THE PARTIAL ORDER DIMENSION PROBLEM IS NP-COMPLETE

by

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ABSTRACT

Given a partial order G, it is shown that the problem to determine the minimum number of total orders whose intersection forms G is NP-complete

1. Introduction

In 1941 Dushnik and Miller [2] introduced the notion of the dimension of a partial order : the smallest number of total orders whose intersection is the original order. Since then the area of problems related to the concept of dimension has been intensively investigated (see e.g. [1] for a literature survey). However, up to now, the inherent complexity of determining the dimension of a partial order remained unsolved and is listed as an open problem in the textbook of Garey and Johnson [3].

Only for the special case of dimension ≤ 2 it is known that we can recognize these graphs in polynomial time. This is possible due to a characterization of partial orders of dimension 2 obtained by Baker, Fishburn and Roberts [1] and an $O(n^3)$ - algorithm for comparability graph recognition (where n is the number of nodes) found by Golumbic [4]: A partial order G is of dimension ≤2 iff a transitive orientation can be assigned to the incomparability graph of G.

In this paper we show that determining the dimension of a partial order is an NP-complete problem and thus with regards to its complexity equivalent to a lot of combinatorial problems like e.g. finding a maximum size clique in a graph or finding a satisfying truth assignment for a boolean formula (for the notion of NP-completeness see Karp [5] and Garey, Johnson [3]). This NP-completeness result gives strong evidence that the task of determining the dimension of a

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partial order is inherently hard, i.e. no polynomial time algorithm is likely to be found.

2. Definitions

Given a finite set V together with a binary relation $<$ on V. If $<$ is asymetric (i.e. $x < y \Rightarrow y \nless x$) and transitive (i.e. $x < y,\, y < z \Rightarrow x < z$) we call $(V,<)$ a "partial order". If we define now a set $E := \{ (x,y) \mid x,y \in V,\ x < y \}$ we can represent the partial order as a directed, acyclic graph $G=(V,E)$ with node set V and edge set E.

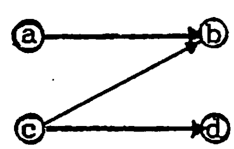
A partial order $G=(V,E)$ is called a "total" (or "linear") order iff *$(x,y) \in E$ or $(y,x) \in E$ for all $x,y \in V$*, i.e. every two elements of V are "comparable". It is easy to see that a total order is the transitive closure of a graph which can be drawn as a chain.

The "dimension" of a partial order $G=(V,E)$ is the minimum number n of total orders $G_1=(V,E_1)$, $G_2=(V,E_2)$, $\cdots$,$G_n=(V,E_n)$ such that $(x,y) \in E \Leftrightarrow (x,y) \in E_i$, $1 \leq i \leq n$.

This defintion suggests that we call G the intersection of $G_1, G_2, \cdots, G_n$.

To illustrate the notion of dimension we give a simple example:

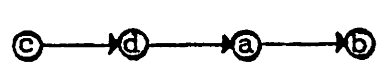
Let G =



then G is of dimension 2 because it is the intersection of G_1 and G_2 with

G_1 =

G_2 =



Like other combinatorial optimizations the problem of minimizing the number of total orders needed for the intersection can be transformed into a language recognition problem, i.e. rather than to determine the dimension of a partial order we are given a partial order G together with a number d and have to answer with "yes" if d is the dimension of G and with "no" otherwise. Because we can deduce the value for the dimension by a series of these yes-no questions (using binary search) it can be argued that, if the language recognition problem is inherently hard then the corresponding optimization problem is also inherently hard.

3. NP-Completeness

We shall design a reduction from the following problem, called SET-PARTITION:

Instance: A collection M of n sets $S_1, S_2, \cdots S_n \subset \{1,2,\dots,m\}$.

Question: Does there exist an index set $I \subset \{1,2,\dots,n\}$ such that

$$\bigcup_{i \in I} S_i = \bigcup_{i \notin I} S_i = \{1,2,\dots,m\},$$

i.e, is there a partition of M into two collections M_1 and M_2, which both cover all the elements ?

