GLOBAL PARAMETRIZATION OF FEEDBACK SYSTEMS

by

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ABSTRACT

For unity-feedback systems with a plant P and controller C, we show that, given a linear or nonlinear stable P, the class of all controllers stabilizing the feedback system is globally parametrized by a stable map Q. We also show how Q is useful in studying the effects of modeling errors on the stability.

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I. Linear Case

In the spirit of Desoer et al. [1], let G be a commutative ring of (causal) transfer functions and let H be the subring of G consisting of stable transfer functions. (e.g. $G = \mathbb{R}_p(s)$ whose elements are analytic in U with $U \supset \mathbb{C}_+$ and U is symmetric with respect to the real axis, $\mathbb{C}_+$ is the closed right half plane). The $m \times n$ matrix M is called H-stable iff $M \in H^{m\times n}$.

Consider on Fig. 1 the linear feedback system Σ_L with inputs (u_1,u_2), errors (e_1,e_2) and outputs (y_1,y_2). Let $P \in H^{n_o \times n_i}$ and $C \in G^{n_i \times n_o}$, we say that Σ_L is well-posed iff the maps $H_{yu} : (u_1,u_2) \mapsto (y_1,y_2)$ and $H_{eu} : (u_1,u_2) \mapsto (e_1,e_2) \in G^{n\times n}$ where $n := n_i+n_o$. Σ_L is said to be H-stable iff H_{eu} and H_{yu} have all their elements in H.

Theorem L Let $P \in H^{n_o \times n_i}$; let $C \in G^{n_i \times n_o}$ be such that Σ_L is well posed; then there is a $C \in G^{n_i \times n_o}$ that H-stabilizes Σ_L if and only if

$$\text{for some } Q \in H^{n_i \times n_o}, \; C = Q(I-PQ)^{-1}. \tag{1}$$

Furthermore, for that C,

$$H_{y_2 u_1} = PQ \tag{2}$$

Comments (a) Eqn. (2) specifies all I/O maps $H_{y_2u_1} \in H^{n_o \times n_o}$ achievable from P using the configuration of Fig. 1. (b) Eqn. (1) shows that $Q \in H^{n_i \times n_o}$ globally parametrizes all H-stabilizing controllers. These controllers are not necessarity H-stable, but have elements in G. (c) Theorem L is a slight extension to the non-square case of Desoer et al. [2] which is based on a parametrization of Zames [7] and a stability theorem of Desoer et al. [3].

(d) Eqn. (2) is equivalent to

$$Q = C(I+PC)^{-1} \tag{3}$$

where the inverse is well defined since $C(I+PC)^{-1} = H_{y_1u_1}$ for the system Σ_L.

II. Nonlinear Case

Let $(L,\|\cdot\|)$ be a normed space of "time-functions" : $T \to V$ where T is the time-set (typically $\mathbb{R}_+$ or $\mathbb{N}$), V is a normed space (typically, $\mathbb{R}$, $\mathbb{R}^n, \mathbb{C}^n, \ldots$) and $\|\cdot\|$ is the chosen norm on L. Let L_e be the corresponding extended space [4,5,6].

Consider the system Σ_N shown in Fig. 1 where now $u_1, e_1, y_2 \in L_e^{n_o}$ and $u_2, e_2, y_1 \in L_e^{n_i}$; C and P are nonlinear causal maps

$$C : L_e^{n_o} \to L_e^{n_i} \; , \; P : L_e^{n_i} \to L_e^{n_o} \tag{4}$$

The system Σ_N is assumed to be well-posed, i.e., H_{eu} and H_{yu} are well defined causal maps from $L_e^n \to L_e^n$. Σ is said to be finite gain (f.g.) stable iff Σ is well-posed and H_{eu} and H_{yu} are f.g. stable; more precisely, H_{yu} is said to be f.g. stable iff $\exists \gamma < \infty$ and $\beta < \infty$ s.t. $\forall T \in T \quad \forall (u_1,u_2) \in L_e^n$,

$$\|H_{yu}(u_1,u_2)\|_T \le \gamma \cdot (\|u_1\|_T + \|u_2\|_T) + \beta. \tag{5}$$

P is said to have finite incremental gain $\tilde{\gamma}$ iff $\forall x, x' \in L_e^{n_i}$, $\forall T \in T$

$$\|Px-Px'\|_T \le \|x-x'\|_T \tag{6}$$

We shall use repeatedly the fact that the sum and the composition of f.g. stable maps is f.g. stable.

