# Errata

1. <u>Delete</u>: from the last line on pg. 9 to end of section 4.2.

   <u>Replace by</u>:

   We know that

$$R_Y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega,z)d\omega \tag{4.7}$$

Define $\tilde{Y}_t$ as the process produced by filtering

$$Y_t \text{ by } H(j\omega) = \begin{cases} 1 : \omega \in [\omega_1,\omega_1+\Delta\omega],\ \omega_1 \in [\omega_{n0},\omega_{nc}-\Delta\omega] \\ 0 : \text{elsewhere} \end{cases} \tag{4.8}$$

Then,

$$R_{\tilde{Y}}(0) = \frac{1}{2\pi} S_Y(\omega_1,z)\Delta\omega \tag{4.9}$$

$$= \frac{1}{2\pi} Q(j\omega_1,z)Q(j\omega_1,z)^*\Delta\omega \tag{4.10}$$

$\tilde{Y}_t$ is a Gaussian random vector, thus

$$P_{\tilde{Y}_t}(y) = \frac{1}{(2\pi)^{m/2}\cdot \det(R_{\tilde{Y}}(0))^{1/2}} \exp[-\frac{1}{2} y^T R_{\tilde{Y}}(0)^{-1} y] \tag{4.11}$$

Now, by (4.10), $R_{\tilde{Y}}(0)$ is Hermitian and positive definite, hence so is $R_{\tilde{Y}}(0)^{-1}$. Thus,

$$y^T R_{\tilde{Y}}(0)^{-1} y \geq \|y\|_2^2\ \lambda_{min}(R_{\tilde{Y}}(0)^{-1})$$

and,

$$\lambda_{min}(R_{\tilde{Y}}(0)^{-1}) = \frac{1}{\lambda_{max}(R_{\tilde{Y}}(0))} = \frac{1}{\bar{\sigma}[Q(j\omega_1,z)]^2}\cdot\frac{2\pi}{\Delta\omega} \tag{4.12}$$

So,

$$y^T R_{\tilde{Y}}(0)^{-1} y \geq \|y\|_2^2 \frac{1}{\bar{\sigma}[Q(j\omega_1,z)]^2}\cdot\frac{2\pi}{\Delta\omega} \tag{4.13}$$

Thus,

$$P_{\tilde{Y}_t}(y) \leq \frac{1}{(2\pi)^{m/2}\cdot \det(R_{\tilde{Y}}(0))^{1/2}} \exp[-\frac{1}{2}\|y\|_2^2 \frac{1}{V_y(j\omega_1,z_1)}] \tag{4.14}$$

where

$$v_y(j\omega_1,z) := \bar{\sigma}[Q(j\omega_1,z)]^2 \cdot \frac{\Delta\omega}{2\pi} \tag{4.15}$$

We know that

$$\det(R_Y(0)) = \left(\frac{\Delta\omega}{2\pi}\right)^m \cdot \prod_{i=1}^{m} [\sigma_i[Q(j\omega_1,z)]]^2 \tag{4.16}$$

So,

$$K(j\omega_1,z) := \frac{v_y(j\omega_1,z)^{m/2}}{\det(R_{\tilde{Y}}(0))^{1/2}} = \frac{\bar{\sigma}[Q(j\omega_1,z)]^m}{\prod_{i=1}^{m} \sigma_i[Q(j\omega_1,z)]} \tag{4.17}$$

Now, by (4.14) and (4.17),

$$P_{\tilde{Y}_t}(y) \le K(j\omega_1,z) \frac{1}{(2\pi)^{m/2} v_y(j\omega_1,z)^{m/2}} \exp\left[-\frac{1}{2} \|y\|_2^2 \frac{1}{v_y(j\omega_1,z)}\right] \tag{4.18}$$

So, if $P_{\tilde{Y}_t}(i)(y_i)$ is the density in the ith output channel, then, $\forall i$

$$P_{\tilde{Y}_t}(i)(y_i) \le K(j\omega_1,z) \frac{1}{\sqrt{2\pi \cdot v_y(j\omega_1,z)}} \exp\left[-\frac{1}{2} \cdot y_i^2 \cdot \frac{1}{v_y(j\omega_1,z)}\right] \tag{4.19}$$

Thus, $\forall i$

$$E[\tilde{Y}_t^{(i)2}] \le K(j\omega_1,z) \cdot v_y(j\omega_1,z) = K(j\omega_1,z)\bar{\sigma}[Q(j\omega_1,z)]^2 \cdot \frac{\Delta\omega}{2\pi} \tag{4.20}$$

So, if (4.2) is satisfied, (4.20) gives an upper bound on the noise power, due to $u_1$, at $\omega = \omega_1 \in [\omega_{no},\omega_{nc}]$, in any plant-output channel.

2. <u>Delete</u>: Equation (5.7) and the next three lines

<u>Replace by</u>:

$$E[Y_t^{(i)2}] \le K(j\omega_1,z)\bar{\sigma}[(I-PQ)(j\omega_1,z)]^2 \cdot \frac{\Delta\omega}{2\pi} \tag{5.7}$$

where $K(j\omega_1,z) := \dfrac{\bar{\sigma}[(I-PQ)(j\omega_1,z)]^m}{\prod_{i=1}^{m} \sigma_i[(I-PQ)(j\omega_1,z)]}$, and $\omega_1 \in [\omega_{do},\omega_{dc}]$

So, if (5.2) is satisfied, (5.7) gives an upper bound on the noise power, due to $d_o$, at $\omega = \omega_1 \in [\omega_{dc}, \omega_{dc}]$ in any closed-loop system-output channel.

3. Delete: Equation (6.5) and the next three lines

Replace by:

$$E[Y_t^{(i)2}] \leq \max_{\omega \in \Omega_f} K(j\omega, z^*)\bar{\sigma}[(I-PQ)(j\omega, z^*)]^2 \cdot \frac{\Delta\omega}{2\pi} \tag{6.5}$$

where

$$K(j\omega, z) := \frac{\bar{\sigma}[(I-PQ)(j\omega, z)]^m}{\prod_{i=1}^{m} \sigma_i[(I-PQ)(j\omega, z)]}$$

So, by (6.5), solution of (6.1)-(6.3) gives a minimal upper bound on the noise power, due to $d_o$, in any closed-loop system-output channel, at any $\omega \in [\omega_{fo}, \omega_{fc}]$.

# CONTROLLER DESIGN FOR LINEAR MULTIVARIABLE FEEDBACK SYSTEMS WITH STABLE PLANTS, USING OPTIMIZATION WITH INEQUALITY CONSTRAINTS

by

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# CONTROLLER DESIGN FOR LINEAR MULTIVARIABLE FEEDBACK SYSTEMS WITH STABLE PLANTS, USING OPTIMIZATION WITH INEQUALITY CONSTRAINTS

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## ABSTRACT

This paper proposes a design methodology for unity-feedback linear multivariable systems with stable plants. A purely algebraic technique shows that the set of all closed-loop I/O maps are parametrized by a matrix Q, and the exponentially stable feedback systems with strictly proper controllers are globally parametrized by all strictly proper and stable Q. Having chosen a set of such Q's, themselves parametrized by $z = (z_1,\ldots,z_m)^T$, the design parameters $z_1,\ldots,z_m$ are then chosen by solution, on a computer, of an optimization problem with inequality constraints. The problem is formulated so that the design satisfies certain practical limitations, which in effect place bounds on the design parameters.

Specifically, examples are worked out, showing how to avoid plant saturation by noise or signal, and how to desensitize the closed-loop response to additive output disturbances or plant perturbations.

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## I. INTRODUCTION

The problem of designing MIMO linear time-invariant finite-dimensional feedback systems has a substantial literature. The well known paper of Youla, et al., was the first to give a parametrized family of designs [You. 1]. Recently, for the case of stable plants, Desoer and Chen proposed a very convenient and flexible design method for unity feedback systems [Des. 1]: we shall exclusively use this method in this report.

This investigation is motivated by the fact that all linear design techniques fail to consider certain practical limitations: for example, bounds on the size of the design parameters.

We propose to express some of these limitations as inequality constraints in a nonlinear programming problem. Using a computer implementation of an appropriate method [Bha. 1], and appropriate objective functions and inequality constraints, the computer determines the best feasible design in the parametrized family.

The effects we intend to consider by use of the inequality constraints are:

(1) avoidance of plant saturation by noise

(2) Desensitization of closed-loop response to additive disturbances at the plant output.

(3) Desensitization of closed-loop response to perturbations (or modeling error) in the plant.

(4) Avoidance of plant saturation by input signal.

The nonlinear programming problem will be solved by the method of Bhatti, Polak and Pister [Bha. 1]. In Appendix A, a summary of this method is given.

The organization of this paper is apparent from the listing of the contents:



Notation:

$a := b$ means "a denotes b" $\mathbb{R}$ := field of real numbers; $\mathbb{C}$ := field of complex numbers; $\mathbb{R}_+$ := set of nonnegative real numbers; $\mathbb{C}_+$, $(\mathbb{C}_-)$ := set of complex numbers such that $\text{Re } z \geq 0$ ($\text{Re } z \leq 0$, resp.). For any set A, $A^{n\times n}$ denotes the class of all n×n arrays with elements in A, and $\overset{\circ}{A}$ denotes the interior of A. Thus $\overset{\circ}{\mathbb{C}}_-$ denotes the open left half-plane. $\mathbb{C}_p(s)$, $(\mathbb{C}_{p,0}(s))$ denotes the class of all proper, (strictly proper, resp.), rational functions with coefficients in $\mathbb{C}$. $\mathcal{R}(0)$, $\mathcal{R}_0(0)$) denotes the class of all elements of $\mathbb{C}_p(s)$ ($\mathbb{C}_{p,0}(s)$ resp.) that are analytic in $\mathbb{C}_+$. If d(s) is polynomial, $\partial d$ := degree of d, $Z[d]$ := set of zeros of d. If $P \in \mathbb{R}(s)^{n\times n}$, $Z[P]$ := set of zeros of transmission of P, $P[P]$ := set of poles of P. For any $A \in \mathbb{C}^{m\times n}$,

$\bar{\sigma}[A] := \sigma_{max}[A]$, the maximum singular value of A, [Stew. 1]. Also, for any $A \in \mathbb{R}(s)^{m\times n}$, $\gamma_j[A]$ denotes the jth column of A, and $\partial\gamma_j[A]$ denotes the largest degree difference between numerator and denominator among the m rational functions in $\gamma_j[A]$.