This problem is also known as SET-SPLITTING or as HYPERGRAPH 2-COLORABILITY (a vertex corresponds to a set S, containing as elements all hyperedges which it is incident to) and has been shown to be NP-complete by Lovasz [1].

From M we will construct an acyclic digraph $G=(V,E)$ and a number d such that :

M has a partition into $M_1, M_2 \iff G$ is of dimension d.

If M has n sets covering m elements then G will have $2n + 4m$ nodes and will be bipartite, i.e. the node set V is the union of two disjoint sets V_1 and V_2 such that $E \subset V_1 \times V_2$. For a better understanding of the following construction we will call the nodes in V_1 "black nodes" (and draw them dark: ●) and call the nodes in V_2 "white nodes" (and draw them light: ○).

Now let $M = S_1, S_2, \dots, S_n \subset \{1,2,\dots,m\}$ be given. Without loss of generality we can assume that for each pair of elements j,k there are sets $S,T \in M$, such that $j \in S, j \notin T, k \in T, k \notin S$, because otherwise we could identify j with k.

Define the following node sets:

The black nodes:

$BS := \{ BS[i] \mid 1 \le i \le n \}$

$BL := \{ BL[i] \mid 1 \le i \le m \}$

$BR := \{ BR[i] \mid 1 \le i \le m \}$

The white nodes:

$WS := \{ WS[i] \mid 1 \le i \le n \}$

$WL := \{ WL[i] \mid 1 \le i \le m \}$

$WR := \{ WR[i] \mid 1 \le i \le m \}$

For $1 \le i \le n$, call $(BS[i], WS[i])$ the "set-pair i", for $1 \le i \le m$, call $(BL[i], WL[i])$ the "left-pair i", for $1 \le i \le m$, call $(BR[i], WR[i])$ the "right-pair i". The set-pairs, left-pairs and right-pairs are also refered to as "vertical pairs".

Define the following edge sets:

$E_1 := BL \times (WS \cup WR)$

$E_2 := BR \times (WS \cup WL)$

$E_3 := \{(BS[i], WS[j]) \mid 1 \le i,j \le n,\ i \ne j\}$

$E_4 := \{(BS[i], WL[j]),\ (BS[i], WR[j]) \mid 1 \le i \le n,\ 1 \le j \le m,\ j \notin S_i \}$

Now set $V := BS \cup BL \cup BR \cup WS \cup WL \cup WR$, $E := E_1 \cup E_2 \cup E_3 \cup E_4$, $d:=n$ and the construction for the acyclic digraph $G=(V,E)$ is completed. (For a visual description see figure 1).

