Copyright © 1981, by the author(s).
All rights reserved.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission.

ON SYNONYMY, ANTONYMY AND NEGATIONS

by

S. V. Ovchinnikov

Memorandum No. UCB/ERL M81/63

20 July 1981

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

ON SYNONYMY, ANTONYMY AND NEGATIONS*

S. V. Ovchinnikov

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

Structures of automorphisms, dual automorphisms and automorphism groups in fuzzy set theory are studied in detail in view of applications to representations of synonymy, antonymy and negations.

* Research sponsored by the National Science Foundation Grant ENG78-23143.

PART I. AUTOMORPHISMS AND AUTOMORPHISM GROUPS

1. Introduction

The aim of Part I of this paper is to suggest an algebraic model which may provide an answer to the following Zadeh's question: how could synonyms and anonyms be represented in fuzzy set theory?

Let us suppose that there is a rule assigning a synonym (or an antonym) to each fuzzy set with a given universe. It is easy to accept a hypothesis that this rule commutes with connectives "and" and "or". For instance, "wealthy and sick" is an antonym to "poor and healthy". It means that the rule in question is actually an automorphism of an algebra of all fuzzy sets. Obviously, there are a lot of possible rules of this kind. Symmetry of synonymity and statements in a colloquial language like "a synonym of a synonym is a synonym" or "an antonym of an antonym is a synonym" show that a proper mathematical model should employ a group structure of a set of automorphisms.

The paper (Part I) studies automorphisms in fuzzy set theory (section 2) and automorphism groups (section 3) with the view of their applications to synonymy and antonymy representations. Only algebraic aspects of the problem in question are considered in this paper. We leave applications to linguistics for further publications.

2. Automorphisms in fuzzy set theory

Let X be a finite set. Fuzzy set theory considers the following model. A *fuzzy set* A with universe X is a mapping $Z: X \to [0;1]$. A function $A(x)$ with domain X and range $[0;1]$ is said to be a *membership function*.

Further we will not distinguish between fuzzy sets and their membership functions. The set of all fuzzy sets with universe X is denoted $\underset{\sim}{P}(X)$ Operations of union and intersection are defined pointwise by

$$(A \cup B)(x) = A(x) \vee B(x) = \max \{A(x), B(x)\},$$

$$(A \cap B)(x) = A(x) \wedge B(x) = \min \{A(x), B(x)\}.$$

The set $\underset{\sim}{P}(X)$ is a complete distributive lattice with respect to operations $\cup$ and $\cap$ and universal bounds 0 and 1 where $0(x) \equiv 0$ and $1(x) \equiv 1$. Considering this lattice as an abstract algebra we denote $L(X) = \langle \underset{\sim}{P}(X); \cup, \cap, 0, 1 \rangle$. Actually, $L(X) = X^{[0;1]}$ where [0;1] is regarded as a lattice with respect to max- and min- operations. An operation of negation $^{-}$ is defined as follows in fuzzy set theory

$$\overline{A}(x) = 1 - A(x), \text{ for all } x \in X.$$

The lattice L(X) endowed with a negation operation defined above is a de Morgan algebra $M(X) = \langle \underset{\sim}{P}(X); \cup, \cap, ^{-}, 0, 1 \rangle$ (see [2] for a general definition of de Morgan algebras).

An automorphism of L(X) is a one-to-one and onto mapping $\Theta : \underset{\sim}{P}(X) \rightarrow \underset{\sim}{P}(X)$ such that

$$\Theta(A \cup B) = \Theta(A) \cup \Theta(B),$$

$$\Theta(A \cap B) = \Theta(A) \cap \Theta(B),$$

$$\Theta(0) = 0 \text{ and } \Theta(1) = 1.$$

We obtain an automorphism of M(X) adding the property

$$\Theta(\overline{A}) = \overline{\Theta(A)}$$

In this section all automorphisms of L(X) and M(X) are completely described. We start with a description of automorphisms of L(X), for any automorphism of M(X) is an automorphism of L(X).

Let us denote P(X) a set of all crisp subsets in X, i.e. fuzzy sets with membership functions taken only values 0 and 1. Then P(X) is a Boolean algebra which is a maximal Boolean subalgebra in $L(X)$ (and $M(X)$).

Lemma 2.1. Let Θ be an automorphism of $L(X)$. Then the restriction of Θ on $P(X)$ is an automorphism of $P(X)$.

