ON THE CONVERGENCE OF FUZZY CONTROL PROCESSES

by

Jerzy Kiszka

Memorandum No. UCB/ERL M81/43

23 February 1981

ON THE CONVERGENCE OF FUZZY CONTROL PROCESSES

by

Jerzy Kiszka

Memorandum No. UCB/ERL M81/43

23 February 1981

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

ON THE CONVERGENCE OF FUZZY CONTROL PROCESSES

Jerzy Kiszka*
Visiting Scholar

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

The feedback system is described by means of fuzzy state equations. Its synthesis is discussed. This leads to a further discussion of convergence of the fuzzy control algorithm. The notions of metric space fuzzy sets and of the convergence of a sequence of fuzzy sets are introduced. A fundamental theorem about convergence of fuzzy systems is proved. Its application is shown. Several examples of convergent and nonconvergent fuzzy control processes are illustrated. The iterative method for the solution of equation $X = X \circ R$ is proposed. The relation between the convergence and controllability of fuzzy control processes is mentioned.

*On leave from the Institute of Automatic Control, Technical University of Kielce, Al. Tysiaclecia Panstwa Polskiego 7, 25-314 Kielce, Poland.

1. INTRODUCTION

Since its introduction [4] fuzzy set theory has successfully been applied to a wide variety of problems. The work by L. A. Zadeh [5] started a new line of the theory of control, namely the theory of fuzzy control. After the introduction of the fuzzy logic controller by E. H. Mamdani [2], the problem of its implementation raised many difficult issues. In particular, the design of a fuzzy logic controller based on a given fuzzy model of a plant and a fuzzy model of a closed loop system remained a central task to be resolved [1,3]. The above mentioned problem requires competence in solving fuzzy relational equations. A method of solution of these fuzzy relational equations based on an iterative approach is proposed in this paper. This method seems to be interesting because the problem of the controllability of fuzzy control processes may be solved if the condition of convergence is fulfilled. The computer method of designing fuzzy control systems seems especially convenient. Section II considers the synthesis of fuzzy control processes and section III shows the proof of a theorem about the convergence of a fuzzy control algorithm and its applications, as well as a discussion of controllability. The concluding section contains a discussion of the remaining problems of transformation and solution of equivalent fuzzy relational equations.

2. SYNTHESIS OF FUZZY CONTROL PROCESSES

Consider the simple system shown in Fig. 1. Let $\mathcal{E} = \{e_j\}$; $j = 1,2,\ldots,p$ be a multidimensional finite discrete space of control error. Let $\mathcal{Y} = \{y_j\}$; $j = 1,2,\ldots,q$ be a multidimensional finite discrete space of outputs. Let $\mathcal{R} = \{r_j\}$; $j = 1,2,\ldots,s$ be a multidimensional finite discrete space of set points. Assume that the error e, the output y, and the set point r are fuzzy sets defined on $\mathcal{E}, \mathcal{Y}$ and $\mathcal{R}$ respectively. Further assume that the dynamic behavior of the comparator is governed by the following discrete time equation:

$$e_t = r_t\, y_t \circ G \qquad (2.1)$$

Here t is a time process. $r_t y_t$ denotes a fuzzy set on the Cartesian product space R × Y, and G a fuzzy relation given by

$$G: R \times Y \times E \to [0.1] \qquad (2.2)$$

where $R = \{r_t\}$, $Y = \{y_t\}$, $E = \{e_t\}$ are finite families of fuzzy sets, dim G = s × q × p, $\circ$ - denotes max-min composition. The error detector may also be described by means of the extension principle: Let $e_t = r_t - y_t$ that is $e_t = f(r_t, y_t)$. Then

$$\mu_G(e) = \mu_{f(r_t,y_t)}(e) = \sup_{r_t, y_t = f^{-1}(e_t)} [\mu_{r_t}(r) \wedge \mu_{y_t}(y)]. \qquad (2.3)$$

