MODAL ANALYSIS OF SYNCHRONOUS MACHINE DYNAMICS

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ABSTRACT

A theory for the asymptotic modal decomposition of a linear multivariable feedback system subject to high gain output feedback has been developed and applied to the linear, ideal synchronous machine by identifying the latter with such a high gain output feedback system. This new conceptualization leading to the asymptotic decoupling of machine modes brings a unified and rigorous understanding to synchronous machine dynamics. The numerical simulation runs show that the asymptotic decoupling approximation is highly accurate for a wide range of situations.

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1. Introduction

As the size and the complexity of an electric power system grow, providing simpler models for the components of the system to be used in numerical or analytical studies becomes an important concern. The synchronous machine is such a component that plays a crucial role in power system dynamics. The relative success of the Liapunov theory developed and applied to multimachine power systems stability assessment [16] owes a lot to the simplicity of the classical transient model of the sychronous machine used within the systemic model. The limitations of the classical transient model [6], however, has been demanding a clear and thorough understanding of other modeling choices for which the theory in question can be extended in meaningful ways.

The example cited above is just one instance, nevertheless important, that renders the development of a unified and rigorous conceptual framework for the model reduction procedures of the synchronous machine a task worth undertaking. The present paper is such an attempt.

The basis for the dynamical reduction of a linear, ideal synchronous machine model is modal (or time scale) selection. The so called subtransient, transient, and steady state models with their variants that include or exclude amortisseur windings, rotor iron effects etc., are, in effect obtained by selecting the appropriate modes of the transformer equations after applying Park's transformation. As a further step in the simplification one can identify the algebraic input-output equations for the quasi steady state of the selected modes by a phasor diagram which, in turn, corresponds to some linear electrical circuit.

A serious theoretical obstacle in deriving the reduced equations has been the absence of an analytical solution or a well defined approximation to the modal decomposition problem of the transformer equations.

Historically this obstacle has partially been circumvented by introducing a series of assumptions, approximations the motivations and justifications of which lies mostly in empirical evidence. In spite of the validity of most of the end results due to the extent of scrutiny involved in empirical verification, the overcomplicated derivation patterns for modal reduction lack conceptual coherence, simplicity, and rigour as can be witnessed in classical texts like [1], [2] as well as relatively modern ones [3], [4].

This paper removes the obstacle in question by solving the modal decomposition problem as a limit case of the synchronous speed. The only empirical verification required is to show that the limit in question is achieved at the nominal synchronous speed for all practical purposes.

The decomposition problem is solved in two steps. First a result is provided within the paper that solves the asymptotic modal decomposition problem for a general linear multivariable feedback system under high gain output feedback. It is then shown that the transformer equations of a synchronous machine can be identified with a linear multivariable feedback system where the d-q axis voltages and fluxes act as the inputs and the outputs respectively, and the electrical rotor speed acts as the feedback gain parameter. It turns out that the nominal synchronous speed of the rotor justifies the "high gain" assumption and the theoretically "asymptotic" result becomes a very good approximation as long as the synchronous machine is operated within the range of its nominal speed.

The problem of tracing the asymptotic behavior of the closed loop eigenvalues of a linear multivariable system under high gain output feedback has received some attention in the last decade [7] - [13]. In spite of the relative abundance of the results for the asymptotic root locus problem, however, no attempt seems to have been made to solve the associated asymptotic modal decomposition problem. Section II of this paper addresses itself to this problem and the main result is presented in a single theorem. In section III after a concise description of the linear, ideal synchronous machine model a transfer function relation is derived. This relation provides a clear understanding of the high gain feedback mechanism involved within the synchronous machine without any reference to the modal decomposition result. The theoretical result of section II is then applied to the synchronous machine and the decoupled state equations as well as the output equations are derived. Making use of the results of Appendix C which relate the classical definitions and approximations to the results of this paper the decomposed model is further simplified and its consequences to model reduction are pointed out.

In section IV three numerical simulation runs are presented in order to assess the accuracy of the asymptotically decoupled model using typical data for a synchronous machine. The numerical study consists of field voltage adjustment, short circuited machine, and stability simulations. The decoupled model used in these simulations is further refined by incorporating a second order adjustment term in the armature modes that diminishes the steady state offset error. In section V prospects for possible extensions of the results of this paper are discussed.

II. Modal Decomposition Under High Feedback Gain

The system to be investigated is described by the following state equation

$$\dot{x} = (A-kBC)x + kBu + Gv \qquad \ldots (2.1)$$

where $x \in R^n$, $u \in R^m$, $v \in R^{\bar{m}}$; A, B, C, and G are constant matrices of appropriate size with rank B = rank C = m. The parameter k denotes the common feedback gain of the multivariable system. The purpose in this section is to obtain an analytical expression for the asymptotically valid modal decomposition of (2.1) for large k, and make precise the nature of the approximation in terms of appropriate limits.

At the outset we make the following assumptions:

A1. det CB $\neq$ 0, and the eigenvalues of CB are distinct.

A2. The zeroes of the system (A,B,C) given by the roots of

$$\det \begin{pmatrix} sI-A & B \\ -C & 0 \end{pmatrix} = 0$$

are distinct.

Remarks

1. It is well known that under assumption A1. (A,B,C) has precisely (n-m) zeroes and as k goes to $+\infty$ (n-m) closed loop poles approach these zeroes and the remaining m closed loop poles go to infinity along directions dictated by the eigenvalues of -CB [7]. This fact will eventually be reproved as a byproduct of the analytical construction of the modal decomposition.

2. The 'distinct'ness assumption both for the zeroes of (A,B,C) and the eigenvalues of CB can be relaxed by allowing for simple repeated roots. For the sake of simplicity, however, we restrict our analysis

subordinate to the assumptions stated above.

The first step in the analysis is the construction of the eigenspaces corresponding to the asymptotic eigenvalues given by the zeroes of (A,B,C). Thus let $\lambda_1,\ldots,\lambda_{n-m}$ denote the zeroes of (A,B,C) and define the nontrivial vectors $(x_i',-q_i')'$ and $(\overline{x}_i',\overline{q}_i')'$ by

$$\begin{pmatrix} \lambda_i I-A & B \\ -C & 0 \end{pmatrix}\begin{pmatrix} x_i \\ -q_i \end{pmatrix} = 0 \quad ,$$

$$\begin{pmatrix} \lambda_i I-A & B \\ -C & 0 \end{pmatrix}'\begin{pmatrix} \overline{x}_i \\ \overline{q}_i \end{pmatrix} = 0 \quad , \; i=1,\ldots,n-m, \qquad \ldots(2.2)$$

and furthermore let

$$\begin{aligned} X &:= (x_1,\ldots,x_{n-m}) \\ Q &:= (q_1,\ldots,q_{n-m}) \\ \overline{X} &:= (\overline{x}_1,\ldots,\overline{x}_{n-m})' \\ \overline{Q} &:= (\overline{q}_1,\ldots,\overline{q}_{n-m})' \end{aligned} \qquad \ldots(2.3)$$

<u>Lemma 1</u> Under the assumption A2. for each i

$$\overline{x}_i'x_i \neq 0, \qquad \ldots(2.4)$$

and if x_i (or $\overline{x}_i$ or both) is normalized so that

$$\overline{x}_i'x_i = 1,$$

then

$$\overline{X}X = I_{n-m} \qquad \ldots(2.5)$$

■

The proof of Lemma 1 is given in Appendix A.

Now define matrices Y and $\overline{Y}$ by requiring that they satisfy the relations

$$CY = Q \qquad \ldots(2.6)$$

$$\bar{Y}B = \bar{Q} \qquad \ldots(2.7)$$

$$\bar{X}Y + \bar{Y}X = 0 \;. \qquad \ldots(2.8)$$

These relations do <u>not</u> specify a <u>unique</u> Y and $\bar{Y}$. In fact

$$Y := B(CB)^{-1}Q + XE$$

$$\bar{Y} := \bar{Q}(CB)^{-1}C - E\bar{X} \qquad \ldots(2.9)$$

is a possible solution to (2.6) - (2.8) where E is any $(n\text{-}m) \times (n\text{-}m)$ matrix. That (2.9) furnishes a solution follows from Lemma 1 and the relations

$$CX = 0 \qquad \ldots(2.10)$$

$$\bar{X}B = 0 \qquad \ldots(2.11)$$

which are true by virtue of (2.2).

The approximation for the eigenvectors corresponding to the asymptotic eigenvalues $\lambda_1,\ldots,\lambda_{n-m}$ are simply the column vectors of the matrix $(X-\frac{1}{k}Y)$. Indeed rewriting (2.2) in matrix notation as

$$X\Lambda - AX = BQ \qquad \ldots(2.12)$$

where

$$\Lambda := \mathrm{diag}(\lambda_1,\ldots,\lambda_{n-m}) \qquad \ldots(2.13)$$

and using (2.6), (2.10), and (2.12) one arrives at

$$(A-kBC)(X-\frac{1}{k}Y) = (X-\frac{1}{k}Y)\Lambda + \frac{1}{k}(Y\Lambda-AY) \qquad \ldots(2.14)$$

in which it is readily observed that the error term is of the order k^{-1}.

Using the dual of the arguments above, an approximation for the left eigenvectors of (A-kBC) follows from

$$(\overline{X}-\frac{1}{k}\overline{Y})(A-kBC) = \Lambda(\overline{X}-\frac{1}{k}\overline{Y})+\frac{1}{k}(\Lambda\overline{Y}-\overline{Y}A). \qquad ...(2.15)$$

Moreover the relation

$$(\overline{X}-\frac{1}{k}\overline{Y})(X-\frac{1}{k}Y) = I_{n-m}+\frac{1}{k^2}\overline{Y}Y \qquad ...(2.16)$$

is a first step in constructing an approximate inverse for an approximate modal matrix of A-kBC, and follows from (2.5) and (2.8)

The second step in the analysis is the computation of approximate eigenvectors for the unbounded eigenvalues. If $\gamma_1,\ldots,\gamma_m$ are the eigenvalues of CB let

$$\gamma := \mathrm{diag}(\gamma_1,\ldots,\gamma_m) \qquad ...(2.17)$$

and P be a modal matrix of CB satisfying

$$CBP = P\gamma \quad . \qquad ...(2.18)$$

We then define

$$U := BP \qquad ...(2.19)$$

$$V := BR + ABP\gamma^{-1} \qquad ...(2.20)$$

where the mxm matrix R is yet to be determined. The approximate eigenvectors are the columns of the matrix $(U-\frac{1}{k}V)$. Establishing this result, however, requires some further work which we take up next.

The unbounded eigenvalues of A-kBC converge to the parametric asymptote loci given by $-\gamma_i k + \alpha_i$ for i = 1,...,m where

$$\alpha_i := P^{-1}_{i.}CABP_{.i}/\gamma_i \qquad ...(2.21)$$

with $P^{-1}_{i.}$ and $P_{.i}$ denoting the ith row and column vectors of P^{-1} and P respectively. As stated earlier this statement will also be reproved as a byproduct of the modal decomposition.

We shall require, for reasons to be revealed in the forthcoming analysis, that R satisfy the matrix equation

$$CBR + CABP\gamma^{-1} = R\gamma + P\alpha \qquad \text{...(2.22)}$$

where

$$\alpha := \mathrm{diag}(\alpha_1, \ldots, \alpha_m) \qquad \text{...(2.23)}$$

Solutions of equation (2.22) for R are characterized by the following result whose proof is straightforward and is therefore omitted.

Lemma 2. Let γ be as in (2.17) and L be a constant matrix, then solutions of the linear matrix equation

$$K\gamma - \gamma K = L \qquad \text{...(2.24)}$$

for K exist iff diagonal entries of L are zero, and are given by

$$K = K_0 + S \qquad \text{...(2.25)}$$

where

$$\begin{aligned} [K_0]_{ij} &= \frac{[L]_{ij}}{\gamma_j - \gamma_i}, \quad i \neq j \\ &= 0 \qquad\quad , \; i = j \end{aligned} \qquad \text{...(2.26)}$$

and S is any diagonal matrix ∎

Post multiply (2.22) by P^{-1} and rearrange terms after using (2.18) to obtain

$$P^{-1}R\gamma - \gamma P^{-1}R = P^{-1}CABP\gamma^{-1} - \alpha \qquad \text{...(2.27)}$$

which upon application of Lemma 2 yields the general solution for R as

$$R = PK_0 + PS \qquad \text{...(2.28)}$$

where K_0 is given by (2.26) with

$$L = P^{-1}CABP\gamma^{-1} - \alpha \qquad \text{...(2.29)}$$

and S is any diagonal matrix.