## II. Summary of Design Algorithm

We consider the unity feedback configuration of Fig. 2.1. We define:

$$\tilde{A} = \mathbb{R}(s)^{m\times m} \qquad \mathbb{B} = R(0)^{m\times m}$$

$$A = \mathbb{R}_p(s)^{m\times m} \qquad \mathbb{B}_s = R_0(0)^{m\times m} \tag{2.1}$$

$$A_s = \mathbb{R}_{p,0}(s)^{m\times m}$$

From Fig. 2.1, assuming P, C $(I+PC)^{-1} \in A$, we obtain

$$H_{yu} = \begin{bmatrix} C(I+PC)^{-1} & -CP(I+CP)^{-1} \\ PC(I+PC)^{-1} & P(I+CP)^{-1} \end{bmatrix} \tag{2.2}$$

and

$$H_{y_2d_o} = (I+PC)^{-1} \tag{2.3}$$

Also, define

$$Q := C(I+PC)^{-1} \tag{2.4}$$

Then,

$$H_{yu} = \begin{bmatrix} Q & -QP \\ PQ & P(I-QP) \end{bmatrix} \tag{2.5}$$

and

$$H_{y_2d_o} = I-PQ \tag{2.7}$$

Theorem [Des. 1; Thm. 1, p. 410]: For the configuration of Fig. 2.1, if $P \in \mathbb{B}_s$, $C \in \tilde{A}$, $H_{yu} \in A^{2\times 2}$, then

$$Q \in \mathbb{B}_s \Leftrightarrow H_{yu} \in \mathbb{B}_s^{2\times 2} \text{ and } C \in A_s$$

Now, using the design algorithm of Desoer and Chen [Des. 1], we find a strictly proper controller such that:

(1) The closed-loop unity-feedback configuration of Fig. 2.1 is exponentially stable. (2.11)

(2) The I/O map $H_{y_2u_1}$ is decoupled and strictly proper. (2.13)

(3) In each diagonal element of $H_{y_2u_1}(s)$, the poles and zeros [in addition to the $\mathbb{C}_+$-zeros of $P(s)$], can be specified by the designer. (2.15)

2.1. Design Algorithm: [Des. 1, p. 412]

Data: $P(s) \in \mathcal{R}_0(0)^{m\times m}$

Step 1: Obtain right coprime factorization of $P(s)$:

$$P(s) = N_{pr}(s)D_{pr}(s)^{-1} \tag{2.17}$$

where $N_{pr}(s), D_{pr}(s) \in \mathbb{R}[s]^{m\times m}$

Step 2: Calculate $[\beta_1(s)\ \beta_2(s)\ \ldots\ \beta_m(s)] := N_{pr}(s)^{-1}$ where $\beta_j(s) \in \mathbb{R}(s)^m$ denotes the jth column

Step 3: If $N_{pr}(s)$ has no $\mathbb{C}_+$-zeros, set $n_{j+}(s) = 1$, $\forall s$, for $j = 1,\ldots,m$, else choose monic polynomials $n_{j+}(s)$, $\forall 1 \le j \le m$, where for each j, $n_{j+}(s)$ is of least degree and such that

$$\beta_j(s)n_{j+}(s) \in \mathbb{R}(s)^m \text{ is analytic in } \mathbb{C}_+ \tag{2.19}$$

Step 4: Choose the polynomials $\tilde{n}_j(s)$, $\forall 1 \le j \le m$, and $d_j(s)$, $\forall 1 \le j \le m$, in

$$H_{y_2u_1}(s) = \text{diag}\left[\frac{n_{j+}(s)\tilde{n}_j(s)}{d_j(s)}\right]_{j=1}^{m} \tag{2.21}$$

such that for $1 \le j \le m$,

a) $Z[d_j] \subset \mathring{\mathbb{C}}_-$ (2.23)

b) $\tilde{n}_j(s)$ is chosen freely (2.25)

c) $\partial d_j > \partial n_{j+} + \partial \tilde{n}_j + \partial \gamma_j[P^{-1}]$ (2.27)

Step 5: Calculate the required controller

$$C(s) = D_{pr}(s)N_{pr}(s)^{-1}\operatorname{diag}\left[\frac{n_j}{d_j(s)-n_j(s)}\right]_{j=1}^{m} \tag{2.29}$$

where $n_j(s) = n_{j+}(s)\tilde{n}_j(s)$, $\forall 1 \le j \le m$ (2.31)

<u>Remarks</u>:

(1) By Step 3, $P[Q] = P[N_{pr}^{-1}H_{y_2u_1}] \subset \mathring{\mathbb{C}}_-$ and, by Step 4, Q is strictly proper. So $Q \in \mathbb{B}_s$, and by the theorem, (2.11) follows

(2) In this paper, we will always choose:

a) $\tilde{n}_j(s) = 1$, $\forall s$, $\forall s$, $\forall 1 \le j \le m$ (2.33)

b) $d_j(s)$ to be a Butterworth polynomial (2.35)

with bandwidth $z_j$, $\forall 1 \le j \le m$. The vector $z = (z_1, z_2, \ldots, z_m)^T$ will be the design vector for the nonlinear programming problem.

2.2 <u>Example 1</u> (no $\mathbb{C}_+$-zeros): Consider

$$P(s) = \frac{1}{(s+2)^2(s+3)}\begin{bmatrix} s^2+8s+10 & 3s^2+7s+4 \\ 2s+2 & 3s^2+9s+8 \end{bmatrix} \in R_0(0)^{2\times 2} \tag{2.37}$$

which has a right coprime factorization

$$P(s) = N_{pr}(s)D_{pr}(s)^{-1} = \begin{bmatrix} s+4 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} s^2+3s+4 & 2 \\ 2 & s+4 \end{bmatrix}^{-1}$$

and

$$Z[P] = Z[N_{pr}] = Z[\det N_{pr}] = \{-2\} \subset \mathring{\mathbb{C}}_- \tag{2.39}$$

Now

$$N_{pr}(s)^{-1} = \frac{1}{3(s+2)}\begin{bmatrix} 3 & -3 \\ -2 & s+4 \end{bmatrix} \tag{2.41}$$

We choose $n_{1+}(s) = n_{2+}(s) = 1$. Then

$$Q(s) = \frac{1}{3(s+2)}\begin{bmatrix} \frac{3s^2+9s+8}{d_1(s)} & \frac{-(3s^2+7s+4)}{d_2(s)} \\ \frac{-2(s+1)}{d_1(s)} & \frac{s^s+8s+10}{d_2(s)} \end{bmatrix} \tag{2.43}$$

To satisfy (2.27), we choose $\partial d_1 = 2$, $\partial d_2 = 2$, i.e.,

$$d_i(s) = (\frac{s}{z_i})^2 + \sqrt{2}(\frac{s}{z_i}) + 1, \text{ for } i = 1,2 \tag{2.45}$$

Thus,

$$H_{y_2u_1}(s) = \text{diag}[\frac{1}{d_1(s)}, \frac{1}{d_2(s)}] \tag{2.47}$$

We have been careful to choose $H_{y_2u_1}(s)\big|_{s=0} = I$, so as to have good output disturbance rejection at low frequencies.

2.3 Example 2 (P has a $\mathbb{C}_+$-zero): Consider [Des. 1, p. 412]

$$P(s) = \frac{1}{(s+2)^2(s+3)}\begin{bmatrix} 3s+8 & 2s^2+6s+2 \\ s^2+6s+2 & 3s^2+7s+8 \end{bmatrix} \in \mathcal{R}_0(0)^{2\times 2}$$

which has a right coprime factorization

$$P(s) = N_{pr}(s)D_{pr}(s)^{-1} = \begin{bmatrix} 3 & 2 \\ s+2 & 3 \end{bmatrix}\begin{bmatrix} s^2+3s+4 & 2 \\ 2 & s+4 \end{bmatrix}^{-1} \tag{2.49}$$

and

$$Z[P] = Z[N_{pr}] = Z[\det N_{pr}] = \{2.5\} \subset \mathbb{C}_+ \tag{2.51}$$

Now

$$N_{pr}(s)^{-1} = \frac{1}{s-2.5}\begin{bmatrix} -1.5 & 1 \\ 0.5(s+2) & -1.5 \end{bmatrix} \tag{2.53}$$

We choose $n_{1+}(s) = n_{2+}(s) = -0.4(s-2.5)$

$$Q(s) = 0.2 \begin{bmatrix} \frac{3s^2+7s+8}{d_1(s)} & \frac{-(2s^2+6s+2)}{d_2(s)} \\ \frac{-(s^2+6s+2)}{d_1(s)} & \frac{3s+8}{d_2(s)} \end{bmatrix} \tag{2.55}$$

To satisfy (2.27), we choose $\partial d_1 = \partial d_2 = 3$, i.e.,

$$d_1(s) = \left(\frac{s}{z_1}\right)^3 + 2\left(\frac{s}{z_i}\right)^2 + 2\left(\frac{s}{z_i}\right) + 1, \text{ for } i = 1,2 \tag{2.57}$$

Thus

$$H_{y_2u_1}(s) = (-0.4)\text{diag}\left[\frac{s-2.5}{d_1(s)}, \frac{s-2.5}{d_2(s)}\right] \tag{2.59}$$

Note that $H_{y_2u_1}(s)\big|_{s=0} = I$, in this case also.

Remark: In each nonlinear programming problem of this paper, we will give two examples - based on the plants specified by (2.37) and (2.49), using $(z_1,z_2)^T$ as the design vector.

## III. General Formulation of Optimization Problem

The nonlinear programming (optimization) problem, with functional inequality constraints, is defined as:

$$\min_z f^o(z) \tag{3.1}$$

subject to:

$$\max_{\omega\in\Omega} \phi^j(z,\omega) \leq 0, \; j \in J_m \tag{3.2}$$

$$g^j(z) \leq 0, \quad j \in J_\ell \tag{3.3}$$

where:

$J_m = \{1,2,\ldots,M\}$

$J_\ell = \{1,2,\ldots,L\}$

$\Omega = [\omega_0,\omega_c] \subset \mathbb{R}$

$z \in \mathbb{R}^p$ is the design vector

Assumptions:

(A1) $f^o : \mathbb{R}^p \to \mathbb{R}$, $g^j : \mathbb{R}^p \to \mathbb{R}$, $j \in J_\ell$, are continuously differentiable in z. (3.6)

(A2) $\phi^j : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$; $\phi^j : (z,\omega) \to \phi^j(z,\omega)$, $j \in J_m$, are continuously differentiable. (3.7)

Remark:

The method of solution is a Phase I - Phase II feasible directions algorithm [Bha. 1]. For an explanation of the algorithm, see Appendix A.

IV. Design for Avoiding Plant Saturation by Noise

4.1 Problem Formulation

We propose the following problem:

$$\min_z -\left(\sum_{i=i}^{m} w_i z_i\right) \text{with} \sum_{i=1}^{m} w_i = 1 \tag{4.1}$$

subject to:

$$\bar{\sigma}[Q(j\omega,z)] \leq L_n, \ \forall\omega \in \Omega, \ \Omega = [\omega_{no},\omega_{nc}] \tag{4.2}$$

$$z_i \geq b_i, \text{ where } b_i \geq 0, \ \forall 1 \leq i \leq m \tag{4.3}$$

Clearly, (4.2) is in the form of (3.2).

4.2 Justification

We justify the use of (4.2) as follows:

Let $N_t = (N_t^{(1)},N_t^{(2)},\ldots,N_t^{(m)})$, where $N_t^{(i)}$, $t \geq 0$, $1 \leq i \leq m$ are independent white noise processes. Apply $N_t$ at $u_1$ in Fig. 2.1. Let $Y_t$ be the vector-valued process seen at $y_1$.

Then

$$S_Y(\omega,z) = Q(j\omega,z)S_N(\omega)Q^*(j\omega,z) = Q(j\omega,z)Q^*(j\omega,z) \tag{4.6}$$

since $S_N(\omega) = I$, for $N_t$ as above.

As is well known, [Stew. 1]

$$\bar{\sigma}[Q(j\omega,z)]^2 = \| Q(j\omega,z)Q^*(j\omega,z)\|_2$$

and since

$$\| Q(j\omega,z)Q^*(j\omega,z)\|_2 \geq \| Q(j\omega,z)Q^*(j\omega,z)\|_\infty$$

it follows that

$$\bar{\sigma}[Q(j\omega,z)]^2 \geq \max_i S_Y^{(j)}(\omega,z) \qquad (4.9)$$

So, if (4.2) is satisfied, (4.9) gives an upper bound on the noise power spectral density in any channel at the plant input over the frequency band of interest.

4.3 Example 1 (based on Example 1, section II)

The values used in (4.1)-(4.3) were:

$$w_1 = 0.8 \quad L_n = 2.5 \quad \omega_{n0} = 0.1 \quad b_1 = b_2 = 1.8$$
$$w_2 = 0.2 \qquad \omega_{nc} = 50 \qquad (4.10)$$

Also, the initial bandwidths, $z_1^0$, $z_2^0$ were chosen as $z_1^0 = z_2^0 = 1$, i.e., $z^0 = (1,1)^T$.

Figure 4.1 details the behavior of z as it converges to z* (the approximate solution to (4.1)-(4.3)), which is:

$$z^* = \begin{bmatrix} 2.52 \\ 1.80 \end{bmatrix} \qquad (4.12)$$

Figure 4.2 contains plots of $\bar{\sigma}[Q(j\omega,z)]$ for $z = z^0$, and $z = z^*$, over [.1,50]. Note that $\max_{\omega\in\Omega} \bar{\sigma}[Q(j\omega,z)]\big|_{z=z^*} \cong L_n$.