Theorem N. Let P and C be defined as above. Call Σ_N the nonlinear system whose configuration is shown on Fig. 1. Assume that Σ_N is well-posed and that P has finite gain and finite incremental gain:

$\gamma(P) < \infty$ and $\tilde{\gamma}(P) < \infty$ (7)

Under these conditions,

a) $H_{yu} : (u_1,u_2) \mapsto (y_1,y_2)$ is finite gain stable (8)

$\Leftrightarrow$

for some f.g. stable $Q : L_e^{n_o} \to L_e^{n_i}$, $C = Q(I-PQ)^{-1}$. (9)

b) Furthermore, in terms of C and P, Q is given by

$$Q = C(I+PC)^{-1}. \tag{10}$$

c) With $u_2 = 0$,

$$H_{y_2u_1} = PQ . \tag{11}$$

<u>Comments.</u> (a) Eqn. (2) and Eqn. (11) have the same form; however in eqn. (2), we have a product of matrix transfer functions whereas in eqn. (11) we have the <u>composition</u> of causal <u>nonlinear</u> maps.

(b) With Σ_N finite gain stable, $Q = H_{y_1u_1}\big|_{u_2=0}$ is also f.g. stable: Eqn. (11) shows that all f.g. stable Q's <u>globally parameterize</u> all f.g. stable I/O maps $H_{y_2u_1}$ of Σ_N. Equations (9) shows that <u>all</u> f.g. stable Q's globally parametrize <u>all</u> compensators C that lead to a f.g. stable Σ_N.

<u>Proof.</u>

In [3], it is shown that given (7), H_{yu} is f.g. stable if and only if $C(I+PC)^{-1}$ is f.g. stable. So if we set $Q := C(I+PC)^{-1}$ we have eqn. (10). It remains to calculate C in terms of P and Q. From the above definition of Q,

$$I-PQ = I-PC(I+PC)^{-1} = (I+PC)^{-1} \tag{12}$$

(where the inverse exists since Σ_N is well posed), note that in (12) we used the fact that the composition of nonlinear maps distributes on the left. Composing both sides of (12) on the left with the nonlinear map C and using (10) we obtain $Q = C(I-PQ)$. Now compose both sides of this last eqn. with $(I-PQ)^{-1}$. Hence $C = Q(I-PQ)^{-1}$. Thus claims a), b) and c) are established.

III. Perturbation

We use the map Q to determine when a nonlinear, not necessarily small, plant perturbation ΔP will maintain stability.

Let Σ be such that $C : L_e^{n_o} \to L_e^{n_i}$, $P : L_e^{n_i} \to L_e^{n_o}$ are linear maps. Let P undergo a nonlinear perturbation ΔP thus becoming $\tilde{P} = P+\Delta P$. The result is a nonlinear system Σ_N with inputs (u_1,u_2) and outputs $(\tilde{e}_1,\tilde{e}_2,\tilde{y}_1,\tilde{y}_2)$. Let both Σ and Σ_N be well-posed.

Σ and Σ_N have the configuration of Fig. 1 hence, in both instances,

$$H_{eu} \text{ f.g. stable} \Leftrightarrow H_{yu} \text{ f.g. stable.} \tag{13}$$

For Σ, we have

$$\begin{bmatrix} I & P \\ -C & I \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \text{ equivalently, } e = Hu \tag{14}$$

where

$$H = M^{-1} = \begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{15}$$

is a linear map. Note that by (3), $H_{21} = C(I+PC)^{-1} = Q$.

For Σ_N, we have

$$\begin{bmatrix} I & \tilde{P} \\ -C & I \end{bmatrix}\begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \text{ equivalently, } \tilde{e} = \tilde{H}u \tag{16}$$

where $\tilde{H}$ is a nonlinear map.

Let Σ_p be the system shown on Fig. 2; Σ_p is described by

$$\begin{bmatrix} I & \Delta P \\ -Q & I \end{bmatrix} \begin{bmatrix} \check{e}_1 \\ \check{e}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ H_{22}u_2 \end{bmatrix} \tag{17}$$

<u>Theorem P</u>. Assume Σ f.g. stable and let $\Delta P : L_e^{n_i} \to L_e^{n_o}$. Then

a) Σ_p is f.g. stable $\Rightarrow \Sigma_N$ is f.g. stable.

b) If, in addition, either P or ΔP is f.g. stable, then Σ_N is f.g. stable $\Rightarrow \Sigma_p$ is f.g. stable.

<u>Comments</u>. (a) Neither P nor ΔP are assumed f.g. stable.