Let $\mathcal{U} = \{u_j\}$; j= 1,2,...,n be a multidimensional finite discrete space of control quantity and $Q = \{q_j\}$; j = 1,2,...,n be a multidimensional finite discrete space of the states of the controller. Assume that the control action u and the state q are fuzzy sets defined on $\mathcal{U}$ and Q respectively. Then the dynamic behavior of the controller may be described as follows:

$$\left.\begin{aligned} q_{t+1} &= q_t e_t \circ H_c \quad &(a) \\ u_t &= q_t \circ G_c \quad &(b) \end{aligned}\right\} \qquad (2.4)$$

where

$$\left.\begin{aligned} H_c&: Q \times E \times Q \to [0,1] \\ G_c&: Q \times U \to [0,1] \end{aligned}\right\} \qquad (2.5)$$

$Q = \{q_t\}$ and $U = \{u_t\}$ are finite families of fuzzy sets, dim $H_c = n \times p \times n$; dim $G_c = n \times m$. The dynamic behavior of the controller-comparator is governed by the following discrete time equation:

$$\left.\begin{array}{lll} q_{t+1} = q_t r_t y_t \circ F_c & & \text{(a)} \\ u_t = q_t \circ G_c & & \text{(b)} \end{array}\right\} \quad (2.6)$$

here:

$q_t r_t y_t$ is a fuzzy set on the Cartesian product space $Q \times R \times Y$;

$$F_c: \; Q \times R \times Y \times Q \rightarrow [0,1] \quad (2.7)$$

dim $F_c = n \times s \times q \times n$.

From equations (2.1) (2.4)

$$F_c = G \circ H_c \quad (2.8)$$

Let $\mathcal{X} = \{x_j\}$; $j = 1,2,\ldots,n$ be a multidimensional finite discrete space of the states of a plant. Assume that the dynamic behavior of the plant is governed by the following equations:

$$\left.\begin{array}{lll} x_{t+1} = x_t u_t \circ F_s & & \text{(a)} \\ y_t = x_t \circ G_s & & \text{(b)} \end{array}\right\} \quad (2.9)$$

Here $X = \{x_t\}$ is a finite family of fuzzy sets defined on $\mathcal{X}$.

$$\left.\begin{array}{ll} F_s : X \times U \times X & \rightarrow [0.1] \\ G_s : X \times Y & \rightarrow [0,1] \end{array}\right\} \quad (2.10)$$

dim $F_s = n \times m \times n$, dim $G_s = n \times q$. The equations of the fuzzy control process are the following:

$$
\left.\begin{aligned}
x_{t+1} &= x_t q_t \circ F_1 && \text{(a)}\\
q_{t+1} &= x_t q_t r_t \circ F_2 && \text{(b)}\\
y_t &= x_t \circ G_1 && \text{(c)}
\end{aligned}\right\} \tag{2.11}
$$

Assume that the relations F_s, G_s, F_c (G,H_c), G_c are given. We would like to find F_1, F_2, G_1 based on these relations. From equations (2.6) (b) and (2.9) (a) we have

$$x_{t+1} = x_t q_t \circ G_c \circ F_c \tag{2.12}$$

Comparing (2.12) with (2.11) (a) we obtain

$$x_t q_t \circ F_1 = x_t q_t \circ G_c \circ F_c \tag{2.13}$$

By virtue of this we have

$$F_1 = G_c \circ F_s \tag{2.14}$$

From (2.9) (b), (2.11) (c)

$$y_t = x_t \circ G_s = x_t \circ G_1 = y_t \tag{2.15}$$

we have

$$G_1 = G_s \tag{2.16}$$

Substituting (2.9) (b) for (2.6) (a) we obtain

$$q_{t+1} = q_t r_t x_t \circ G_s \circ F_s \tag{2.17}$$

Equations (2.17), (2.11) (b)

$$x_t q_t r_t \circ F_2 = x_t q_t r_t \circ G_s \circ F_s \tag{2.18}$$

imply

$$F_2 = G_s \circ F_s \tag{2.19}$$

Conclusion: <u>If we have the fuzzy relations of the controller and the plant, then the model of the fuzzy control process is given by equations (2.11) and (2.14), (2.16), (2.19)</u>. We need only execute the composition of fuzzy relations (2.14), (2.16) respectively. Now let us suppose that we have the contrary situation, namely, we are given the fuzzy equations of the closed loop system (F_1,F_2,G_1) and the plant. We would like to find the fuzzy equation of the controller (F_cG_c). This problem is very important from the practical point of view.