The resulting compensator (substituting (4.12) in (2.29)) is:

$$C(s) = \frac{1}{s(s+2)} \begin{bmatrix} \frac{2.95}{s+3.56}(3s^2+9s+8) & \frac{-1.52}{s+2.54}(3s^2+7s+4) \\ \frac{-2.95}{s+3.56}(2s+2) & \frac{1.52}{s+2.54}(s^2+8s+10) \end{bmatrix} \qquad (4.13)$$

4.4 Example 2 (based on Example 2, section II)

The values used in (4.1)-(4.3) were:

$$w_1 = 0.8 \quad L_n = 3.5 \quad \omega_{n0} = 0.1 \quad b_1 = b_2 = 1.8$$
$$w_2 = 0.2 \qquad\qquad \omega_{nc} = 50 \tag{4.15}$$
$$z^0 = (1,1)^T.$$

Figure 4.3 details the behavior of z as it converges to z*, which is:

$$z^* = \begin{bmatrix} 2.26 \\ 1.80 \end{bmatrix} \tag{4.16}$$

Figure 4.4 contains plots of $\bar{\sigma}[Q(j\omega,z)]$ for $z = z^0$, and $z = z^*$, over [.1,50]. Note that $\max_{\omega\in\Omega} \bar{\sigma}[Q(j\omega,z)]\big|_{z=z^*} \cong L_n$.

The resulting compensator (substituting (4.16) in (2.29)) is:

$$C(s) = \frac{1}{s}\begin{bmatrix} \dfrac{-5.77(3s^2+7s+8)}{s^2+4.52s+9.72} & \dfrac{2.91(2s^2+6s+2)}{s^2+3.6s+5.57} \\ \dfrac{5.77(s^2+6s+2)}{s^2+4.52s+9.72} & \dfrac{-2.91(3s+8)}{s^2+3.6s+5.57} \end{bmatrix} \tag{4.17}$$

Remarks:

The compensators (4.13) and (4.17) are such that:

(1.) (2.11) and (2.13) are true

(2.) (4.2) is satisfied

## V. Design for Desensitization to Output Disturbances and Plant Perturbations

### 5.1 Problem Formulation

We propose the following problem:

$$\min_z \left(\sum_{i=1}^{m} w_i z_i\right) \text{with} \sum_{i=1}^{m} w_i = 1 \tag{5.1}$$

subject to:

$$\bar{\sigma}[(I-PQ)(j\omega,z)] \leq L_d, \ \forall\omega \in \Omega = [\omega_{do},\omega_{dc}] \tag{5.2}$$

$$z_i \geq b_i, \text{ where } b_i \geq 0, \ \forall 1 \leq i \leq m \tag{5.3}$$

5.2 Justification

We justify the use of (5.2) as follows:

(1.) For Densitization to output disturbances

Define a vector-valued stochastic process $N_t$ as we did in section IV. Apply $N_t$ and $d_o$ in Fig. 2.1. Let $Y_t$ be the vector-valued process seen at $y_2$.

Now,

$$H_{y_2 d_o} = (I+PC)^{-1} = I-PQ \tag{5.6}$$

Thus, reasoning as in section IV,

$$\bar{\sigma}[(I-PQ)(j\omega,z)]^2 \geq \max_j S_Y^{(j)}(\omega,z) \tag{5.7}$$

So, if (5.2) is satisfied, (5.7) gives an upper bound on the noise power spectral density, in each channel at the closed-loop system output, over the frequency band of interest.

(2.) For Desensitization to plant perturbations.

Consider the unity feedback system of Fig. 2.1. Assume it is exponentially stable. Let P be perturbed into $\tilde{P}$ (and $H_{y_2u_1}$ into $\tilde{H}_{y_2u_1}$), such that the closed loop system remains exponentially stable. Define

$$\Delta H_{y_2u_1} := \tilde{H}_{y_2u_1} - H_{y_2u_1} \tag{5.8}$$

$$\Delta P := \tilde{P}-P \tag{5.9}$$

then [Saf. 1],

$$\Delta H_{y_2u_1} \cdot H_{y_2u_1}^{-1} = (I-PQ)\Delta P \cdot \tilde{P}^{-1} \qquad (5.10)$$

since [Stew. 1],

$$\| I-PQ \|_2 = \bar{\sigma}[I-PQ]$$

then

$$\| \Delta H_{y_2u_1} \cdot H_{y_2u_1}^{-1}(j\omega,z) \|_2 \leq \bar{\sigma}[(I-PQ)(j\omega,z)] \cdot \| \Delta P \cdot \tilde{P}^{-1}(j\omega,z) \|_2, \ \forall \omega \in \Omega \qquad (5.11)$$

So if (5.2) is satisfied for $L_d << 1$, it follows that

$$\| \Delta H_{y_2u_1} \cdot \tilde{H}_{y_2u_1}(j\omega,z) \|_2 << \| \Delta P \cdot \tilde{P}^{-1}(j\omega,z) \|_2, \ \forall \omega \in \Omega \qquad (5.12)$$

Thus, the closed loop I/O map from $u_1$ to $y_2$ is insensitive to perturbations in the plant, over the band $\Omega$.

5.3 Example 1 (based on Example 1, section II)

The values used in (5.1)-(5.3) were:

$w_1 = 0.8 \quad L_d = 0.3 \quad \omega_{do} = 0.01 \quad b_1 = b_2 = 0.5$

$w_1 = 0.2 \qquad \omega_{dc} = 0.5$

$z^0 = (1,1)^T$

Figure 5.1 details the behavior of z as it converges to z*, which is:

$$z^* = \begin{bmatrix} 2.38 \\ 2.38 \end{bmatrix} \qquad (5.14)$$

Figure 5.2 contains plots of $\bar{\sigma}[(I-PQ)(j\omega,z)]$ for $z = z^0$, and $z = z^*$ over [0.1,5]. Note that $\max_{\omega \in \Omega} \bar{\sigma}[(I-PQ)(j\omega,z)]\big|_{z=z^*} \cong L_d$.

The resulting compensator is:

$$C(s) = \frac{2.67}{s(s+2)(s+3.36)} \begin{bmatrix} 3s^2+9s+8 & -(3s^2+7s+4) \\ -(2s+2) & s^2+8s+10 \end{bmatrix} \qquad (5.15)$$

5.4 Example 2 (based on Example 2, section II)

The values used in (5.1)-(5.2) were:

$$w_1 = 0.8 \quad L_d = 0.6 \quad \omega_{do} = 0.01 \quad b_1 = b_2 = 0.5$$
$$w_2 = 0.2 \qquad\qquad \omega_{dc} = 0.5 \tag{5.17}$$

Figure 5.3 details the behavior of z as it converges to z*, which is:

$$z^* = \begin{bmatrix} 2.48 \\ 2.48 \end{bmatrix} \tag{5.18}$$

Figure 5.4 contains plots of $\bar{\sigma}[(I-PQ)(j\omega,z)]$ for $z = z^0$, and $z = z^*$, over [.01,.5]. Note that $\max_{\omega\in\Omega} \bar{\sigma}[(I-PQ)(j\omega,z)]\big|_{z=z^*} \cong L_d$.

The resulting compensator is:

$$C(s) = \frac{7.62}{s(s^2+4.96s+12.3)} \begin{bmatrix} -(3s^2+7s+8) & 2s^2+6s+2 \\ s^2+6s+2 & -(3s+8) \end{bmatrix} \tag{5.19}$$

Remarks:

The compensators (5.15) and (5.19) are such that:

(1.) (2.11) and (2.13) are true

(2.) (5.2) is satisfied

## VI. Design for Densitization to output Disturbances and Avoiding Plant Saturation by Noise

### 6.1 Problem Formulation

We propose the following problem:

$$\min_z\{\max_\omega \bar{\sigma}[(I-PQ)(j\omega,z)]: \omega \in \Omega_f = [\omega_{fo},\omega_{fc}]\} \tag{6.1}$$

subject to:

$$\bar{\sigma}[Q(j\omega,z)] \leq L_n, \ \forall\omega \in \Omega = [\omega_{no},\omega_{nc}] \tag{6.2}$$

$$z_i \geq b_i,\ b_i \geq 0,\ \forall 1 \leq i \leq m \tag{6.3}$$

6.2 Justification

By (4.9), (6.2) gives an upper bound on the noise power spectral density in any channel at the plant input, over $[\omega_{no}, \omega_{nc}]$.

Now, let z* be a solution to (6.1)-(6.3). Define the stochastic processes $N_t$ and $Y_t$ as we did in section V. Then, by (5.7),

$$\max_{\omega \in \Omega_f} \sigma[(I-PQ)(j\ ,z^*)]^2 \geq \max_j S_Y^{(j)}(\omega, z^*),\ \forall \omega \in \Omega_f \qquad (6.5)$$

So, by (6.5), solution of (6.1)-(6.3) gives a minimal upper bound on the noise power spectral density at $y_2$ in Fig. 2.1, over $[\omega_{fo}, \omega_{fc}]$.

By solution of (6.1)-(6.3), we also minimize the sensitivity of the closed-loop sinusoidal steady-state response to plant perturbations. This is clear from (5.11).

6.3 Example 1 (based on Example 1, section II)

The values used in (6.1)-(6.3) were:

$\omega_{fc} = 0.01 \quad L_n = 2.5 \quad \omega_{no} = 0.1 \quad b_1 = 2.1$

$\omega_{fc} = 0.5 \quad \omega_{nc} = 50 \quad b_2 = 1.7$

$z^0 = (1,1)^T$.

Figure 6.1 details the behavior of z as it converges to z*, which is:

$$z^* = \begin{bmatrix} 2.10 \\ 1.95 \end{bmatrix} \qquad (6.7)$$

Figure 6.2 contains plots of $\bar{\sigma}[(I-PQ)(j\omega,z)]$ for $z = z^0$, and $z = z^*$ over [.01,.5]. For this design, $\max\{\bar{\sigma}[(I-PQ)(j\omega,z)]\big|_{z=z^*} : \omega \in \Omega_f\} \cong .367$.

Figure 6.3 contains plots of $\sigma[Q(j\omega,z)]$ for $z = z^0$, and $z = z^*$, over [.1,50]. Note that $\max_{\omega \in \Omega} \bar{\sigma}[Q(j\omega,z)]\big|_{z=z^*} \cong L_n$.

The resulting compensator is:

$$C(s) = \frac{1}{s(s+2)} \begin{bmatrix} \frac{2.07}{s+2.96}(3s^2+9s+8) & \frac{-1.79}{s+2.75}(3s^2+7s+4) \\ \frac{-2.07}{s+2.96}(2s+2) & \frac{1.79}{s+2.75}(s^2+8s+10) \end{bmatrix} \qquad (6.8)$$

6.4 Example 2 (based on Example 2, section II)

The values used in (6.1)-(6.3) were:

$\omega_{fo} = 0.01 \quad L_n = 6.0 \quad \omega_{no} = 0.1 \quad b_1 = 3.1$

$\omega_{fc} = 0.5 \quad \omega_{nc} = 50 \quad b_2 = 2.5$

$z^0 = (1,1)^T$

Figure 6.4 details the behavior of z as it converges to z*, which is:

$$z^* = \begin{bmatrix} 3.10 \\ 2.87 \end{bmatrix} \qquad (6.10)$$

Figure 6.5 contains plots of $\bar{\sigma}[(I-PQ)(j\omega,z)]$ for $z = z^0$, and $z = z^*$, over [.01,5]. For this design, $\max\{\bar{\sigma}[(I-PQ)(j\omega,z)]\big|_{z=z^*} : \omega \in \Omega_f\} \cong .547$.

Figure 6.6 contains plots of $\bar{\sigma}[Q(j\omega,z)]$ for $z = z^0$, and $z = z^*$, over [.1,50]. Note that $\max_{\omega\in\Omega} \bar{\sigma}[Q(j\omega,z)]\big|_{z=z^*} \cong L_n$.

