} i \\ v \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here, $u\ (\stackrel{\Delta}{=} \frac{di}{dt})$ is a newly defined variable introduced solely for the purpose of setting up state equations.† We need u as

†There are many other devices which require a similar treatment. For example, the higher-order elements defined in [9] with $\beta > 0$ or $\alpha < \beta < 0$ belong to this class.

a "signal of interest" in the state-space description. For a port variable description, though, all we need is $s(t) = [v(t)\ i(t)]^T$, so $S = V$ in this case. ¤

(2.3) Example. All our examples so far have used voltages and currents as signals, but this is not essential. If we wanted to use a scattering description for a circuit, then a more likely choice would be something like $s(t) = [v(t)-i(t), v(t)+i(t)]^T$. ¤

For any consideration of passivity, an important quantity is the energy entering the device. This is a function of the time interval over which measurements are taken, and of the signals at the ports. Let the energy supplied to the device over time interval $[t_0,t_1]$ be denoted $E(s,t_0,t_1)$, where $s \in S$ is the signal. For an n-port, an explicit formula is

$$E(s,t_0,t_1) = \int_{t_0}^{t_1} v(t)^T i(t)dt$$

where $v(t)$ and $i(t)$ are vectors of port voltages and currents, with the usual sign conventions. To complete the definition, it is necessary to specify how v and i depend on s.

Let $(V(s(t)), I(s(t)))$ be a pair of functions such that

$$(v(t),i(t)) = (V(s(t)),I(s(t))).$$

The pair $(V(s(t)),I(s(t)))$ will be called the port voltage-current readout map.

(2.4) Example. For the nonlinear two-port in Example (2.1), if we had chosen

$$s(t) = [v_1(t)\ v_2(t)\ i_1(t)\ i_2(t)]^T,$$

we would have

$$V(s(t)) = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix} s(t)$$

and

$$I(s(t)) = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} s(t).$$

For the choice of s(t) in the scattering description of Example (2.3), we have:

$$V(s(t)) = [\ 1/2 \quad 1/2]s(t)$$

and

$$I(s(t)) = [-1/2 \quad 1/2]s(t)$$

¤

When the port readout map exists, the energy entering the device in the time interval $[t_0,t_1]$ is given by

$$E(s,t_0,t_1) = \int_{t_0}^{t_1} V^T(s(t))I(s(t))dt$$

A standard assumption is that for the device under consideration, the quantity in the integrand is integrable over any finite interval in $\mathbb{R}$. This simply means that in any finite time interval, the energy entering the device is finite, and is not at all a strong restriction.

Most generally, though, we cannot guarantee the existence of a readout map of the above form. Two practical examples are:

(a) If $s(t)$ is a vector of voltmeter and ammeter readings, then at even moderately high frequencies the meter dynamics cannot be neglected; so s, v and i are related by differential equations. In this case a readout map can still be defined (although not precisely in the form introduced above), but $v(t)$ and $i(t)$ depend on the time evolution of s, rather than on the signal $s(t)$ at one point in time.

(b) In high-frequency circuits, and also in LSI circuits, the interconnections between components are distributed in space, and there are no well-defined "ports" or "terminals". In this case the voltages and currents must be written in the form $v(z,t)$ and $i(z,t)$, where z is a vector of space co-ordinates. Correspondingly, $s(t)$ will lie in some infinite-dimensional space. The above energy integral must then be replaced by an integration over both space and time co-ordinates.

It would be a tedious matter to write down a single energy formula which covered all such cases (together, perhaps, with cases which have not occurred to the authors). Fortunately, the results of this paper do not depend on the precise form of $E(\cdot,\cdot,\cdot)$. We do require, however, that the following properties hold.

(2.5) <u>Assumption</u>. For every $s \in \mathcal{S}$, and every $t_0, t_1, \in \mathbb{R}$ such that $t_0 \leq t_1$, $E(s, t_0, t_1)$ is a well-defined number in $\mathbb{R} \cup \{-\infty, +\infty\}$, and

(a) $E(s, t_0, t_0) = 0$

(b) $E(s, t_0, t_1) = E(\hat{s}, t_0, t_1)$, for any $\hat{s}$ such that $\hat{s}(t) = s(t)$ for all $t \in [t_0, t_1]$

(c) $E(s, t_0, t_2) = E(s, t_0, t_1) + E(s, t_1, t_2)$ whenever $t_0 \leq t_1 \leq t_2$.

Assumption (2.5) is trivially satisfied when a port readout map exists, and is also satisfied in every other case known to the authors.

Another important quantity is the available energy:

$$E_A((x_0,p),t_0) = \sup_{\substack{s \in X((x_0,p),t_0) \\ t_1 \geq t_0}} \{-E(s,t_0,t_1)\}$$

where $X((x_0,p),t_0) \subset \mathcal{S}$ is the set of all admissible signals consistent with the initial condition $x(t_0) = x_0$ and parameter setting p. (The extra parentheses around (x_0,p) are for consistency with a more formal definition to be given later). It is readily seen that this method of writing E_A is equivalent to the more usual definitions -- see for example [1,2] -- where the signal space is split into separate input and output spaces. The properties of E_A are intimately linked with passivity [1,2], and in fact the passivity definition in [2] is stated directly in terms of finiteness of E_A.

Most of what has been said, so far, works equally well in a state-space setting or an input-output setting. However E_A is a function of initial state; and input-output models lack the concept of initial state.