Let us consider the equation (2.9) (b). Suppose that

$$G_s \circ K_s = I \tag{2.20}$$

where, I is the unit matrix. Then

$$x_t = y_t \circ K_s \quad . \tag{2.21}$$

Substituting (2.21) for (2.11) (b), we have

$$q_{t+1} = q_t r_t y_t \circ K_s \circ F_2 \tag{2.22}$$

Comparing (2.22) with (2.6) (a), we obtain

$$q_{t+1} = q_t r_t y_t \circ K_s \circ F_2 = q_t r_t y_t \circ F_c = q_{t+1} \tag{2.23}$$

From (2.23) we have

$$F_c = K_s \circ F_2 \tag{2.24}$$

In order to obtain F_c, we must first solve equation (2.20) with respect to K_2 and next compose it with F_2 (2.24). Now consider equation (2.14)

Let

$$F_s \circ L_s = I \tag{2.25}$$

then

$$G_c = F_1 \circ L_s \tag{2.26}$$

If we solve equation (2.25) with respect to L_s and compose it with F_1, then we obtain the fuzzy relation G_c.

Conclusion: If we are able to solve the fuzzy relational equations (2.20) and (2.25) then the problem of the synthesis of a fuzzy controller will be resolved. Generally speaking, we should show a method of solving the relational equation

$$X \circ A = B. \tag{2.27}$$

For the moment, we will concentrate on a different equation, namely

$$X = X \circ R. \tag{2.28}$$

The connection between the two will be discussed. For convenience, we will investigate the equation

$$x_{t+1} = x_t u_t \circ R \tag{2.29}$$

where, dim $x_t u_t = n \times m$ and dim $R = n \times m \times n$. If we make the substitution $t + 1 = n$, then

$$x_n = x_{n-1} u_{n-1} \circ R; \quad n = 1,2,3,... \tag{2.30}$$

We will also investigate the possibility of carrying out process (2.29) from a given initial point $X_0 \in X$ to a final point $X^* \in X$. This problem is very important from the practical point of view because in many cases for a given R a fuzzy control action sequence under which the fuzzy control process is controllable does not exist.

3. THE CONVERGENCE OF A FUZZY CONTROL PROCESS

Let $X = \{X_i\}$, i = 1,2,...,m be a family of fuzzy sets defined on a multidimensional dinite discrete space $\mathcal{X} = \{x_j\}$; j = 1,2,...,n.

Definition 1. A fuzzy metric space is the pair of elements (X,ρ) such that $\rho: X \times X \quad R_*$ is a mapping (to be called metric) which satisfy the following conditions:

(i) $\rho(X_1,X_2) = 0 \Leftrightarrow X_1 = X_2$ (in a fuzzy sense)

(ii) $\rho(X_1,X_2) = \rho(X_2,X_1)$; $X_1,X_2 \in X$

(iii) $\rho(X_1,X_2) \leq \rho(X_1,X_3) + \rho(X_3,X_2)$; $X_1,X_2,X_3 \in X$.

where

$\times$ - Cartesian product in the classical sense.

R_* - set of real nonnegative numbers.

Example 1

Let $X_1 = [\mu_{X1}(x_1), \mu_{X2}(x_2),\ldots,\mu_{X2}(x_n)]$; $X_2 = [\mu_{X2}(x_1), \mu_{X2}(x_2),\ldots, \mu_{X2}(x_n)]$ denote two fuzzy sets defined on $\mathcal{X}$. The expressions

$$\rho_1(X_1,X_2) = \sqrt{\sum_{j=1}^{n} |\mu_{X_1}(x_j) - \mu_{X_2}(x_j)|^2}$$

$$\rho_2(X_1,X_2) = \max_{i\leq j\leq n} |\mu_{X_1}(x_j) - \mu_{X_2}(x_j)|$$

are metrics.