The resulting compensator is:

$$C(s) = \frac{1}{s} \begin{bmatrix} \frac{-14.9(3s^2+7s+8)}{s^2+6.2s+21.5} & \frac{11.8(2s^2+6s+2)}{s^2+5.74s+17.7} \\ \frac{14.9(s^2+6s+2)}{s^2+6.2s+21.5} & \frac{-11.8(3s+8)}{s^2+5.74+17.7} \end{bmatrix} \qquad (6.11)$$

Remarks:

The compensators (6.7) and (6.11) are such that:

(1.) (2.11) and (2.13) are true

(2.) (6.2) is satisfied, and an upper bound exists for $\bar{\sigma}[(I-PQ)(j\omega,z)]\big|_{z=z^*}$ over $\Omega_f$.

# VII. Design for Avoiding Plant Saturation by Input Signal

## 7.1 Problem Formulation

Consider the signals: $v_1(t)$ and $v_2(t)$ as shown in Fig. 7.1. In Fig. 2.1, denote $u_1(t) \in \mathbb{R}^2$, $y_1(t) \in \mathbb{R}^2$ as:

$$u_1(t) = \begin{bmatrix} u_{11}(t) \\ u_{12}(t) \end{bmatrix} \quad y_1(t) = \begin{bmatrix} y_{11}(t) \\ y_{12}(t) \end{bmatrix} \tag{7.1}$$

Let $u_{11}(t) = v_1(t)$, $u_{12}(t) = v_2(t)$, $\forall t \geq 0$. Call the <u>zero-state</u> response to this input $\bar{y}_1(t)$. Similarly, let $u_{11}(t) = v_2(t)$ and $u_{12}(t) = v_1(t)$, and call this zero-state response $\tilde{y}_1(t)$.

We propose the following problem:

$$\min_z -\left(\sum_{i=1}^{m} w_i z_i\right) \text{with} \sum_{i=1}^{m} w_i = 1 \tag{7.3}$$

subject to:

$$|\bar{y}_{11}(t)| \leq L_{s1},\ |\bar{y}_{12}(t)| \leq L_{s2},\ \forall t \in [0,t_c] \tag{7.4}$$

$$|\tilde{y}_{11}(t)| \leq L_{s1},\ |\tilde{y}_{12}(t)| \leq L_{s2},\ \forall t \in [0,t_c] \tag{7.5}$$

$$z_i \geq b_i,\ b_i \geq 0,\ \forall 1 \leq i \leq m \tag{7.6}$$

## 7.2 Justification

By placing small enough upper bounds on $|y_{11}(t)|$ and $|y_{12}(t)|$, the responses to some appropriate test signals, over an appropriate time interval, we can insure that the plant will not be saturated for these inputs. We have chosen $v_1(t)$, and $v_2(t)$ as in Fig. 7.1, as the test signals, for this application.

7.3 <u>Example 1</u> (based on Example 1, section II)

The values used in (7.3)-(7.6) were:

$$w_1 = 0.8 \quad L_{s1} = 2.5 \quad t_c = 6 \text{ sec.} \quad b_1 = b_2 = 1.1$$

$$w_2 = 0.2 \quad L_{s2} = 2.5 \tag{7.8}$$

$$z^0 = (2,2)^T$$

Figure 7.2 details the behavior of z as it converges to z*, which is:

$$z^* = \begin{bmatrix} 2.35 \\ 1.52 \end{bmatrix} \tag{7.10}$$

Figure 7.3 contains plots of $\bar{y}_{11}(t)$, $\bar{y}_{12}(t)$, $\tilde{y}_{11}(t)$, and $\tilde{y}_{12}(t)$, for $z = z^0$ and $z = z^*$, over [0,6].

The resulting compensator is:

$$C(s) = \frac{1}{s(s+2)} \begin{bmatrix} \frac{2.60}{s+3.32}(3s^2+9s+8) & \frac{-1.08}{s+2.14}(3s^2+7s+4) \\ \frac{-2.60}{s+3.32}(2s+2) & \frac{1.08}{s+2.14}(s^2+8s+10) \end{bmatrix} \tag{7.11}$$

7.4 <u>Example 2</u> (based on Example 2, section II)

The values used in (7.3)-(7.6) were:

$$w_1 = 0.8 \quad L_{s1} = 7.5 \quad t_c = 6 \text{ sec.} \quad b_1 = b_2 = 1.1$$

$$w_2 = 0.2 \quad L_{s2} = 7.5$$

$$z^0 = (2,2)^T$$

Figure 7.4 details the behavior of z as it converges to z*, which is

$$z^* = \begin{bmatrix} 1.20 \\ 1.19 \end{bmatrix} \tag{7.14}$$

Figure 7.5 contains plots of $\bar{y}_{11}(t)$, $\bar{y}_{12}(t)$, $\tilde{y}_{11}(t)$, and $\tilde{y}_{12}(t)$ for $z = z^0$, and $z = z^*$, over [0,6].

The resulting compensator is:

$$C(s) = \frac{1}{s}\begin{bmatrix} \frac{-0.86(3s^2+7s+8)}{s^2+2.4s+2.13} & \frac{0.84(2s^2+6s+2)}{s^2+2.38+2.09} \\ \frac{0.86(s^2+6s+2)}{s^2+2.4s+2.13} & \frac{-0.84(3s+8)}{s^2+2.38s+2.09} \end{bmatrix} \qquad (7.15)$$

Remarks:

The compensators (7.11) and (7.15) are such that:

(1.) (2.11) and (2.13) are true

(2.) (7.4) and (7.5) are satisfied

## VIII. Conclusions

The thrusts of this paper are the following:

(1.) The algebraic design theory provides us with a conveniently parametrized family of exponentially stable I/O maps; furthermore, to each of these I/O maps there corresponds a unique strictly proper controller.

(2.) The design is viewed as an optimization problem: to select the optimal parameters by maximizing or minimizing an objective function subject to some inequality constraints. Objective functions which minimize a weighted sum of normalized I/O bandwidths, maximize a similar sum, or minimize $\bar{\sigma}[(I-PQ)(j\omega,z)]$ over $\omega$ in a given frequency band, are used. Constraint functions are used which place an upper bound on $\bar{\sigma}[Q(j\omega,z)]$, or on $\bar{\sigma}[(I-PQ)(j\omega,z)]$ over specified frequency bands. Also, time domain constraint functions, which put an upper bound on the response at the plant input to certain test signals, are used.

(3.) We firmly believe that linear methods <u>alone</u> are inadequate for design purposes, because, for example, they pay no attention to the size of certain gains and certain signals. Hence, we believe that computer aided design, using inequality constraints (as in (2.) above), makes linear theory more realistic, and hence more useful.

The purpose of this experiment has been to demonstrate the feasibility of the design methodology represented by the ideas above. It is clear that in a more realistic environment, additional inequality constraints and other objective functions would have to be considered.

## IX. Acknowledgements

The authors would like to thank Dr. E. Polak for his aid and his time. The influence of his design concepts on this paper is clear. Many thanks also to Ming-Jeh Chen whose comments and suggestions were indispensable.

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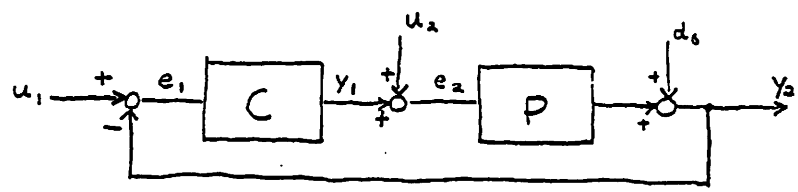


Figure 2.1

LEGEND: ( used for all subsequent functional plots )