Here we must face a fundamental distinction between a state-space model and an input-output mapping: a state-space description implicitly describes the admissible signals <u>for every possible initial state</u>. Input-output descriptions, on the other hand, describe what happens <u>for a fixed initial state</u>. So, for a device $\mathcal{D}$ for which we have a set of state-space descriptions $\{S(p) = p \in \mathcal{P}\}$, the usual input-output model describes, not $\mathcal{D}$, but something like $\mathcal{D}(x_0,p)$ for fixed x_0 and p. Frequently, only the case $x_0 = 0$ is considered.

When no state-space model is available, what substitutes for initial state? The answer is that the admissible signals of a device depend on a variety of factors -- for example, temperature, initial capacitor charges, manual knob settings -- which we can lump together under the general name of "parameters." Let Γ denote the set of all possible parameters, and let $\gamma \in \Gamma$ denote the parameter value in any one experiment. When we have a state-space model, it will usually be possible to write $\Gamma = \sum \times \mathcal{P}$ and correspondingly to partition $\gamma = (x_0,p)$, where $\sum$ is the state space and $\mathcal{P}$ accounts for every parameter which is not an initial state. Even when no state-space model is available, it will sometimes be possible to divide the parameters into "initial conditions" and "everything else." In general, though, we cannot always expect to be able to make this distinction.

If $\mathcal{D}$ is the device, then we can denote that instance of the device with parameter setting γ, and with observations starting at time t_0, as $\mathcal{D}(\gamma,t_0)$. (Initial time might, of course, also be a component of γ; but it will simplify our notation if t_0 is displayed explicitly, even if it already occurs as a component of γ.) Mathematically, $\mathcal{D}(\gamma,t_0)$ is just a set of signals -- i.e., it is a subset of $\mathcal{S}$. It will be convenient for us to call $\mathcal{D}(\gamma,t_0)$ a <u>frame</u> of device $\mathcal{D}$. The set of all admissible signals for device $\mathcal{D}$ is, of course, the set $\{\mathcal{D}(\gamma,t_0): \gamma \in \Gamma\ ,\ t_0 \in \mathbb{R}\}$.

(2.6) <u>Example</u>. Consider the linear capacitor described by

$$\mathcal{D}((q_0,C),t_0) = \{(v,i):Cv(t) = q_0 + \int_{t_0}^{t} i(\tau)d\tau\}$$

Here a signal in $\mathcal{S}$ has two components, the voltage v and current i, and their relationship (i.e., the admissible set of (v,i) pairs) is affected by the parameters q_0 and C. If we suppose that

q_0 is allowed to take any real value, and that -- perhaps via a manual knob setting -- C can take any real value in the range $[C_1, C_2]$, then $\Gamma = \mathbb{R} \times [C_1, C_2]$. (With one special proviso: if $C_1 < 0 < C_2$, then the parameter combinations $q_0 \neq 0$, $C = 0$ should be be excluded from Γ.) If, in some application, it is desirable to make the parameter space a linear space, we can simply embed Γ in the larger space $\hat{\Gamma} = \mathbb{R}^2$. (If this is done, then it is important to insist that $\Gamma \subseteq \hat{\Gamma}$, and that parameters are allowed to take values only in Γ, not in the whole of $\hat{\Gamma}$.)

Ignoring the possibility $C = 0$, for simplicity, there are at least two obvious state-space models for this device:

(i) The equations $\dot{q} = i$ and $v = C^{-1}q$. Here we have $\Sigma = \mathbb{R}$ and $P = [C_1, C_2]$.

(ii) The equations $\dot{q} = i$, $\dot{C} = 0$, and $v = C^{-1}q$. In this case

$\Sigma = \mathbb{R} \times [C_1, C_2]$ and P is the empty set. ¤

(2.7) Example. For the time-varying capacitor

$$\mathcal{D}(q_0, t_0) = \{(v,i): (2+\sin t)v(t) = q_0 + \int_{t_0}^{t} i(\tau)d\tau\}$$

an obvious set of state equations is

$$\dot{q} = i$$

$$v = \frac{1}{2+\sin t} q$$

An equally plausible choice is the second-order (nonlinear but time-invariant) set

$$\dot{q} = i$$
$$\dot{C} = \sqrt{(1-(C-2)^2)}$$
$$v = q/C$$

In this latter case we must of course restrict the initial condition such that $C(t_0) = 2 + \sin t_0$. That is, the set of permissible initial conditions is not the entire state-space. ¤

(2.8) Example. The multi-valued resistor

$$\mathcal{D}(\gamma,t_0) = \{(v,i):v^2(t)+i^2(t)=1 \text{ for all } t\}$$

has no state-space representation in the usual sense. Also, the parameter γ has no effect, so the choice of Γ is irrelevant. For simplicity, we can just pick $\Gamma = \{0\}$. ¤

Examples (2.6) and (2.7) illustrate the point that the choice of Σ and P can be somewhat subjective. The parameter space Γ is more fundamental, because it does not depend on the choice of a state-space. The examples also demonstrate that an initial state can be interpreted as a parameter, or vice-versa. This is the reason why, even when a state-space model is available, we lump x_0 and p together as $\gamma = (x_0,p)$, without making a strong distinction between "initial condition parameters" and "other parameters."

Earlier, a semi-formal definition of the available energy E_A was given. It is now possible to give a formal definition, in a form which does not depend on existence of a state-space.

(2.9) Definition

The available energy for device $\mathcal{D}$ is

$$\boxed{E_A(\gamma,t_0) = \sup_{\substack{s\in\mathcal{D}(\gamma,t_0)\\ t_1\geq t_0}} \{-E(s,t_0,t_1)\}}$$ ¤

Notice that this reduces to the previously given definition, in the case where a state-space model is available and $\Gamma = \Sigma \times P$. Thus, we have a framework which works equally well whether or not a state-space model can be written down. Although γ cannot properly be called a "state," it serves essentially the same purpose.