Definition 2. $\underset{\sim}{Q}$ is said to be a fuzzy neighborhood of fuzzy set $X_i \in X$ if $X_i \subset \underset{\sim}{Q}$, $(\mu_{X_i} \leq \mu_{\underset{\sim}{Q}})$.

Definition 3. The sequence of fuzzy sets $\{X_n\}$ converges to fuzzy set (point) X_o if and only if for each neighborhood $\mathcal{Q}$ almost all terms of $\{X_n\}$ satisfy $X_n \subset \mathcal{Q}$.

Notation:

$\lim X_n = X_o$, or $X_n \to X_o$,

in other words,

$[X_n \to X_o] \Leftrightarrow \forall\ \mathcal{Q}$ of X_o, $\exists\ k \in \underline{N}$, $\forall\ n > k\ (X_n \subset \mathcal{Q})$ where $\underline{N}$-set of natural numbers exclusive of zero.

Definition 4. Fuzzy sequence $\{X_n\}$ of terms of fuzzy metric space X is said to be a Cauchy fuzzy sequence iff $\forall\ \varepsilon > 0 \exists\ k \in \underline{N}$, $\forall\ n > k\ \forall\ m > k$, $\rho(X_n,X_m) < \varepsilon$ or $\lim\limits_{n,m\to\infty} \rho(X_n,X_m) = 0$, or $\forall\ \varepsilon > 0 \exists\ k \in \underline{N}$, $\forall\ n > k\ \forall\ m > k$, $\rho(X_n,X_m) \leq \varepsilon$.

Theorem 1. Every fuzzy convergent sequence is a Cauchy sequence.

Proof.

Let $X_n \to X_o$.

Based on (iii) def. 1 we may write:

$\rho(X_n,X_m) \leq \rho(X_n,X_o) + \rho(X_m,X_o)$.

Because $\rho(X_n,X_o) \to 0$ and $\rho(X_m,X_o) \to 0$ then $\rho(X_n,X_m) \to 0$. ¤

Observation 1. Definition 4 may be rewritten as: $\rho(X_n,X_{n+p}) \to 0$, $\forall_{p,n} \to \infty$.
This means that $\forall\ \varepsilon > 0 \ \exists\ k \in \underline{N}$, $\forall_p \in \underline{N}\ \forall\ n > k$, $\rho(X_n,X_{n+p}) < \varepsilon$. (*)

Definition 5. If in fuzzy metric space (X,ρ) every Cauchy fuzzy sequence converges to an element of this space then it is called a fuzzy complete metric space.

Theorem 2. Let (X,ρ) be a fuzzy complete metric space and $R : X \times U \times X \to [0,1]$ be a fuzzy relation such that

$$X_n = X_{n-1}\ U_{n-1} \circ R;\ n = 1,2,3,... \tag{3.1}$$

and

$$\rho(X_1U_1\circ R, X_2U_2\circ R) \leq \lambda\rho(X_1,X_2);\ \forall X_1,X_2 \in X;\ \forall U_1,U_2 \in U,\ \lambda \in [0,1) \tag{3.2}$$

then there exists one and only one fuzzy point (set) $X^* \in X$ and $U^* \in U$, that

$$X^* = X^* U^* \circ R \tag{3.3}$$

Proof.

Let $X_o \in X$, $U_o \in U$ be initial fuzzy points of fuzzy complete metric spaces (X,ρ), (U,ρ) respectively. We will demonstrate that expression (3.1) is a Cauchy fuzzy sequence. First of all we must prove that

$$\rho(X_{n-1},X_n) \leq \lambda^{n-1} \rho(X_o,X_o,U_o\circ R) \text{ for } n = 1,2,\ldots \tag{3.4}$$

Condition (3.4) is true for $n = 1$, $X_1 = X_oU_o\circ R$, $\lambda^o = 1$; $\rho(X_o,X_1) = \lambda^o \rho(X_o,X_oU_o\circ R)$. Suppose that it is true for $n = k$, we will prove that (3.4) is satisfied for $n = k + 1$,

$$\rho(X_{k-1},X_k) \leq \lambda^{k-1} \rho(X_o,X_oU_o\circ R), \ (n = k) \tag{3.5}$$