Now letting R given by (2.28) be a solution of (2.22) the required relation for the eigenvectors are derived as follows

$$\begin{aligned}(A-kBC)(U-\tfrac{1}{k}V) &= -kBCU + AU + BCV - \tfrac{1}{k}AV \\ &= -kBCBP + ABP + BC(BR+ABP\gamma^{-1}) - \tfrac{1}{k}AV \\ &= -kBP\gamma + ABP + B(CBR+CABP\gamma^{-1}) - \tfrac{1}{k}AV .\end{aligned}$$

Using (2.22) for replacing the term within the parenthesis

$$\begin{aligned}(A-kBC)(U-\tfrac{1}{k}V) &= -kU\gamma + ABP + B(R\gamma+P\alpha) - \tfrac{1}{k}AV , \\ &= -kU\gamma + (BR+ABP\gamma^{-1})\gamma + U\alpha - \tfrac{1}{k}AV ,\end{aligned}$$

$$(A-kBC)(U-\tfrac{1}{k}V) = (U-\tfrac{1}{k}V)(-k\gamma+\alpha) + \tfrac{1}{k}(V\alpha-AV) \qquad \text{...(2.30)}$$

The proofs for the dual formulas are done similarly. Here we simply summarize the results by stating the definitions and the relevant relations below†.

$$\bar{U} := \gamma^{-1}P^{-1}C \qquad \text{...(2.31)}$$

$$\bar{V} := \gamma^{-1}(\bar{R}C + \gamma^{-1}P^{-1}CA) \qquad \text{...(2.32)}$$

$$\bar{R}CB + \gamma^{-1}P^{-1}CAB = \gamma\bar{R} + \alpha P^{-1} \qquad \text{...(2.33)}$$

$$\bar{R} = \bar{K}_0P^{-1} + \bar{S}P^{-1} \qquad \text{...(2.34)}$$

†The asymmetry of dual definitions arise from the fact that normalization of eigenvectors are achieved by scaling dual vectors.

where $\bar{S}$ is any diagonal matrix, and $\bar{K}_0$ solves

$$\gamma\bar{K}_0 - \bar{K}_0\gamma = \gamma^{-1}P^{-1}CABP - \alpha \ , \qquad \ldots(2.35)$$

and

$$(\bar{U}-\frac{1}{k}\bar{V})(A-kBC) = (-k\gamma+\alpha)(\bar{U}-\frac{1}{k}\bar{V}) + \frac{1}{k}(\alpha\bar{V}-\bar{V}A) \qquad \ldots(2.36)$$

Next, we show that by an appropriate choice of S and/or $\bar{S}$ (in (2.28) or (2.34)) one can set

$$\bar{U}V + \bar{V}U = 0 \qquad \ldots(2.37)$$

In order to prove (2.37) the following result is required whose proof is given in Appendix A.

Lemma 3

If S_1 and S_2 are square matrices that solve the equations

$$S_1\gamma - \gamma S_1 = C_0\gamma^{-1}$$

$$\gamma S_2 - S_2\gamma = \gamma^{-1}C_0$$

where γ is given by (2.17) and C_0 is a constant matrix, then $S_1\gamma + \gamma S_2$ is a diagonal matrix ∎

Now, forming

$$\bar{U}V + \bar{V}U = \gamma^{-1}P^{-1}C(BR+ABP\gamma^{-1}) + \gamma^{-1}(\bar{R}C+\gamma^{-1}P^{-1}CA)BP$$

$$= \gamma^{-1}P^{-1}(CBR+CABP\gamma^{-1}) + \gamma^{-1}(\bar{R}CB+\gamma^{-1}P^{-1}CAB)P \ ,$$

using (2.22) and (2.33)

$$= \gamma^{-1}(P^{-1}R\gamma+\alpha) + \bar{R}P + \gamma^{-1}\alpha \ ,$$

and substituting for R and $\bar{R}$ from (2.28) and (2.34)

$$\bar{U}V + \bar{V}U = \gamma^{-1}(K_0\gamma+\gamma\bar{K}_0) + S + \bar{S} + 2\gamma^{-1} \qquad \ldots(2.38)$$

where

$$K_0\gamma - \gamma K_0 = [P^{-1}CABP-\gamma\alpha]\gamma^{-1}$$

$$\gamma\bar{K}_0 - \bar{K}_0\gamma = \gamma^{-1}[P^{-1}CABP-\alpha\gamma]$$

Therefore Lemma 3 applies and the right side of (2.38) is a diagonal matrix which can be zeroed by a suitable choice of S and/or $\bar{S}$ and thus (2.37) is proved.

Finally the relations

$$\bar{U}U = I_m \quad . \qquad \ldots(2.39)$$

$$\bar{X}U = 0 \ , \ \bar{U}X = 0 \qquad \ldots(2.40)$$

$$\bar{U}Y + \bar{V}X = 0 \ , \quad \bar{X}V + \bar{Y}U = 0 \qquad \ldots(2.41)$$

hold, where it is straightforward to prove (2.39) and (2.40). In order to prove the first equation of (2.41) note that

$$\bar{U}Y + \bar{V}X = \gamma^{-1}P^{-1}CY + \gamma^{-1}(\bar{R}C+\gamma^{-1}P^{-1}CA)X$$

and using (2.6), (2.10) and (2.12) for AX

$$\begin{aligned}\bar{U}Y + \bar{V}X &= \gamma^{-1}P^{-1}Q + \gamma^{-1}(CX\Lambda) + \gamma^{-2}P^{-1}(CBQ) \\ &= \gamma^{-1}P^{-1}Q - \gamma^{-1}P^{-1}Q \\ &= 0\end{aligned}$$

and the second equation of (2.41) is proved similarly.

All the necessary relations have now been established in defining the algebraic transformation for the modal decomposition. We define a new set of variables by the transformation

$$x := T\xi \qquad \text{...(2.42)}$$

where

$$T := (X \,\vdots\, U) - \frac{1}{k}(Y \,\vdots\, V) \qquad \text{...(2.43)}$$

If we further define

$$\bar{T} := \left(\frac{\bar{X}}{\bar{U}}\right) - \frac{1}{k}\left(\frac{\bar{Y}}{\bar{V}}\right) \qquad \text{...(2.44)}$$

then using the previously derived results of this section one can show that

$$\bar{T}T = I_n + \frac{1}{k^2}\left(\begin{array}{c|c} \bar{Y}Y & \bar{Y}V \\ \hline \bar{V}Y & \bar{V}V \end{array}\right) \qquad \text{...(2.45)}$$

The equation corresponding to (2.1) in the transformed variables can then be written as

$$\dot{\xi} = (\bar{T}T)^{-1}\bar{T}\,(A-kBC)T\xi + (\bar{T}T)^{-1}\bar{T}\,(kBu+Gv) \qquad \text{...(2.46)}$$

After partitioning the modal state variable ξ according to the partitioning of T and performing the necessary simplifications (2.46) can be written as below

$$\dot{\xi}_1 = \Lambda\xi_1 + \frac{1}{k}(\theta_{11}(k)\xi_1 + \theta_{12}(k)\xi_2)$$

$$- \bar{Q}u + \frac{1}{k}\tilde{\theta}_{11}(k)u + \bar{X}Gv + \frac{1}{k}\tilde{\theta}_{12}(k)v \qquad \text{...(2.47a)}$$

$$\dot{\xi}_2 = (-k\gamma+\alpha)\xi_2 + \frac{1}{k}(\theta_{21}(k)\xi_1 + \theta_{22}(k)\xi_2)$$

$$+ kP^{-1}u + \tilde{\theta}_{21}(k)u + \tilde{\theta}_{22}(k)v \qquad \text{...(2.47b)}$$

where the matrices $\theta_{ij}(k)$ and $\tilde{\theta}_{ij}(k)$ i, j = 1,2, are bounded in k for large k. Equations (2.47a) and (2.47b) are exact equations which are the transformed versions of (2.1). We define the approximate modally decoupled equations as

$$\dot{\hat{\xi}}_1 = \Lambda\hat{\xi}_1 - \bar{Q}u + \bar{X}Gv \qquad \ldots(2.48a)$$

$$\dot{\hat{\xi}}_2 = (-k\gamma+\alpha)\hat{\xi}_2 + kP^{-1}u \qquad \ldots(2.48b)$$

and the approximate initial condition as

$$\hat{\xi}(0) := \begin{pmatrix} \hat{\xi}_1(0) \\ \hat{\xi}_2(0) \end{pmatrix} := \begin{pmatrix} \bar{X}\, x(0) \\ \bar{U}\, x(0) \end{pmatrix} \qquad \ldots(2.49)$$

and finally the approximate original trajectory as

$$\hat{x}(t) := X\hat{\xi}_1(t) + U\hat{\xi}_2(t) \qquad \ldots(2.50)$$

Before we state the main result of this section we make a final assumption related to the unbounded eigenvalues of the closed loop system.

A3. All eigenvalues of the matrix CB are on the closed right half plane.

Theorem Consider the system described by equation (2.1) and suppose that assumptions A1. and A2. are satisfied. Let X, $\bar{X}$, U, and $\bar{U}$ be matrices defined by (2.3), (2.13), and (2.31) respectively. Under these conditions there exists matrices Y, $\bar{Y}$, V, $\bar{V}$, (not necessarily unique) such that if T and $\bar{T}$ are as defined in (2.43) and (2.44), they satisfy (2.45) and the transformed differential equation with respect to the linear transformation given by (2.42) enjoys the structure given by (2.47a) and (2.47b) where all θ matrices are bounded in k for large k.

If in addition to the above hypotheses u(•) and v(•) are vector valued functions whose components are absolutely integrable on <u>any</u> finite interval [0,t], $\dot{u}(\cdot)$ has components which are integrable functions plus a <u>finite</u> number of shifted delta functions multiplied by <u>finite</u> constants, x(0) is <u>any</u> given initial condition, and assumption A3 is satisfied then for each t > 0 the error vector satisfies

$$\lim_{k\to+\infty} \|x(t,k) - \hat{x}(t,k)\| = 0 \,, \qquad \ldots(2.51)$$

where x and $\hat{x}$ (the dependence on k has been emphasized) are given by (2.42) and (2.50) respectively, and $\|\cdot\|$ denotes any norm on R^n.

<u>Proof of Theorem</u>

The first part of the Theorem has already been proved. In order to prove (2.51) note that it is enough to prove the case where x and $\hat{x}$ are replaced by ξ and $\hat{\xi}$ since the transformation defined by (2.42) converges in the limit to that defined in (2.50).