—— ⇒ $z = z^*$

...... ⇒ $z = z_0$

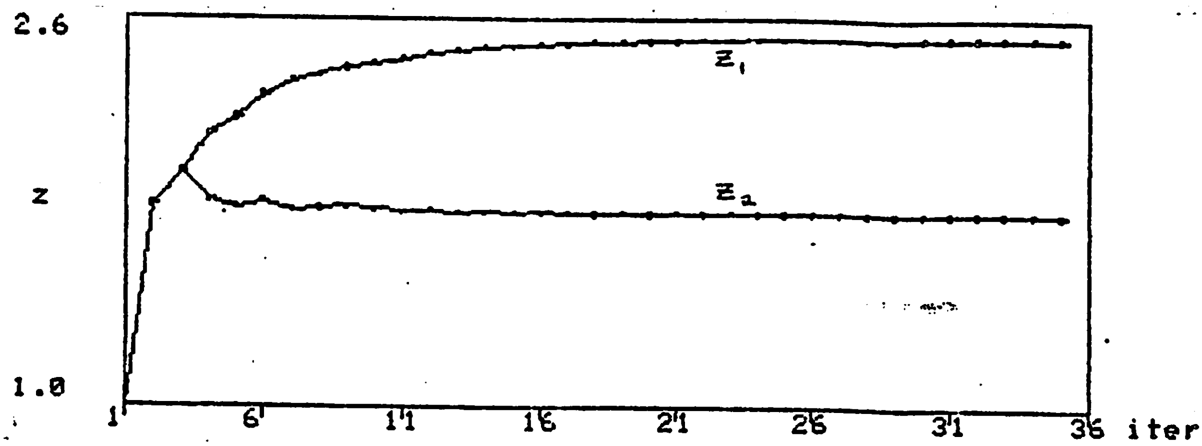


Figure 4.1

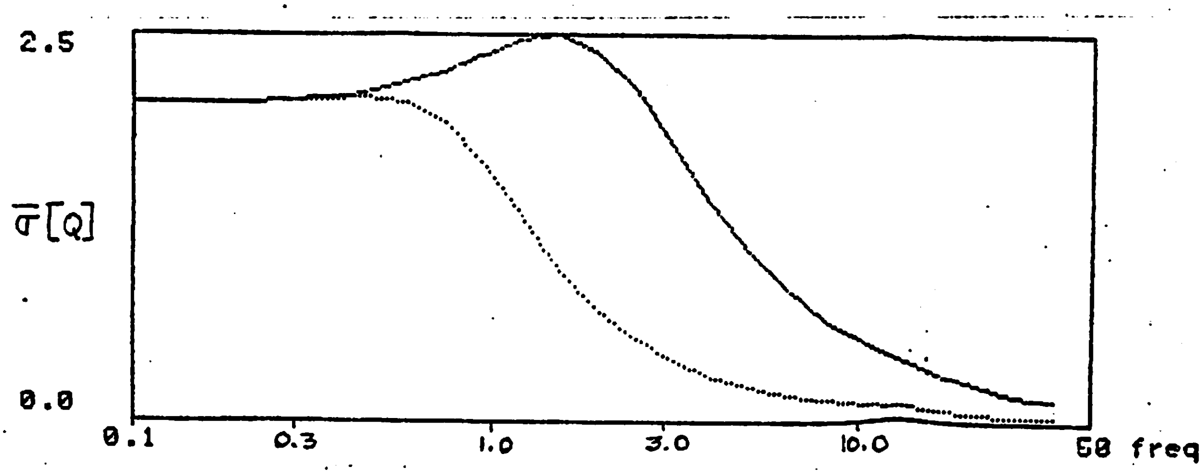


Figure 4.2

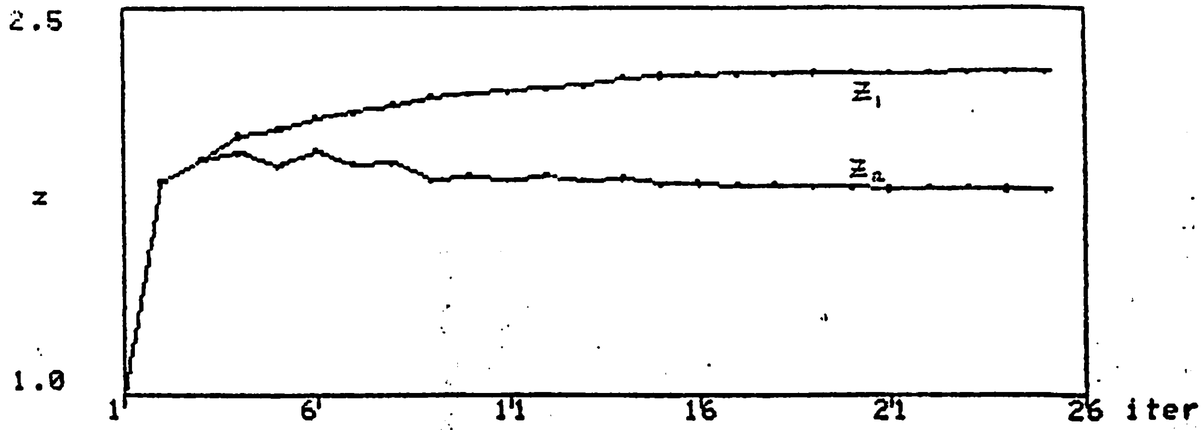


Figure 4.3

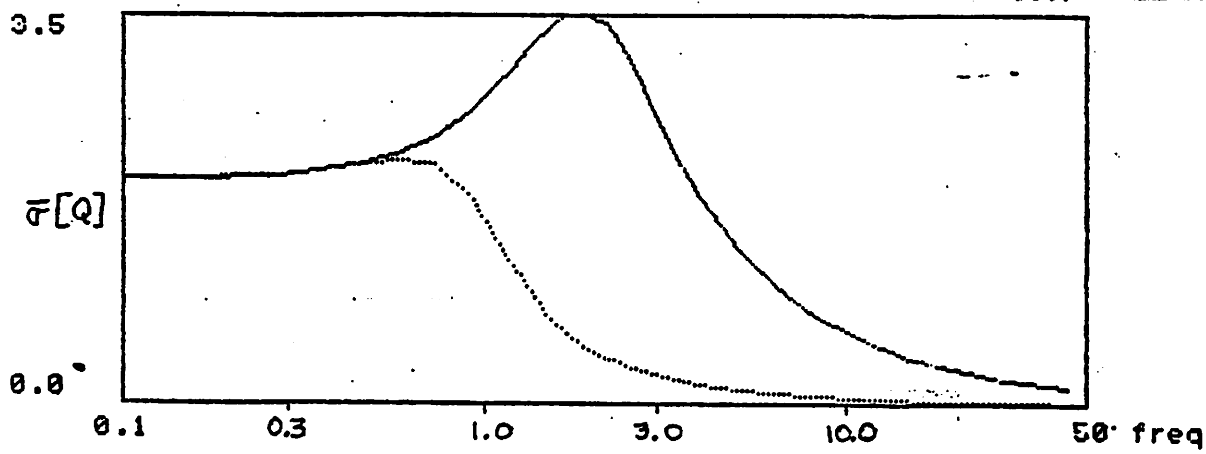


Figure 4.4

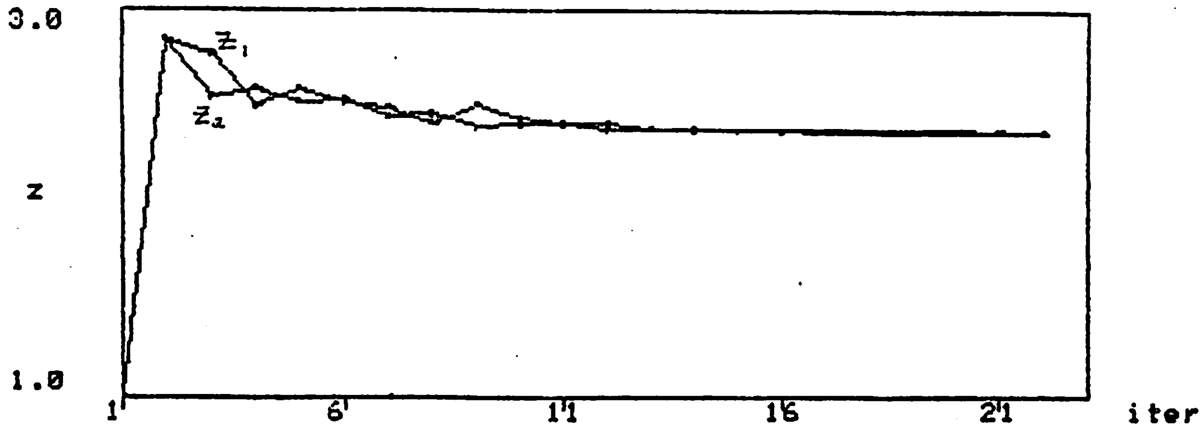


Figure 5.1

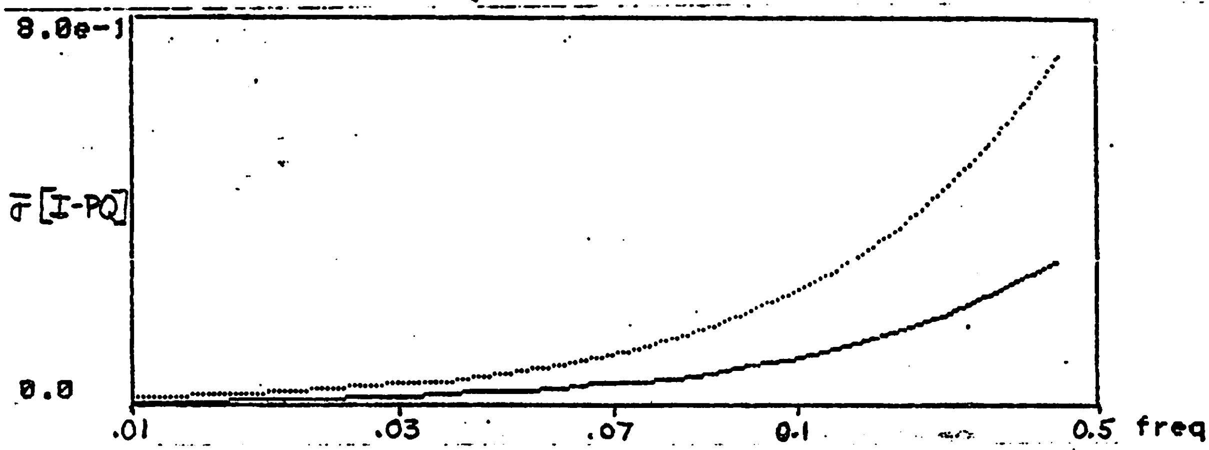


Figure 5.2

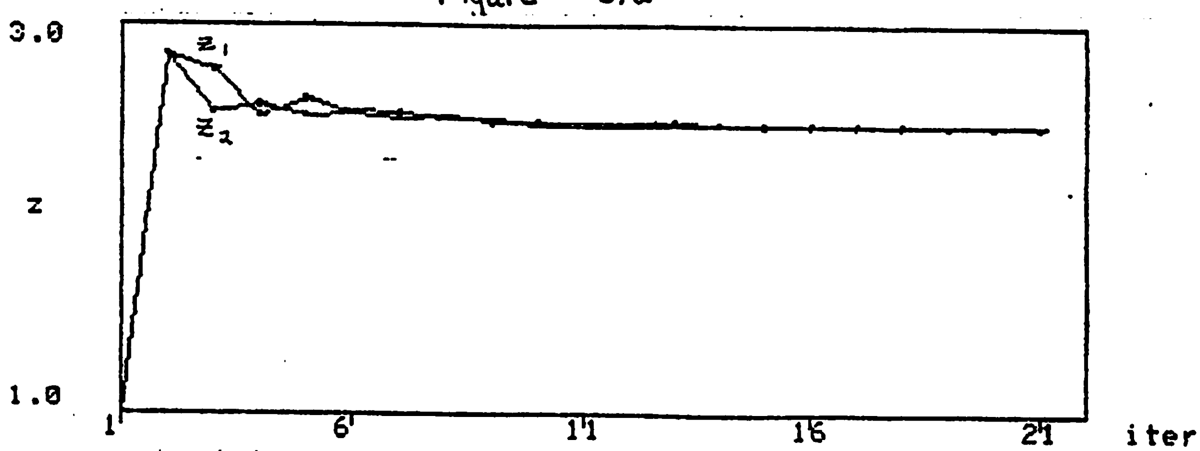


Figure 5.3

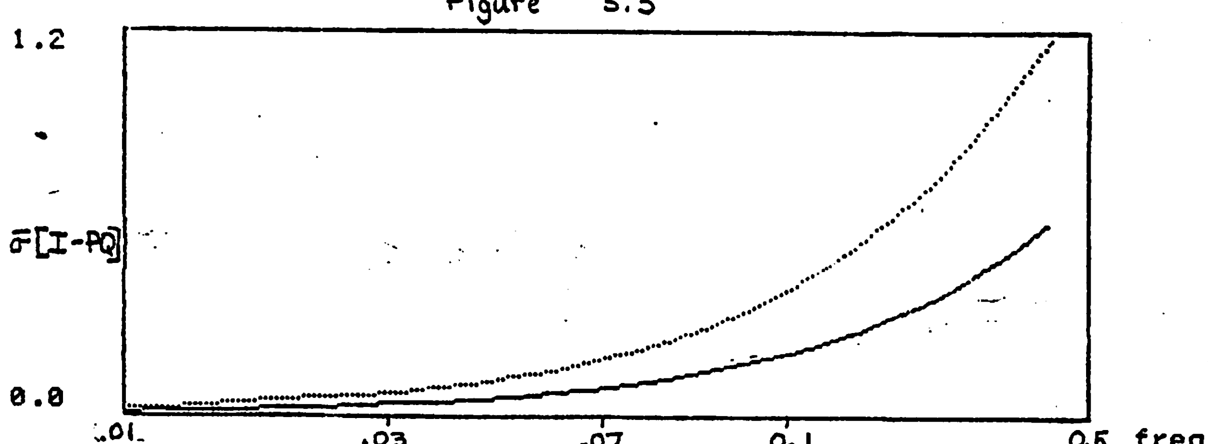


Figure 5.4

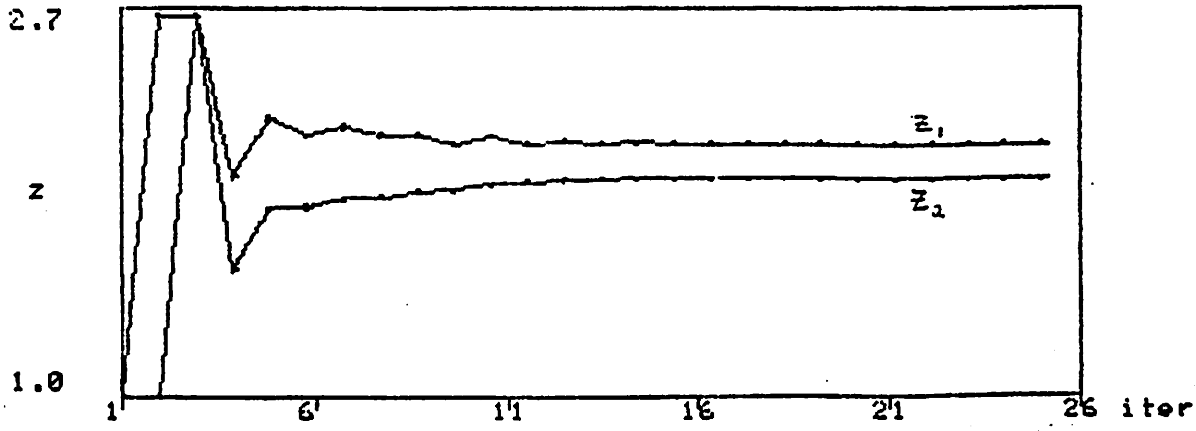


Figure 6.1

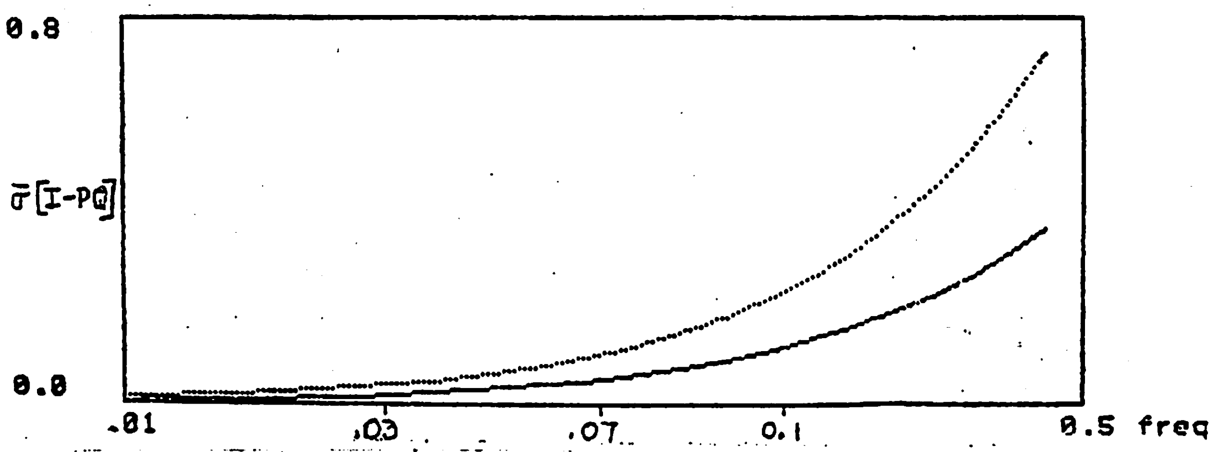


Figure 6.2

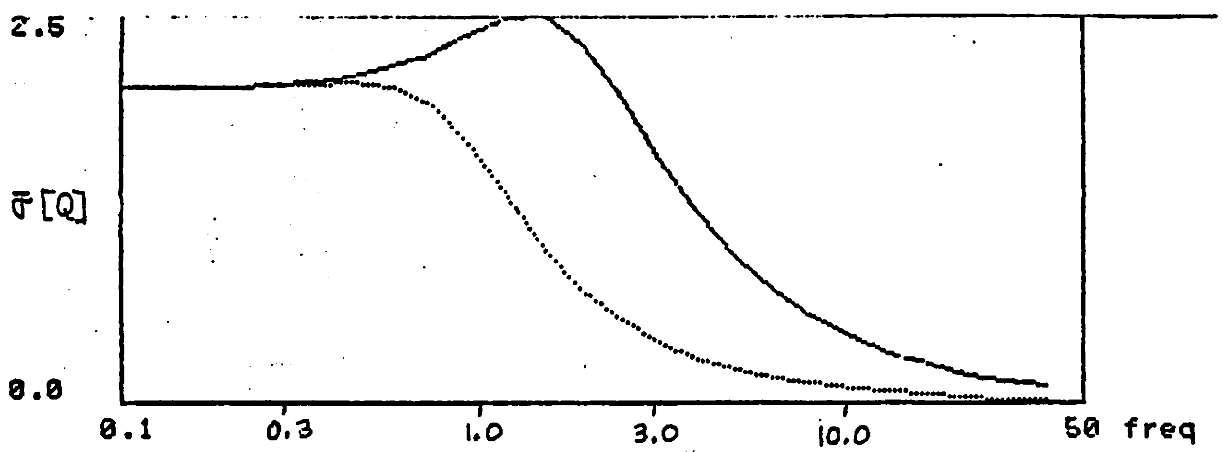


Figure 6.3

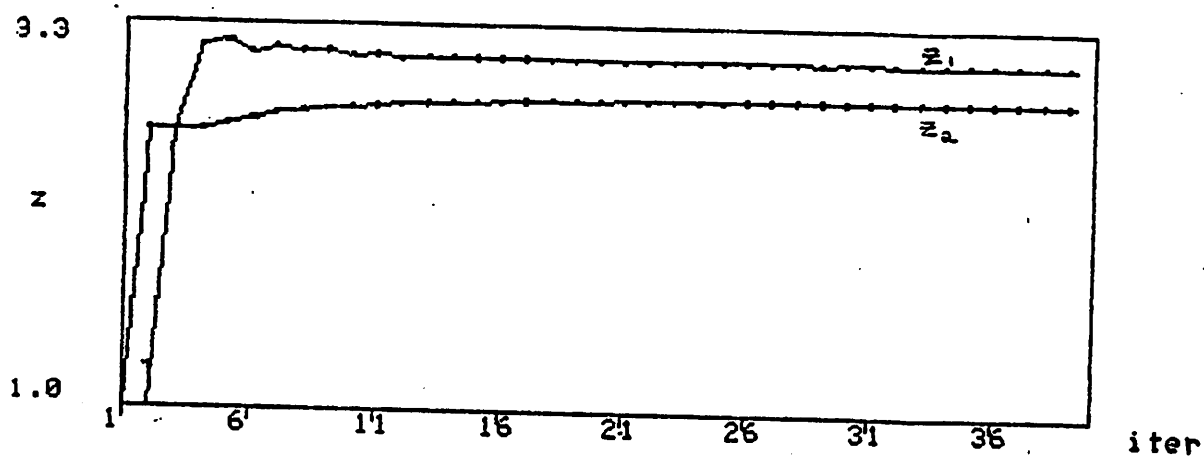


Figure 6.4

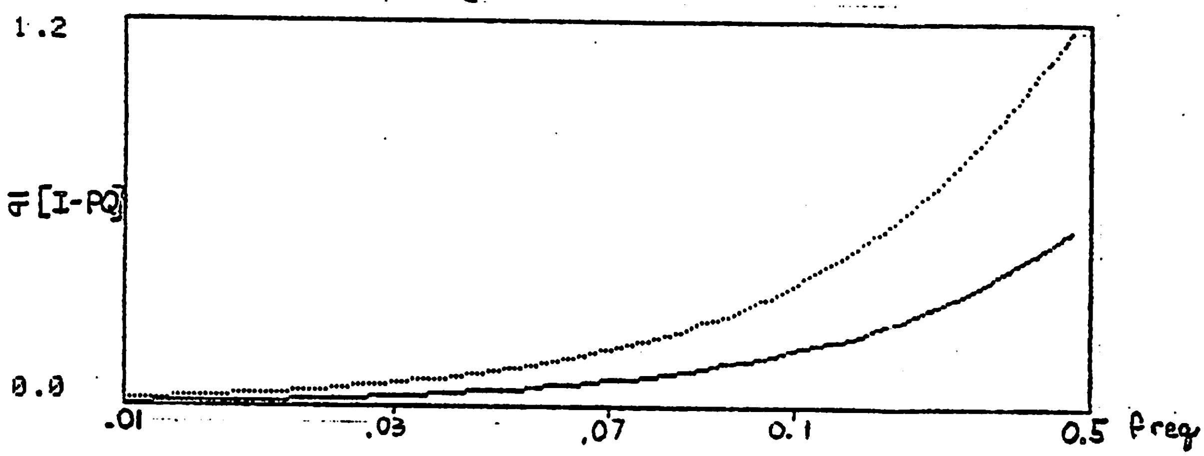


Figure 6.5

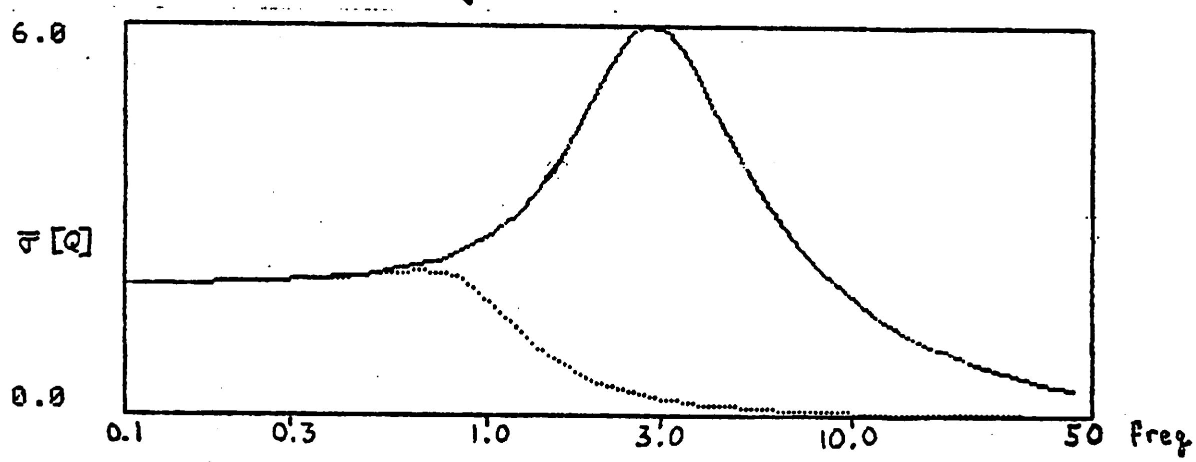


Figure 6.6

