THE REPRODUCIBILITY OF FUZZY CONTROL SYSTEMS

by

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Memorandum No. UCB/ERL M81/37

9 June 1981

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ABSTRACT

The need for research of reproducibility of fuzzy systems has been established. The definition of reproducibility has been given. The necessary and sufficient conditions of reproducibility of regular fuzzy sets are given. The concept of fuzzy interval has been defined. The theorems stating the necessary and sufficient conditions of reproducibility of fuzzy intervals have also been given. The connection between reproducibility of regular fuzzy sets as well as fuzzy intervals is illustrated by a theorem. The above theorems are illustrated by examples.

Key words: fuzzy sets, fuzzy systems, fuzzy control, fuzzy modelling fuzzy relation, fuzzy intervals

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Research sponsored partially by the National Science Foundation Grant ENG78-23143 and partially by the National Science Foundation Grant MCS75-10376.

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1. INTRODUCTION

One of the basic problems in the theory of fuzzy control is the construction of fuzzy models for objects and controllers. This issue is based on the building of fuzzy relations. In the papers [20,21] the definition of fuzzy relation, the means of fuzzy model constructions and possible applications have been shown.

The problem of fuzzy relation properties with respect to fuzzy controllers has been discussed in [1,16,17,19]. The quality assessment of fuzzy models as well as the technique of such model construction for actual objects have been researched in [13,14,15].

The issue of fuzzy implications, composition rule of inference and fuzzy relations have been investigated in [2,3,5,10,11,12,18].

The good mapping property, stability, sensitivity, controllability and convergence of fuzzy relations in light of analysis and synthesis of fuzzy systems have been studied in [4,6,7,8,9].

On the basis of the results cited above, intuition and experience in designing of controllers and fuzzy models it has been established that in many cases fuzzy relation is an adequate description of an object if it has property of reproducibility with respect to certain classes of fuzzy sets.

In an informal way one can say, that by reproducibility we mean such feature of fuzzy relation which is based on maintenance by output sets of certain properties of fuzzy sets being inputs.

It is known that the form of fuzzy sets which constitute labels of certain physical quantities is connected to their qualitative aspects. For example, if we say that the temperature of boiler is "about 150°C", we mean fuzzy set as on Fig. 1.1a. The current intensity of d.c. motor described as "medium" is illustrated in Fig. 1.1b. The pressure of steam boiler described as "negative big" is shown in Fig. 1.1c. To show that the speed of d.c. motor is equal to "1250 r.p.m." we use fuzzy singleton as on Fig. 1.1d.

Let us notice certain common properties of fuzzy sets shown on Fig. 1.1 which are linguistic description of physical quantitites. All of membership functions are unimodals and achieve the maximal value of one.

If the membership function of a fuzzy set achieves the maximal value of one then such a set we call a regular set.

If the membership function of a fuzzy set is an unimodaled one, then such a set we call a fuzzy interval.

If the relation does not have reproducibility property with respect to a certain class of fuzzy sets though from an intuitive point of view it should have, it can either indicate an improper interpretation of linguistic description of object or errors made during the construction of relation.

One can assume that such a relation will not be a realistic description of the object.

If we take under consideration the system described by the equation $Y = X \cdot R$ and allow that the input and output of such a system are as in Fig. 1.2a then in view of earlier observation the relation R has property of reproducibility with respect to fuzzy intervals and regular sets.

The situation shown on Fig. 1.2b, i.e. input is a fuzzy singleton and output is a fuzzy interval (an especially interesting case from the point of view of applications) is desired phenomenon. If input of system is fuzzy set X_3 and output is Y_3 then relation does not have property of reproducibility with respect to regular sets. Fuzzy set which is not regular set is difficult to interpret linguistically i.e. is quite difficult to prescribe to it a linguistic label naming physical quantity which this set represents. One thinks that there has occured a loss of information and the description of the object is incomplete. The situation in this case becomes even more complicated when irregular set is the input for another system e.g. control action for object. Of course linguistic description of cooperation of objects becomes extremely difficult.

Suppose that for input X_4 (Fig. 1.2d) the system reacted as Y_4. It is clear that input is fuzzy interval and regular set, but the output of system does not have those properties, i.e. fuzzy relation R has reproducibility properties neither with respect to regular sets nor fuzzy intervals. In this case it is quite difficult to find the determined value of fuzzy sets. The search for determined value through the center of area of Figure Y_4 is not convincing. As in the case of irregular sets it is difficult to attribute a linguistic label to a set which is not an interval.

The property of reproducibility of fuzzy intervals is an analogue to continuity as shown by the following

Lemma 0

If a function $f : \mathbb{R} \to \mathbb{R}$ has not discontinuities of second kind then it is continuous if and only if the image of each interval is an interval.

Example 1. The following function has discontinuoities of first kind and image of the interval $[x_0,x_1]$ is the union of disjoint intervals $[y_0,y_1]$ and $[y_2,y_3)$. (Fig. 1.3)

Figure 1.2e shows another example of reproducibility of desired characteristics of input set. Questions concerning other features of system like stability, sensitivity etc. become very doubtful when fuzzy relation R does not have reproducibility property.

The subject of this paper will be fuzzy dynamic systems described by equations $X_{t+1} = X_t \cdot R$, $X_{t+1} = X_t U_t \cdot R$ and fuzzy static systems described by $Y = X \cdot R$; where X_t-fuzzy state at instant t, U_t-fuzzy input at instant t; X,Y-input and output of static system.

There have been given the definitions of reproducibility of regular sets and fuzzy intervals in Section II.

Theorems giving conditions for reproducibility of fuzzy intervals and regular sets and connections between reproducibility of regular sets and fuzzy intervals have been shown in the same section.

II. THE REPRODUCIBILITY PROPERTY OF FUZZY RELATIONS

F_X will denote the family of all fuzzy sets defined on a set X. All fuzzy sets we consider later will be defined on finite sets. To better illustrate theorems we will draw pictures using fuzzy sets defined on infinite sets.

Definition 1

A fuzzy set X is called regular if $\mu_X(x) = 1$ for some $x \in X$. ¤

The family of all regular fuzzy sets on X will be denoted by F_X^R.

Lemma 1

The Cartesian product of fuzzy sets $X_1,\dots,X_n$ is a regular fuzzy set if and only if all fuzzy sets $X_1,\dots,X_n$ are regular.

Proof

If $X = X_1 \times \ldots \times X_n$ is a regular fuzzy set then $\mu_X(x) = 1$ for some $x = (x^1, \ldots, x^n)$.

$$1 = \mu_X(x) = \min(\mu_{X_1}(x^1), \mu_{X_2}(x^2), \ldots, \mu_{X_n}(x^n)).$$

$\mu_{X_k}(x^k) = 1$ for $k = 1,2,\ldots,n$.

Thus all fuzzy sets $X_1, \ldots, X_n$ are regular.

Assume that all fuzzy sets $X_1, \ldots, X_n$ are regular. Then there exist points $x^1, x^2, \ldots, x^n$ such that $\mu_{X_k}(x^k) = 1$ for $k = 1,2,\ldots,n$.