From equations (2.47) and (2.48) it follows that

$$\xi(t') - \hat{\xi}(t') = \Bigg\{ e^{M(k)t'}(\xi(0)-\hat{\xi}(0)) + \frac{1}{k}\int_0^{t'} e^{M(k)(t'-t'')}\theta(k)\hat{\xi}(t'')dt''$$

$$+ \int_0^{t'} e^{M(k)(t'-t'')}\begin{pmatrix} \frac{1}{k}\,\tilde{\theta}_{11}(k) \\ \tilde{\theta}_{21}(k) \end{pmatrix} u(t'')dt''$$

$$+ \int_0^{t'} e^{M(k)(t'-t'')}\begin{pmatrix} \frac{1}{k}\,\tilde{\theta}_{12}(k) \\ \tilde{\theta}_{22}(k) \end{pmatrix} v(t'')dt'' \Bigg\}$$

$$+ \frac{1}{k}\int_0^{t'} e^{M(k)(t'-t'')}\theta(k)(\xi(t'')-\hat{\xi}(t''))dt'' \,,$$

$$\forall\, 0 \le t' \le t \,, \qquad \ldots(2.52)$$

where

$$\theta(k) := \begin{pmatrix} \theta_{11}(k) & \theta_{12}(k) \\ \theta_{21}(k) & \theta_{22}(k) \end{pmatrix}$$

$$M(k) := \mathrm{diag}(\Lambda, -\gamma k+\alpha) \ .$$

Because of assumption A3. the matrix $e^{M(k)}$ is bounded in norm with respect to k. This follows since the only unbounded entries of the diagonal matrix M(k) are those for which the coefficient of k has non-positive real part. Furthermore because of the assumption on the input $v(\cdot)$ and A3, $\hat{\xi}_2(\cdot)$ is bounded on the interval [0,t]. In fact for the case at the edge whereby γ has some purely imaginary diagonal entries, the corresponding critical components of $\hat{\xi}_2(t)$ can be identified by the Fourier transform of the derivative of a function which is zero outside the interval [0,t] and has a finite number of jumps on this interval. Since the Fourier variable is identified by k and the Fourier transform is bounded in k the assertion on $\hat{\xi}(\cdot)$ follows.

The observation above together with the fact that $\xi(0)$ converges to $\hat{\xi}(0)$ as $k \to +\infty$ imply that the first two terms within the parenthesis in (2.52) tend to zero. The fact that the remaining two terms tend to zero is obvious except for the situation where γ has some purely imaginary diagonal entries. In this case the result follows by the well known Riemann-Lebesque lemma [15] using the absolute integrability assumptions on $u(\cdot)$ and $v(\cdot)$ (the proof for the assertion that the convergence of the terms within the parenthesis to zero is uniform for $t' \in [0,t]$ is omitted).

In view of the arguments above the proof of the theorem follows by applying the Bellman-Gronwall Lemma to (2.52) after taking norms on both sides. ¤

Remarks

1. The Theorem makes precise what is meant by "asymptoticity" of the approximation equations. (2.51) shows that the error approaches to zero as k goes to $+\infty$. Also observe that by adding higher order terms to the approximation equations (2.48a) and (2.48b) nothing will be gained theoretically because of the compatibility of the order of the coupling terms in the original equations (2.47a) and (2.47b).

2. If the constant gain parameter k is replaced by $k_0 + k(t)$ where $k(t)$ is a given differentiable function of t, then it can be shown that the main result of the Theorem extends to this case provided that the term $\dot{k}(t)/(k_0+k(t))$ is bounded in t. The relevance of this observation stems from the application of the theory to the synchronous machine where k_0 is identified with the synchronous electrical rotor speed and $k(t)$ with the deviation from the synchronous speed.

3. If det CB = 0 or the system (A,B,C) has (nonsimple) repeated zeros, or CB has (nonsimple) repeated eigenvalues then the analysis of this section does not apply. It is a fact that under such circumstances, unlike the case treated here, asymptotic expansions of closed loop eigenvalues (hence eigenvectors) take place in noninteger powers of k^{-1}[12]. Extensions of the theory to such cases remain an open problem.

III. Modal Analysis of the Synchronous Machine

A. Synchronous Machine Model

We shall be dealing with the ideal, linear synchronous machine model where the effects of iron core saturation and space MMF harmonics are omitted. After applying Park's reference frame transformation and disregarding zero sequence quantities (since zero sequence equations are totally isolated from the others and can be treated separately) the relevant machine equations can be written, in the p.u. notation employed in [3], as follows[†],

$$\dot{\phi}_d = -ri_d - \omega\phi_q - v_d \;,$$
$$\dot{\phi}_F = -r_F i_F + v_F \;,$$
$$\dot{\phi}_D = -r_D i_D \;,$$
$$\dot{\phi}_q = -ri_q + \omega\phi_d - v_q \;,$$
$$\dot{\phi}_q = -r_Q i_Q \qquad \text{...(3.1)}$$

$$\begin{pmatrix} \phi_d \\ \phi_F \\ \phi_D \end{pmatrix} = \begin{pmatrix} L_d & L_{AD} & L_{AD} \\ L_{AD} & L_F & L_{AD} \\ L_{AD} & L_{AD} & L_D \end{pmatrix} \begin{pmatrix} i_d \\ i_F \\ i_D \end{pmatrix} \;,$$

$$\begin{pmatrix} \phi_q \\ \phi_Q \end{pmatrix} = \begin{pmatrix} L_q & L_{AQ} \\ L_{AQ} & L_Q \end{pmatrix} \begin{pmatrix} i_q \\ i_Q \end{pmatrix} \qquad \text{...(3.2)}$$

$$\ddot{\theta} + D\dot{\theta} = P_m - P_e \qquad \text{...(3.3)}$$

$$P_e = \omega(i_q\phi_d - i_d\phi_q) \qquad \text{...(3.4)}$$

Equations (3.1), (3.2) are called the <u>transformer</u> equations, (3.3) is called the <u>swing</u> equation where θ is the electrical angle of the

[†] These equations are for a salient pole machine modeled by a rotor possessing three separate windings corresponding to field and amortisseur circuits. Models for cylindrical rotor machines that employ additional rotor windings to represent eddy currents (see for example [5]) can be treated by the similar techniques developed in this paper.

rotor, $\omega = \dot{\theta}$, P_m is the mechanical power applied, and P_e given by (3.4) is the instantaneous counter electrical power induced when the machine is operated in the generator mode.

We shall adopt the commonly utilized "small deviation in speed" assumption so that the rotor speed ω is taken constant that is equal to the synchronous speed in the analysis of the transformer equations. The voltages v_d, v_q, and v_F are treated as the inputs. Using (3.1) and (3.2) one can write the associated state and the flux equations separately as

$$\dot{i}^d = -(L^d)^{-1}R^d i^d - \omega(L^d)^{-1}e_1e_1'L^q i^q - (L^d)^{-1}e_1 v_d + (L^d)^{-1}e_2 v_F \ ,$$

$$\dot{i}^q = -(L^q)^{-1}R^q i^q + \omega(L^q)^{-1}e_1e_1'L^d i^d - (L^d)^{-1}e_1 v_q \qquad \ldots(3.5)$$

$$\phi^d = L^d i^d$$

$$\phi^q = L^q i^q \qquad \ldots(3.6)$$

where

$$i^d := (i_d, i_F, i_D)' \ ,$$

$$i^q := (i_q, i_Q)' \ ,$$

$$\phi^d := (\phi_d, \phi_F, \phi_D)' \ ,$$

$$\phi^q := (\phi_q, \phi_Q)' \qquad \ldots(3.7)$$

$$L^d := \begin{pmatrix} L_d & L_{AD} & L_{AD} \\ L_{AD} & L_F & L_{AD} \\ L_{AD} & L_{AD} & L_D \end{pmatrix} \ ,$$

$$L^q := \begin{pmatrix} L_q & L_{AQ} \\ L_{AQ} & L_Q \end{pmatrix} \qquad \ldots(3.8)$$

$$R^d := \mathrm{diag}(r, r_F, r_D) \ ,$$

$$R^q := \mathrm{diag}(r, r^Q) \qquad \ldots(3.9)$$

and e_i is the ith unit vector of appropriate dimension.

We define new input voltage variables $\bar{v}_d$ and $\bar{v}_q$ by

$$v_d = \omega \bar{v}_d$$

$$v_q = \omega \bar{v}_q \ . \qquad (3.10)$$

The reason for this new adjustment is to set the synchronous machine transformer equations in the form given by (2.1). It is to be stressed that our purpose is to view the transformer equations as an <u>instance</u> in a limiting process with respect to the synchronous speed. This specific instance is when $\omega = 1$, the normalized nominal speed, and therefore a practical meaning to the above definition need not be enforced. If the necessity for such a practical interpretation is forced however, it is possible to reason by stating that <u>all</u> the voltages in a power system change in proportion to the synchronous speed of the system since they arise from the induced e.m.f. of the generators within the system. In the following derivations we shall use the new variables $\bar{v}_d$ and $\bar{v}_q$ only within the limit expressions.

From the above equations it can be seen that the model given by (2.1) applies with:

$$A := \mathrm{diag}(A_1, A_2)\ ,$$

$$A_1 := -(L^d)^{-1}R^d\ ,$$

$$A_2 := -(L^q)^{-1}R^q\ ,$$

$$B := \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix},$$

$$C := \begin{pmatrix} 0 & c_2 \\ -c_1 & 0 \end{pmatrix},$$

$$b_1 := (L^d)^{-1}e_1\ ,$$

$$b_2 := (L^q)^{-1}e_1\ ,$$

$$c_1 := e_1'L^d\ ,$$

$$c_2 := e_1'L^q\ ,$$

$$u(t) := \begin{pmatrix} -\overline{v}_d(t) \\ -\overline{v}_q(t) \end{pmatrix}$$

$$G := ((L^d)^{-1}e_2)',\ 0)'$$

$$v(t) := v_F(t)\ ,$$

$$k := \omega \qquad \ldots(3.11)$$

The high gain assumption turns out to be valid for this example. In fact, this empirical evidence is frequently reiterated in the literature as the dominance of the speed voltages over the transformer voltages. The <u>open loop</u> system (ω=0) corresponds to the blocked rotor case and if the armature and field voltages are taken as the inputs then the open loop system consists of two (d and q axes) mutually coupled passive R-L circuits. The feedback effect is due to the speed voltages induced in the armature circuits and the high feedback gain is the electrical speed of the rotor.

Before we apply the results of section II it will be instructive to give a transfer function solution of the transformer equations in which the high gain feedback structure becomes transparent. First we define the following transfer functions:

$$
\begin{aligned}
h_1(s) &:= e_1' L^d M_d(s)(L^d)^{-1} e_1 \ , \\
h_2(s) &:= e_1' L^q M_q(s)(L^q)^{-1} e_1 \ , \\
\bar{h}_1(s) &:= e_1' M_d(s)(L^d)^{-1} e_1 \ , \\
\bar{h}_2(s) &:= e_1' M_q(s)(L^q)^{-1} e_1 \ , \\
h_{dF}(s) &:= e_1' L^d M_d(s)(L^d)^{-1} e_2 \ , \\
\bar{h}_{dF}(s) &:= e_1' M_d(s)(L^d)^{-1} e_2 \ , \\
M_d(s) &:= (sI+(L^d)^{-1}R^d)^{-1} \ , \\
M_q(s) &:= (sI+(L^q)^{-1}R^q)^{-1}
\end{aligned}
\qquad \ldots(3.12)
$$

where h_1, h_2 are the $v_d - \phi_d$ and $v_q - \phi_q$ open loop (blocked rotor) transfer functions; $\bar{h}_1$, $\bar{h}_2$ are the $v_d - i_d$ and $v_q - i_q$ open loop transfer functions; and h_{dF}, $\bar{h}_{dF}$ are the $v_F - \phi_d$ and $v_F - i_d$ open loop transfer functions.