Proof. Let A be a crisp set, i.e. $A \in P(X)$. Then

$$A \cup \overline{A} = 1 \text{ and } A \cap \overline{A} = 0$$

imply

$$\Theta(A) \cup \Theta(\overline{A}) = 1 \text{ and } \theta(A) \cap \Theta(\overline{A}) = 0.$$

Hence, $\Theta(A)$ is a crisp set and $\Theta(\overline{A}) = \overline{\Theta(A)}$. ¤

Lemma 2.2. Let Θ be an automorphism of $P(X)$. There is a permutation $s: X \to X$ such that

$$\Theta(A)(x) = A(s(x)) \text{ for any } A \in P(X).$$

Proof. Atoms in $P(X)$ are singletons in X. An image and an inverse image of any atom are atoms again, for Θ is an automorphism. Hence, Θ defines a permutation on the set X. Note now that any $A \in P(X)$ is a union of atoms. ¤

Remark. The group of all automorphisms of $P(X)$ is isomorphic to the symmetric group S_n for $n = |X|$.

In order to describe automorphisms of $L(X)$ we introduce the following families of elements in $L(X)$:

$$\delta_a(x) = \begin{cases} 1 \text{ if } x = a, \\ 0 \text{ is } x \neq a, \text{ for } a \in X \end{cases}$$

and

$$\sigma_\alpha(x) \equiv \alpha \qquad \text{for } \alpha \in [0;1].$$

Note that $\delta_a(x)$ is an atom in $P(X)$ for any $a \in X$.

Let Θ be an automorphism of $L(X)$. We define

$$\Theta_x(\alpha) = \Theta(\sigma_\alpha)(x) \quad \text{for } x \in X,\ \alpha \in [0;1].$$

Lemma 2.3. Θ_x is an automorphism of $[0;1]$ for any $x \in X$.

Proof. We have

$$\Theta_x(\alpha \wedge \beta) = \Theta(\sigma_{\alpha\wedge\beta})(x) = \Theta(\sigma_\alpha \wedge \sigma_\beta)(x) =$$

$$\Theta(\sigma_\alpha)(x) \wedge \Theta(\sigma_\beta)(x) = \Theta_x(\alpha) \wedge \Theta_x(\beta).$$

In the same way $\Theta_x(\alpha \vee \beta) = \Theta_x(\alpha) \vee \Theta_x(\beta)$. Finally,

$$\Theta_x(0) = 0 \text{ and } \Theta_x(1) = 1.$$ ¤

Now we have the following

Theorem 2.1. A mapping Θ: $\underset{\sim}{P}(X) \to \underset{\sim}{P}(X)$ is an automorphism of the lattice $L(X)$ iff there is a family $\{\Theta_x\}$, $x \in X$ of automorphisms of $[0;1]$ and a permutation s: $X \to X$ such that

$$\Theta(A)(x) = \Theta_x(A(s(x))) \tag{2.1}$$

for any $A \in \underset{\sim}{P}(X)$.

Proof. It is easy to verify that (2.1) defines an automorphism of $L(X)$ for any family $\{\Theta_x\}$ and any permutation s.

Conversely, let $A \in \underset{\sim}{P}(X)$. Then we have a decomposition

$$A(x) = \bigcup_{a \in X} \{\delta_a(x) \wedge \sigma_{A(a)}(x)\}.$$

Hence,

$$\Theta(A)(x) = \bigcup_{a \in X} \{\Theta(\delta_a)(x) \wedge \Theta(\sigma_{A(a)})(x)\}.$$

By the definition of Θ_x and by lemma 2.2 we infer

$$\Theta(A)(x) = \bigcup_{a \in X} \{\delta_{s^{-1}(a)}(x) \wedge \Theta_x(A(a))\} = \Theta_x(A(s(x)))$$

for some permutation s^{-1} on X. The proof is over. ¤

The following theorem describes all automorphisms of a de Morgan algebra A(X).

<u>Theorem 2.2.</u> A mapping Θ: $\underset{\sim}{P}(X) \to \underset{\sim}{P}(X)$ is an automorphism of M(X) iff there is a family $\{\Theta_x\}$, $x \in X$ of automorphisms of [0;1] fulfilling the equation

$$\Theta_x(\alpha) + \Theta_x(1 - \alpha) = 1 \tag{2.2}$$

for all $x \in X$, $\alpha \in [0;1]$, and a permutation s: $X \to X$ such that

$$\Theta(A)(x) = \Theta_x(A(s(x))) \tag{2.3}$$

for all $A \in \underset{\sim}{P}(X)$.