$\mu_X(x) = \mu_X((x^1, x^2, \ldots, x^n)) = \min(\mu_{X_1}(x^1), \mu_{X_2}(x^2), \ldots, \mu_{X_n}(x^n)) = 1$, what implies that X is regular. ¤

Let R be a "maxmin" fuzzy relation represented by a matrix $\{r_{xy}\}$.

$$R : F_X \to F_Y \qquad X \to Y = X \cdot R,\ \mu_Y(y) = \max_{x \in X} \min\{\mu_X(x), r_{xy}\}.$$

Let $F \subset F_X$ and $G \subset F_Y$ be some families of fuzzy sets.

Definition 2

A fuzzy relation R has the reproducibility property with respect to (F,G) (abbreviated as (F,G)-RP) if $X \cdot R \in G$ for all $X \in F$. ¤

Theorem 1

A fuzzy relation R has (F_X^R, F_Y^R)-RP if and only if

$$\max_{y \in Y} r_{xy} = 1 \text{ for all } x \in X. \tag{2.1}$$

Proof

(i) Necessity.

Let us suppose that there exists $x \in X$ such that $\max_{y \in Y} r_{xy} = \alpha < 1$.

The fuzzy singleton $\{x\}$ (denoted also $1/x$) is regular. If $1/x \cdot R = Y$, then $\max_{y \in Y} u_Y(y) = \max_{y \in Y} r_{xy} = \alpha < 1$.

This proves the necessity of condition (2.1).

(ii) Sufficiency.

Let $X \in F_X^R$. There exists $x_0 \in X$ such that $\mu_X(x_0) = 1$.

By condition (2.1) there exists $y_0 \in Y$ such that $r_{x_0 y_0} = 1$.

Let $Y = X \cdot R$. Then

$$\mu_Y(y_0) = \max_{x \in X} \min(\mu_X(x), r_{xy_0}) \geq \min(\mu_X(x_0), r_{x_0 y_0}) = 1$$

Thus Y is regular. ¤

Remark 1

The theorem 1 states sufficient and necessary conditions for the reproducibility of fuzzy regular sets. Each row of the fuzzy matrix R must contain 1 to satisfy this property.

Example 2

Let

$$R = \begin{bmatrix} 1. & .5 & .3 & .1 \\ .9 & .4 & .2 & 1. \\ .6 & 1. & .7 & .0 \\ .3 & .4 & 1. & .2 \end{bmatrix}$$

If for example $X = [.2\ 1.\ .4\ .3]$ then $Y = X \cdot R = [.9\ .4\ .4\ 1.]$ and Y is a regular set. ¤

Let us assume that X bears the structure of linearly ordered set. Let $\leq$ denote the order in X.

Definition 3

A fuzzy set $X \in F_X$ is called a fuzzy interval if $\mu_X(x)$ is a nondecreasing function of x for $x \leq x_0$ and a nonincreasing function of x for $x_0 \leq x$, where x_0 is some point in X.

Remark 2

A fuzzy set representing classic interval is also a fuzzy interval.

Example 3

A fuzzy interval is shown in Fig. 2.1.

Let $X = X_1 \times \ldots \times X_n$, where $\{X_k\}$ are linearly ordered sets. The orders will be denoted by the same sign $\leq$. If $x_0, x_1 \in X$ then the set of all points $x = (x^1, x^2, \ldots, x^n) \in X$ such that x^k lies between x_0^k and x_1^k for all k (i.e. either $x_0^k \leq x^k \leq x_1^k$ or $x_1^k \leq x^k \leq x_0^k$) will be denoted $\overline{x_0, x_1}$. $\overline{x_0, x_1}$ is an n-dimensional interval in classic sense.

Definition 4

A fuzzy set $X \in F_X$ is called a fuzzy n-dimensional interval (or simply a fuzzy interval) if for all $x_0, x_1 \in X$ and all $x \in \overline{x_0, x_1}$ we have $\mu_X(x) \geq \min(\mu_X(x_0), \mu_X(x_1))$.

Example 4

A fuzzy 2-dimensional interval is shown in Fig. 2.2.

Remark 3

It is easy to see that Definition 3 is a special case of Definition 4.

Remark 4

A fuzzy set represeting a classic n-dimensional interval is a fuzzy n-dimensional interval.

The families of all fuzzy n-dimensional intervals and all n-dimensional intervals will be denoted by F_X^I and F_X^i, respectively.

Lemma 2

The Cartesian product of fuzzy intervals is a fuzzy interval.

Proof

Let $X_1, X_2, \ldots, X_n$ be fuzzy intervals and $X = X_1 \times X_2 \times \ldots \times X_n \in F_X$. Let also $x_0, x_1, x \in F_X$ and $x \in \overline{x_0, x_1}$.

By definition of $\overline{x_0, x_1}$ and assumption of the lemma we have $\mu_{X_k}(x^k) \geq \min(\mu_{X_k}(x_0^k), \mu_{X_k}(x_1^k))$ for all k.

By definition of Cartesian product of fuzzy sets we obtain

$$\mu_X(x) = \min_k(\mu_{X_k}(x^k)) \geq \min_k(\min(\mu_{X_k}(x_0^k),\mu_{X_k}(x_1^k))$$
$$= \min(\min_k(\mu_{X_k}(x_0^k)), \min_k(\mu_{X_k}(\mu(x_1^k))) = \min(\mu_X(x_0),\mu_X(x_1)) \quad ¤$$

Remark 5

All fuzzy intervals involved in Lemma 2 can be taken fuzzy multi-dimensional intervals.

Remark 6

The converse to Lemma 2 is also true: an n-dimensional interval is a product of some fuzzy intervals. ¤

Let $X = X_1 \times \ldots \times X_n$ be the Cartesian product of ordered sets, $\mathcal{Y}$-an ordered set and let $R : F_X \to F_{\mathcal{Y}}$ be a "maxmin" fuzzy relation.

Lemma 3

A fuzzy relation R has the $(F_X^I, F_{\mathcal{Y}}^I)$-RP if and only if it has $(F_X^i, F_{\mathcal{Y}}^I)$-RP.

Proof

Necessity is obvious since $F_X^i \subset F_X^I$.

To prove sufficiency let us assume that there exists a fuzzy interval $X \in F_X^I$ such that $Y = X \cdot R \notin F_{\mathcal{Y}}^I$.

There exist y_0, y_1 and $y \in \mathcal{Y}$ such that $y_0 \leq y \leq y_1$ and $\mu_Y(y) < \min(\mu_Y(y_0),\mu_Y(y_1)) = \alpha$.

It implies existance of $x_0, x_1 \in X$ such that

$\min(\mu_X(x_0), r_{x_0y_0}) \geq \alpha$ and $\min(\mu_X(x_1), r_{x_1y_1}) \geq \alpha$.

$\mu_X(x_0) \geq \alpha$ and $\mu_X(x_1) \geq \alpha$.