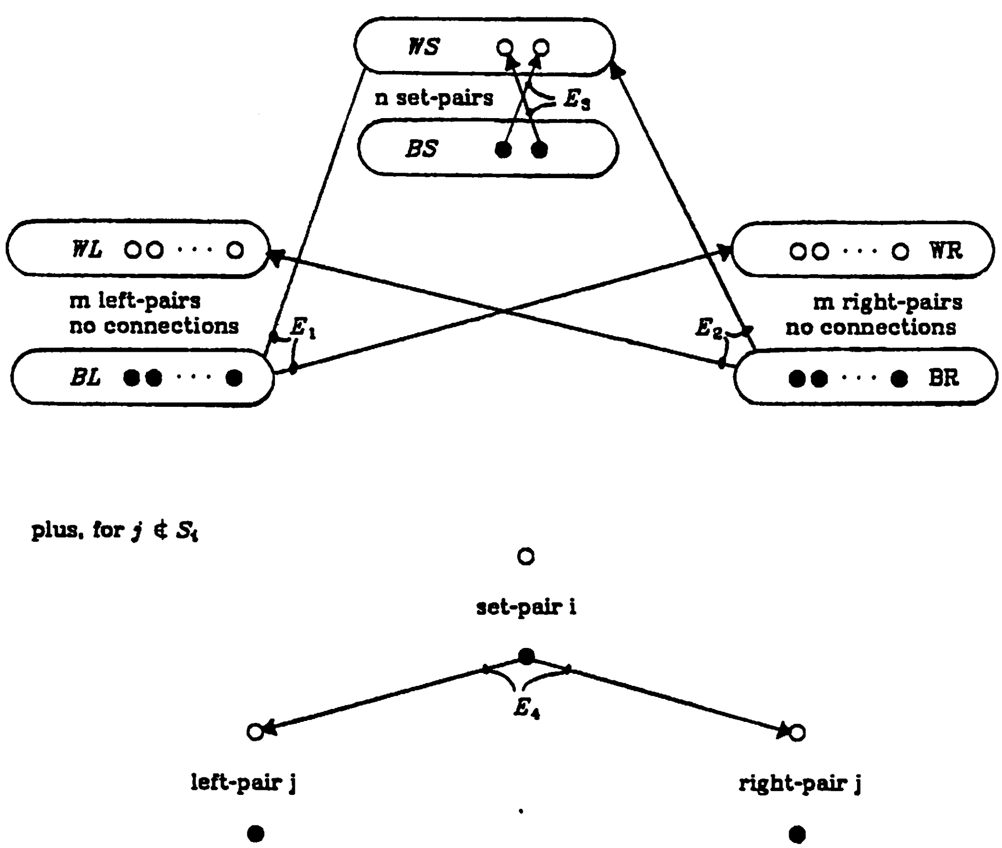


Figure 1: The graph G, constructed from the set collection M

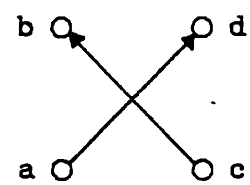
The following terms are useful to formulate the proof:

Given a partial order $G=(V,E)$.

A pair of nodes (a,b) is in "forward relation" iff $(a,b) \in E$, it is in "backward relation" iff $(b,a) \in E$.

A pair of nodes is "comparable" iff it is either is forward relation or in backward relation, otherwise it is "incomparable".

Two pairs of nodes (a,b) and (c,d) are "connected by a crossover", iff $(a,d) \in E$ and $(c,b) \in E$:



All edges in the graph $G=(V,E)$ constructed from the set collection M are called "fixed edges", in contrast to the "specifying edges", which have to be introduced (in addition to the fixed edges) within the total orderings to make the incomparable pairs from G also incomparable within the intersection.

Notice now, that two incomparable pairs (a,b) and (c,d) from G which are connected by a crossover, must within the total orderings

1.) both once be set into forward relation and once be set into backward relation

2.) not be set in backward relation in the same total ordering, since this would create a cycle.

This is the basic tool in our construction and it applies to two set-pairs, to a left-pair with regard to a right-pair and to the set-pair i with regard both to the left-pair j and the right-pair j if $j \notin S_i$.

Now we are ready for the proof. Given the set collection M and the constructed digraph $G=(V,E)$. We show first:

G is of dimension n => M has a partition into M_1 and M_2

Proof: If G is of dimension n, there are n total orderings $G_1, G_2, \ldots, G_n$ whose intersection forms G. These total orderings make all incomparable pairs from G incomparable in the intersection and therefore especially each vertical pair in G is set once into forward relation and once set into backward relation. Since all (incomparable) n set-pairs are pairwise connected by a crossover in G, it follows that in each of the total orderings G_1, G_2, ..., G_n exactly one set-pair is set into backward relation. Without loss of generality let it be the set-pair i which is in backward relation in G_i.

We will now interpret the backward relation of the left-pair j in G_i as the meaning, that the set S_i has been chosen for the M_1 collection to cover the element j and the backward relation of the right-pair j in G_i means that the set S_i has been chosen for the M_2 collection to cover the element j. Since each left-pair is connected to each right-pair by a crossover, it follows that in each of the linear orderings a left-pair and a right-pair cannot be in backward relation at the same time. Since the set-pair i is connected by a crossover to each left-pair j and right-pair j with $j \notin S_i$, all these (left- and right-) pairs are forced to be in forward relation when the set-pair i is in backward relation.

This implies, that in G_i only those left-pairs j or right-pairs j (but not both) can be in backward relation, which correspond to an element j with $j \in S_i$.