<u>Proof.</u> A mapping Θ defined by (2.3) is an automorphism of L(X). To prove that it is an automorphism of M(X) it suffices to show that

$\Theta(\overline{A}) = \overline{\Theta(A)}$. We have

$$\Theta(\overline{A})(x) = \Theta_x(1 - A(s(x))) = 1 - \Theta_x(A(s(x))) =$$

$$1 - \Theta(A)(x) = \overline{\Theta(A)}(x)$$

by (2.2) and (2.3).

Conversely, let Θ be an automorphism of M(X). Then, by theorem 2.1, Θ is represented by (2.3). Let us prove (2.2) in this case. We have $\Theta(\overline{A}) = \overline{\Theta(A)}$ for any $A \in \underset{\sim}{P}(X)$. Let $A = \sigma_\alpha$. Then $\overline{A} = \sigma_{1-\alpha}$ and we obtain $\Theta(\sigma_{1-\alpha}) = 1 - \Theta(\sigma_\alpha)$, or, by (2.3),

$\Theta_x(1 - \sigma) = 1 - \Theta_x(\alpha)$, Q.E.D. ¤

3. <u>Automorphism groups</u>

Only lattices L(X) are considered in this section, because all statements concerning automorphisms of these lattices are easily extended to the case of de Morgan algebras by theorem 2.2.

The set of all automorphisms of a given algebra is a group with respect to a composition of automorphisms. We use a symbol o as a denotation for any composition operation. Aut(L), Aut(X) and Aut([0;1]) denote, respectively, automorphism groups of a lattice L(X), a set X and an interval [0;1]. X is supposed to be a finite set with cardinality n. Aut(X) is a symmetric group S_n and Aut([0;1]) is an automorphism group or the unit interval considered as a lattice with universal bounds (the latter group is studied in [11]).

By theorem 2.1 any automorphism of L(X) is determined by a pair $<\{\Theta_x\}, s>$ where $\Theta_x \in Aut([0;1])$ for all $x \in X$ and $s \in Aut(X)$.

(For simplicity sake, we denote $\{\Theta_x\}$ a family $\{\Theta_x\}_{x \in X}$.)

The composition law in Aut(L) is given by

$$<\{\Theta'_x\}, s'> \circ <\{\Theta''_x\}, s''> = <\{\Theta'_x \circ \Theta''_{s'(x)}\}, s' \circ s''>$$

For instance,

$$<\{\Theta_x\}, s>^{-1} = <\{\Theta^{-1}_{s^{-1}(x)}\}, s^{-1}>$$

and an identity element id_L in Aut(L) is $<\{id_{[0;1]}\}, id_X>$ where $id_{[0;1]}$ and id_X are identity elements in Aut([0;1]) and Aut(X), respectively.

Let us denote $K = \{<\{\Theta_x\}, id_X>\}$ and $H = \{<\{id_{[0;1]}\}, s>\}$.

It is easy to verify that K and H are subgroups of Aut(L) such that $K \cong Aut^n([0;1])$ and $H \cong Aut(X)$. Moreover, we have the following

<u>Theorem 3.1.</u> The group Aut(L) is a semidirect product of K by H.

<u>Proof.</u> Obviously, $K \cap H = \{id_L\}$. Hence, it suffices to prove that K is a normal subgroup and $K \cup H = Aut(L)$ (see theorem 6.5.3 in [7]).

We have

$$<\{\Theta_x\}, s>^{-1} o <\{\Theta'_x\}, id_x> o <\{\Theta_x\}, s> =$$

$$<\{\Theta^{-1}_{s^{-1}(x)} \circ \Theta'_{s^{-1}(x)} \circ \Theta_{s^{-1}(x)}\}, id_X>$$

Hence, K is a normal subgroup.

Further,

$$<\{\Theta_x\}, s> = <\{\Theta_x\}, id_X> o <\{id_{[0;1]}\}, s>.$$

Hence, K and H generate Aut(L). The proof is over. ¤

The following definition introduces some particular auromorphisms which are important in applications to representations of synonymy and antonymy.

Definition 3.1. 1) An automorphism Θ of L(X) is said to be an S-automorphism if $\Theta(A) = A$ for any crisp set A;

2) An automorphism Θ of L(X) is said to be an A-automorphism if Θ^2 is an S-automorphism and there is a crisp set A such that $\Theta(A) \neq A$.