(b) Suppose that 1) we are given a <u>nonlinear</u> plant $\tilde{P}$; 2) we approximate it with a <u>linear</u> P (hence, error ΔP); 3) we obtain a family of designs using a <u>linear</u> controllers C (see [1], or [2] if P is stable). Then using the corresponding Q and Σ_p with Theorem P, we can verify the stability of the <u>actual nonlinear</u> system.

<u>Proof</u> (I) Consider Σ_N: from $\tilde{P} = P+\Delta P$, (16) and (15), and using the linearity of $H = M^{-1}$, we see that $(\tilde{e}_1,\tilde{e}_2)$ is a solution of (16) iff it is a solution of the following

$$\begin{bmatrix} I & (I+PC)^{-1}\Delta P \\ 0 & I+C(I+PC)^{-1}\Delta P \end{bmatrix} \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{18}$$

(II) Consider Σ_p: since Q is linear, we perform row (2) $\leftarrow$ row (2) + Q $\cdot$ row (1) on eqn. (17) then (17) is equivalent to

$$\begin{bmatrix} I & \Delta P \\ 0 & I+Q\Delta P \end{bmatrix} \begin{bmatrix} \check{e}_1 \\ \check{e}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ H_{22}u_2+Qu_1 \end{bmatrix} \tag{19}$$

<u>(III) Proof of (a)</u>: Σ_p f.g. stable $\Rightarrow \Sigma_N$ f.g. stable. By assumption Σ_p is f.g. stable; hence, by (17), the map $(u_1,u_2) \mapsto (\check{e}_1,\check{e}_2)$ is f.g. stable; by (19), this implies $(u_1,u_2) \mapsto (\Delta P\check{e}_2,\check{e}_2)$ is f.g. stable.

Comparing with (18) and noting that Σ is f.g. stable and that $\tilde{e}_2 = \check{e}_2$ we conclude that the map $(u_1,u_2) \mapsto (\tilde{e}_1,\tilde{e}_2)$ of Σ_N is f.g. stable.

(IV) <u>Proof of (b)</u>: Σ_N is f.g. stable $\Rightarrow \Sigma_P$ is f.g. stable.

<u>Case 1</u>. ΔP f.g. stable. Since the second eqn. of (18) and (19) are identical and since Σ_N is f.g. stable, $(u_1,u_2) \mapsto \check{e}_2$, defined by (19), is f.g. stable. By inspection of (19), since ΔP is f.g. stable, $(u_1,u_2) \mapsto (\check{e}_1,\check{e}_2)$ is f.g. stable.

<u>Case 2</u>. P is f.g. stable. As in case 1, we have $(u_1,u_2) \mapsto \check{e}_2$ is f.g. stable. From (16) and (17) and using $\tilde{e}_2 = \check{e}_2$, we have

$$\check{e}_1 = u_1 - \Delta P\tilde{e}_2 = e_1 + P\tilde{e}_2 \tag{20}$$

Thus, since $(u_1,u_2) \rightarrow (\tilde{e}_1,\tilde{e}_2)$ and P are f.g. stable, so is $(u_1,u_2) \mapsto \check{e}_1$. Hence we have proved: $(u_1,u_2) \mapsto (\check{e}_1,\check{e}_2)$ is f.g. stable.

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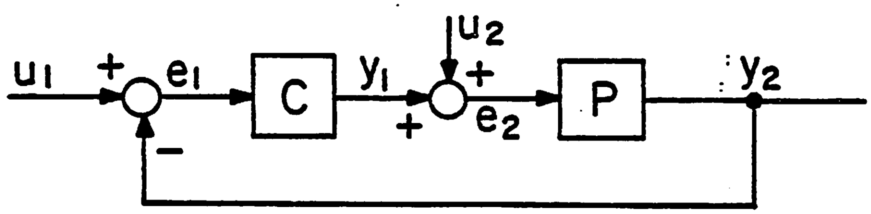

Fig 1

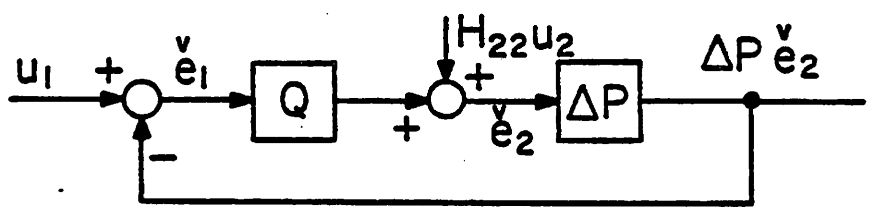

Fig 2

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