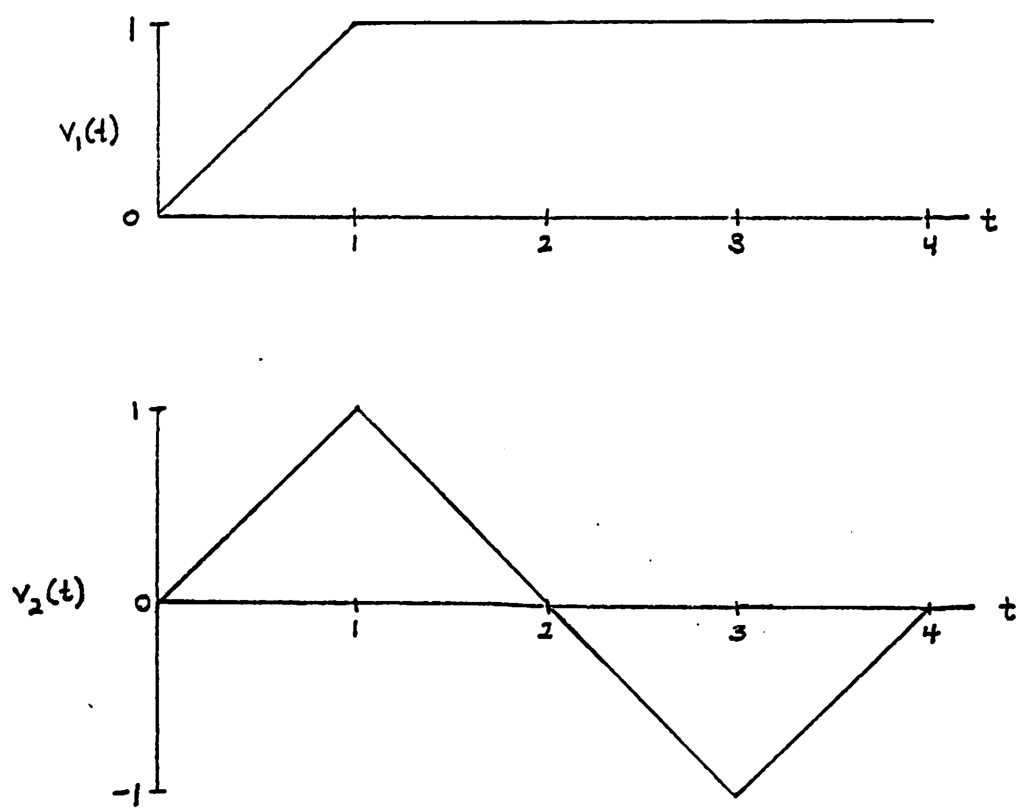


Figure 7.1

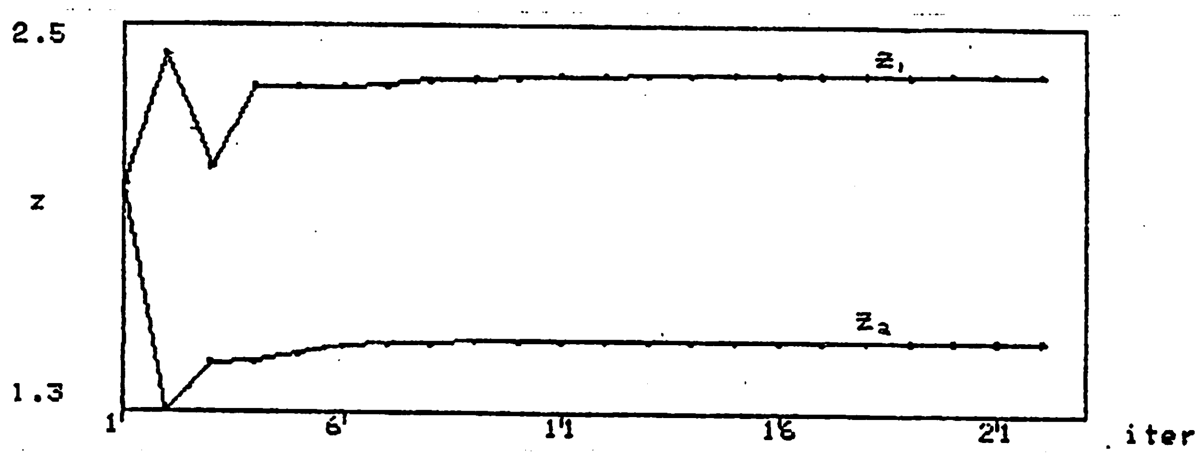


Figure 7.2

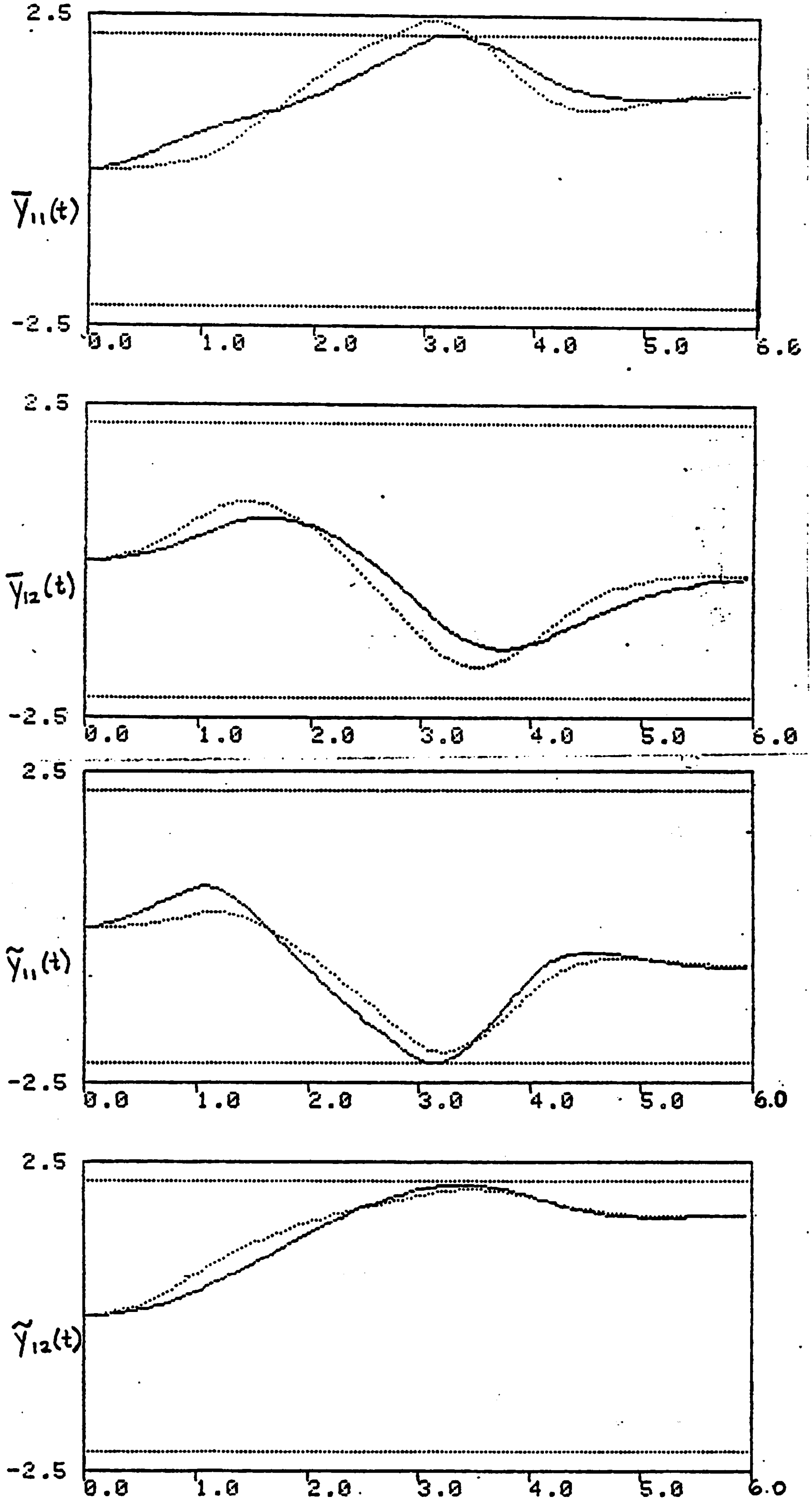
2.5
-2.5
$\bar{y}_{11}(t)$
$\bar{y}_{12}(t)$
$\tilde{y}_{11}(t)$
$\tilde{y}_{12}(t)$
0.0
1.0
2.0
3.0
4.0
5.0
6.0

Figure 7.3

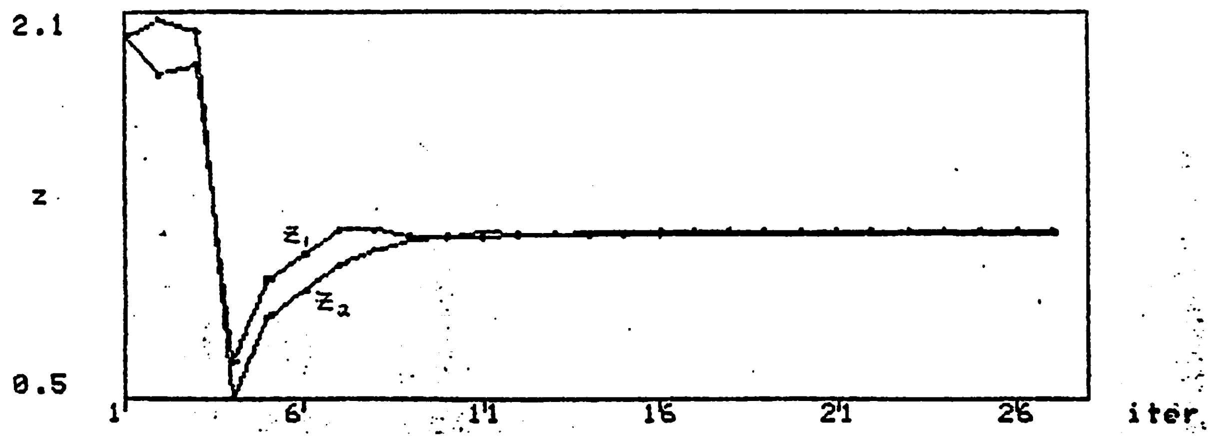


Figure 7.4

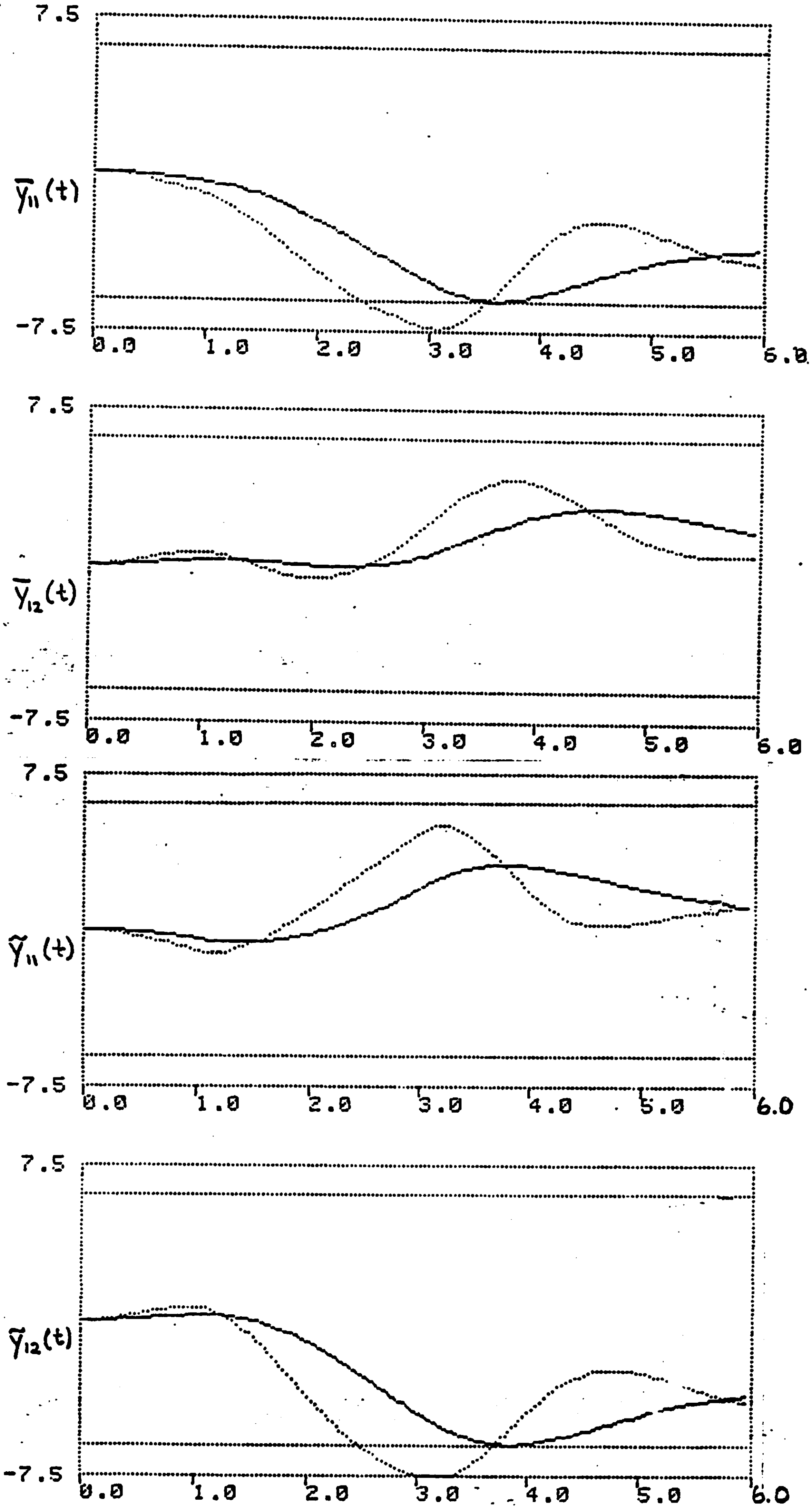
7.5
$\overline{y}_{11}(t)$
-7.5
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.5
$\overline{y}_{12}(t)$
-7.5
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.5
$\tilde{y}_{11}(t)$
-7.5
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.5
$\tilde{y}_{12}(t)$
-7.5
0.0
1.0
2.0
3.0
4.0
5.0
6.0

Figure 7.5

APPENDIX A

## I. Problem formulation

The nonlinear programming problem, with functional inequality constraints is defined as:

$$\min_{z} f^0(z) \tag{1.1}$$

subject to:

$$\max_{\omega \in \Omega} \phi^j(z,\omega) \leq 0, \ j \in J_m \tag{1.2}$$

$$g^j(z) \leq 0, \ j \in J_\ell \tag{1.3}$$

where

$$J_m = \{1,2,\ldots,M\}$$

$$J_\ell = \{1,2,\ldots,L\}$$

$$\Omega = [\omega_0,\omega_c], \text{ specified interval} \tag{1.5}$$

$z \in \mathbb{R}^p$ is the design vector

Assumptions:

(A1) $f^0 : \mathbb{R}^p \to \mathbb{R}$, $g^j : \mathbb{R}^p \to \mathbb{R}$, $j \in J_\ell$ are continuously differentiable in z.

(A2) $\phi^j : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$, $j \in J_m$ are continuously differentiable in z and $\omega$.

Each inequality in (1.2) is transformed into q+1 inequalities by fixing $\omega$ at q distinct points in $\Omega$. Denote this set as $\Omega_q$.