Using the definitions given above and the Laplace transformed version of equations (3.5) and (3.6), after a straightforward computation one arrives at the following equations where the symbol '~' stands for the Laplace transformed version of a variable.

$$
\begin{aligned}
\tilde{\phi}_d &= -\omega h_1 \tilde{\phi}_q - h_1 \tilde{v}_d + h_{dF} \tilde{v}_F \ , \\
\tilde{\phi}_q &= \omega h_2 \tilde{\phi}_d - h_2 \tilde{v}_q
\end{aligned}
\qquad \ldots(3.13)
$$

$$\tilde{i}_d = -\omega\bar{h}_1\tilde{\phi}_q - \bar{h}_1\tilde{v}_d + \bar{h}_{dF}\tilde{v}_F ,$$

$$\tilde{i}_q = \omega\bar{h}_2\tilde{\phi}_d - \bar{h}_2\tilde{v}_q \qquad \ldots(3.14)$$

Solving for $\tilde{\phi}_d$ and $\tilde{\phi}_q$ from (3.13) and substituting the result in (3.14) the following input-output transfer function relations are obtained

$$\begin{pmatrix} \tilde{\phi}_d \\ \tilde{\phi}_q \end{pmatrix} = \frac{1}{1+\omega^2 h_1 h_2} \begin{pmatrix} -h_1 , & \omega h_1 h_2, & h_{dF} \\ -\omega h_1 h_2, & -h_2 , & \omega h_2 h_{dF} \end{pmatrix} \begin{pmatrix} \tilde{v}_d \\ \tilde{v}_q \\ \tilde{v}_F \end{pmatrix} , \qquad \ldots(3.15)$$

$$\begin{pmatrix} \tilde{i}_d \\ \tilde{i}_q \end{pmatrix} = \frac{1}{1+\omega^2 h_1 h_2} \begin{pmatrix} -\bar{h}_1 , & \omega\bar{h}_1 h_2, & \bar{h}_{dF}+\omega^2 h_2(h_1\bar{h}_{dF}-\bar{h}_1 h_{dF}) \\ -\omega\bar{h}_2 h_1, & -\bar{h}_2 , & \omega h_2 h_{dF} \end{pmatrix} \begin{pmatrix} \tilde{v}_d \\ \tilde{v}_q \\ \tilde{v}_f \end{pmatrix} \qquad \ldots(3.16)$$

If $n_i(s)$ and $d_i(s)$ denote the numerator and denominator polynomials of $h_i(s)$, $i = 1,2$, then it can be shown (as is almost obvious from (3.16)) that the closed loop eigenvalues are the roots of

$$d_1(s)d_2(s) + \omega^2 n_1(s)n_2(s) = 0$$

It is interesting that the multivariable root locus problem has reduced to a single variable one due to the cross coupled structure of the feedback. We shall postpone any further discussion on the above transfer function relations and compare them later with the approximate expressions obtained via the modal analysis. At this stage it suffices to remark that the equations (3.15) and (3.16) do not involve any approximating assumptions based on high feedback gain and thus are exact in this sense.

B. Modal Equations for the Synchronous Machine

The results of section 2 are now applied to the example of the synchronous machine described by (2.1) and (3.11). First it is noted that the zeroes of the representation (A,B,C) are simply the roots of the polynomials $n_1(s)$ and $n_2(s)$ (numerators of $h_1(s)$ and $h_2(s)$) since

$$C(sI-A)^{-1}B = \begin{pmatrix} 0 & h_2(s) \\ -h_1(s) & 0 \end{pmatrix} \qquad \text{...(3.17)}$$

If we let λ_d' and λ_d'' denote the distinct zeroes of $n_1(s)$, and λ_q'' the zero of $n_2(s)$ then the following relations are a straightforward consequence of the definitions of section II and (3.11).

$$X = \begin{bmatrix} \frac{(\lambda'_d I-A_1)^{-1}b_1}{\lambda_d' h_1'(\lambda'_d)} , & \frac{(\lambda_d'' I-A_1)^{-1}b_1}{\lambda_d'' h_1'(\lambda_d'')} , & 0 \\ 0 \quad , & 0 \quad , & \frac{(\lambda_q'' I-A_2)^{-1}b_2}{\lambda_q'' h_2'(\lambda_q'')} \end{bmatrix}$$

...(3.18)

$$\overline{X} = \begin{bmatrix} -\lambda_d' c_1(\lambda_d' I-A_1)^{-1} & , & 0 \\ -\lambda_d'' c_1(\lambda_d'' I-A_1)^{-1} & , & 0 \\ 0 & , & -\lambda_q'' c_2(\lambda_q'' I-A_2)^{-1} \end{bmatrix}$$

...(3.19)

$$Q = \begin{bmatrix} \frac{1}{\lambda_d' h_1'(\lambda_d')} & , & \frac{1}{\lambda_d'' h_1'(\lambda_d'')} & , & 0 \\ 0 & , & 0 & , & \frac{1}{\lambda_q'' h_2'(\lambda_q'')} \end{bmatrix}$$

...(3.20)

$$\overline{Q} = \begin{bmatrix} 0 & \lambda_d' \\ 0 & \lambda_d'' \\ -\lambda_q'' & 0 \end{bmatrix} \qquad \ldots(3.21)$$

$$CB = \begin{bmatrix} 0 & c_2b_2 \\ -c_1b_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \ldots(3.22)$$

$$\gamma = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}, \quad P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$$

$$\ldots(3.23)$$

$$\alpha = \frac{1}{2}\,(c_1A_1b_1+c_2A_2b_2) \cdot I_2 \qquad \ldots(3.24)$$

$$U = BP = \begin{bmatrix} b_1 & b_1 \\ jb_2 & -jb_2 \end{bmatrix} \qquad \ldots(3.25)$$

$$\overline{U} = \gamma^{-1}P^{-1}C = \frac{1}{2} \begin{bmatrix} c_1 & -jc_2 \\ c_1 & jc_2 \end{bmatrix} \qquad \ldots(3.26)$$

where h_1' and h_2' denote the derivatives of h_1 and h_2. Expressions for Y, $\overline{Y}$, V, $\overline{V}$, and the auxiliary matrices R, $\overline{R}$, K_0, $\overline{K}_0$, S, $\overline{S}$, that are used in computing V and $\overline{V}$, and the details of these computations are given in Appendix B.

Finally we apply the transformation P^{-1} to the last two variables (the complex modes) in the approximate modal equations so that the associated Jordan matrix is real. The resulting modal equations after this modification are given by

$$\dot{z}'_d = \lambda'_d z'_d + \frac{\lambda'_d}{\omega} v_q - \lambda'_d h_{dF}(\lambda'_d) \cdot v_F \quad , \qquad \ldots(3.27a)$$

$$\dot{z}''_d = \lambda''_d z''_d + \frac{\lambda''_d}{\omega} v_q - \lambda''_d h_{dF}(\lambda''_d) \cdot v_F \quad , \qquad \ldots(3.27b)$$

$$\dot{z}''_q = \lambda''_q z''_q - \frac{\lambda''_q}{\omega} v_d \quad , \qquad \ldots(3.27c)$$

$$\dot{z}_{sd} = \lambda_a z_{sd} - \omega z_{sq} - v_d \quad , \qquad \ldots(3.27d)$$

$$\dot{z}_{sq} = \omega z_{sd} + \lambda_a z_{sq} - v_q \qquad \ldots(3.27e)$$

where

$$\lambda_a := \frac{1}{2}(c_1 A_1 b_1 + c_2 A_2 b_2) \qquad \ldots(3.28)$$

and h_{dF} is given in (3.12). The corresponding asymptotic expressions for the d and q axes current and flux variables denoted by $\hat{i}_d$, $\hat{i}_q$, $\hat{\phi}_d$, $\hat{\phi}_q$ are obtained using (2.50) as below.

$$\hat{i}_d = \frac{\bar{h}_1(\lambda'_d)}{\lambda'_d h'_1(\lambda'_d)} \cdot z'_d + \frac{\bar{h}_1(\lambda''_d)}{\lambda''_d h'_1(\lambda''_d)} \cdot z''_d + e'_1 b_1 \cdot z_{sd} \qquad \ldots(3.29)$$

$$\hat{i}_q = \frac{\bar{h}_2(\lambda''_q)}{\lambda''_q h'_2(\lambda''_q)} \cdot z''_q + e'_1 b_2 \cdot z_{sq} \qquad \ldots(3.30)$$

$$\hat{\phi}_d = z_{sd} \qquad \ldots(3.31)$$

$$\hat{\phi}_q = z_{sq} \qquad \ldots(3.32)$$

where $U = BP$ was replaced by B in (2.50) and $\bar{u} = \gamma^{-1}P^{-1}C$ was replaced by $P\gamma^{-1}P^{-1}C = (CB)^{-1}C$ in (2.49) because of the modification leading to the real Jordan form as mentioned above.

The approximate version of the induced counter electrical power is given by

$$\hat{P}_e = \omega(\hat{i}_q\hat{\phi}_d-\hat{\phi}_q\hat{i}_d) \qquad \ldots(3.33)$$

Equations (3.27a) - (3.27e) and (3.29) - (3.33) are the asymptotic modal equations for the synchronous machine. The sense in which these equations may be called "approximate" has already been made precise by the Theorem of section II. Moreover because of Remark 2 following the proof of the Theorem in section II the rotor speed ω may be allowed to vary with time (i.e. $\omega(t) = \omega_0 + \dot{\theta}(t)$ where $\dot{\theta}(t)$ is the instantaneous deviation from the synchronous speed) and if $\ddot{\theta}(t)/(\omega_0+\dot{\theta}(t))$ is bounded in t as $\omega_0 \to +\infty$ then our approximation remains valid when ω is replaced by $\omega(t)$.

We shall now present a view that gives another related interpretation to our modal approximation by using Laplace transforms. Assume that $\bar{v}_d$, $\bar{v}_q$, and v_F are L-transformable functions of time on $[0,+\infty)$. Let $s \in C$ and $s \neq \lambda_d', \lambda_d'', \lambda_q''$. Again using '~' for the transformed variable one computes the transform of the $\hat{i}_d$ using (3.10), (3.27a) - (3.27e) and (3.29) as

$$\tilde{\hat{i}}_d(s) = \left(\frac{\bar{h}_1(\lambda_d')}{h_1'(\lambda_d')}\frac{1}{s-\lambda_d'} + \frac{\bar{h}_1(\lambda_d'')}{h_1'(\lambda_d'')}\frac{1}{s-\lambda_d''}\right)\tilde{\bar{v}}_q(s)$$

$$+ e_1'b_1\left[\frac{\tilde{\bar{v}}_q(s)\omega^2}{(s-\lambda_a)^2+\omega^2} - \frac{\tilde{\bar{v}}_d(s)\omega(s-\lambda_a)}{(s-\lambda_a)^2+\omega^2}\right]$$

$$-\left[\frac{h_{dF}(\lambda_d')\,\bar{h}_1(\lambda_d')}{h_1'(\lambda_d')}\frac{1}{s-\lambda_d'} + \frac{h_{dF}(\lambda_d'')\,\bar{h}_1(\lambda_d'')}{h_1'(\lambda_d'')}\frac{1}{s-\lambda_d''}\right]\tilde{v}_F(s)$$

and therefore

$$\lim_{\omega \to +\infty} \tilde{\hat{i}}_d(s) = \frac{\bar{h}_1(s)}{h_1(s)} \tilde{\bar{v}}_q(s) + (h_{dF}(s) - \frac{\bar{h}_1(s) h_{dF}(s)}{h_1(s)}) \tilde{v}_F(s) \quad \ldots(3.34)$$

using (D1) and (D3) of Appendix D. But using the exact expression given by (3.16) with $\tilde{v}_q = \omega\tilde{\bar{v}}_q$ it follows that

$$\begin{aligned} \lim_{\omega \to +\infty} \tilde{i}_d(s) &= \frac{\bar{h}_1(s)}{h_1(s)} \tilde{\bar{v}}_q(s) + (\bar{h}_{dF}(s) - \frac{h_1(s) h_{dF}(s)}{h_1(s)}) \tilde{v}_F(s) \quad \ldots(3.35) \\ &= \lim_{\omega \to +\infty} \tilde{\hat{i}}_d(s) \end{aligned}$$

which shows that the transformed error approaches zero in the limit. Using similar reasoning and (D2) one can show that $\tilde{\hat{i}}_q(s)$, $\tilde{\hat{\phi}}_d(s)$, and $\tilde{\hat{\phi}}_q(s)$ approach in the limit to $\tilde{i}_q(s)$, $\tilde{\phi}_d(s)$, and $\tilde{\phi}_q(s)$ respectively.

C. Comparison with Classical Models

In Appendix C an account of the classical definitions and their relations to the model developed in this section are presented. The approximations involved in the derivation of this presentation stem from the range of the values of the machine parameters and not from the main and the only source of approximations used so far, namely high rotor speed. Our purpose for using these parameter approximations and the associated terminology is to relate our work to the existing body of literature.