Since each left-pair is at least once in backward relation, it follows that each element j is covered by a set chosen for M_1 and since each right-pair is at least once in backward relation, it follows that each element is covered by a set chosen for M_2. Since a left-pair and a right-pair can not be in backward relation at the same time, it follows that no set S is chosen both for M_1 and M_2.

Thus we can define a partition for M by

$M_1 := \{ S_i \mid$ there is a left-pair in backward relation in $G_i \}$

$M_2 := \{ S_i \mid$ there is a right-pair in backward relation in $G_i \}$.

Now we show:

M has a partition into M_1, M_2 => G is of dimension n

Proof: Let the two partitions M_1 and M_2 and the constructed digraph $G=(V,E)$ be given. We have to construct n total orderings $G_1=(V,E_1)$, $G_2=(V,E_2)$,..., $G_n=(V,E_n)$ whose intersection forms G. This means the n total orderings must have the property

I) $(a,b) \in E \Rightarrow (a,b) \in E_i$ for $1 \leq i \leq n$

II) $(a,b) \notin E$ and $(b,a) \notin E \Rightarrow$ there are $i,j \in \{1,2,\dots,n\}$, $i \neq j$ such that $(a,b) \in E_i$, $(b,a) \in E_j$.

First we describe how the total orderings G_1, G_2, ..., G_n can be obtained and then we show that their intersection is G.