X is a fuzzy interval so $\mu_X(x) \geq \alpha$ for all $x \in \overline{x_0, x_1}$.

$$\alpha > \mu_Y(y) = \max_{x \in X} \min(\mu_X(x), r_{xy}) \geq \max_{x \in \overline{x_0,x_1}} \min(\mu_X(x), r_{xy})$$
$$\geq \max_{x \in \overline{x_0,x_1}} \min(\alpha, r_{xy}) = \min(\alpha, \max_{x_0 \in \overline{x_0,x_1}} r_{xy}).$$

The last inequality can be satisfied only if $\max_{x_0 \in \overline{x_0,x_1}} r_{xy} < \alpha$.

Let $Y_1 = \overline{x_0,x_1} \cdot R$.

$\mu_{Y_1}(y_0) = \max_{x \in X} \min(\mu_{\overline{x_0,x_1}}(x_0), r_{xy_0}) = \max_{x \in \overline{x_0,x_1}} r_{xy_0} \geq r_{x_0y_0} \geq \alpha$.

Similarly $\mu_{Y_1}(y_1) \geq \alpha$.

$$\mu_{Y_1}(y) = \max_{x \in X} \min(\mu_{\overline{x_0,x_1}}(x), r_{xy}) = \max_{x \in \overline{x_0,x_1}} r_{xy} < \alpha.$$

Thus $Y_1 \notin F_y^I$, $Y_1 = \overline{x_0,x_1} \cdot R$ and $\overline{x_0,x_1} \in F_X^i$. The assumption that R has not (F_X^I,F_y^I)-RP leads us to the conclusion that R has not (F_X^i,F_y^I)-RP. ¤

Before formulation of next theorem we need some more definitions and notation.

Let $X = X_1 \times \ldots \times X_n$ and $X_0 \subset X$.

<u>Definition 5</u>

Points $x_0, x_1 \in X_0$ are called X_0-close ($x_0 \sim x_1 (\mathrm{mod}\ X_0)$) if $\overline{x_0,x_1} \cap X_0 = \{x_0,x_1\}$.

<u>Example 5</u>

In this case we have

$X_0 = \{x_0,x_1,x_2,x_3\}$

$x_0 \sim x_1(\mathrm{mod}\ X_0)$

$x_1 \nsim x_3(\mathrm{mod}\ X_0)$. (Fig. 2.3) ¤

Let R be a fuzzy relation, $R : F_X \to F_y$, where $X = X_1 \times \ldots \times X_n$. $X_1, X_2, \ldots, X_n$ and Y are linearly ordered.

Let us denote $\bar{r}_x = \max_{y \in Y} r_{xy}$. The set of all numbers $\bar{r}_x$, $x \in X$ is finite.

Let they be called $r^1, r^2, \ldots, r^k$ and assume that $r^1 > r^2 > \ldots > r^k$.

Now for each $s = 1,2,\ldots,k$, let $X^{(s)}$ be the set of all points $x \in X$ such that $\bar{r}_x \geq r^s$.

Theorem 2

A fuzzy relation R has the (F_X^I, F_Y^I)-RP if and only if

(i) for all $x \in X$ the fuzzy set $Y \in F_y$ defined by

$$\mu_Y(y) = r_{xy} \tag{2.2}$$

is a fuzzy interval and

(ii) for all $s = 1,2,\ldots,k$ and all $x_0, x_1 \in X^{(s)}$ which are $X^{(s)}$-close the fuzzy set $Y \in F_y$ defined by

$$\mu_Y(y) = \max(r_{x_0 y}, r_{x_1 y}) \tag{2.3}$$

is a fuzzy interval.

Example 6

In order to check if a fuzzy relation R has the (F_X^I, F_Y^I)-RP one should check whether images of fuzzy singletons (e.g. X_1) are fuzzy intervals and images of sets defined in part (ii) of theorem 2 (e.g. X_2) are also fuzzy intervals. (see Fig. 2.4)

Remark 7

If condition (i) is satisfied then (ii) is equivalent to the following condition

(iii) for all $s = 1,2,\ldots,k$ and $x_0, x_1 \in X^{(s)}$ which are $X^{(s)}$-close the fuzzy set $Y_1 \in F_y$ defined by

$$\mu_{Y_1}(y) = \begin{cases} r^s & \text{if } \max(r_{x_0 y}, r_{x_1 y}) \geq r^s \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

is a fuzzy interval.

Proof

(ii) $\Rightarrow$ (iii)

If there exist $y_0 \leq y \leq y_1$ with $\mu_{Y_1}(y) < \min(\mu_{Y_1}(y_0), \mu_{Y_1}(y_1))$ then $\mu_Y(y) < r^s = \min(\mu_{Y_1}(y_0), \mu_{Y_1}(y_1)) \leq \min(\mu_Y(y_0), \mu_Y(y_1))$.

(iii) $\Rightarrow$ (ii)

If there exist $y_0 \leq y^* \leq y_1$ with $\mu_Y(y^*) < \min(\mu_Y(y_C), \mu(y_1))$ then by (i) and assumption $x_0, x_1 \in X^{(s)}$ the maximum of $r_{x_0 y}$ and $r_{x_1 y}$ with respect to y are attained on opposite sides of y^*, at y_2 and y_3, say. Then $\mu_{Y_1}(y_2) = \mu_{Y_1}(y_3) = r^s$ and $\mu_{Y_1}(y^*) = 0$. ¤

To prove theorem 2 we will need the following

Lemma 4

If $x_0, x_1 \in X_0 \subset X$ then there exists a set of points (maybe empty) $z_1, z_2, \ldots, z_m \in X_0$ such that $x_0 \sim z_1 (\mathrm{mod}\ X_0)$, $z_1 \sim z_2 (\mathrm{mod}\ X_0), \ldots$, $z_m \sim x_1 (\mathrm{mod}\ X_0)$. Moreover, $z_1, z_2, \ldots, z_m$ can be chosen from $X_0 \cap \overline{x_0, x_1}$.

Proof

If x_0 is not X_0-close to x_1 then $\overline{x_0, x_1} \cap X_0 = X_1 \neq \{x_0, x_1\}$. It is easy to see that if $x_3 \neq x_4$ and $x_4 \in \overline{x_0, x_3}$ then $x_3 \notin \overline{x_0, x_4}$. Thus, by finiteness of X_1 there exists z_1 such that $\overline{x_0, z_1} \cap X_1 = \{x_0, z_1\}$.

We can construct by induction a sequence of points $z_1, z_2, \ldots$, such that $\overline{z_s, z_{s+1}} \cap (\overline{z_s, x_1} \cap X_0) = \{z_s, z_{s+1}\}$.

It is not hard to see that the induction process must terminate and z's constitute the desirable set of points. ¤

Proof of theorem 2

Necessity.

At first assume that (i) is not satisfied, i.e. there exists $x_0 \in X$ such that Y defined by (2.2) is not a fuzzy interval. $1/x_0$ is a fuzzy interval and $1/x_0 \cdot R = Y$. This proves the necessity of (i).

Now assume that (iii) is not satisfied. (Notice Remark 7). Let $x_0, x_1 \in X^{(s)}$, $x_0 \sim x_1 (\mathrm{mod}\ X^{(s)})$ and let Y_1 defined by (2.4) be not a fuzzy interval.

We will show that $Y_2 = \overline{x_0, x_1} \cdot R \notin F^I_{\mathcal{Y}}$.