Based on (3.1), (3.2) and (3.5) we have $\rho(X_k,X_{k+1}) = \rho(X_{k-1}U_{k-1}\circ R, X_kU_k\circ R) \leq \lambda \rho(X_{k-1}, X_k) \leq \lambda\cdot\lambda^{k-1} \rho(X_o,X_oU_o\circ R) = \lambda^k \rho(X_o,X_oU_o\circ R)$.
Therefore,

$$\rho(X_{k1}X_{k+1}) \leq \lambda^k \rho(X_o,X_oU_o\circ R) \tag{3.6}$$

This means that (3.4) is true for $n = k + 1$. By induction we prove that (3.4) is true $\forall\, n \in \underline{N}$. Now consider $\rho(X_n,X_{n+p})$ for $n = 1,2,\ldots$ Based on def. 1 cond. (iii) and (3.6) we have

$$\rho(X_nX_{n+p}) \leq \rho(X_n,X_{n+1}) +\ldots+ \rho(X_{n+p-1},X_{n+p}) \leq$$

$$\leq \lambda^n \rho(X_oX_oU_o\circ R) + \ldots + \lambda^{n+p-1} \rho(X_o,X_o,U_o\circ R) =$$

$$= \rho(X_o,X_oU_o\circ R)\ [\lambda^n + \ldots + \lambda^{n+p-1}] =$$

$$= \lambda^n \frac{1-\lambda^p}{1-\lambda} \rho(X_o,X_oU_o\circ R) \leq \frac{\lambda^n}{1-\lambda} \rho(X_o,X_oU_o\circ R).$$

From here

$$\rho(X_n, X_{n+p}) \to 0 \text{ if } n \to \infty \text{ for } \forall_p.$$

The condition (*) of observation 1 is satisfied and this neans that $\{X_n\}$ is a Cauchy fuzzy sequence. Since our metric space is complete, then $X_n \to X^*$. Considering (3.2) we may write

$$\rho(X_n U_n \circ R,\ X^* U^* \circ R) \leq \lambda\ \rho(X_n, X^*)$$

$$X_n U_n \circ R \to X^* U^* \circ R.$$

Because

$$X_{n+1} \to X^*$$

and

$$\rho(X_{n+1}, X^*) \leq \rho(X_n X_{n+1}) + \rho(X_n, X^*) \to 0$$

then

$$\rho(X^*, X^* U^* \circ R) \leq \rho(X^*, X_{n+1}) + \rho(X_{n+1}, X^* U^* \circ R) =$$

$$= \rho(X^*, X_{n+1}) + \rho(X_n U_n \circ R,\ X^* U^* \circ R).$$

we know that

$$\rho(X^*, X_{n+1}) \to 0$$

and

$$\rho(X_n U_n \circ R,\ X^* U^* \circ R) \to 0.$$

From here

$$X^* = X^* U^* \circ R.$$

Now we prove that there exists one and only one point X^* (proof by contradiction). Let $X_1 \neq X_2$) and $X_1 = X_1 U_1 \circ R$, $X_2 = X_2 U_2 \circ R$. We have

$$\rho(X_1, X_2) = \rho(X_1 U_1 \circ R,\ X_2 U_2 \circ R \leq \lambda\ \rho(X_1, X_2) \tag{3.7}$$

Because $X_1 \neq X_2$ then $\rho(X_1, X_2) \neq 0$. Dividing (3.7) by $\rho(X_1, X_2)$ we have $1 \leq \lambda$. This contradicts assumption (3.2). Similarly we demonstrate that for a given X_0 and an attainable X^* there exists one and only one sequence of fuzzy control quantity $U_0, U_1, U_2, \ldots, U^*$ and $U_0', U_1', U_2', \ldots, U^{*'}$ which carry the fuzzy control process from X_0 to X^*.