(2.10) Example. For the capacitor of Example (2.6), a straightforward calculation gives $E(s,t_0,t_1) = \frac{1}{2}Cv^2(t_1) - \frac{1}{2C}q_0^2$ if $C\neq 0$, or zero if C is zero. Therefore

$$E_A((q_0,C),t_0) = \begin{cases} \frac{1}{2C} q_0^2 & , \text{ if } C>0 \\ 0 & , \text{ if } C=0 \\ \infty & , \text{ if } C<0 \end{cases}$$

Example (2.7) is a little more complicated. Consider the case $q_0 = 0$, $t_0 = 0$. If we consider the admissible voltage waveform

$$v(t) = \begin{cases} 0 & , \quad 0 \le t \le \frac{\pi}{2} \\ -\cos t, & \frac{\pi}{2} \le t \le \frac{3\pi}{2} \\ 0 & , \quad \frac{3\pi}{2} \le t \le 2\pi \end{cases}$$

and calculate the corresponding current, the result is $E(s,0,2\pi) = -\frac{2}{3}$. If the same waveform is repeated periodically for N periods, then $E(s,0,2N\pi)$ can be made as negative as desired by taking large enough N. This means that $E_A(0,0) = \infty$. A more detailed analysis will show that $E_A(q_0,t_0) = \infty$ for all q_0 and t_0.
¤

The fact, illustrated above, that $E_A(\gamma,t_0)$ need not be finite (for some or all γ and t_0) should be borne in mind in the following section.

3. <u>ATTAINABILITY</u>

A fundamental concept in state-space theories is the notion of <u>reachability</u> of one state from another. To say that x_1 is reachable from x_0 on the time interval $[t_0,t_1]$ means that there exists a state trajectory $x(t)$ such that $x(t_0) = x_0$ and $x(t_1) = x_1$. (A more precise definition is given in the Appendix.) If we consider the parameter γ in $\mathcal{D}(\gamma,t_0)$ as being comparable to a state, is there any sense in which we can say γ_1 is reachable from γ_0? The immediate answer is no, because γ is a <u>constant</u> in each experiment. There is no $\gamma(t)$ to connect γ_0 to γ_1. That is, we can interpret γ as being like an <u>initial</u> state, but there is no analog of "state trajectory" in the input-output model. However, we shall show that it is possible to define something close to reachability. Before doing this, it is necessary to establish some notation.

(3.1) Notation

For any $s \in S$, $s_{[t_0,t_1]}$ denotes the signal such that

$$s_{[t_0,t_1]}(t) = \begin{cases} s(t) & \text{, for } t_0 \le t \le t_1 \\ 0 & \text{, otherwise} \end{cases}$$

For $s \in S$ and $T \in \mathbb{R}$, $Q_T s$ means† the same as $s_{[T,\infty)}$.

For a set $X \subset S$, $Q_T X$ means $\{Q_T s : s \in X\}$.

(3.2) Notation

For a given signal $\hat{s} \in S$,

$$\mathcal{D}(\gamma,t_0)\Big|\hat{s}_{[t_0,T]} = \{s \in \mathcal{D}(\gamma,t_0) : s(t) = \hat{s}(t)\ \forall\ t \in [t_0,T]\} \qquad ¤$$

That is, $\mathcal{D}(\gamma,t_0)\Big|\hat{s}_{[t_0,T]}$ is that subset (it might be empty) of $\mathcal{D}(\gamma,t_0)$ such that all signals are equal to $\hat{s}$ on the interval $[t_0,T]$, but are otherwise unconstrained. Figure 2 shows a diagrammatic representation, for the slightly oversimplified case where $s(t)$ is a scalar. For some fixed $\gamma_0 \in \Gamma$, the solid lines are intended to represent signals in $\mathcal{D}(\gamma_0,t_0)\Big|\hat{s}_{[t_0,t_1]}$, and the dotted lines are intended to represent all other signals in $\mathcal{D}(\gamma_0,t_0)$. Now, it might well happen that that portion of the solid lines from t_1 onwards coincides with the signals in $\mathcal{D}(\gamma_1,t_1)$ for some $\gamma_1 \in \Gamma$ which may be different from γ_0. This motivates the following definition.

(3.3) Definition

For device $\mathcal{D}$, we say that (γ_1,t_1) <u>is attained from</u> (γ_0,t_0) <u>via</u> $\hat{s}$ iff $Q_{t_1}\mathcal{D}(\gamma_1,t_1) = Q_{t_1}(\mathcal{D}(\gamma_0,t_0)\Big|\hat{s}_{[t_0,t_1]})$ ¤

(3.4) Definition

For device $\mathcal{D}$ and energy measure E, (γ_1,t_1) is <u>attainable</u> from (γ_0,t_0) iff $\exists\ \hat{s} \in \mathcal{D}(\gamma_0,t_0)$ such that (γ_1,t_1) is attained from (γ_0,t_0) via $\hat{s}$, and $E(\hat{s},t_0,t_1)$ is finite. ¤

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† Q_T is often called the <u>anti-causal truncation operator</u>.

If one thinks of γ as the "state" of the device, then this is somewhat like state-space reachability. The precise connection is given in the Appendix. With the aid of this newly introduced concept, we are now able to state one of the most important results of this paper.