We first write the field excitation voltage v_F as

$$v_F = v_{F0} + \Delta v_F$$

where it is assumed that v_{F0} has been applied for a long time and Δv_F is the incremental adjustment for possible terminal voltage regulation. The steady state values for $\hat{i}_q$, $\hat{\phi}_d$, and $\hat{\phi}_q$ corresponding to the input v_{F0} are all zero except for $\hat{i}_d$ which is

$$\hat{i}_{dss} = \left[\frac{\bar{h}_1(\lambda_d')\cdot h_{dF}(\lambda_d')}{\lambda_d' h_1'(\lambda_d')} + \frac{\bar{h}_1(\lambda_d'')\cdot h_{dF}(\lambda_d'')}{\lambda_d'' h_1'(\lambda_d'')}\right] v_{F0}$$

$$= \left[\bar{h}_{dF}(0) - \frac{\bar{h}_1(0)\cdot h_{dF}(0)}{h_1(0)}\right] v_{F0}$$

$$= \left[0 - \frac{(1/r)\cdot L_{AD}/r_F}{L_d/r}\right] v_{F0}$$

$$= -\frac{L_{AD}}{L_d}\cdot\frac{v_{F0}}{r_F}, \qquad \text{...(3.36)}$$

where (3.27a), (3.27b), (3.29), (D3), and (3.12) have been used. If we define

$$E_f := \omega L_{AD}\frac{v_{F0}}{r_F} \qquad \text{...(3.37)}$$

$$E_f' := E_f \cdot \frac{L_d'}{L_d} \qquad \text{...(3.38)}$$

$$E_f'' := E_f \cdot \frac{L_d''}{L_d} \qquad \text{...(3.39)}$$

then

$$\hat{i}_{dss} = -\frac{E_f}{\omega L_d} = -\frac{E_f'}{\omega L_d'} = -\frac{E_f''}{\omega L_d''}, \qquad \text{...(3.40)}$$

Now also using the approximations of Appendix C a new modally decoupled model is obtained as presented below.

$$\tau_d' \dot{u}_d' = -u_d' - v_q + \omega\beta'\Delta v_F \quad , \qquad \text{...(3.41a)}$$

$$\tau_d'' \dot{u}_d'' = -u_d'' - v_q + \omega\beta''\Delta v_F \quad , \qquad \text{...(3.41b)}$$

$$\tau_q'' \dot{u}_q'' = -u_q'' + v_d \quad , \qquad \text{...(3.41c)}$$

$$\frac{(\tau_a \dot{u}_{sd} + u_{sd})}{\omega\tau_a} = - u_{sq} - v_d \quad , \qquad \text{...(3.41d)}$$

$$\frac{(\tau_a \dot{u}_{sq} + u_{sq})}{\omega\tau_a} = u_{sd} - v_q \quad , \qquad \text{...(3.41e)}$$

$$\hat{i}_d \cong \left(\frac{1}{\omega L_d'} - \frac{1}{\omega L_d}\right) u_d' + \left(\frac{1}{\omega L_d''} - \frac{1}{\omega L_d'}\right) u_d'' + \frac{1}{\omega L_d''} u_{sd} - \frac{E_f}{\omega L_d} \, , \qquad \text{...(3.42)}$$

$$\hat{i}_q \cong \left(\frac{1}{\omega L_q''} - \frac{1}{\omega L_q}\right) u_q'' + \frac{1}{\omega L_q''} u_{sq} \quad , \qquad \text{...(3.43)}$$

$$\omega\hat{\phi}_d \cong u_{sd} \quad , \qquad \text{...(3.44)}$$

$$\omega\hat{\phi}_q \cong u_{sq} \quad , \qquad \text{...(3.45)}$$

$$P_e \cong (u_{sd} \cdot \hat{i}_q - u_{sq} \hat{i}_d) \quad , \qquad \text{...(3.46)}$$

where

$$\beta' := h_{dF}(\lambda_d') \quad ,$$

$$\beta'' := h_{dF}(\lambda_d'') \quad , \qquad \text{...(3.47)}$$

and the term $E_f/\omega L_d$ in (3.48) can be replaced by $E_f'/\omega L_d'$ or $E_f''/\omega L_d''$ because of (3.41).

The model presented above is central to the reduced models of different temporality derived in literature. If we substitute $u_d' \equiv u_d'' \equiv u_q'' \equiv 0$, (d and q axis modes yet unexcited) $u_{sq} = -v_d$, $u_{sd} = v_q$ (the armature complex mode given by (3.41d) and (3.41e) is in its guasi steady state, i.e. LHS = 0), and $\Delta v_F \equiv 0$ (voltage regulation dynamics not excited), then we obtain the so called <u>subtransient model</u> for the synchronous machine. If we substitute $u_d' \equiv 0$ (d axis transient mode not excited) and $\tau_d''\dot{u}_d'' = 0$, $\tau_q''\dot{u}_q'' = 0$, $u_{sq} = -v_d$, $u_{sd} = v_q$ (d axis subtransient mode, q axis subtransient mode and armature modes follow their respective quasi steady state values) and $\Delta v_F \equiv 0$ we obtain the <u>transient model</u>, and finally the steady state model is obvious. If in addition we assume that in each of these models $L_d'' = L_q''$, $L_d' = L_q$, and $L_d = L_q$ respectively then the quasi steady state values substituted in (3.42) and (3.43) can be identified by a(complex)equality of phasors which, in turn, can be identified by an equivalent single phase linear circuit in steady state consisting of a reactance and an independent source behind it delivering an average AC power given by (3.46).

One observes that the d and q axis flux (or induced voltage = $\omega\cdot$flux) variables coincide with the armature <u>modal</u> flux variables as seen from (3.44) and (3.45). This complex mode with eigenvalues $-\frac{1}{\tau_a} \pm j^{\omega}$ contributes to the armature DC offset mode in 3 phase operation. This is because the difference frequency is zero after performing inverse Park transformation. The rather intriguing fact that, it is because an asymptotic natural frequency of the closed loop system equals the synchronous speed that there is an exponentially decaying DC offset mode in armature currents, has apparently remained unrecognized. For various nonanalytical explanations of this phenomenon (involving various definitions of negative sequence reactances) the reader is referred to Kimbark [1]pp. 39, 68 or Concordia [2] pp. 87-95.

A considerable amount of effort is spent by the classical treatment in explaining and justifying the assumption of the "constancy of the field flux linkage" (see for example Kimbark [1] pp. 13-18). This assumption admits a very elementary interpretation in modal formulation. Indeed it sums up to saying that the component of the field flux along all the quick modes are negligible and therefore it can approximately be identified with the d axis transient mode which is slow. In order to quantify this statement we first write the field flux as

$$\hat{\phi}_F = \frac{e_2' L^d (\lambda_d' L^d + R^d)^{-1} e_1}{\lambda_d' h_1'(\lambda_d')} \cdot z_d' + \frac{e_2' L^d (\lambda_d'' L^d + R^d)^{-1} e_1}{\lambda_d'' h_1'(\lambda_d'')} \cdot z_d''$$

which follows from the definition of $\hat{\phi}_F$ and (2.50). It is already seen from this formula that components along the complex armature mode and the q axis subtransient mode are zero. Now observe

$$|\lambda_d''| >> \frac{r}{L_d}, \frac{r_F}{L_F}, \frac{r_D}{L_D}$$

which in practice is valid, and consider

$$\lambda_d'' L^d + R^d \simeq \lambda_d'' L^d,$$

$$e_2' L^d (\lambda_d'' L^d + R^d)^{-1} e_1 \simeq e_2' L^d (\lambda_d'' L^d)^{-1} e_1 = \frac{1}{\lambda_d''} e_2' e_1 = 0 ,$$

implying that the component along the d axis substransient mode is negligible.

Finally one observes that a simple interpretation for the d and q axis subtransient and transient reactances can be given without even any reference to the modal decomposition result. Assuming that τ_a is small enough[†] (otherwise the given definitions on subtransient and transient phenomena are devoid of any meaningful interpretation) one can write from (3.15)

$$\tilde{\phi}_d(s) \approxeq \tilde{v}_q(s)$$
$$\tilde{\phi}_q(s) \approxeq -\tilde{v}_d(s)$$

which upon substitution in (3.16) yield

$$\tilde{i}_d(s) \approxeq \frac{1}{\omega}\frac{\bar{h}_1(s)}{h_1(s)} \cdot \tilde{\phi}_d(s)$$
$$\tilde{i}_q(s) \approxeq \frac{1}{\omega}\frac{\bar{h}_2(s)}{h_2(s)} \tilde{\phi}_q(s) ,$$

and using (C31), (C34), (C37)-(C39) in (D1) and (D2)

$$\tilde{i}_d(s) \approxeq \left\{\frac{1}{L''_d} + \left(\frac{1}{L''_d} - \frac{1}{L'_d}\right)\frac{\lambda''_d}{s-\lambda''_d} + \left(\frac{1}{L'_d} - \frac{1}{L_d}\right)\frac{\lambda'_d}{s-\lambda'_d}\right\}\tilde{\phi}_d(s)$$

$$\tilde{i}_q(s) \approxeq \left\{\frac{1}{L''_q} + \left(\frac{1}{L''_q} - \frac{1}{L_q}\right)\frac{\lambda''_q}{s-\lambda''_q}\right\}\tilde{\phi}_q(s)$$

which gives the dynamical relations between the flux and the current variables. For a unit step input in v_d (or equivalently ϕ_d) and $v_q(\phi_q)$ the corresponding currents are

$$i_d(t) \approxeq \frac{1}{L''_d} + \left(\frac{1}{L'_d} - \frac{1}{L''_d}\right)\left(1-e^{-\frac{t}{\tau'_d}}\right) + \left(\frac{1}{L_d} - \frac{1}{L'_d}\right)\left(1-e^{-\frac{t}{\tau''_d}}\right) ,$$

[†] τ_a is <u>not</u> small if the resistance r in (C.10) is taken as the armature resistance of the machine which would be the case if machine is short circuited at its terminals. It is the line resistance added to r that makes τ_a small even within the subtransient time scale.

$$i_q(t) \cong \frac{1}{L_q''} + \left(\frac{1}{L_q} - \frac{1}{L_q''}\right)\left(1 - e^{-\frac{t}{\tau_q''}}\right)$$

which require no further explanation.

IV Simulation Results

In this section we present three numerical simulation experiments in order to assess the accuracy of the asymptotically decoupled model using typical synchronous machine data. The experiments in question are field voltage adjustment, short circuit analysis, and stability analysis.

First let us emphasize the fact that the decoupled model given by (3.27)-(3.33) becomes identical to the model given by (3.41)-(3.47) provided that we replace the approximate time constants τ_d', and τ_d'' by their counterparts obtained from the roots of $n_1(s)$, and replace L_d' by L_{dmod}' where

$$L_{dmod}' := 1/\left(\frac{1}{L_d''} - \frac{\bar{h}_1(\lambda_d'')}{\lambda_d'' h_1'(\lambda_d'')}\right), \tag{4.1}$$

in the latter model. In order that no other source of approximation contaminates we shall use our original model given by (3.27)-(3.33). There is, however, a final empirical adjustment to be made so as to diminish the steady state error contributed by second order terms. In particular we replace (3.27d) and (3.27e) by

$$\dot{z}_{sd} = \lambda_a z_{sd} - \omega z_{sq} - v_d - \frac{\lambda_a}{\omega} v_q \tag{4.2a}$$

$$\dot{z}_{sq} = \omega z_{sd} + \lambda_a z_{sq} - v_q + \frac{\lambda_a}{\omega} v_d \, . \tag{4.2b}$$

so that in the steady state the relations

$$z_{sq} = -\frac{v_d}{\omega}\,,$$
$$z_{sd} = \frac{v_q}{\omega}\,, \tag{4.3}$$

hold irrespective of the relative magnitude of λ_a. If the armature resistance is relatively large (when line resistance is added) the contribution of the offset error terms to i_d or i_q may be of the order of 30% unless the above adjustment is made. On the other hand our adjustment is theoretically neutral since addition of second order terms to the right side of (2.48) do not influence the theoretical conclusions.

The machine data used for the simulations is almost identical to the one given in [3] p. 176 and is given in Appendix E. In Table 1 the closed loop poles are given as a function of ω. This tabulation exhibits the validity of the high gain assumption as far as the asymptotic root convergence is concerned.