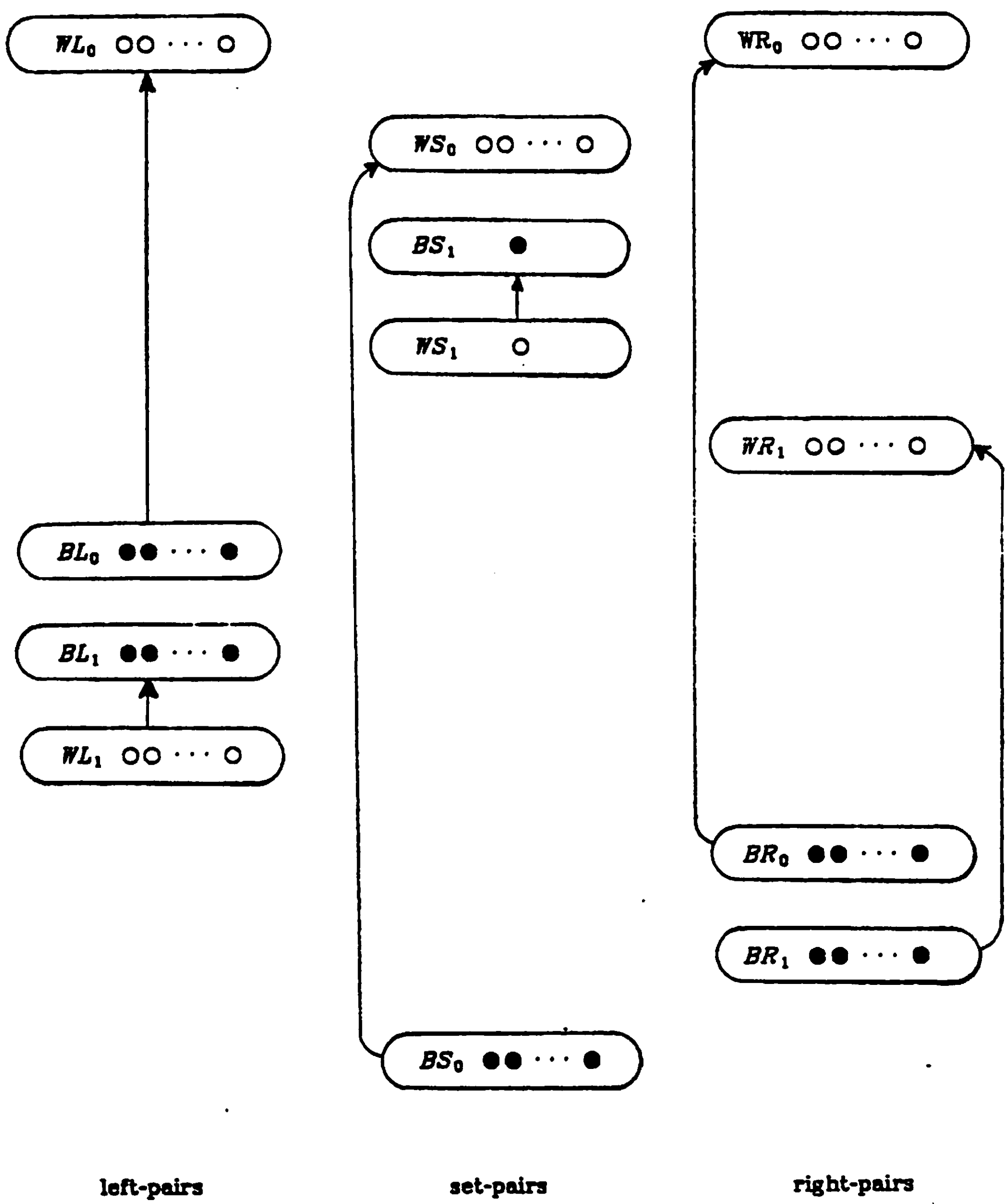


Figure 2 : The relations for the vertical pairs are shown by arrows

To construct G_i, we will augment G according to the diagram shown in figure 2, i.e. we will add specifying edges to the fixed edges of G. Thus E_i will contain E and property I is fulfilled automatically.

The total order G_i will represent the fact, that the set S_i has been chosen either for M_1 or for M_2 and that S_i, within its partition, covers some elements.

In order to construct G_i assume now, that S_i belongs to partition M_1. (If S_i belongs to M_2, the diagram can be used after changing the role of left and right, i.e. after substituting the letter "R" for "L" and "L" for "R").

Define the following subsets of V, which all depend on the index i:

$BS_1 := \{ BS[i] \}$
$BS_0 := \{ BS[j] \mid 1 \leq j \leq n, j \neq i \}$
$BR_1 := \{ BR[j] \mid j \in S_i \}$
$BR_0 := \{ BR[j] \mid j \notin S_i \}$
$BL_1 := \{ BL[j] \mid j \in S_i \}$
$BL_0 := \{ BL[j] \mid j \notin S_i \}$

$WS_1 := \{ WS[i] \}$
$WS_0 := \{ WS[j] \mid 1 \leq j \leq n, j \neq i \}$
$WR_1 := \{ WR[j] \mid j \in S_i \}$
$WR_0 := \{ WR[j] \mid j \notin S_i \}$
$WL_1 := \{ WL[j] \mid j \in S_i \}$
$WL_0 := \{ WL[j] \mid j \notin S_i \}$

Clearly, this is a partition of the nodes of V into 12 disjoint sets. Now, using the diagram shown in figure 2, construct a digraph $G_i' := (V, E_i')$ by defining

E_i' := $\{ (x,y) \mid x,y \in V$, the set which contains x appears in the diagram below the set which contains $y \}$.

Obviously G_i' is an acyclic graph and, moreover, E_i' contains E, since it can be verified that all fixed edges from G lead in the diagram from a lower level to a higher level.

Now transform the partial order G_i' into a total order G_i by introducing any set of additionel edges which creates no cycles (e.g. by sorting G_i' topologically).

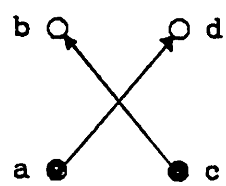
It remains to show that property II is fulfilled, i.e. we have to show, that each incomparable pair from G is once set into forward relation and once set into backward relation.

First, as the arrows in the diagram show, this is true for the vertical pairs:

The set-pair i is in backward relation in G_i, otherwise it is in forward relation.

Since for each element j there is a set $S_i \in M_1$ and a different set $S_k \in M_2$ which both cover j, it follows: The left-pair j is in backward relation in G_i and in forward relation in G_k. The right-pair j is in backward relation in G_k and in forward relation in G_i.