(3.5) Theorem. The available energy E_A has the properties

(i) $E_A(\gamma,t_0) \geq 0$

for all (γ,t_0) such that $\mathcal{D}(\gamma,t_0)$ is nonempty.

(ii) For any $t_0,t_1 \in \mathbb{R}$, any $\gamma_0,\gamma_1 \in \Gamma$, and any $\hat{s} \in \mathcal{D}(\gamma_0,t_0)$ such that (γ_1,t_1) is attained from (γ_0,t_0) via $\hat{s}$,

(3.6) $$\boxed{E_A(\gamma_0,t_0) + E(\hat{s},t_0,t_1) \geq E_A(\gamma_1,t_1)}$$ ¤

A proof is given in the Appendix.

Inequality (3.6) is reminiscent of the well-known inequality

$$E_s(x(t_0),t_0) + \int_{t_0}^{t_1} v(t)^T i(t)dt \geq E_s(x(t_1),t_1)$$

for a passive circuit, where $E_s(x,t)$ is the stored energy of the circuit, and $x(t)$ is the state at time t. Suppressing the technical details, which may be found in [1,2,7], we simply note that inequalities of the above form are fundamental to any careful discussion of passivity. The significance of Theorem (3.5) is that, for the first time, it states this dissipation inequality without requiring the existence of a state-space model.

Note that Theorem (3.5) does not require any sort of passivity assumption. It turns out, though, that finiteness of $E_A(\gamma,t)$ will require some sort of passivity property to hold (see Section 4). Obviously inequality (3.6) is interesting and useful only when all terms are finite.

4. DEFINING PASSIVITY

The following definitions are taken from [1,2], with minor and non-significant changes to be consistent with our present notation.

(4.1) Definition

$\mathcal{D}(\gamma,t_0)$ is externally passive iff $E(s,t_0,t_1) \geq 0$ for all $s \in \mathcal{D}(\gamma,t_0)$ and all $t_1 \geq t_0$. ¤

(4.2) Definition

$\mathcal{D}(\gamma,t_0)$ is externally weakly passive iff there exists some finite $\beta \in \mathbb{R}$ such that $E(s,t_0,t_1) + \beta \geq 0$ for all $s \in \mathcal{D}(\gamma,t_0)$ and all $t_1 \geq t_0$. ¤

(4.3) Definition

A state representation $S(p)$ with state-space $\sum$ is internally passive iff $E_A((x,p),t_0) < \infty$ for all $x \in \sum$ and all $t_0 \in \mathbb{R}$. ¤

(4.4) Definition

A relaxed state of $S(p)$ at t_0 is any x^* such that $E_A((x^*,p),t_0) = 0$. ¤

(4.5) Definition

A state representation $S(p)$ with state-space $\sum$ is internally strongly passive if it is internally passive, and there exists some relaxed state $x^* \in \sum$.

In a very crude sense, external weak passivity is comparable to internal passivity, and external passivity is comparable to internal strong passivity. However an exact comparison would be pointless, because the "internal" and "external" versions really refer to different concepts. Definitions (4.1) and (4.2) do not define passivity of a device; they refer only to frames $\mathcal{D}(\gamma,t_0)$ of $\mathcal{D}$. Definitions (4.3) and (4.5) refer to the whole state-space, and therefore to the whole device in the special case $\Gamma = \sum$, but are only applicable when a state-space model is available. Our aim in this section is to supply a passivity definition which applies to $\mathcal{D}$ rather than just one of its frames, and which is applicable whether or not a state-space model is available.

The historical background to Definitions (4.1) and (4.2) is easy to deduce. The inequality $E(s,t_0,t_1) \geq 0$ says that $\mathcal{D}(\gamma,t_0)$ can never deliver more energy to the external world than it has received; this clearly agrees with the generally accepted intuitive notion of "passivity." On the other hand, it takes no account of initially stored energy. The term[1] β in Definition (4.2) allows for the possibility that some energy is stored in $\mathcal{D}(\gamma,t_0)$ at time t_0. Trivially, external passivity implies external weak passivity. A more significant connection between the two definitions is given in the following theorem.

(4.6) Theorem. If $\mathcal{D}(\gamma_0,t_0)$ is externally passive, then $\mathcal{D}(\gamma_1,t_1)$ is externally weakly passive, for every (γ_1,t_1) such that (γ_1,t_1) is attainable from (γ_0,t_0).

Proof. Choose any $\hat{s}$ such that (γ_1,t_1) is attained from (γ_0,t_0) via $\hat{s}$. (The choice is not in general unique, but an arbitrary choice will suffice for the proof.) From Definition (4.1) and Assumption 2.5 it follows that

$$E(s,t_1,t_2) + E(\hat{s},t_0,t_1) \geq 0$$

for all $s \in \mathcal{D}(\gamma_0,t_0)\Big|\hat{s}_{[t_0,t_1]}$. From Definition (3.3), and the fact (Assumptions 2.5 again) that $E(s,t_1,t_2)$ does not depend on $s(t)$ for any $t < t_1$, we can deduce that the same inequality must hold for all $s \in \mathcal{D}(\gamma_1,t_1)$. Finally, note that $E(\hat{s},t_0,t_1)$ is fixed once s has been chosen, so we can set $\beta = E(\hat{s},t_0,t_1)$. ¤

Another important result is the following.

(4.7) Theorem. $\mathcal{D}(\gamma,t_0)$ is externally weakly passive iff $0 \leq E_A(\gamma,t_0) < \infty$, and externally passive iff $E_A(\gamma,t_0) = 0$.