The following theorem yields a description of S- and A-automorphisms.

Theorem 3.2. An automorphism $\Theta = <\{\Theta_x\}, s>$ of L(X) is an S-automorphism (resp. A-automorphism) iff $s = id_X$ (resp. $s^2 = id_X$ and $s \neq id_X$).

Proof. 1) Let $s = id_X$. Then $\Theta(A) = A$ for any crisp set A, by theorem 2.1. Conversely, let $\Theta(A) = A$ for any crisp set A. We have

$$\delta_a(x) = \Theta(\delta_a)(x) = \delta_a(s(x)) = \delta_{s^{-1}(a)}(x).$$

Hence, $s(a) = a$ for all $a \in X$.

2) Let $s^2 = id_X$ and $s \neq id_X$. Then $\Theta^2 = <\{\Theta_x \circ \Theta_{s(x)}\}, id_X>$ is an S-sutomorphism by previous arguments. Obviously, there is a crisp set A such that $\Theta(A) \neq A$ for $s \neq id_X$. Conversely, let Θ^2 is an S-automorphism and there is a crisp set A such that $\Theta(A) \neq A$. We have $\Theta^2 = <\{\Theta_x \circ \Theta_{s(x)}\}, s^2>$ which implies $s^2 = id_X$, by previous arguments. Finally, $s \neq id_X$, since $\Theta(A) \neq A$.

Corollary. The set of all S-automorphisms is a subgroup K.

Permutations s such that $s^2 = id_X$ will be called *symmetries*.

We define below a special class of subgroups of Aut(L), namely, SA-subgroups. If G is an SA-subgroup, then elements of G may be regarded as representations of synonymy and antonymy. The following definition is based on an observation that a synonym of a synonym is a synonym again and an antonym of an antonym should be a synonym.

Definition 3.2. A subgroup $G \subseteq Aut(L)$ is said to be an *SA-subgroup* if

1) any element of G is either an S-automorphism or an A-automorphism;

2) G contains at least one A-automorphism;

3) composition of any two A-automorphisms is an S-automorphism.

The structure of SA-subgroups is established in the following

Theorem 3.3. Let G be an SA-subgroup of Aut(L). Then

1) there is a symmetry s such that

$$G \cap H = \{<\{id_{[0;1]}\}, id_X>, <\{id_{[0;1]}\}, s>\} \cong Z_2;$$

2) G is a semidirect product of $G \cap K$ by $G \cap H$.

Proof. 1) Let $<\{id_{[0;1]}\}, s_1>$ and $<\{id_{[0;1]}\}, s_2>$ are in $G \cap H$ and different from id_L. Then they are A-sutomorphisms, i.e. $s_1^2 = s_2^2 = id_X$. On the other hand $s_1 o s_2 = id_X$, by definition 3.2,3). Hence, $s_1 = s_1^{-1} = s_2$, i.e. $G \cap H$ contains only one automorphism, say $<\{id_{[0;1]}\}, s>$, which is different from the identity element. We have $G \cap H = Z_2$, for $s^2 = id_X$.

2) $G \cap K$ is a subgroup of G. Moreover, G is generated by $G \cap K$ and $G \cap H$. Indeed, any element $<\{\theta_x\}, s>$ of G is an S- or an A-automorphism, i.e. $s^2 = id_X$. We have

$$<\{\theta_x\}, s> = <\{\theta_x\}, id_X> o <\{id_{[0;1]}\}, s>$$

where $<\{\theta_x\}, id_X> \in G$, since $<\{id_{[0;1]}\}, s> \in G$. Hence, $G = (G \cap K) \cup (G \cap H)$. The rest of the proof is the same as the proof of theorem 3.1. ¤

Corollary. G is a union of a normal subgroup $G \cap K$ of S-automorphisms and a unique coset of all A-automorphisms in G.

There are two special kinds of SA-subgroups which are useful in applications.

Definition 3.3. An SA-subgroup G is said to be

1) a full SA-subgroup if $K \subseteq G$;

2) a homogeneous SA-subgroup if $G \cap K$ is a diagonal in $K \doteqdot Aut^n([0;1])$.

Note, that G is a homogeneous SA-subgroup if and only if $G \cap K = \{<\{\theta_x\}, id_X> \mid \theta_x \equiv \theta \text{ for some } \theta \in Aut([0;1])\}$.