In all the three simulations the synchronous machine is connected to an infinite busbar of voltage magnitude V_b. If θ is the electrical angle between the voltage of the phase a and the quadrature axis of the rotor, then using the definition of Park's transformation it can be shown that v_d and v_q are given by

$$v_d = -V_b \sin\theta$$
$$v_q = V_b \cos\theta \quad . \tag{4.4}$$

It has been assumed that the connecting line resistances and reactances are incorporated in L_d, L_q, and r for the sake of simplicity. Only in the short circuit simulation we replace r by

$$r = .0021 \text{ pu}$$

to observe the effect of relatively larger time constant for the armature modes (machine short circuited at its terminals). For the simulations we do not use the assumption of small speed deviation, and therefore use the actual rotor speed variable both within the original and decoupled transformer equations. The differences arising from this approximation, however, are observed to be insignificant in all our simulations.

The starting conditions for all the simulations are:

$$V_b = 1 \text{ pu}$$

$$v_{F0} = .5 \times 10^{-3} \text{ pu } (E_f = 1.044 \text{ pu}) \;.$$

$$P_m = .3 \text{ pu.}$$

The corresponding equilibrium angle is given by

$$\theta = 29.23°.$$

Fourth order Runge-Kutta routine is used in solving the original and the decoupled equations.

In the first simulation a 10% step increase is applied to v_F and the resulting i_d and i_q are plotted both within the fast and slow time scales as shown in figures 1-4. The oscillatory response of the original model in the fast time scale is due to the second order effect of the step increase in exciting the armature modes. The dynamical error is seen to be within 5% from the slow time scale plots.

In the short circuit simulation the resulting d and q axis transient currents exhibit an error low enough not to be visible from the plots of figures 5 and 6. The discrepancy between the angles of the original and the decoupled model given in figure 6' is of interest. Since the angle is proportional to twice the integral of the difference between the mechanical power and the transient electrical power given in figure 5' it is seen that due to the large amplitude of the electrical power transients higher order terms are important in the determination of the angle deviation. Although in the more realistic situation the effect of saturation limits the electrical power swing to a somewhat lower level the practice of using only the swing equation for computing the angle deviation during a short circuit fault interval may well give rise to erroneous results in computing critical clearing times for stability analysis. A full model for the short circuited machine that incorporates electrical transients and saturation effects may be advisable under such circumstances.

The final simulation deals with the stability of the synchronous machine. In all the cases the initial speed is the synchronous speed and the initial angles are chosen as -120°, -150°, -180°, and -160°; and the resulting swing curves are given in figures 7-10 respectively. These figures exhibit the fact that the decoupling approximation deteriorates when the initial angle is within the boundary of stable region. This is to be expected since the sensitivity of the solution with respect to the initial angle becomes high in the vicinity of the stability boundary and higher order terms dominate. All the curves show the dramatic damping effect of the subtransient modes recalling that the mechanical damping is taken as zero. As a comparison of interest we state the interval of stability computed by the equal area criterion; and for the decoupled, and original models as observed in the simulation. These intervals are [-168°, 176°], [-175°, 184°], and [-160°, 198°] respectively.

V. Conclusions

A theory for the asymptotic modal decomposition of a linear multivariable feedback system subject to high gain output feedback has been developed and applied to the linear, ideal synchronous machine by identifying the latter with such a high gain output feedback system. This new conceptualization leading to the asymptotic decoupling of machine modes brings a unified and rigorous understanding to synchronous machine dynamics.

The simulation results confirm that the asymptotically decoupled model is an excellent approximation for the linear ideal synchronous machine for a wide range of situations. Possible exceptions are pointed out in section IV. What is more important, however, is that the decoupled model is a theoretical starting point for model reduction methods and is related to the original model through an analytically well defined approximation based only on high rotor speed.

The theoretical result of section II can possibly be extended to the case where CB is singular; or zeros of (A,B,C) or eigenvalues of CB are nonsimple repeated. Another possibility that can be exploited is to retain higher order terms in the asymptotic expansions of the eigenvalues and the eigenvectors for greater accuracy. These questions remain open and efforts along these directions would be a welcome contribution to the theory.

An extension of asymptotic modal decoupling results to the linear but the nonideal synchronous machine looks promising. The recent work of Youla and Bongiorno [14] depicts the existence of a Floquet frame under nonideal circumstances and moreover they conjecture a very simple dependence of this frame on the rotor speed. These results have an important bearing to the modal analysis of the nonideal machine and are therefore worth a careful scrutiny.

If machine saturation is treated in a post-Park manner, which seems to be the usual practice (and an unjustified one), then our results are valid for the linearized machine with saturation. If, however, saturation is modeled before Park's transformation, situation is more difficult. The existence of a Park-like transformation for a saturated machine becomes a problem in itself. If at a given solution linearization is performed, the resulting linear system is periodic and coincides with a linear nonideal machine. One therefore concludes that an asymptotic modal analysis for the pre-Park saturated linearized machine is reduced to the case of a linear, nonideal machine.

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Appendix A

1. Proof of Lemma 1

First we establish the fact that

$$\bar{x}_i x_j = 0 \quad , \quad i \neq j \qquad \text{...(A1)}$$

Using (2.2) we may write

$$\bar{x}_j'(\lambda_i I - A)x_i = \bar{x}_j' B q_i = 0 \ ,$$

$$\bar{x}_j'((\lambda_i - \lambda_j)I + (\lambda_j I - A))x_i = 0 \ ,$$

$$(\lambda_i - \lambda_j)\bar{x}_j' x_i + \bar{x}_j'(\lambda_j I - A)x_i = 0 \ ,$$

$$(\lambda_i - \lambda_j)\bar{x}_j' x_i + q_j' C x_i = 0 \ ,$$

$$(\lambda_i - \lambda_j)\bar{x}_j' x_i = 0 \ ,$$

which implies that $\bar{x}_j' x_i = 0$ by assumption A2. We prove the statement $x_i' x_i \neq 0$ by contradiction. If we assume that $\bar{x}_i' x_i = 0$ for some i, then by (2.2) we can write

$$\bar{x}_i'(\lambda_i I - A)x_i = 0 \ ,$$

or

$$\bar{x}_i' A x_i = 0 \ .$$

Now consider the product of matrices

$$M \triangleq \begin{bmatrix} \bar{x}'_i & \bar{q}_i' \\ \bar{X}_i' & \bar{Q}_i' \end{bmatrix} \begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} x_i & X_i \\ -q_i & Q_i \end{bmatrix}$$

where $\bar{X}_i'$, $\bar{Q}_i'$, X_i, and Q_i are constant matrices of appropriate size so that the first and third matrices in the above product are nonsingular.

Multiplying out we obtain

$$M = \begin{bmatrix} \bar{x}_i'(sI-A)x_i & \{\bar{x}_i'(sI-A)-\bar{q}_i'C\}X_i \\ \bar{X}_i'\{(sI-A)-Bq_i\} & * \end{bmatrix}$$

where we have used (2.10) and (2.11) repeatedly. Finally using

$\bar{x}_i'x_i = \bar{x}_i'Ax_i = 0$

$$M = \begin{bmatrix} 0 & \{\bar{x}_i'(sI-A)-\bar{q}_i'C\}X_i \\ X_i'\{(sI-A)x_i-Bq_i\} & * \end{bmatrix}$$

But because of (2.2) the first column and row vectors of M are polynomial vectors which vanish at $s = \lambda_i$. Thus M can be factored as

$$M = \mathrm{diag}((s-\lambda_i),1,\ldots,1)\, P(s)\, \mathrm{diag}((s-\lambda_i),1,\ldots,1)$$

where P(s) is a polynomial matrix. Therefore

$$\det M = (s-\lambda_i)^2 \det P(s) = \text{Const.} \det\begin{pmatrix} sI-A & B \\ -C & 0 \end{pmatrix}$$

which clearly contradicts assumption A2. Hence $\bar{x}_i'x_i \neq 0$ and the rest follows easily. ∎

2. <u>Proof of Lemma 3</u>

For $i \neq j$

$$[S_1]_{ij} = \frac{[C_0]_{ij}}{\gamma_j(\gamma_j-\gamma_i)}$$

$$[S_2]_{ij} = \frac{[C_0]_{ij}}{\gamma_i(\gamma_i-\gamma_j)}$$

therefore

$$[S_1\gamma+\gamma S_2]_{ij} = [S_1]_{ij}\gamma_j + [S_2]_{ij}\gamma_i = \frac{[C_0]_{ij}}{\gamma_j-\gamma_i} + \frac{[C_0]_{ij}}{\gamma_i-\gamma_j} = 0 \quad \blacksquare$$

Appendix B

First define:

$$\delta_1 := c_1A_1b_1 \qquad \ldots(B1)$$

$$\delta_2 := c_2A_2b_2 \qquad \ldots(B2)$$

Then one easily computes

$$P^{-1}CABP = \frac{1}{2}\begin{pmatrix} \delta_1+\delta_2, & \delta_2-\delta_1 \\ \delta_2-\delta_1, & \delta_1+\delta_2 \end{pmatrix} \gamma = \frac{1}{2}\gamma \begin{pmatrix} \delta_1+\delta_2, & \delta_1-\delta_2 \\ \delta_1-\delta_2, & \delta_1+\delta_2 \end{pmatrix}$$

$$\ldots(B3)$$

Thus using (2.27) and (2.28)

$$P^{-1}R = \frac{1}{4}\begin{pmatrix} s_1 & j(\delta_2-\delta_1) \\ -j(\delta_2-\delta_1) & s_2 \end{pmatrix},$$

$$R = \frac{1}{4}\begin{pmatrix} s_1-j(\delta_2-\delta_1), & s_2+j(\delta_2-\delta_1) \\ j(s_1+j(\delta_2-\delta_1)), & -j(s_2-j(\delta_2-\delta_1)) \end{pmatrix},$$

$$\ldots(B4)$$

from which using (2.20) we obtain

$$V = \begin{pmatrix} -jA_1b_1+\frac{1}{4}(s_1-j(\delta_2-\delta_1))b_1, & jA_1b_1+\frac{1}{4}(s_2+j(\delta_2-\delta_1)b_1) \\ A_2b_2+\frac{j}{4}(s_1+j(\delta_2-\delta_1))b_2, & A_2b_2-\frac{j}{4}(s_2-j(\delta_2-\delta_1))b_2 \end{pmatrix}$$

$$\ldots(B5)$$

The dual relations are computed similarly and are given below

$$\overline{R}P = \frac{1}{4}\begin{pmatrix} \overline{s}_1 & j(\delta_2-\delta_1) \\ -j(\delta_2-\delta_1), & \overline{s}_2 \end{pmatrix}, \qquad \ldots\text{(B6)}$$

$$\overline{R} = \frac{1}{8}\begin{pmatrix} \bar{s}_1+j(\delta_2-\delta_1), & -j(\bar{s}_1-j(\delta_2-\delta_1)) \\ \bar{s}_2-j(\delta_2-\delta_1), & j(\bar{s}_2+j(\delta_2-\delta_1)) \end{pmatrix} \qquad \ldots\text{(B7)}$$

$$\overline{V} = \frac{1}{2}\begin{pmatrix} -jc_1A_1+\frac{1}{4}(\overline{s}_1-j(\delta_2-\delta_1))c_1, & -c_2A_2-\frac{j}{4}(\overline{s}_1-j(\delta_2-\delta_1))c_2 \\ jc_1A_1+\frac{1}{4}(\overline{s}_2+j(\delta_2-\delta_1))c_1, & -c_2A_2+\frac{j}{4}(\overline{s}_2-j(\delta_2-\delta_1))c_2 \end{pmatrix}$$

$$\ldots\text{(B8)}$$

If we choose

$$s_1 = 2j(\delta_1+\delta_2),\ s_2 = -2j(\delta_1+\delta_2)$$

$$\overline{s}_1 = s_1,\ \overline{s}_2 = s_2\ ,$$

then it can easily be shown that (2.37) is satisfied. Thus the corresponding matrices V and $\overline{V}$ are

$$V = \begin{pmatrix} -jA_1b_1+j(\frac{3}{4}\delta_1+\frac{1}{4}\delta_2)b_1, & jA_1b_1-j(\frac{3}{4}\delta_1+\frac{1}{4}\delta_1)b_2 \\ A_2b_2-(\frac{3}{4}\delta_2+\frac{1}{4}\delta_1)b_2, & A_2b_2-(\frac{3}{4}\delta_2+\frac{1}{4}\delta_1)b_2 \end{pmatrix},$$

$$\overline{V} = \frac{1}{2}\begin{pmatrix} -jc_1A_1+j(\frac{3}{4}\delta_1+\frac{1}{4}\delta_2)c_1, & -c_2A_2+(\frac{3}{4}\delta_2+\frac{1}{4}\delta_1)c_2 \\ jc_1A_1-j(\frac{3}{4}\delta_1+\frac{1}{4}\delta_2)c_1, & -c_2A_2+(\frac{3}{4}\delta_2+\frac{1}{4}\delta_1)c_2 \end{pmatrix}$$

$$\ldots\text{(B9)}$$

Finally we may use (2.9) with $E = 0$ in order to compute a Y and a $\overline{Y}$ matrix as

$$Y = \begin{pmatrix} 0 & 0 & \frac{1}{\lambda_q'' h_2'(\lambda_q')} b_1 \\ \frac{1}{\lambda_d' h_1'(\lambda_d')} b_2 & \frac{1}{\lambda_d'' h_1'(\lambda_d'')} b_2 & 0 \end{pmatrix},$$

$$\bar{Y} = \begin{pmatrix} 0 & \lambda_d' c_2 \\ 0 & \lambda_d'' c_2 \\ -\lambda_q'' c_1 & 0 \end{pmatrix} . \qquad \ldots(B10)$$