Proof. It is obvious that external weak passivity provides an upper bound for the "sup" in Definition (2.9); in fact we have $E_A(\gamma,t_0) \leq \beta$. For the converse, note that Definition (2.9) implies that $E_A(\gamma,t_0) \geq -E(s,t_0,t_1)$ for any $t_1 \geq t_0$ and any $s \in \mathcal{D}(\gamma,t_0)$. Thus we can set $\beta = E_A(\gamma,t_0)$. For the final assertion of the theorem, simply repeat the above proof for the case $\beta = 0$. ¤

These results suggest that the correct generalization of the "relaxed state" idea is a pair (γ^*,t_0) such that $E_A(\gamma^*,t_0) = 0$. This is certainly a logical extension of the definition of "relaxed operating point" in

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[1] Note that, although β is a constant for each γ and t_0, it will in general vary with γ and t_0.

[8]. If a device $\mathcal{D}$ is to be called passive, then, we ought to have the properties

(i) $\mathcal{D}(\gamma^*,t_0)$ is externally passive, for all relaxed (γ^*,t_0); and

(ii) $\mathcal{D}(\gamma,t_0)$ is externally weakly passive, for all $\gamma \in \Gamma$ and all $t_0 \in \mathbb{R}$.

There is, unfortunately, a complication. A linear circuit always has a relaxed state, but this property does not extend to the general nonlinear case.

(4.8) Example. Consider the capacitor whose voltage-charge characteristic is given by $v = e^q$. That is, we have

$$\mathcal{D}(q_0,t_0) = \{(v,i) : v(t) = \exp(q_0 + \int_{t_0}^{t} i(\tau)\, d\tau)\}$$

The available energy is $E_A(q,t) = e^q$; so $\mathcal{D}(q_0,t_0)$ is externally weakly passive for all $q_0 \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. But there is no state of "zero stored energy." No matter what the initial q_0 is, some finite amount of energy can be extracted from this capacitor. ¤

The capacitor in Example (4.8) should, one could argue, be called passive on the grounds that one can never extract more than a finite amount of energy at its terminals. But there are at least two grounds for suggesting that it should perhaps not be called passive:

(a) How can one synthesize such a device? Whatever method is used, some form of initial energy storage mechanism must be built into it. This runs counter to the idea that one ought to be able to build any passive device, at least in principle, using no energy other than that which is dissipated in a non-electrical form (heat, etc.) during manufacture.

(b) If the capacitor in question is connected in parallel with a 1 ohm linear resistor, the resulting charge tends to $-\infty$ as time increases. This is an example of the result in [6], which states that the stability properties, which are generally believed to be associated with passive circuits, do not extend to the (externally) weakly passive case.

To get around this dilemma, we propose replacing the traditional passive/active distinction by a three-way classification: strongly passive, passive, and active. An active device can supply unlimited energy to the external world. A strongly passive device can supply no energy, other than -- at most -- the energy which was supplied to it in driving it from a relaxed condition. The borderline case, which we simply call passive, is where the device can supply energy to the external world, but only a finite amount. The precise definitions are given below.

(4.9) Definition

$\mathcal{D}$ is passive iff $\mathcal{D}(\gamma,t_0)$ is externally weakly passive for all $\gamma \in \Gamma$ and all $t_0 \in \mathbb{R}$. ¤

(4.10) Definition

$\mathcal{D}$ is active if there exists $\gamma \in \Gamma$ and $t_0 \in \mathbb{R}$ such that $\mathcal{D}(\gamma,t_0)$ is not externally weakly passive. ¤

(4.11) Definition

$\mathcal{D}$ is strongly passive iff, for every $\gamma_1 \in \Gamma$ and $t_1 \in \mathbb{R}$, there exists $\gamma_0 \in \Gamma$ and $t_0 \in \mathbb{R}$ such that (γ_1,t_1) is attainable from (γ_0,t_0), and $\mathcal{D}(\gamma_0,t_0)$ is externally passive. ¤

Notice that, by Theorem (4.6), strong passivity implies passivity.

One feature of the above definitions, which is perhaps not immediately obvious, is illustrated in the following example.

(4.12) Example. Consider the two devices described by

$$\mathcal{D}_1(v_0,t_0) = \{(v,i):i(t) = \frac{dv(t)}{dt} \text{ and } v(t_0) = v_0\}$$

$$\mathcal{D}_2(i_0,t_0) = \{(v,i):i(t) = \frac{dv(t)}{dt} \text{ and } i(t_0) = i_0\}$$

Device $\mathcal{D}_1$ is just a one-Farad capacitor, and it is strongly passive. Its available energy is $E_A(v_0,t_0) = \frac{1}{2} v_0^2$. However the available energy of $\mathcal{D}_2(i_0,t_0)$ is $+\infty$ for any i_0 and t_0; from Theorem (4.7), then, $\mathcal{D}_2$ is active. The reason is that the energy that can be extracted depends on $v(t_0)$, and, for any fixed i_0, $\mathcal{D}_2$ contains (v,i) pairs with arbitrarily large $v(t_0)$. ¤

The above result is not particularly surprising if one notes that whereas $\mathcal{D}_1$ is just a one-Farad capacitor, $\mathcal{D}_2$ is really a $v^{(2)}$-$i^{(1)}$ "higher-order element" [9] having a constitutive relation

$$\frac{di(t)}{dt} = \frac{d^2v(t)}{dt^2}$$

subject to the <u>initial condition</u>:

$$i(t_0) = i_0 \quad v(t_0) = \text{arbitrary}, \quad \left.\frac{dv(t)}{dt}\right|_{t=t_0} = i_0$$

This element can be realized by the <u>linear</u> "active" 1-port shown in Fig. 3. To verify this, note that

$$v_2 = \dot{v}_{c_2} = \ddot{v}_{c_3} = \frac{d^2v}{dt^2}$$

and

$$\frac{di}{dt} = \frac{dv_{c_1}}{dt} = i_{c_1} = -i_2 = v_2$$

Equating these two equations, we obtain the desired constitutive relation.