Theorem 3.4. Any homogeneous SA-subgroup F of Aut(L) is isomorphic to a direct product of Aut([0;1]) on Z_2.

Proof. G is a semidirect product of $G \cap K \cong$ Aut([0;1]) by $G \cap H \cong Z_2$, by the previous theorem. Let $<\{\Theta\}, id_X> \in G \cap K$ and $<\{id_{[0;1]}\}, s> \in G \cap H$. We have

$$<\{\Theta\}, id_X> o <\{id_{[0;1]}\}, s> = <\{id_{[0;1]}\}, s> o <\{\Theta\}, id_X>$$

i.e., any two elements of G K and G H commute. Hence, G is a direct product of $G \cap K$ on $G \cap H$ (see Section 6.5 in [7]). ¤

Corollary. Any element $<\{\Theta\}, s>$ in a homogeneous SA-subgroup has a unique representation as a composition

$$<\{\Theta\}, s> = <\{\Theta\}, id_X> o <\{id_{[0;1]}\}, s> = <\{id_{[0;1]}\}, s> o <\{\Theta\}, id_X>.$$

4. Representations by ultrafuzzy sets

Let A be a given fuzzy set and Θ - an S-automorphism. We consider $B = \Theta(A)$ as a synonym of A and define a degree of synonymity of B with respect to A by

$$\Sigma_A(B) = 1 - d(A,B) \tag{4.1}$$

where d is any normed distance function on $\underset{\sim}{P}(X)$. We set $\Sigma_A(B) = 0$ iff B is not a synonym of A. $\Sigma_A(B)$ thus defined may be regarded as a value of a membership function of an ultrafuzzy set Σ_A (an ultrafuzzy set is a fuzzy set with universe $\underset{\sim}{P}(X)$). This set is considered as a fuzzy set of all synonyms of a given fuzzy set A.

We have $\Sigma_A(A) = 1$ which implies $\bigcup_{A \in \underset{\sim}{P}(X)} \Sigma_A = \underset{\sim}{P}(X)$. Hence, the family $\{\Sigma_A\}_{A \in \underset{\sim}{P}(X)}$ is a covering of $\underset{\sim}{P}(X)$.

Actually, this covering is a partition of $\underset{\sim}{P}(X)$ if a max-Δ composition law is employed in ultrafuzzy set theory, where Δ is a connective defined by

$$x \,\Delta\, y = \max(x+y-1,\ 0).$$

(See [10] for definitions of coverings, partitions and related results and [5] and [3] where a detailed study of a connective Δ may be found.) Then a resemblance relation

$$I(A,B) = \bigvee_{C \in \underset{\sim}{P}(X)} \Sigma_C(A) \,\Delta\, \Sigma_C(B)$$

generated by the covering $\{\Sigma_A\}_{A \in \underset{\sim}{P}(X)}$ is a similarity relation. Simple calculations yield

$$I(A,B) = \begin{cases} 1 - d(A,B), & \text{if } A \text{ and } B \text{ and synonyms,} \\ 0, & \text{otherwise.} \end{cases}$$

The relation I may be regarded as a synonymity relation on $\underset{\sim}{P}(X)$. Classes of this similarity relation are ultrafuzzy sets of synonyms.

PART II. GENERAL NEGATIONS

1. Introduction

Let $\underset{\sim}{P}(X) = [0;1]^X$ denote the set of all fuzzy sets with a universe X. It is possible to define logical connectives "or" and "and" for fuzzy sets by different ways [1]. We use standard definitions due to Zadeh [14] in this paper. Then $\underset{\sim}{P}(X)$ is a complete distributive lattice under operations of union and intersection with universal bounds $\emptyset$ and X. The lattice $\underset{\sim}{P}(X)$ is a noncomplemented lattice, i.e., there is no operation $A \to \bar{A}$ in $\underset{\sim}{P}(X)$ such that

$$A \cap \bar{A} = \emptyset, \text{ and} \tag{1}$$

$$A \cup \bar{A} = X. \tag{2}$$

Nevertheless, a number of "complement" operations, usually called negations, are studied in current papers (see, for example [8], [12-14]). Naturally, these negations violate at least one of the properties (1) - (2). Definitions of most negations suggested are pointwise ones. That means that $\bar{A}(x) = n(A(x))$, where $n: [0;1] \to [0;1]$ is any "negation function".