Let $y_0, y, y_1 \in \mathcal{Y}$, $y_0 \le y \le y_1$ and

$$0 = \mu_{Y_1}(y) < \min(\mu_{Y_1}(y_0), \mu_{Y_1}(y_1)) = r^s.$$

$$\mu_{Y_2}(y_0) = \max_{x \in X} \min(\mu_{\overline{x_0,x_1}}(x), r_{xy_0}) = \max_{x \in \overline{x_0,x_1}} r_{xy_0}$$

$$\ge \max(r_{x_0 y_0}, r_{x_1 y_0}) \ge \mu_{Y_1}(y_0) = r^s.$$

Similarly $\mu_{Y_2}(y_1) \ge r^s$.

$$\mu_{Y_2}(y) = \max_{x \in X} \min(\mu_{\overline{x_0,x_1}}(x), r_{xy}) = \max_{x \in \overline{x_0,x_1}} r_{xy}$$

$$= \max(\max_{x \in \overline{x_0,x_1} \setminus \{x_0,x_1\}} r_{xy}, r_{x_0 y}, r_{x_1 y})$$

$$\le \max(\max_{x \in \overline{x_0,x_1} \setminus \{x_0,x_1\}} \bar{r}_x, r_{x_0 y}, r_{x_1 y})$$

$$\le \max(r^{s+1}, \max(r_{x_0 y}, r_{x_1 y})) < \max(r^s, r^s) = r^s.$$

This implies that $Y_2 \notin F^I_{\mathcal{Y}}$.

Sufficiency.

Let us assume that there exists $X \in F^I_X$ and $Y_1 = X \cdot R \notin F^I_{\mathcal{Y}}$.

Let $y_0, y_1, y \in \mathcal{Y}$, $y_0 \le y \le y_1$ be such that

$$\mu_{Y_1}(y) < \min(\mu_{Y_1}(y_0), \mu_{Y_1}(y_1)) = \alpha.$$

$$\alpha \le \mu_{Y_1}(y_0) = \max_{x \in X} \min(\mu_X(x), r_{xy_0})$$

$$= \min(\mu_X(x_0), r_{x_0 y_0}) \le r_{x_0 y_0} \text{ for suitable } x_0.$$

Similarly $\alpha \le r_{x_1 y_1}$ for suitable $x_1 \in X$.

$r^s = \min(\bar{r}_{x_0}, r_{x_1}) \geq \min(r_{x_0y_0}, r_{x_1y_1}) \geq \alpha$, for some s. $x_0, x_1 \in X^{(s)}$.

$$\alpha > \mu_{Y_1}(y) = \max_{x\in X} \min(\mu_X(x), r_{xy}) \geq \max_{x\in \overline{x_0,x_1}} \min(\mu_X(x), r_{xy})$$

$$\geq \max_{x\in \overline{x_0,x_1}} \min(\min(\mu_X(x_0), \mu_X(x_1)), r_{xy})$$

$$\geq \max_{x\in \overline{x_0,x_1}} \min(\alpha, r_{xy}) = \min(\alpha, \max_{x\in \overline{x_0,x_1}} r_{xy}).$$

It follows that

$$\max_{x\in \overline{x_0,x_1}} r_{xy} < \alpha \tag{2.5}$$

Let $z_1, z_2, \ldots, z_m$ be picked as in Lemma 4. $z_1, z_2, \ldots, z_m \in X^{(s)}$. The maximum of characteristic function of any of fuzzy sets corresponding to $x_0, z_1, \ldots, z_k, x_1$ by (2.2) is not less than $r^s \geq \alpha$.

But the values of these characteristic functions computed at y are all less than α by (2.5). Thus the maxima can be attained either to the left or to the right from y.

It is easy to see that the maxima of r_{x_0u} and r_{x_1u} (with respect to u) are attained at $y_3 < y$ and $y_4 > y$ respectively.

There exist two points among $x_0, z_1, \ldots, x_1$ which are $X^{(s)}$-close and such that the maxima of the characteristic functions of the fuzzy sets corresponding to them are attained on the opposite sides of y. The fuzzy set corresponding to this pair of points by (2.3) is not a fuzzy interval. This contradiction proves sufficiency of conditions (i) and (ii). ¤

The condition (ii) of theorem 2 is complicated in this general setting. It becomes, however, much simpler in some special cases, which seem to be interesting and important for applications.

Example 7

Let

$$R = \begin{bmatrix} 1. & .5 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ .5 & .5 & .5 & .5 & 0. & 0. & 0. & 0. & 0. \\ 0. & .5 & 1. & .5 & 0. & 0. & 0. & 0. & 0. \\ 0. & .5 & .5 & .5 & .5 & .5 & 0. & 0. & 0. \\ 0. & 0. & 0. & .5 & 1. & .5 & .5 & 0. & 0. \\ 0. & 0. & 0. & .5 & .5 & 1. & .5 & 0. & 0. \\ 0. & 0. & 0. & 0. & .5 & .5 & .5 & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix}$$

Suppose that X = [.8 1. .8 .0 .0 .0 .0 .0 .0]. It is a regular set and a fuzzy interval. Then Y = X • R = [.8 .5 .8 .5 .0 .0 .0 .0 .0] is neither fuzzy interval nor regular set. Relation R has the reproducibility property neither with respect to fuzzy intervals nor regular sets.

Example 8

Let

$$R = \begin{bmatrix} 1. & .5 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ .5 & .5 & .5 & .5 & .5 & 0. & 0. & 0. & 0. \\ 0. & 1. & 1. & .5 & 0. & 0. & 0. & 0. & 0. \\ 0. & .5 & .5 & .5 & .5 & .5 & 0. & 0. & 0. \\ 0. & 0. & .5 & 1. & 1. & .5 & .5 & 0. & 0. \\ 0. & 0. & 0. & .5 & .5 & 1. & .5 & 0. & 0. \\ 0. & 0. & 0. & 0. & .5 & .5 & .5 & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix}$$

Let X be as above (ex. 7).

Then Y = X • R = [.8 .8 .8 .5 .0 .0 .0 .0 .0] is a fuzzy interval, but is not a regular set. Relation does not have the reproducibility property with respect to regular sets, but it reproduces fuzzy intervals. ¤

Two points $x_0, x_1 \in X = X_1 \times \ldots \times X_n$ will be called adjacent if $\overline{x_0, x_1} = \{x_0, x_1\}$.

Theorem 3

If a fuzzy relation R has the (F_X^R, F_y^R)-RP then it has (F_X^I, F_y^I)-RP if and only if

(i) for all $x \in X$ the fuzzy set $Y \in F_y$ defined by

$$\mu_Y(y) = r_{xy}$$

is a fuzzy interval and

(ii) the fuzzy set Y defined by

$$\mu_Y(y) = \max(r_{x_0 y}, r_{x_1 y})$$

if a fuzzy interval for all pairs x_0, x_1 of adjacent points.

Example 9

Sets X_1 and X_2 correspond to pairs of adjacent points and are the examples of inputs for which one need to check if their images are fuzzy intervals (see Fig. 2.5).