Now notice, that the two incomparable black nodes and the two (incomparable) white nodes of two vertical pairs (a,b) and (c,d) which are connected by a crossover, are therefore also made incomparable:



(a,b) in backward relation => $c < a$ and $b < d$
(c,d) in backward relation => $a < c$ and $d < b$

We will classify the remaining incomparable pairs from G according to their membership in the node sets BS, BL, BR, WS, WL, WR. These sets are represented in figure 3 as nodes and an edge is provided between two classes if they contain some of the remaining incomparable pairs. Referring to the label of the edge from a set A to a set B we will decscribe how an incomparable pair (a,b) with $a \in A$ and $b \in B$ is made incomparable in the intersection. (In the following we write $X < Y$ for sets X,Y, iff $x < y$ for all $x \in X, y \in Y$).

1) For each $1 \leq j \leq m$ holds:
There is $S_i \in M_1$ with $j \in S_i$. => in G_i : $WL[j] \in WL_1$, and $WL_1 < WS$.
There is $S_k \in M$ with $j \notin S_k$. => in G_k : $WL[j] \in WL_0$, and $WL_0 > WS$

2,4) For all $1 \leq k,l \leq m$ holds:
There are $S_i, S_j \in M$ with $k \in S_i, k \notin S_j, l \notin S_i, l \in S_j$. =>
in G_i : $WL[k] \in WL_1 < WL_0 \ni WL[l]$ and $BL[k] \in BL_1 < BL_0 \quad BL[l]$
in G_j : $WL[k] \in WL_0 > WL_1 \ni WL[l]$ and $BL[k] \in BL_0 > BL_1 \quad BL[l]$

3) For each $1 \leq j \leq m$ holds:
There is $S_i \in M_1$ with $j \in S_i$. => in G_i : $WL[j] \in WL_1$, and $WL_1 < BL$.
There is $S_k \in M_1$ with $j \notin S_k$. => in G_k : $WL[j] \in WL_0$, and $WL_0 > BL$.

5) (Only meaningful for $BS[i]$ versus $WL[j]$ if $j \in S_i$, because otherwise $BS[i]$ and $WL[j]$ are comparable). So define $L(i) := WL[j] \mid j \in S_i$.
For each $BS[i]$ holds:
In G_i : $BS[i] \in BS_1 > WL_1 = L(i)$
In each G_j with $j \neq i$: $BS[i] \in BS_0 < WL \supset L(i)$

6) For each $BS[i]$ holds:
In G_i : $BS[i] \in BS_1 > BL$
In each G_j with $j \neq i$: $BS[i] \in BS_0 < BL$

Cases 7 to 12 are analogous to cases 1 to 6 and can be considered by substituting the letter "R" for "L" and "M_2" for "M_1". This completes the proof.

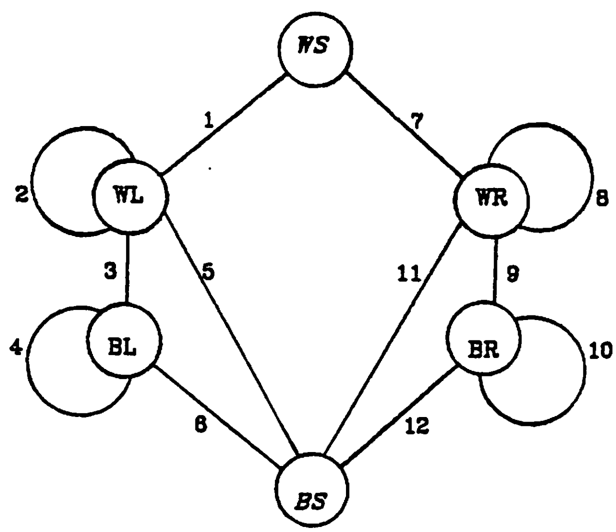


Figure 3 : Node classes containing unrelated pairs

4. Conclusion

We have shown that, given a partial order G and a number d, it is an NP-complete problem to decide whether the dimension of G is d. Since in our reduction the value n depends on the size of the set system M this proofs the NP-completeness of the dimension problem only for an arbitrary value of n. So besides the dimension 2 problem (which has a polynomial time solution) the complexity for the dimension k problem if k is fixed (e.g. k=3) remains open.

5. References

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