This part of the paper is concerned with a general (not necessarily pointwise) negation in fuzzy set theory. At first, all "optimal" in some precise sense negations are completely described. They turn out to be only involutions, intuitionistic negations and dual intuitionistic negations. Then a structure of involutions in $\underset{\sim}{P}(X)$ is studied. It is proven that each involution in $\underset{\sim}{P}(X)$ is a variable pointwise involution generated by a family of negation functions, i.e. Lowen's fuzzy complement [8]. In conclusion some possible generalizations are discussed.

2. General Negations

We begin with some common examples.

Example 1. Let η be a decreasing function η: $[0;1] \to [0;1]$ such that $\eta(\eta(\alpha)) = \alpha$ for all $\alpha \in [0;1]$ ("strong negation function" in [12]. Then an *involutionary negation* in $\underset{\sim}{P}(X)$ (see [13]) is defined by

$$A(x) = \eta(A(x)) \text{ for all } x \in X.$$

Example 2. Let $\{\eta_x\}_{x \in X}$ be a family of strong negation functions. Lowen in [8] defines a "fuzzy complement" on X by $\overline{A}(x) = \eta_x(A(x))$. This operation generalizes the previous example.

Recall the reader (see [4], p. 3) that an *involution* in the lattice $\underset{\sim}{P}(X)$ is a mapping Θ: $\underset{\sim}{P}(X) \to \underset{\sim}{P}(X)$ such that: 1) $A \subseteq B$ iff $\Theta(A) \supseteq \Theta(B)$, and 2) Θ^2 is an identity in $\underset{\sim}{P}(X)$. Any fuzzy complement on X (and, therefore, any involuntary negation) is an involution in $\underset{\sim}{P}(X)$. Fuzzy complements in Lowen's sense are called *variable pointwise involutions* in this paper.

Example 3. $\underset{\sim}{P}(X)$ is a completely distributive lattice. Hence, each element $A \in \underset{\sim}{P}(X)$ has a pseudo complement $\overline{A}$ (see [4]). We have, by definition,

$$\overline{A} = \vee\{B \mid A \cap B = \emptyset\}, \text{ or}$$

$$\overline{A}(x) = \begin{cases} 0, & \text{if } A(x) > 0, \\ 1, & \text{if } A(x) = 0 \end{cases}$$

in $\underset{\sim}{P}(X)$. This negation is said to be an *intuitionistic negation* in [13].

Example 4. By duality, we define a *dual intuitionistic negation* by

$$\overline{A}(x) = \begin{cases} 1, & \text{if } A(x) < 1, \\ 0, & \text{if } A(x) = 1. \end{cases}$$

All the negations defined fulfill the following

Extension Principle. ([15]) The restriction of a negation on the set of all crisp subsets of X is a usual complement.

Only negations satisfied this Principle are considered in this paper.

In addition to properties (1) and (2) the crisp complement fulfills the following DeMorgan's laws

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \tag{3}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \text{ and} \tag{4}$$

$$\overline{\overline{A}} = A. \tag{5}$$

It is easy to verify that any involution satisfies (3) - (5) and does not satisfy (1) - (2). On the other hand an intuitionistic negation fulfills (2) - (4) and does not fulfill (1) and (5).

We will use the following general definition of a negation in $\underset{\sim}{P}(X)$:

Definition 1. An operation $\ominus$: $\underset{\sim}{P}(X) \to \underset{\sim}{P}(X)$ is said to be a *negation* if it violates as few as possible of properties (1) - (5). We denote $A = \ominus(A)$ in this case.

One can consider negations thus defined as "optimal" negations in the sense that they are notions nearest to the crisp one.

Lemma 1. A negation in $\underset{\sim}{P}(X)$ fulfills exactly three of properties (1) - (5).

Proof. Since $\underset{\sim}{P}(X)$ is a noncomplemented lattice, any negation $A \to \overline{A}$ violates (1) or (2). Let, for example, (1) is violated and $A \cap \overline{A} = B \neq \emptyset$ for some $A \in \underset{\sim}{P}(X)$. Suppose that all the rest properties (2) - (5) are fulfilled. Then we have

$$\overline{B} = \overline{A \cap \overline{A}} = \overline{A} \cup A = X, \text{ by (3), (5) and (2),}$$

and $B = \emptyset$, by (5) and the Extension Principle. This contradiction shows that at least two of properties (1) - (5) are violated. As it was mentioned above, for example, an intuitionistic negation violates exactly two of properties (1) - (5) which completes the proof. ¤

<u>Corollary</u>. Any involution in $\underset{\sim}{P}(X)$ and both intuitionistic and dual intuitionistic negations are negations in the sense of definition 1.