Proof of Theorem 3

By Theorem 1 $\max\limits_{x \in X} r_{xy} = 1$ for all $x \in X$. Thus k = 1 and the condition (ii) of Theorem 2 must be checked for all x_0, x_1 X-close. It is easy to see that x_0 and x_1 are X-close if and only if x_0 and x_1 are adjacent. ¤

Remark 8

If X is a set of L elements, linearly ordered then there are L-1 pairs of adjacent points. If R has the (F_X^R, F_y^R)-RP then in order to check whether it has (F_X^I, F_y^I)-RP one needs to check if 2L-1 fuzzy sets described by conditions (i) and (ii) of theorem 3 are transformed by R to fuzzy intervals.

Example 10

Let

$$
R = \begin{bmatrix}
1. & .5 & .5 & .0 & .0 & .0 & .0 & .0 & .0 \\
.5 & 1. & 1. & .5 & .0 & .0 & .0 & .0 & .0 \\
0. & 0. & 1. & 1. & .5 & .5 & 0. & 0. & 0. \\
0. & 0. & 0. & .5 & 1. & .5 & 0. & 0. & 0. \\
0. & 0. & .5 & 1. & 0. & 0. & 0. & 0. & 0. \\
0. & .5 & 1. & 1. & .5 & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & .5 & 1. & .5 & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 1. & 1. & .5 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & .5 & 1. & .5 & 0.
\end{bmatrix}
$$

Suppose that X = [.0 .0 .2 .9 .8 .7 .2 .1 .0], then

$Y = X \circ R$ = [.0 .5 .7 .8 .9 .5 .1 .0 .0] is a regular set and a fuzzy interval.

Relation has the reproducibility property with respect to regular sets and fuzzy intervals.

Example 11

Consider the process $X_{t+1} = X_t U_t \cdot R$, where

$$
R = \quad
\begin{bmatrix}
0. & .3 & .2 & .2 & 0. & 0. \\
.3 & .7 & .2 & .7 & .8 & 0. \\
.7 & .6 & .7 & 1. & .6 & 0. \\
0. & 0. & 1. & 5. & .2 & 0.
\end{bmatrix}
\begin{bmatrix}
.0 & 1. & .2 & 0. & 0. & 0. \\
.7 & 1. & .8 & 1. & 1. & 0. \\
.7 & 1. & 1. & 1. & .7 & 0. \\
0. & 1. & .7 & 1. & .7 & 0.
\end{bmatrix}
\begin{bmatrix}
.4 & 1. & 1. & .8 & .2 & .2 \\
1. & .6 & 1. & 1. & 1. & .4 \\
1. & .6 & .7 & .6 & .7 & .1 \\
.2 & 1. & .5 & .5 & 1. & .4
\end{bmatrix}
\begin{bmatrix}
.7 & 1. & .7 & 1. & .8 & .2 \\
1. & .4 & 1. & .7 & .1 & .7 \\
1. & .4 & .3 & .2 & .5 & 1. \\
.7 & 1. & .2 & .2 & 1. & .7
\end{bmatrix}
\begin{bmatrix}
1. & .7 & .4 & .8 & 1. & .8 \\
.8 & .2 & 1. & .3 & .5 & 1. \\
.7 & .0 & .2 & .0 & .2 & .7 \\
1. & .7 & .0 & .0 & .2 & 1.
\end{bmatrix}
\begin{bmatrix}
1. & .0 & .0 & .2 & .8 & 1. \\
.4 & .0 & .8 & .2 & .2 & .7 \\
.4 & .0 & .0 & .0 & .0 & .0 \\
1. & .2 & .0 & .0 & .0 & 1.
\end{bmatrix}
$$

Suppose that X_t = [.0 .0 .0 .0 1. .0] and U_t = [.0 .0 1. 0.] are fuzzy singletons, then

$$X_tU_t = \begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix}$$

Therefore X_{t+1} = [.0 .2 .5 .7 .7 .6] is a fuzzy interval, but is not a regular set.

This 3-dimensional relation has the reproducibility property with respect to fuzzy intervals, but it does not have the reproducibility property with respect to regular sets.

Example 12

Let

$$R = \begin{bmatrix} 1. & .0 & .0 & .2 & .8 & 1. \\ .4 & .0 & .8 & .2 & .2 & .7 \\ .4 & .0 & .0 & .0 & .0 & .0 \\ 1. & .2 & 0. & 0. & 0. & 1. \end{bmatrix} \begin{bmatrix} 1. & .7 & .4 & .8 & 1. & .8 \\ .8 & .2 & 1. & .3 & .5 & 1. \\ .7 & .0 & .2 & .0 & .2 & .7 \\ 1. & .7 & .0 & .0 & .2 & .1 \end{bmatrix} \begin{bmatrix} .7 & 1. & .7 & 1. & .8 & .2 \\ 1. & .4 & 1. & .7 & 1. & .7 \\ 1. & .4 & .3 & .2 & .5 & 1. \\ .7 & 1. & .2 & .2 & 1. & .7 \end{bmatrix} \begin{bmatrix} .4 & 1. & 1. & .8 & .2 & .2 \\ 1. & .6 & 1. & 1. & 1. & .4 \\ 1. & .6 & .7 & .6 & 1. & .7 \\ .2 & 1. & .5 & .5 & 1. & .4 \end{bmatrix} \begin{bmatrix} 0. & .8 & 1. & .2 & 0. & 0. \\ .7 & 1. & .8 & 1. & 1. & 0. \\ .7 & 1. & 1. & .7 & 1. & 0. \\ .0 & 1. & .7 & 1. & .7 & 0. \end{bmatrix} \begin{bmatrix} .0 & .3 & .2 & .2 & 0. & 0. \\ .3 & .7 & .2 & .7 & .8 & 0. \\ .7 & .6 & .7 & 1. & .6 & 0. \\ .0 & .0 & 1. & .5 & .2 & 0. \end{bmatrix}$$

Suppose that X_t and U_t are as above (ex. 11). Then $X_{t+1} = X_tU_t \cdot R$ = [.0 .2 .5 1. 1. .6] is a fuzzy interval and a regular set.

This relation reproduces fuzzy intervals and regular sets.

III. Concluding Remarks

Intention of this paper is to provide the method of checking correctness of fuzzy relation being the model for a real system.

The algebraic criteriae of theorems given in this article allow us to test if a fuzzy relation proposed by designer has the reproducibility property with respect to fuzzy intervals and regular sets. Those criteriae should be applied to test the relation correctness when intuition or experience are prompting that real system has appropriate properties.

The authors suggest that fuzzy relation describing the object characterized by a continuous mapping of input to output should have the reproducibility property with respect to fuzzy intervals.

IV. Acknowledgements

We would like to express our very warm thanks for the scientific attention given to us in this study by Professor L. A. Zadeh. and to thank R. M. Tong for interesting discussion.