The point of Example (4.12) is that <u>two devices with the same constitutive relation</u> (here $i = \frac{dv}{dt}$) <u>can be totally different devices</u>. A device is specified in terms of its frames $\mathcal{D}(\gamma,t_0)$, and <u>the parametrization</u> γ <u>forms part of the description of the device</u>.

When a state-space model is available, the above complications do not arise. Suppose that $\Gamma = \Sigma \times P$ where Σ is the state space. It is immediately evident, from Theorem (4.7), that

(a) $\mathcal{D}$ is passive iff $S(p)$ is internally passive for all $p \in P$;

(b) $\mathcal{D}$ is active iff there exists some $p \in P$ such that $S(p)$ is not internally passive.

It should also be clear, from the details given in the Appendix, that $\mathcal{D}$ is strongly passive iff for each $p \in P$: (i) $S(p)$ is internally passive; (ii) there exists at least one relaxed state of $S(p)$; (iii) for every $x_1 \in \Sigma$, there exists $x_2 \in \Sigma$ which is equivalent to x_1 and reachable from a relaxed state. With the obvious observability assumption, condition (iii) simplifies to requiring that every state is reachable from at least one of the relaxed states. If every state is reachable from every other state, then (without requiring any observability assumption) $\mathcal{D}$ is strongly passive iff, for every $p \in P$, $S(p)$ is internally strongly passive.

If the entire state space is reachable from some relaxed state, there is no difficulty in seeing that Definition (4.11) agrees with commonly accepted intuitive ideas about passivity. For an interesting example, then, we need to look at a situation where some states are not reachable from a relaxed state.

(4.13) Example. An ideal diode in series with a 1F linear capacitor can be described by

$$\mathcal{D}(q_0,t_0) = \{(v,i) : v(t) = q(t) \text{ and } i(t) \geq 0, \\ \text{or } v(t) < q(t) \text{ and } i(t) = 0, \\ \text{where } q(t) = q_0 + \int_{t_0}^{t} i(\tau)d\tau\}$$

(A state-space description is also possible, but is a little more complicated). Suppose $\Gamma = \mathbb{R}$; that is, we permit the initial charge q_0 to have any real value. The available energy is easily computed to be

$$E_A(q_0,t_0) = \begin{cases} \frac{1}{2} q_0^2, & \text{if } q_0 < 0 \\ 0, & \text{if } q_0 \geq 0 \end{cases}$$

Note that this is not quite the same as what one would usually call the "stored energy." (If $q_0 > 0$ then some energy is presumably stored, but this can never be verified without breaking open the device; the stored energy cannot be extracted at the terminals). Every $q_0 \geq 0$ is a relaxed state. It is also easy to see that (q_1,t_1) is attainable from (q_0,t_0) iff $q_1 = q_0$ and $t_1 \geq t_0$, or $q_1 > q_0$ and $t_1 > t_0$. Therefore no $q_0 < 0$ is reachable from a relaxed state. The conclusion is that the device is passive, but not strongly passive. ⌑

Is this a reasonable classification? We believe so, for the following reason: it is impossible to make the charge negative by any choice of voltage and/or current at the terminals. The only way to achieve $q_0 < 0$ is to charge the capacitor during construction of the device (i.e., before the diode is placed in series with the capacitor). Electrical energy is necessarily expended during construction of the device, and this is why we choose not to call it strongly passive.

Suppose we disallowed charging of the capacitor before assembly of the device. This would give us a different device, with the same rules for deciding which (v,i) pairs are admissible, but with $\Gamma = \{0\}$ or $\Gamma = \mathbb{R}^+$, depending on precisely what was allowed during construction. In either case the above problem would disappear, and this new device would be called strongly passive.

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APPENDIX

A. Attainability and Reachability

When a state-space description exists, and $\Gamma = \Sigma \times P$, then for each $p \in P$ there exists a state transition function ψ_p and a readout map r_p such that

(A.1) $$x(t) = \psi_p(t, t_0, x(t_0), s_{[t_0,t]})$$

(A.2) $$s(t) = r_p(t, x(t), s(t))$$

where $x(t) \in \Sigma$ is the state at time t. This formulation is a little unconventional in that it is more common to separate the signal s(t) into two components called the input u(t) and output y(t); but this change makes no difference to what follows, it is merely a notational convenience. With $\gamma = (x(t_0), p)$, and some specified $s \in S$, equations (A.1) and (A.2) can be solved for x if and only if $s \in \mathcal{D}(\gamma, t_0)$.

The following two definitions are, modulo a change of notation, completely standard.

(A.3) Definition. Two states $x_1, x_2 \in \Sigma$ are equivalent at t_0 iff $Q_{t_0}\mathcal{D}((x_1,p),t_0) = Q_{t_0}\mathcal{D}((x_2,p),t_0)$ ¤

(A.4) Definition. State x_1 is reachable from x_0 on $[t_0,t_1]$ iff there exists $\hat{s} \in \mathcal{D}((x_0,p),t_0)$ such that $x_1 = \psi_p(t_1, t_0, x_0, \hat{s}_{[t_0,t_1]})$. ¤

Note that equivalence of x_1 and x_2 implies that there is no way, from input-output measurements alone, to tell whether the initial state was x_1 or x_2. An observable state-space model has the property that no two states are equivalent; that is, equivalence of x_1 and x_2 at any t_0 implies that $x_1 = x_2$.