Appendix C

We first state definitions of certain parameters of classical importance:

$$L'_d := L_d - \frac{L_{AD}^2}{L_F} = \text{d axis transient reactance,}^{\dagger} \qquad \ldots(C1)$$

$$L''_d := L_d - \frac{L_{AD}^2((L_D+L_F)-2L_{AD})}{L_D L_F - L_{AD}^2}$$

$$= \text{d axis subtransient reactance,} \qquad \ldots(C2)$$

$$L''_q := L_q - \frac{L_{AQ}^2}{L_Q} = \text{q axis subtransient reactance,} \qquad \ldots(C3)$$

$$\tau'_{do} := \frac{L_F}{r_F} = \text{d axis transient open circuit time constant,} \qquad \ldots(C4)$$

$$\tau''_{do} := \frac{L_F L_D - L_{AD}^2}{L_F r_D} = \text{d axis subtransient open circuit time constant,} \qquad \ldots.(C5)$$

$$\tau''_{qo} := \frac{L_Q}{r_Q} = \text{q axis subtransient open circuit time constant,} \qquad \ldots(C6)$$

$$\tau'_d := \tau'_{do} L'_d / L_d = \text{d axis transient time constant,} \qquad \ldots(C7)$$

$$\tau''_d := \tau''_{do} L''_d / L'_d = \text{d axis subtransient time constant,} \qquad \ldots(C8)$$

$$\tau''_q := \tau''_{qo} L''_q / L_q = \text{q axis subtransient time constant,} \qquad \ldots(C9)$$

$$\tau_a = \frac{2(L''_d \cdot L''_q)}{r(L''_d + L''_q)} = \text{armature dc offset time constant,} \qquad \ldots(C10)$$

---

† In the pu system used the synchronous frequency ω is taken as 1, so that there is no distinction between pu inductances and reactances.

If $d_1(s)$ and $d_2(s)$ denote the denominator polynomials of the transfer functions $h_1(s)$ and $h_2(s)$, a straightforward computation shows that:

$$d_1(s) = \det(s.L^d+R^d)/\det L^d ,$$

$$\det L^d = L_d''(L_D L_F - L_{AD}^2) , \qquad \ldots(C11)$$

$$d_1(s) = [L_d''(L_D L_F - L_{AD}^2)s^3 + \{L_d' L_F r_D + L_D(L_d r_F + L_F r) - L_{AD}^2(r_F + r\ )\}s^2 + \{(L_d r_F + L_F r)r_D + L_D r r_F\}s + r_D r_F r]/\det L^d , \qquad \ldots(C12)$$

$$d_2(s) = \det(sL^q+R^q)/\det L^q ,$$

$$\det L^q = L_q'' L_Q , \qquad \ldots(C13)$$

$$d_2(s) = [L_q'' L_Q s^2 + (L_q r_Q + L_Q r)s + r r_Q]/\det L^q . \qquad \ldots(C14)$$

One can similarly compute the numerator polynomials $n_1(s)$ and $n_2(s)$ as

$$n_1(s) = [L_d''(L_F L_D - L_{AD}^2)s^2 + \{L_d' L_F r_D + (L_d L_D - L_{AD}^2)r_F\}s + L_d r_F r_D]/\det L^d , \qquad \ldots(C15)$$

$$n_2(s) = (L_q'' L_Q s + L_q r_Q)/\det L^q . \qquad \ldots(C16)$$

If one makes use of the approximation

$$r_D >> r_F \qquad \ldots(C17)$$

where both r_D and r_F are measured in pu quantities†, then the following

†(C17) is to be attributed to the strong coupling between the field and armature circuits and is not true for the actual resistance values.

approximate expressions can be written for $d_1(s)$ and $n_1(s)$:

$$d_1(s) \cong [L_d''(L_FL_D-L_{AD}^2)s^3 + (L_d'L_Fr_D+(L_DL_F-L_{AD}^2)r)s^2 + (L_dr_F+L_Fr)r_Ds$$

$$+ r_Dr_Fr]/\det L^d \qquad \text{...(C18)}$$

$$n_1(s) \cong [L_d''(L_FL_D-L_{AD}^2)s^2 + L_d'L_Fr_Ds + L_dr_Fr_D]/\det L^d \; , \qquad \text{...(C19)}$$

Furthermore again using (C17) and its application into the first order approximate factorization of $as^2 + bs + c$ when c is small (which is: $as^2+bs+c \approx a(s+\frac{b}{a})(s+\frac{c}{b})$) yields the following approximation for $n_1(s)$

$$n_1(s) \cong (s+\frac{1}{\tau_d'})(s+\frac{1}{\tau_d''}) \quad , \qquad \text{...(C20)}$$

whereas

$$n_2(s) = (s+\frac{1}{\tau_q''}) \quad . \qquad \text{...(C21)}$$

If $\bar{n}_1(s)$ and $\bar{n}_2(s)$ denote the numerator polynomials of the transfer functions $\bar{h}_1(s)$ and $\bar{h}_2(s)$ then

$$\bar{n}_1(s) = [(L_FL_D-L_{AD}^2)s^2 + (L_Dr_F+L_Fr_D)s + r_Fr_D]/\det L^d \quad , \qquad \text{...(C22)}$$

$$\bar{n}_2(s) = (L_Qs+r_Q)/\det L^q \quad , \qquad \text{...(C23)}$$

which by virtue of (C17), (C11) and the same approximation that led to (C20) becomes

$$\bar{n}_1(s) \cong [(L_FL_D-L_{AD}^2)s^2 + L_Fr_Ds+r_Fr_D]/\det L^d \quad , \qquad \text{...(C24)}$$

$$\bar{n}_1(s) \cong \frac{1}{L_d''}(s+\frac{1}{\tau_{do}''})(s+\frac{1}{\tau_{do}'}) \; , \qquad \text{...(C25)}$$

whereas

$$\bar{n}_2(s) = \frac{1}{L_q''}(s+\frac{1}{\tau_{qo}''}) \; . \qquad \text{...(C26)}$$

Finally

$$-c_1A_1b_1 = e_1'L^d(L^d)^{-1}R^d(L^d)^{-1}e_1 = \frac{r}{L_d''} \quad , \qquad \text{...(C27)}$$

$$-c_2A_2b_2 = e_1'L^q(L^q)^{-1}R^q(L^q)^{-1}e_1 = \frac{r}{L_q''} \quad , \qquad \text{...(C28)}$$

and therefore

$$\frac{1}{2}(c_1A_1b_1+c_2A_2b_2) = -\frac{r}{2}\left(\frac{1}{L_d''}+\frac{1}{L_q''}\right) = -\frac{1}{\tau_a} \quad , \qquad \text{...(C29)}$$

The definitions of the decoupled model of section III are related to the classical definitions as follows

$$\lambda_d' \cong -\frac{1}{\tau_d'}$$

$$\lambda_d'' \cong -\frac{1}{\tau_d''}$$

$$\lambda_q' = -\frac{1}{\tau_q''}$$

$$\lambda_a = -\frac{1}{\tau_a} \quad , \qquad \text{...(C30)}$$

$$\frac{h_1(\lambda_d')}{\lambda_d'h_1'(\lambda_d')} = \frac{\bar{n}_1(\lambda_d')}{\lambda_d'n_1'(\lambda_d')} \cong \frac{1}{L_d'} - \frac{1}{L_d} \qquad \text{...(C31)}$$

where (C20), (C25), (C30) and the approximation

$$\frac{\left(\frac{1}{\tau_{do}''}-\frac{1}{\tau_d'}\right)}{\left(\frac{1}{\tau_d''}-\frac{1}{\tau_d'}\right)} \cong \frac{\tau_d''}{\tau_{do}''} = \frac{L_d''}{L_d'} \quad , \qquad \text{...(C32)}$$

which in practice is valid since

$$\tau_d' >> \tau_d'', \tau_{do}'' \qquad \text{...(C33)}$$

are used in deriving (C31). In a similar way:

$$\frac{\bar{h}_1(\lambda_d'')}{\lambda_d'' h_1'(\lambda_d'')} = \frac{\bar{n}_1(\lambda_d'')}{\lambda_d'' n_1'(\lambda_d'')} \cong \left(\frac{1}{L_d''} - \frac{1}{L_d'} \right) \qquad \text{...(C34)}$$

where the approximation used is

$$\frac{\left(\frac{1}{\tau_{do}'} - \frac{1}{\tau_d''} \right)}{\left(\frac{1}{\tau_d'} - \frac{1}{\tau_d''} \right)} \cong 1 \qquad \text{...(C35)}$$

because of (C33) and

$$\tau_{do}' >> \tau_d'' \quad . \qquad \text{...(C36)}$$

In addition to the approximations above we have the exact expressions

$$\frac{\bar{h}_2(\lambda_q')}{\lambda_q' h_2'(\lambda_q')} = \frac{\bar{n}_2(\lambda_q')}{\lambda_q' n_2'(\lambda_q')} = \left(\frac{1}{L_q''} - \frac{1}{L_q} \right) \qquad \text{...(C37)}$$

and

$$e_1' b_1 = e_1' (L^d)^{-1} e_1 = \frac{1}{L_d''} \quad , \qquad \text{...(C38)}$$

$$e_1' b_2 = e_1' (L^q)^{-1} e_1 = \frac{1}{L_q''} \quad . \qquad \text{...(C39)}$$

Appendix D

We shall prove the formulas:

$$\frac{\bar{h}_1(s)}{h_1(s)} = e_1'b_1 + \frac{\bar{h}_1(\lambda_d')}{h_1'(\lambda_d')}\frac{1}{s-\lambda_d'} + \frac{\bar{h}_1(\lambda_d'')}{h_1'(\lambda_d'')}\frac{1}{s-\lambda_d''} \qquad \ldots(D1)$$

$$\frac{\bar{h}_2(s)}{h_2(s)} = e_1'b_2 + \frac{\bar{h}_2(\lambda_q')}{h_2'(\lambda_q')}\cdot\frac{1}{s-\lambda_q'} \qquad \ldots(D2)$$

and

$$\bar{h}_{dF}(s) - \frac{\bar{h}_1(s)h_{dF}(s)}{h_1(s)}$$

$$= -\left[\frac{\bar{h}_1(\lambda_d')\cdot h_{dF}(\lambda_d')}{h_1'(\lambda_d')}\cdot\frac{1}{s-\lambda_d'} + \frac{\bar{h}_1(\lambda_d'')\cdot h_{dF}(\lambda_d'')}{h_1'(\lambda_d'')}\,\frac{1}{s-\lambda_d''}\right] \qquad \ldots(D3)$$