$$\left.\begin{aligned} X_1 &= X_0 U_0 \circ R \\ X_2 &= X_1 U_1 \circ R \\ &\vdots \\ X^* &= X^* U^* \circ R \end{aligned}\right\} \quad (3.8) \qquad \left.\begin{aligned} X_2 &= X_0 U_0' \circ R \\ X_2 &= X_1 U_1' \circ R \\ &\vdots \\ X^* &= X^* U^{*'} \circ R \end{aligned}\right\} \quad (3.9)$$

$$U_0 \neq U_0',\ U_1 \neq U_1', \ldots, U^* \neq U^{*'} \qquad (3.10)$$

Based on (3.8) and (3.9) we have

$$\left.\begin{aligned} X_0 U_0 \circ R &= X_0 U_0' \circ R \\ X_1 U_1 \circ R &= X_1 U_1' \circ R \\ &\vdots \\ X^* U^* \circ R &= X^* U^{*'} \circ R \end{aligned}\right\} \qquad (3.11)$$

Equations (3.11) are true for a special case when $R = I$. This means that

$$U_0 = U_0',\ U_1 = U_1', \ldots, U^* = U^{*'}$$

and there exists one and only one sequence of fuzzy control quantity. ¤

<u>Conclusions</u>:

(1) The theorem 2 is also true when the sequence of fuzzy control quantity and/or the sequence of states are <u>determined</u>, that is (see Fig. 2.)

$$\mu_U(u) = \begin{cases} 1, & u = u_t \\ \text{otherwise} & \end{cases} \quad ; \qquad \mu_X(x) = \begin{cases} 1, & x = x_t \\ \text{otherwise} & \end{cases}$$

(2) For given $X_0, X^* \in X$ we may construct a fuzzy relation R (fuzzy controller) such that there exists one and only one sequence of fuzzy

control quantity $U_0, U_1, U_2, \ldots, U^*$ which carries the fuzzy control process from an initial fuzzy point $X_0 \in X$ to a final fuzzy point $X^* \in X$. Theorem 2 gives the condition of controllability of fuzzy control process (2.28).

(3) If the conditions of theorem 2 are satisfied then the fuzzy relational equation $X = X \circ R$ has one and only one solution. This solution may be found by means of successive approximation.

Example 1: Consider the process $X_{t+1} = X_t \circ R$. where

$$R = \begin{bmatrix} 0. & 0. & .5 & 1. & .5 & .5 & 1. & .5 & 0. & 0. & 0. & 0. \\ 0. & 0. & .5 & .5 & .5 & .5 & .5 & .5 & 0. & 0. & 0. & 0. \\ 0. & 0. & .5 & .5 & .5 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & .5 & 1. & .5 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & .5 & 1. & .5 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & .5 & .5 & .5 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & .5 & 1. & .5 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & .5 & .5 & .5 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & .5 & .5 & .5 & .5 & .5 & .5 & 0. & 0. & 0. & 0. \\ 0. & 0. & .5 & 1. & .5 & .5 & 1. & .5 & 0. & 0. & 0. & 0. \\ 0. & 0. & .5 & .5 & .5 & .5 & .5 & .5 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix} \tag{3.12}$$

Suppose that for $t = 0$ we have X_0=[0. 0. 0. 0. 0. 0. 0. 0. 0. .5 1. .5 0.]

then $X_1 = X_0 \circ R$ = [0. 0. .5 1. .5 .5 1. .5 0. 0. 0. 0.]

$X_2 = X_1 \circ R$ = [0. 0. .5 1. .5 0. 0. 0. 0. 0. 0. 0.]

$X_3 = X_2 \circ R$ = [0. 0. .5 1. .5 0. 0. 0. 0. 0. 0. 0.]

$X_2 = X_3 = X^*$ = [0. 0. .5 1. .5 0. 0. 0. 0. 0. 0. 0.]

Let for $t = 0$ be X_0 = [1. .5 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.] - then

$X_1 = X_0 \circ R$ = [0. 0. .5 1. .5 .5 1. .5 0. 0. 0. 0.]

$X_2 = X_1 \circ R$ = [0. 0. .5 1. .5 .5 1. .5 0. 0. 0. 0.]

$X_3 = X_2 \circ R$ = [0. 0. .5 1. .5 .5 1. .5 0. 0. 0. 0.]

$X_2 = X_3 = X^*$ = [0. 0. .5 1. .5 .5 1. .5 0. 0. 0. 0.]