Suppose now that $((x_1,p),t_1)$ is attained from $((x_0,p),t_0)$ via $\hat{s}$. Then

$$\begin{aligned} Q_{t_1}\mathcal{D}((x_1,p),t_1) &= Q_{t_1}(\mathcal{D}((x_0,p),t_0) \,|\, \hat{s}_{[t_0,t_1]}) \\ &= Q_{t_1}\mathcal{D}((x_2,p),t_1) \end{aligned}$$

where $x_2 = \psi_p(t_1, t_0, x_0, \hat{s}_{[t_0,t_1]})$. This means that x_2 is a state which is reachable from x_0 on $[t_0,t_1]$, and equivalent to x_1 at t_1. But equivalent states are "effectively identical" as far as external measurements are concerned. This means that what we have called "attainability" is the closest one can come, in an input-output setting, to the state-space notion of reachability. For observable state representations, attainability is the same as reachability.

B. Proof of Theorem (3.5)

Property (i) of the theorem follows trivially from the definition of E_A and from assumptions 2.5. For property (ii), the definition of E_A ensures that

$$E_A(\gamma_0,t_0) \geq -E(s,t_0,t_2)$$
$$= -E(s,t_0,t_1) - E(s,t_1,t_2)$$

(where the second line follows from the assumptions on $E(\cdot,\cdot,\cdot)$) for all $s \in \mathcal{D}(\gamma_0,t_0)$, whenever $t_0 \leq t_1 \leq t_2$. In particular, then, we have

$$E_A(\gamma_0,t_0) + E(\hat{s},t_0,t_1) \geq -E(s,t_1,t_2)$$

for all $s \in \mathcal{D}(\gamma_0,t_0)\Big|\hat{s}_{[t_0,t_1]}$. We also have

$$E_A(\gamma_1,t_1) = \sup_{\substack{s\in\mathcal{D}(\gamma_1,t_1)\\ t_2 \geq t_1}} \{-E(s,t_1,t_2)\}$$
$$= \sup_{\substack{s\in X\\ t_2\geq t_1}} \{-E(x,t_1,t_2)\},$$

where $X = \mathcal{D}(\gamma_0,t_0)\Big|\hat{s}_{[t_0,t_1]}$. This last replacement is possible because of Definition (3.3) and Assumption 2.5(b). This leads immediately to inequality (3.6). ¤

FIGURE CAPTIONS

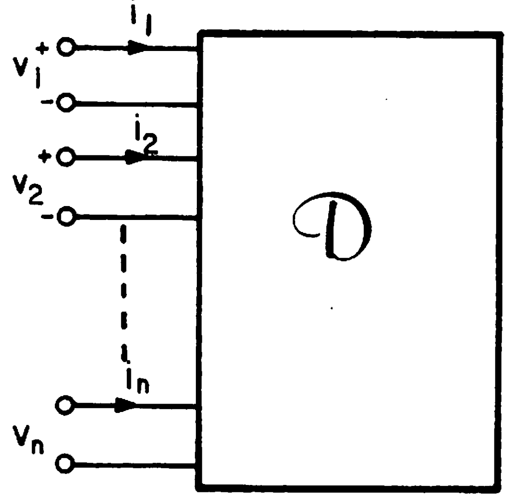
Fig. 1. (a) An electrical n-port.

(b) An (n+1)-terminal device.

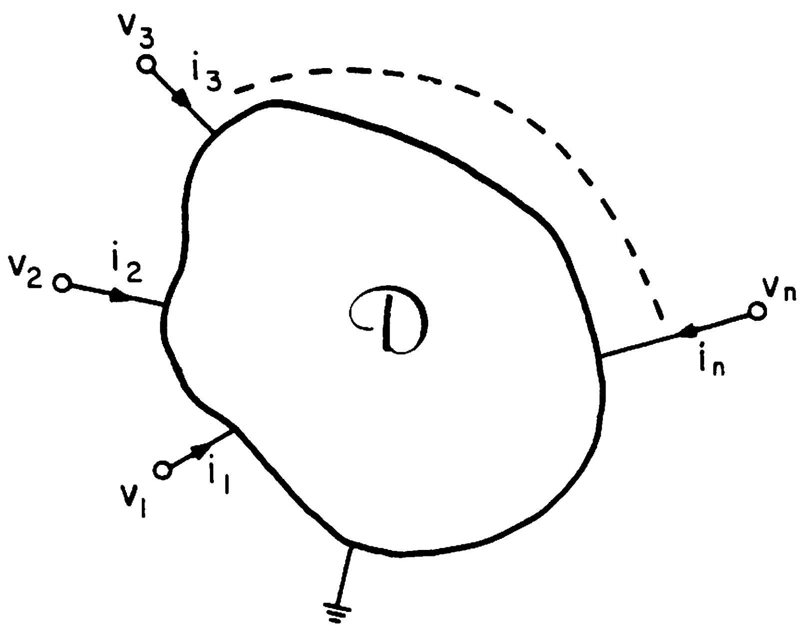
Fig. 2. Illustration of $\mathcal{D}(\gamma_0,t_0)\Big|\hat{s}_{[t_0,t_1]}$

Fig. 3. Circuit for realizing the constitutive relation

$$\frac{di(t)}{dt} = \frac{d^2v(t)}{dt^2} .$$



(a)



(b)