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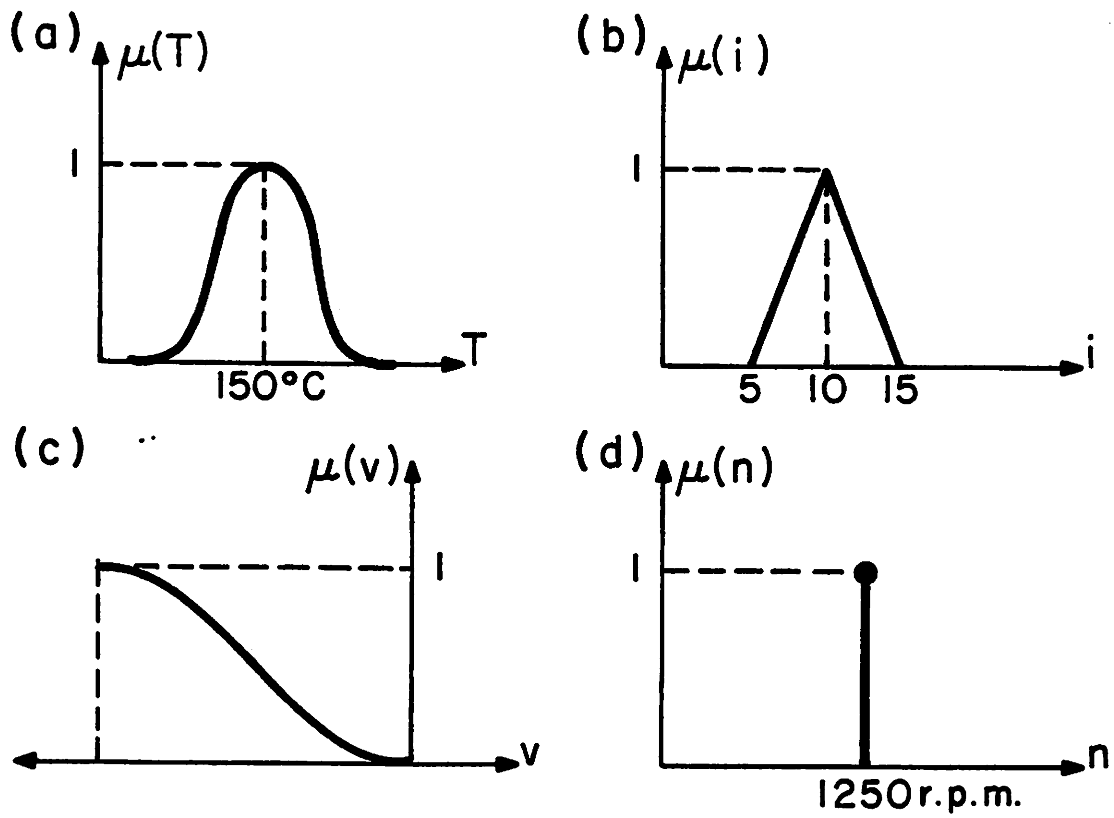


Fig. 1.1

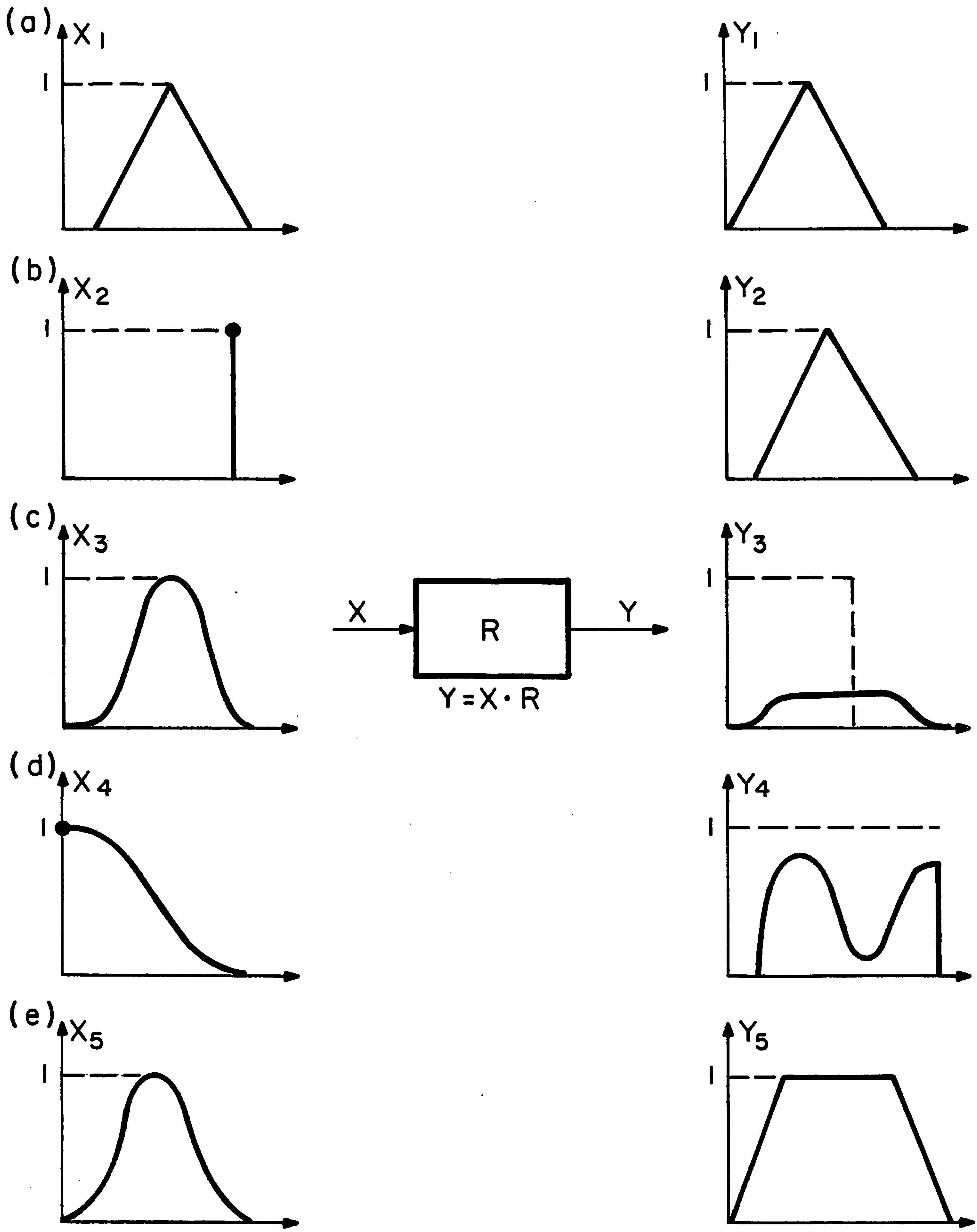
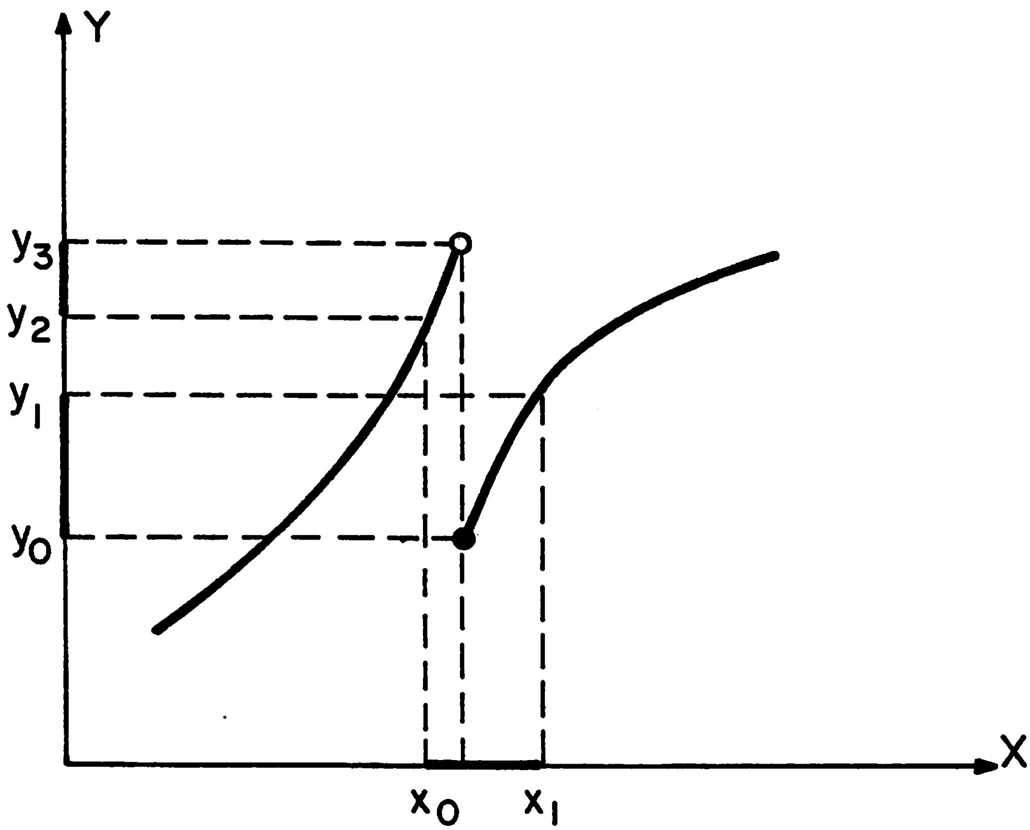