## II. Optimality Function Definition - Active Constaints

Define:

$$\tilde{\psi}_q(z) = \max\{\phi^j(z,\omega), \ j \in J_m, \ \omega \in \Omega_q; \ g^j(z), \ j \in J_\ell\} \tag{2.1}$$

$$\psi_q(z) = \max\{0,\tilde{\psi}_q(z)\} \tag{2.2}$$

The set of points for which a functional constraint is active is denoted by:

$$\bar{\Omega}^j_{q,\varepsilon}(z) \triangleq \{\omega \in \Omega_q \mid \phi^j(z,\omega) - \psi_q(z) \geq -\varepsilon\},\ j \in J_m \tag{2.3}$$

We define the "ε-active constaint index" set for functional constaints by choosing the local maximum of each subinterval of $\Omega_q$, whose points are all active.

For example, in figure I, we define two active leftmost local maxima $(\omega_2, \omega_8)$.

Let $\Omega^j_{q,\varepsilon}$ = {all active local maxima for $\phi^j(z,\omega)$, $j \in J_m$}.

Now, the "ε-active constaint index" set for functional constaints is defined as follows:

$$J^\phi_{\varepsilon,q}(z) = \{(j,\omega) \mid j \in J_m,\ \omega \in \Omega^j_{q,\varepsilon}(z)\} \tag{2.5}$$

The ε-active constaint index set for conventional constraints is defined by:

$$J^g_{\varepsilon,q} = \{j \mid g^j(z) - \psi_q(z) \geq -\varepsilon,\ j \in J_\ell\} \tag{2.6}$$

The optimality function $\theta_{\varepsilon,q}(z)$: $\mathbb{R}^p \to \mathbb{R}$ for the nonlinear programming problem is:

$$\theta_{\varepsilon,q}(z) = \min_{h \in \mathbb{R}^p} \{\tfrac{1}{2}\|h\|_2^2 + \max\{\langle \nabla f^0(z), h\rangle - \gamma\psi_q(z);\ \langle \nabla g^j(z), h\rangle,\ j \in J^g_{\varepsilon,q}(z);\ \langle \nabla_z \phi^j(z,\omega), h\rangle,\ (j,\omega) \in J^\phi_{\varepsilon,q}(z)\}\} \tag{2.7}$$

<u>Theorem 1</u>:

If z is optimal for (1.1)-(1.3), then $\theta_{0,q}(z) = 0$.

The optimality function $\theta_{\varepsilon,q}(z)$ in (2.7) generalizes the Zoutendijk optimality function for finding Fritz-John points, by adding:

(a) Constaint boundary smearing - when a constaint becomes ε-active, it's gradient is included in (2.7), which rotates h away from the constant boundary. This prevents jamming.

(b) Quadratic term: $\frac{1}{2}\|h\|_2^2$ - This prevents deterioration of $h(\langle \nabla f^0(z),h\rangle \cong 0)$ for feasible z. Also, minimization in (2.7) need not be done on a compact set.

(c) The term: $-\gamma\psi_q(z)$, $\gamma \geq 1$. For infeasible z, this term effectively removes $\langle \nabla f^0(z),h\rangle$ from (2.7). Thus h will depend only on gradients of active constraints.

$\theta_{\varepsilon,q}(z)$ is actually solved in it's dual form:

$$\min_{\mu \geq 0}\{\frac{1}{2}\mu^T Q\mu + D^T\mu \mid R^T\mu = 1\} \tag{2.9}$$

where

$$Q \triangleq AA^T,\ A \in \mathbb{R}^{(1+k+v)\times p},\ \mu \in \mathbb{R}^{1+k+v}$$

$$A \triangleq \left[\frac{\nabla f^0(z)^T}{s_0} \,\middle|\, \frac{\nabla g^{k_1}(z)^T}{s_g^{k_1}} \,\middle|\, \cdots \,\middle|\, \frac{\nabla g^{k_k}(z)^T}{s_g^{k_k}} \,\middle|\, \frac{\nabla\phi^{j_1}(z,\omega_\ell)^T}{s_\phi^{j_1\ell_1}} \,\middle|\, \cdots \,\middle|\, \frac{\nabla\phi^{j_v}(z,\omega_{\ell v})^T}{s_\phi^{j_v\ell_v}}\right]^T \tag{2.10}$$

with

$$s_0 \triangleq \|\nabla f^0(z)\|_\infty \tag{2.11}$$

$$s_g^j \triangleq \|\nabla g^j(z)\|_\infty,\ j \in J_\ell \tag{2.12}$$

$$s_\phi^{j\ell} \triangleq \|\nabla\phi^j(z,\omega_\ell)\|_\infty,\ j \in J_m,\ \omega_\ell \in \Omega_q \tag{2.13}$$

$$k \triangleq \text{number of elements of } J^g_{\varepsilon,q}(z) \tag{2.14}$$

$$v \triangleq \text{number of elements of } J^\phi_{\varepsilon,q}(z) \tag{2.15}$$

and

$$D^T = [\gamma\psi_q(z)/s_0,0,0,\ldots,0] \in \mathbb{R}^{1+k+v} \tag{2.16}$$

$$R^T = [\rho_0, \rho_g^{k_1}, \ldots, \rho_g^{k_k}, \rho_\phi^{j_1 \ell_1}, \ldots, \rho_\phi^{j_v \ell_v}] \quad (2.16)$$

for "push factors":

$$\rho_0 = \xi_0(1/s_0 - 1) \quad (2.18)$$

$$\rho_g^j = \xi_g^j + \eta\left[1 + \frac{g^j(z)-\psi_q(z)}{\varepsilon}\right]^2; \; j \in J_\ell \quad (2.19)$$

$$\rho_\phi^{j,\ell} = \xi_\phi^j + \eta\left[1 + \frac{\phi^j(z,\omega_\ell)-\psi_q(z)}{\varepsilon}\right]^2 ; \; j \in J_m, \; \omega_\ell \in \Omega_q \quad (2.20)$$

with input parameters: (to the optimization algorithm)

$$\xi_0; \; \xi_g^j, \; j \in J_\ell; \; \xi_\phi^j, \; j \in J_m; \; \eta \quad (2.21)$$

From $\mu \in \mathbb{R}^{1+k+v}$, we calculate the direction vector h:

$$h_{\varepsilon,q}(z)^T = -\mu^T A \quad (2.22)$$

Solving (2.7) and (2.22) yields a vector h which is a convex combination of the gradients of the active constraints and the cost gradient. The h chosen is specifically, the convex combination with minimum Euclidean norm. The set of convex combinations can be changed by varying the parameters (2.21). Increasing $\rho_0$ in (2.17), for example, will "push" h closer to $-\nabla f^0(z)$. For an example with $p = 2$ ($z \in \mathbb{R}^2$) see figure 2 below.

In figure II, $g_1$, $g_2$, $g_f$ are gradients at $z^*$, of active constraints $f_1$, $f_2$ and cost $f^0$ respectively. G is the convex hull of $g_1$, $g_2$, $g_f$.

## III. Algorithm

A Phase I - Phase II feasible directions algorithm is used to solve (1.1)-(1.3).

DATA: $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \geq 1$, $\delta \in (0,1]$, $\varepsilon_0 > 0$, $\mu_1 > 0$, $\mu_2 > 0$, $M > 0$, $q_0$, $q_{max}$, $z_0 \in \mathbb{R}^p$.

STEP 0: Set $i = 0$, $q = q_0$

STEP 1: Set $\varepsilon = \varepsilon_0$

STEP 2: Compute $[\theta_{\varepsilon,q}(z^i), h_{\varepsilon,q}(z^i)]$ from (2.7), (2.22)

STEP 3: If $\theta_{\varepsilon,q}(z^i) \leq -2\varepsilon d$ go to step 6.

STEP 4: First, set $\varepsilon = \frac{\varepsilon}{2}$. If $\varepsilon < \varepsilon_0 \frac{\mu_1}{q}$ and $\psi_q(z^i) < \frac{\mu_2}{q}$, set $q = 2q$ and go to step 5. Else, go to step 2.

STEP 5: If $q > q_{max}$, STOP. Else, go to step 1.

STEP 6: Compute the largest step size $\lambda_i = \beta^k \in (0, M^*]$, where

$M^* = \max 1, \frac{M}{\|h_{\varepsilon,q}(z^i)\|_\infty}$ and k an integer such that:

(i) if $z^i \in F^c$ (not feasible)

$$\psi_q\{z^i + \lambda_i h_{\varepsilon,q}(z^i)\} - \psi_q(z^i) \leq -\alpha\lambda_i\delta\varepsilon$$

(ii) if $z^i \in F$ (feasible)

$$f^0\{z^i + \lambda_i h_{\varepsilon,q}(z^i)\} - f^0(z^i) \leq -\alpha\lambda_i\delta\varepsilon$$

$$z^{i+1} \in F \text{ (feasible)}$$

STEP 7: Set $z^{i+1} = z^i + \lambda_i h_{\varepsilon,q}(z^i)$. Set $i = i+1$, go to step 2.

If $z^i$ is feasible, an Armijo step is taken on the cost, with $z^{i+1}$ constrained to be feasible.

For $z^i$ not feasible, an Armijo step is taken on $\psi_q(z^i)$. As $\theta_{\varepsilon,q}(z^i)$ grows small, $\varepsilon$ is halved. As $\varepsilon$ is decreased, q will eventually increase, depending on $\mu_1$, $\mu_2$. For $q > q_{max}$, execution ceases.

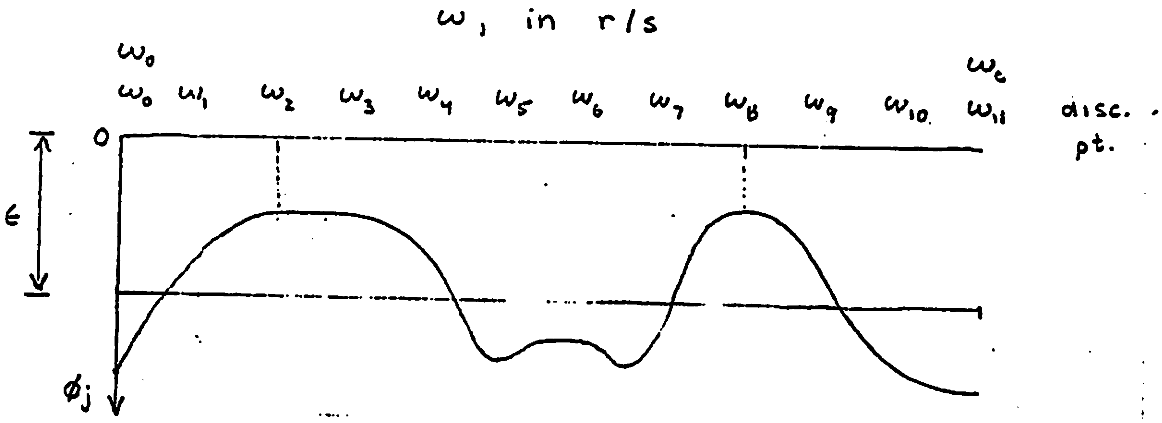


Figure I

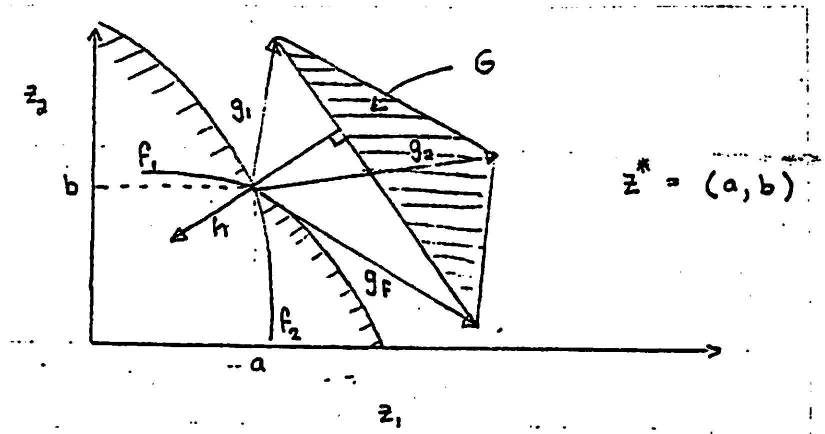


Figure II