Let for $t = 0$ to be X_0 = [0. 0. 0. 0. 0. .5 1. .5 0. 0. 0. 0.] then

$$X_1 = X_o \circ R = [0.\ 0.\ .5\ 1.\ .5\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.]$$
$$X_2 = X_1 \circ R = [0.\ 0.\ .5\ 1.\ .5\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.]$$
$$X_1 = X_2 = X^* = [0.\ 0.\ .5\ 1.\ .5\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.]$$

Process (3.12) is convergent. If we start from any initial point $X_o \in X$ then the final point X^* is always obtained. The conditions of theorem 2 are satisfied. This example shows the iterative method of solving the equation $X = X \circ R$. In this case the solution is X^*. This means that there exists a control action (in this case non-fuzzy) which carries the process from any initial point $X_o \in X$ to a final point $X^* \in X$. This process is controllable (in the sense of conclusion (2)).

<u>Example 2</u>. Let

$$R = \begin{bmatrix} 1. & .5 & .5 & 1. & .5 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ .5 & .5 & .5 & .5 & .5 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & .5 & .5 & .5 & .5 & .5 & .5 & .5 & .5 & .5 & 0. \\ 0. & 0. & .5 & 1. & .5 & .5 & 1. & .5 & .5 & 1. & .5 & 0. \\ 0. & 0. & .5 & .5 & .5 & .5 & .5 & .5 & .5 & .5 & .5 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & .5 & .5 & .5 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & .5 & 1. & .5 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & .5 & .5 & .5 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix} \qquad (3.13)$$

Suppose that for t = 0 we have $X_o = [1.\ .5\ .5\ 1.\ .5\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.]$ then

$$X_1 = X_o \circ R = [0.\ 0.\ .5\ 1.\ .5\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.]$$
$$X_2 = X_1 \circ R = [0.\ 0.\ 0.\ 0.\ 0.\ .5\ 1.\ .5\ 0.\ 0.\ 0.\ 0.]$$
$$X_3 = X_2 \circ R = [0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.] = \phi$$ - empty fuzzy set.

Let for t = 0 be $X_o = [0.\ 0.\ .5\ 1.\ .5\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.]$ then

$$X_1 = X_0 \circ R = [0.\ 0.\ 0.\ 0.\ 0.\ .5\ 1.\ .5\ 0.\ 0.\ 0.\ 0.]$$
$$X_2 = X_3 = X_1 \circ R = [0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.]$$ - empty fuzzy set.

Let for $t = 0$ be $X_0 = [0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ .5\ 1.\ .5\ 0.]$ then

$X_2 = X_1 = X_0 \circ R = [0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.\ 0.]$ empty fuzzy set.

This process is nonconvergent. Any initial point $X_0 \in X$ leads to an empty fuzzy set which is not a solution for given relation R. The conditions of theorem 2 there are not satisfied. This means that there does not exist any sequence of control quantity which carries the process from any $X_0 \in X$ to X^*. Process (3.13) is noncontrollable (in the sense of conclusion (2)).

4. CONCLUSION

The problem of designing a fuzzy controller based on a given fuzzy model of plant and closed loop system has been shown. It requires the solution of a fuzzy relational equation. Under conditions of theorem 2 the solution of equation $X = X \circ R$ may be found. In order to apply this successive approximation to equation $X \circ A = B$ we must transform the latter to the form $X = X \circ R$, but under the condition that both equations have the same solution. Theorem 2 seems to be useful from a practical point of view because it enables the construction of a fuzzy controller which has the controllability property. If conditions of the theorem are fulfilled, then the process is carried from the initial point to a final point under an action of fuzzy sequence of control.

ACKNOWLEDGEMENT

I would like to express my very warm thanks for the scientific attention given to me in this study by Professor L. A. Zadeh, and to Dr. V. Konrad for interesting discussions.

Research sponsored by National Science Foundation Grant ENG78-32143.

REFERENCES

[1] M. Braae, D. A. Rutherford, "Theoretical and Linquistic Aspects of the Fuzzy Logic Controller," Automatica, vol. 15, pp. 553-573, 1979.