The following theorem shows that the converse is also true.

<u>Theorem 1</u>. Any negation in $\underset{\sim}{P}(X)$ is an involution or an intuitionistic negation or a dual intuitionistic negation.

<u>Proof</u>. 1) Let Θ be a negation which satisfies (5). Then, by lemma 1, it also satisfies (3) or (4). By (5), Θ^2 is an identical mapping which implies bijectiveness of Θ. Let, for example, (3) is true (the second case is dual to this one). Then $A \leq B$ implies $A = A \cap B$ implies $\overline{A} = \overline{A} \cup \overline{B}$ implies $\Theta(A) \geq \Theta(B)$, and $\Theta(A) \geq \Theta(B)$ implies $\overline{B} = \overline{A} \cap \overline{B}$ implies $B = A \cup B$ implies $A \leq B$. Hence, Θ is an involution.

2) Let Θ violate (5). Then it violates (1) or (2). Let (1) be true and $A \cap B = \emptyset$. Then $\overline{A} \cup \overline{B} = X$ which implies $B = B \cap (\overline{A} \cup \overline{B}) = (B \cap \overline{A}) \cup (B \cap \overline{B}) = B \cap \overline{A}$, by (1). Hence, $B \leq \overline{A}$ and we have, by (1),

$$\overline{A} = \vee\{B \mid B \cap A = \emptyset\}$$

i.e. Θ is an intuitionistic negation.

3) A dual argument shows that Θ is a dual intuitionistic negation in case when (5) and (1) are violated. ¤

Note, that an intuitionistic negation Θ_i and a dual intuitionistic negation Θ_{di} can be considered as limit cases of negations because we have $\Theta_i \leq \Theta \leq \Theta_{di}$ for any negation Θ.

3. Involutions in $\underset{\sim}{P}(X)$

It follows from section 2 that, generally speaking, most negations in $\underset{\sim}{P}(X)$ are involutions. The following theorem shows that variable pointwise involutions (see Example 2) turn out to be the most general form of an involution in $\underset{\sim}{P}(X)$ fulfilled the Extension Principle.

Theorem 2. Any involution in $\underset{\sim}{P}(X)$ which fulfills the Extension Principle is a variable pointwise involution.

Proof. Let us define the following fuzzy sets:

$$\delta_a(x) = \begin{cases} 1, & x = a \\ 0, & x \neq a, \ a \in X, \end{cases}$$

and

$$\sigma_\alpha(x) \equiv \alpha, \alpha \in [0;1].$$

Then

$$A = \bigcup_a \{\delta_a \cap \sigma_{A(a)}\}.$$

Let $A \to \bar{A}$ be an involution in $\underset{\sim}{P}(X)$. Then

$$\bar{A} = \bigcap_a \{\bar{\delta}_a \cup \bar{\sigma}_{A(a)}\}.$$

Let us denote $\eta_x(\alpha) = \bar{\sigma}_\alpha(x)$. Then, by Extension Principle,

$$\bar{A}(x) = \bigwedge_a \{\bar{\delta}_a(x) \vee \eta_x(A(a))\} = \eta_x(A(x)).$$

For $A = \bar{\sigma}_\alpha$ the last formula gives $\alpha = \eta_x(\eta_x(\alpha))$, i.e. η_x^2 is an identity. Let $\alpha \le \beta$. Then $\sigma_\alpha \le \sigma_\beta$ or $\sigma_\alpha = \sigma_\alpha \cap \sigma_\beta$ which implies $\bar{\sigma}_\alpha = \bar{\sigma}_\alpha \cup \bar{\sigma}_\beta$ or $\bar{\sigma}_\alpha \ge \bar{\sigma}_\beta$. Hence, $\eta_x(\alpha) \ge \eta_x(\beta)$. By dual arguments, $\eta_x(\alpha) \ge \eta_x(\beta)$ implies $\alpha \le \beta$. Hence $\eta_x(\alpha)$ is a strong negation function and $A \to \bar{A}$ is a variable pointwise involution. ¤

Note that the statement of this theorem is strongly based on the Extension Principle because there are a lot of involutions in $\underset{\sim}{P}(X)$ which are different from variable pointwise involutions (see [9] where all involutions in $\underset{\sim}{P}(X)$ are described).