Fig. 1.2



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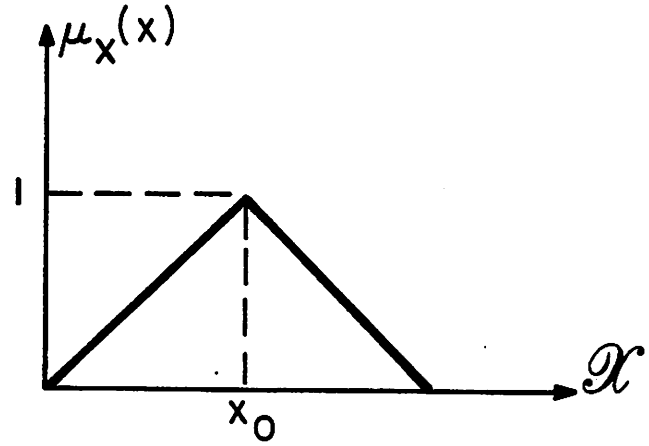


Fig. 2.1

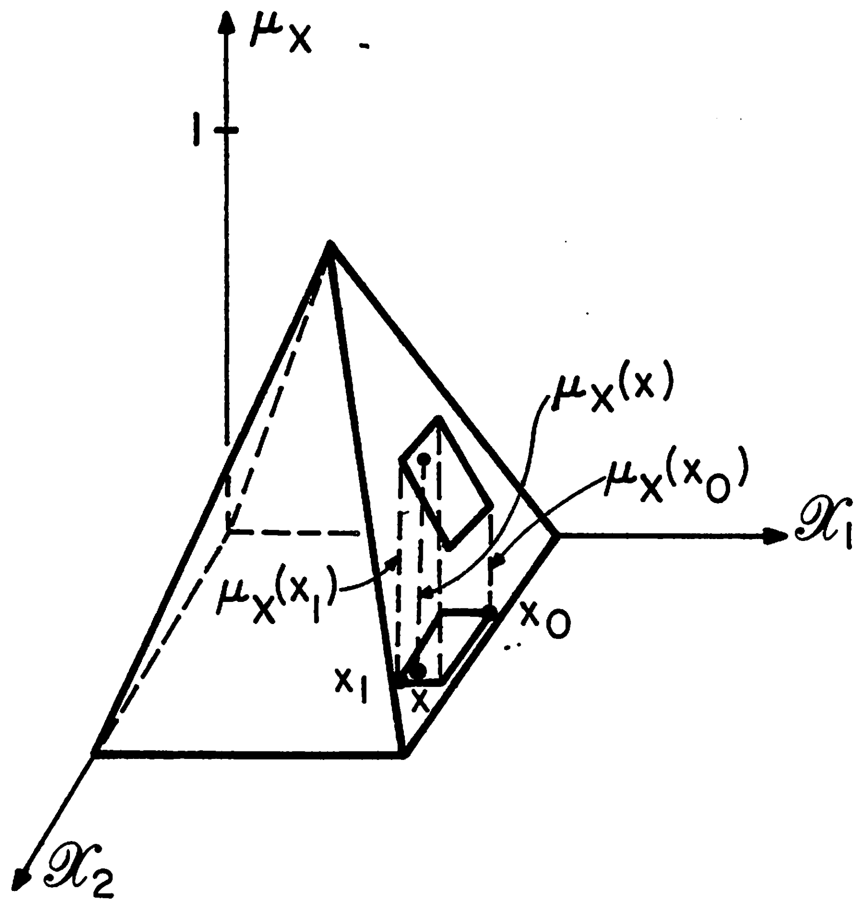


Fig. 2.2

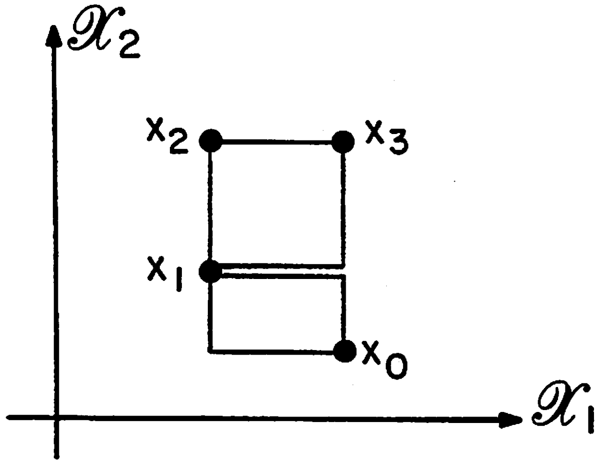
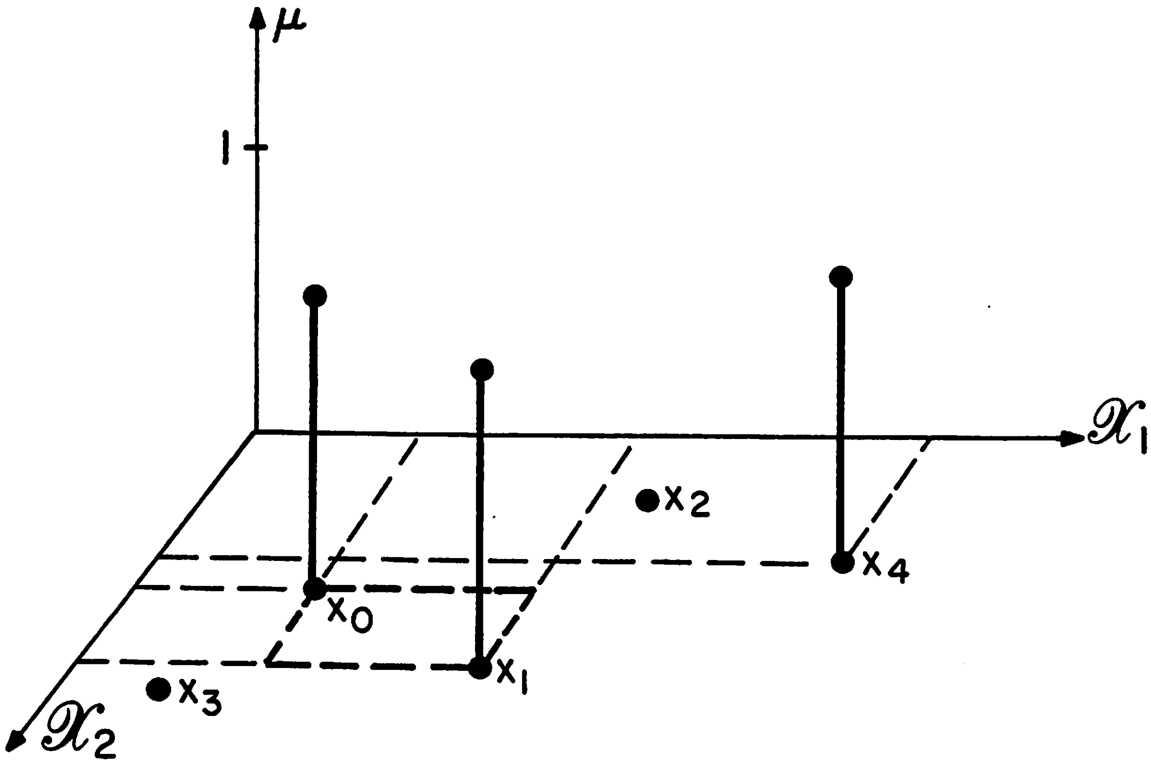


Fig. 2.3



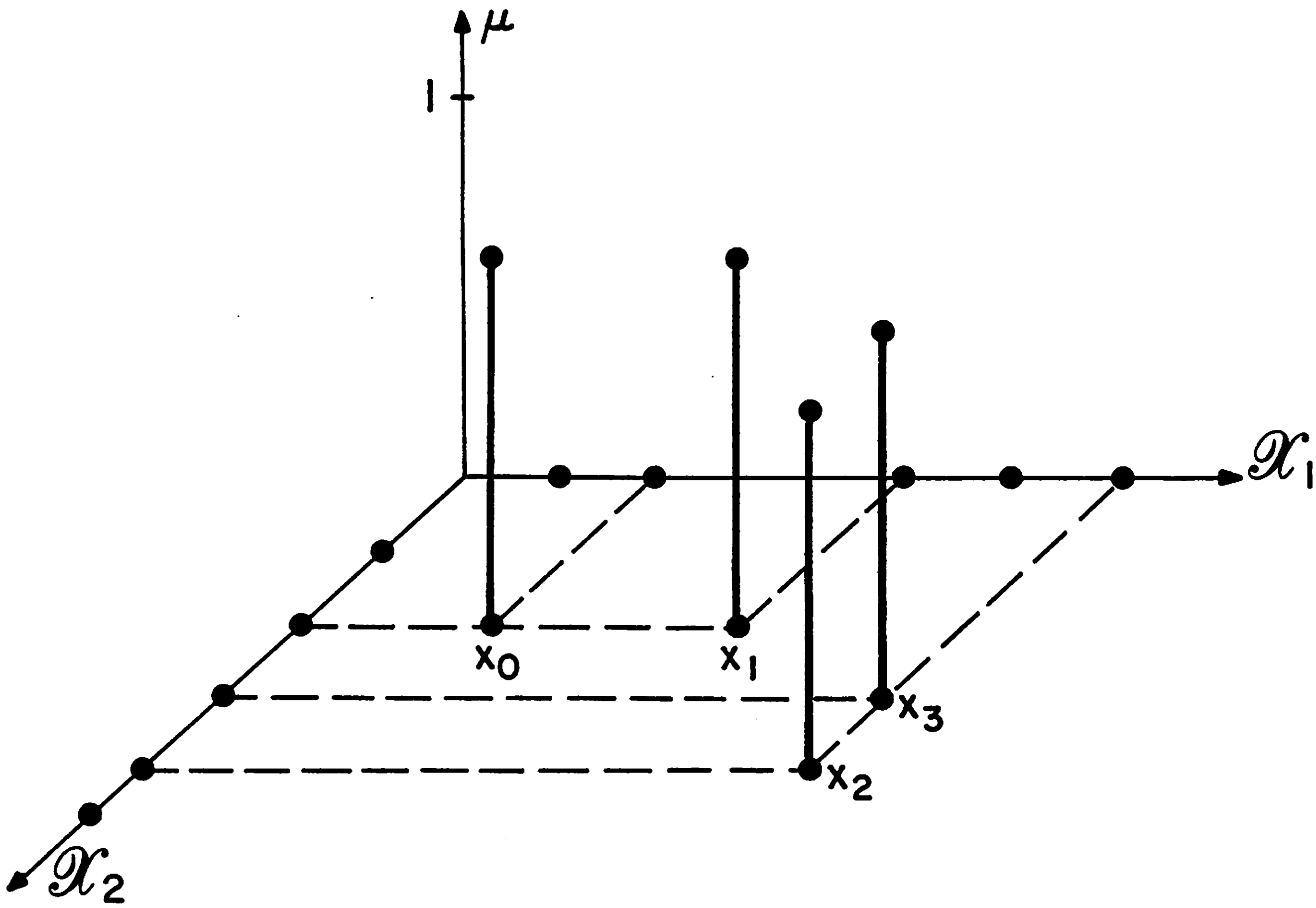
$X_1 \longleftrightarrow \mu_I(x_4) \longleftrightarrow \{x_4\}$

$X_2 \longleftrightarrow \mu_I(x_0) \cup \mu_I(x_1) \longleftrightarrow \{x_0, x_1\}$

$\mathscr{X}^{(3)} = \{x_0, x_1, x_2, x_3\}$

$x_0 \sim x_1 \ (\mathrm{mod}\ \mathscr{X}^{(3)})$

Fig. 2.4



$\mathscr{X}_1$ and $\mathscr{X}_2$ are represented by heavy dots.

$$X_1 \longleftrightarrow \mu_I(x_0) \cup \mu_I(x_1) \longleftrightarrow \{x_0, x_1\}$$
$$X_2 \longleftrightarrow \mu_I(x_2) \cup \mu_I(x_3) \longleftrightarrow \{x_2, x_3\}$$

Fig. 2.5