Fig. 1

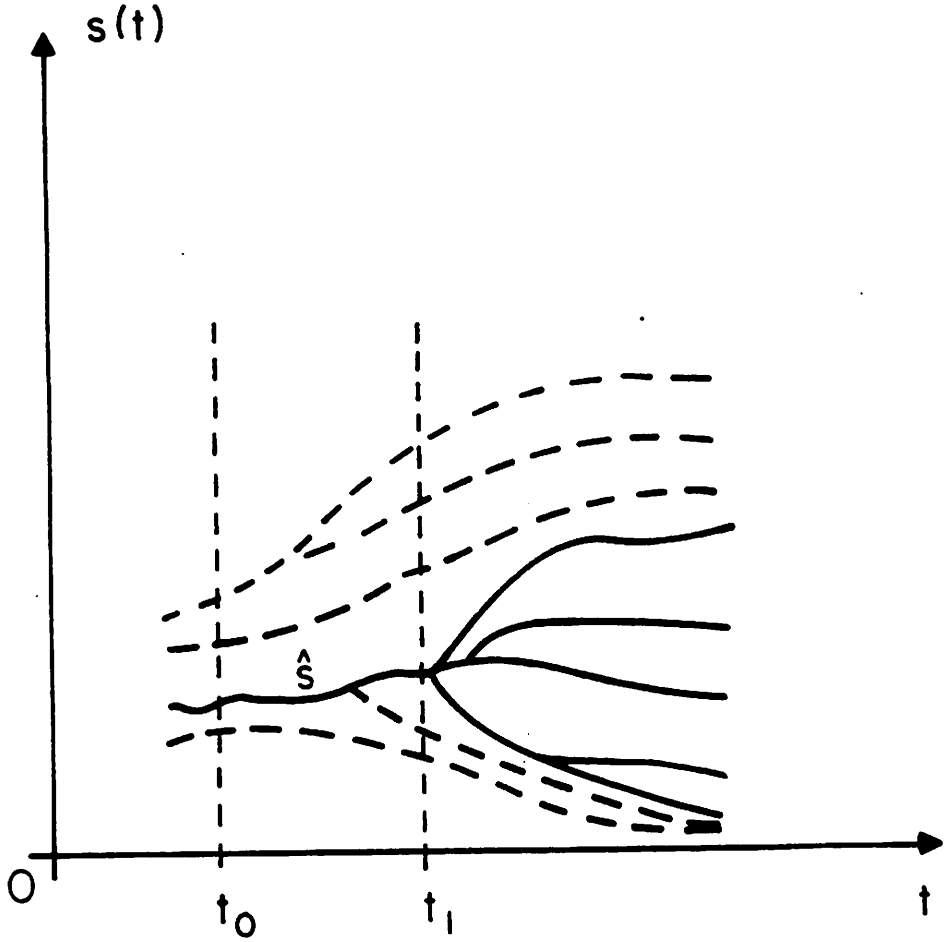


Fig. 2

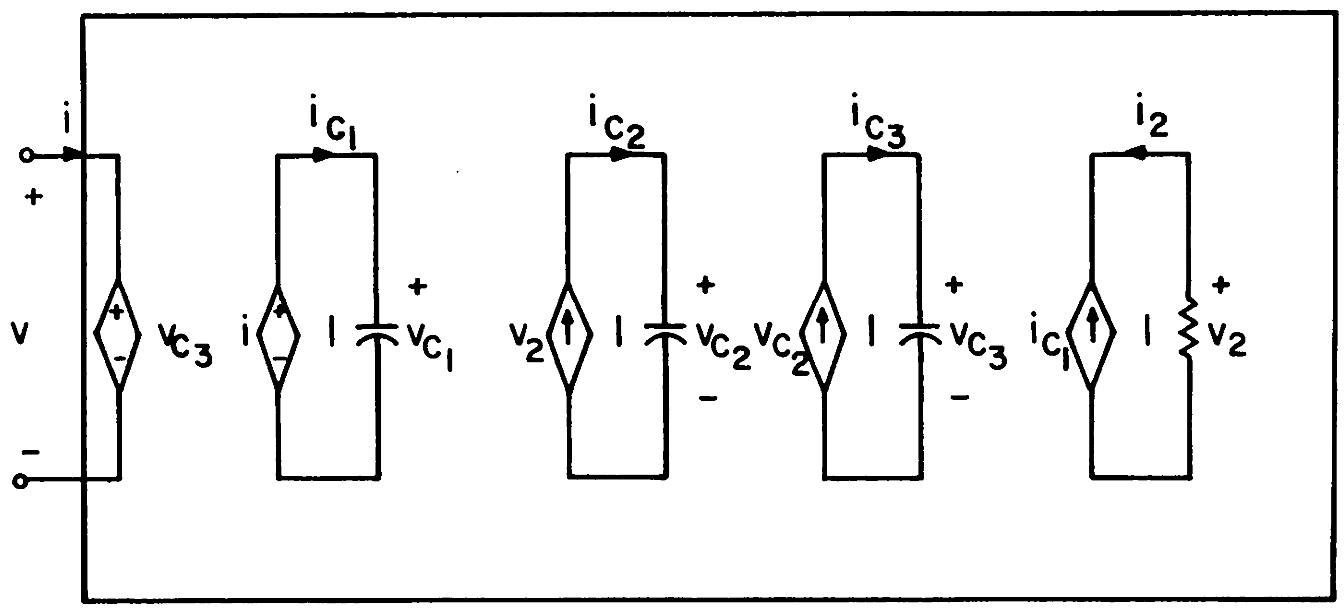


Fig. 3