4. Concluding Remarks

It follows from theorems 1 and 2 that negations from Examples 2-4 present all possible optimal negations in the sense of definition 1. All these negations can be extended on arbitrary L-sets (see [6] for definitions) where L is any complete distributive lattice. The main problem here is an existence problem: whether or not a given negation can be defined on a given lattice. Let us consider, for example, a lattice L from Fig. 1. It is easy to prove that this lattice does not admit an involution and both intuitionistic and dual intuitionistic negations are not optimal in L. On the other hand it can be proven that theorem 1 is true in any L-set theory if L has irreducible universal bounds, in particular, when L is a chain. Also theorem 2 is true in any L-set theory if L is a complete distributive lattice. Nevertheless, it should be noted that involutions could not exist even in the case when L is a chain. It becomes clear if we consider $L = \{0\} \cup [\frac{1}{2};1]$ with a natural order.

Acknowledgement

The author is grateful to Professor L.A. Zadeh for his suggestion to study a problem of a synonymy representation and numerous fruitful discussions. In addition, this work was stimulated by discussions with J. Dombi, T. Riera and E. Trillas.

References:

1. Alsina, C., Trillas, E., Valverde, L. (1981), "On non-distributive logical connectives for fuzzy set theory", (to appear in J. Math. Anal. Appl.)
2. Balbes, R., Dwinger, P. (1974), "Distributive Lattices", University of Missouri Press.
3. Bezdek, J.C., Harris, J.D. (1978), "Fuzzy partitions and relations; an axiomatic basis for clustering", Fuzzy Sets and Systems, 1, 111-127.
4. Birkhoff, G. (1967), "Lattice Theory", AMS Colloquim Publ., v. 25.
5. Giles, R. (1976), "Lukasiewicz logic and fuzzy set theory", Int.J. Man-Machine Studies, 8, 313-327.
6. Goguen, J.A.(1967), "L-fuzzy sets", J. Math. Anal. Appl., 18, 145-174.
7. Hall, M., (1959), "The Theory of Groups," The Macmillan Company, N.Y.
8. Lowen, R. (1978), "On fuzzy complements", Inform. Sci., 14, 107-113
9. Ovchinnikov, S.V. (1980), "Involutions in fuzzy set theory", Stochastica, v.IV, no.3, 227-231.
10. Ovchinnikov, S.V., Riera, T. (1981), "On fuzzy classifications", (to appear in: R.R.Yager, ed., Recent Developments in Fuzzy Set and Possibility Theory, Pergamon Press, N.Y.).
11. Riera, T., Trillas, E. (1980), "Towards a representation of synonyms and antonyms by fuzzy sets", (unpublished).
12. Trillas, E. (1979), "Sobre functones de negacion en la teoria de conjuntos difusos", Stochastica, III, no. 1, 47-60.
13. Yager, R.R. (1980), "On the measure of fuzziness and negation II. Lattices", Inform. Control, 44, 236-260.

14. Zadeh, L.A. (1965), "Fuzzy sets", <u>Inform. Control</u>, 8, 338-353.

15. Zadeh, L.A. (1975), "The concept of linguistic variable and its application to approximate reasoning I", <u>Inform. Sci.</u>, 8, 199-249.

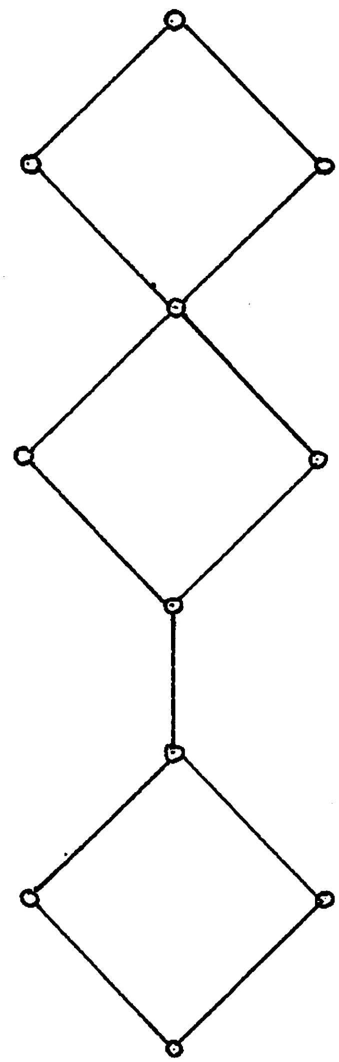

Fig.1