[2] E. H. Mamdani, "Application of Fuzzy Algorithms for Control of Simple Dynamic Plant," Proc. IEEE, 121, 12, 1974.

[3] R. M. Tong, "Some Properties of Fuzzy Feedback Systems," IEEE Trans. Systems, Man, Cyber., vol. SMC-10, no. 6, June 1980.

[4] L. A. Zadeh, "Fuzzy Sets," Inform. Control, 8, 1965.

[5] L. A. Zadeh, "A Rationale for Fuzzy Control," Journal of Dynamic Systems, Measurement and Control, 3, 1972.

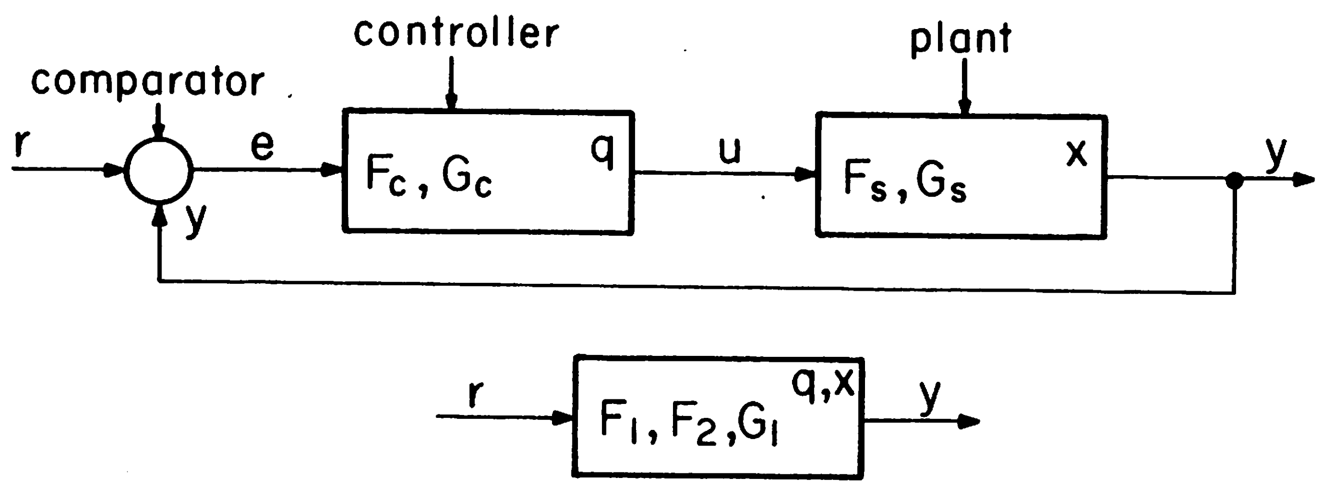
controller
plant
comparator
r
e
F_c, G_c
q
u
F_s, G_s
x
y
y
r
F_1, F_2, G_1
q,x
y

Fig. 1.

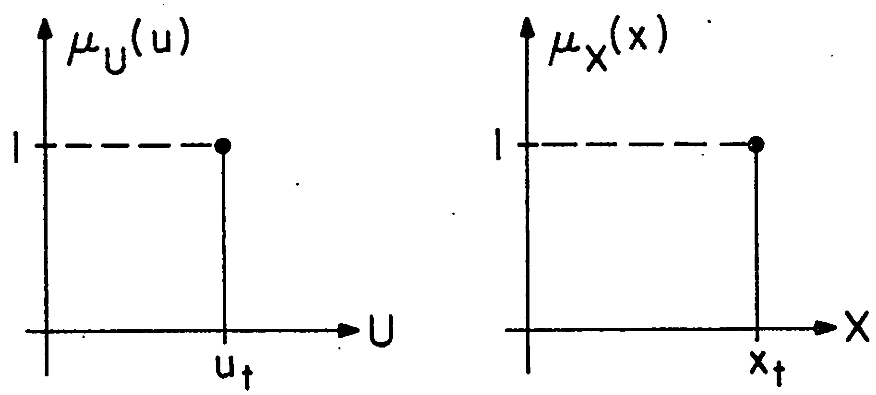

Fig. 2.