We have

$$\frac{\bar{h}_1(s)}{h_1(s)} = \frac{\bar{n}_1(s)}{n_1(s)} = K + \frac{\bar{n}_1(s)-Kn_1(s)}{n_1(s)}$$

where

$$K := \lim_{|s|\to+\infty}\frac{\bar{n}_1(s)}{n_1(s)} = \lim_{|s|\to+\infty}\frac{\bar{h}_1(s)}{h_1(s)} = \frac{1}{L_d''} = e_1'b_1$$

Therefore

$$\frac{\bar{h}_1(s)}{h_1(s)} = e_1'b_1 + \frac{\bar{n}_1(s)-e_1'b_1n_1(s)}{(s-\lambda_d')(s-\lambda_d'')}$$

$$= e_1'b_1 + \frac{\bar{n}_1(\lambda_d')}{\lambda_d'-\lambda_d''}\frac{1}{s-\lambda_d'} + \frac{\bar{n}_1(\lambda_d'')}{(\lambda_d''-\lambda_d')}\frac{1}{s-\lambda_d''}$$

$$= e_1'b_1 + \frac{\bar{n}_1(\lambda_d')}{n_1'(\lambda_d')}\frac{1}{s-\lambda_d'} + \frac{\bar{n}_1(\lambda_d'')}{n_1'(\lambda_d'')}\frac{1}{s-\lambda_d''}$$

from which (D1) follows since $n_1(\lambda_d') = n_1(\lambda_d'') = 0$ and thus

$$\frac{\bar{n}_1(\lambda_d')}{n_1'(\lambda_d')} = \frac{\bar{h}_1(\lambda_d')}{h_1'(\lambda_d')}\ ,\ \frac{\bar{n}_1(\lambda_d'')}{n_1'(\lambda_d'')} = \frac{\bar{h}_1(\lambda_d'')}{h_1'(\lambda_d'')}\ .$$

The proof for (D2) is similar. In order to prove (D3) we first write the left side of (D3) after cancelling the common terms in the numerator and the denominator as

$$\bar{h}_{dF}(s) - \frac{\bar{h}_1(s)\cdot h_{dF}(s)}{h_1(s)}$$

$$= \frac{[\bar{n}_{dF}(s)n_1(s)-\bar{n}_1(s)n_{dF}(s)]}{n_1(s)d_1(s)}\ , \qquad \ldots(D4)$$

where the symbol 'n' is used for denoting the numerators of the corresponding transfer functions. The partial fraction expansion of (D4) can be written as

$$\bar{h}_{dF}(s) - \frac{\bar{h}_1(s)h_{dF}(s)}{h_1(s)}$$

$$= \sum_{i=1}^{3} \frac{\{\bar{n}_{dF}(\lambda_i)n_1(\lambda_i)-\bar{n}_1(\lambda_i)n_{dF}(\lambda_i)\}}{n_1(\lambda_i)d_1'(\lambda_i)}\ \frac{1}{s-\lambda_i}$$

$$- \left[\frac{\bar{h}_1(\lambda_d')\cdot h_{dF}(\lambda_d')}{h_1'(\lambda_d')}\ \frac{1}{s-\lambda_d'} + \frac{\bar{h}_1(\lambda_d'')h_{dF}(\lambda_d'')}{h_1'(\lambda_d'')}\ \frac{1}{s-\lambda_d''}\right]$$

where $\lambda_1,\lambda_2,\lambda_3$ are the roots of $d_1(s)$. It therefore suffices to prove the relation

$$\frac{\bar{n}_{dF}(\lambda_i)}{n_{dF}(\lambda_i)} = \frac{\bar{n}_1(\lambda_i)}{n_1(\lambda_i)}\ ,\ i = 1,2,3\ , \qquad \ldots(D5)$$

in order to prove (D3). The proof of (D5) requires brute force computation making use of $d_1(\lambda_i) = 0$ $i = 1,2,3$.

Appendix E

Simulation Data and Results

Synchronous Machine Data (in pu)

$L_d=1.7$, $r=.01096$

$L_f=1.65$, $r_F=.000742$

$L_D=1.605$, $r_D=.0131$

$L_{AD}=1.55$, $r_Q=.054$

$L_q=1.64$, $M=1800.0$

$L_Q=1.526$, $D=0.0$

$L_{AQ}=1.490$

Derived Transient Parameters:

$L_d'=.2439$, $L_{dmod}'=.2383$, $L_d''=.1847$, $L_q''=.1852$

$T_d'=325.3$ pu(or .8628 sec.)

(classical approx.=319.1 pu(or .8464 sec.))

$T_d''=8.4$ pu(or .0224 sec.)

(classical approx.=8.6 pu(or .0228 sec.))

$T_q''=3.2$ pu(or .0085 sec.)

$T_a=16.9$ pu(or .0448 sec.)

(for the short circuit sim. $T_a=84.5$ pu)

w	λ'_d	λ''_d	λ''_q	$\lambda_a \pm jw$
.2	$-3.0648*10^{-3}$	$-.1232$	$-.351$	$-.0378 \pm j.18$
.4	$-3.0719*10^{-3}$	$-.1198$	$-.332$	$-.0492 \pm j.38$
.6	$-3.0732*10^{-3}$	$-.1190$	$-.323$	$-.0540 \pm j.59$
.8	$-3.0737*10^{-3}$	$-.1189$	$-.319$	$-.0562 \pm j.79$
1.	$-3.0739*10^{-3}$	$-.1186$	$-.317$	$-.0572 \pm j.99$
∞	$-3.0742*10^{-3}$	$-.1184$	$-.313$	$-.0593 \pm jw$

TABLE 1 Roots of $d_1(s).d_2(s)+w.n_1(s).n_2(s)=0$

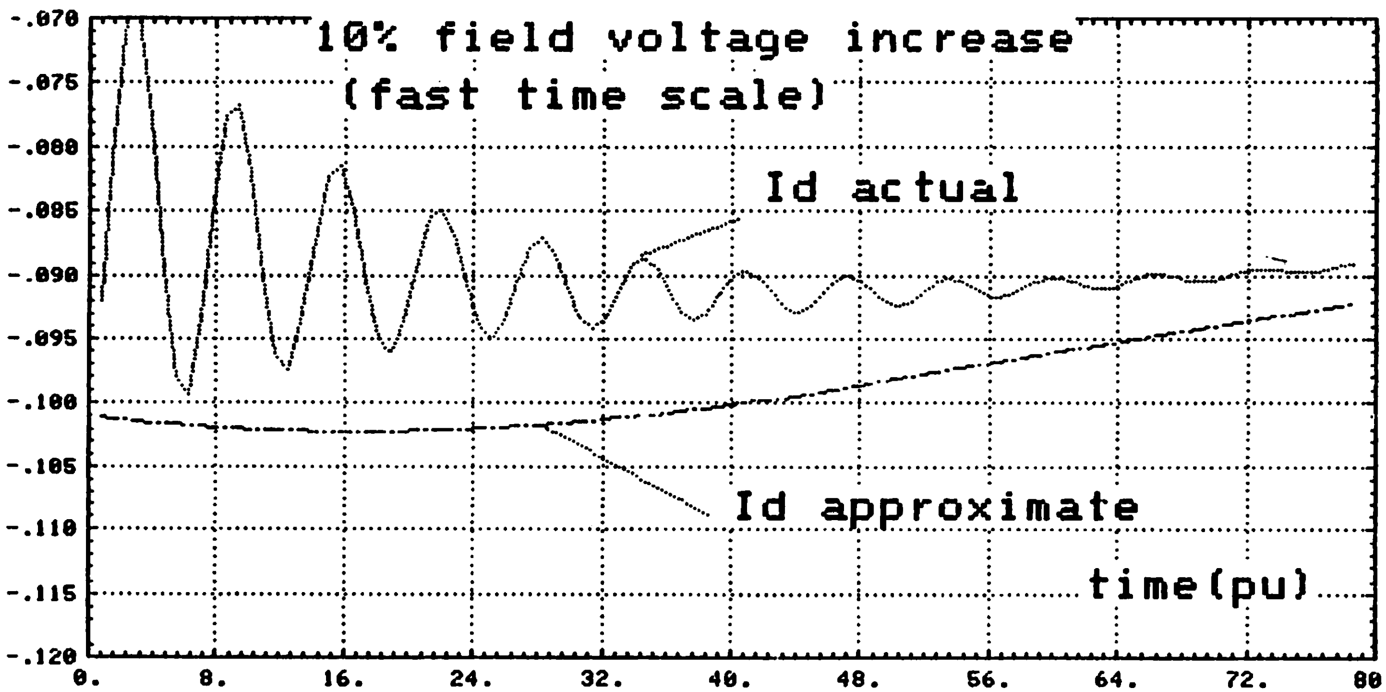

Figure 1 d-axis current

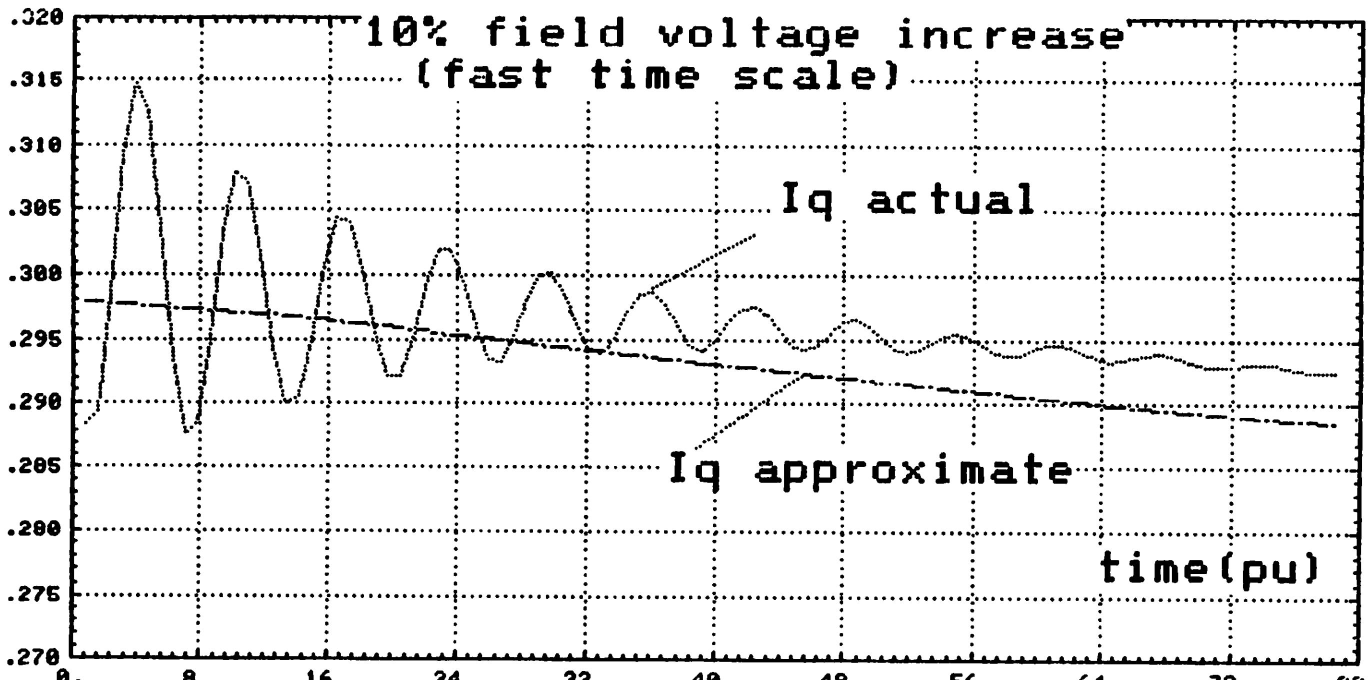

Figure 2 q-axis current

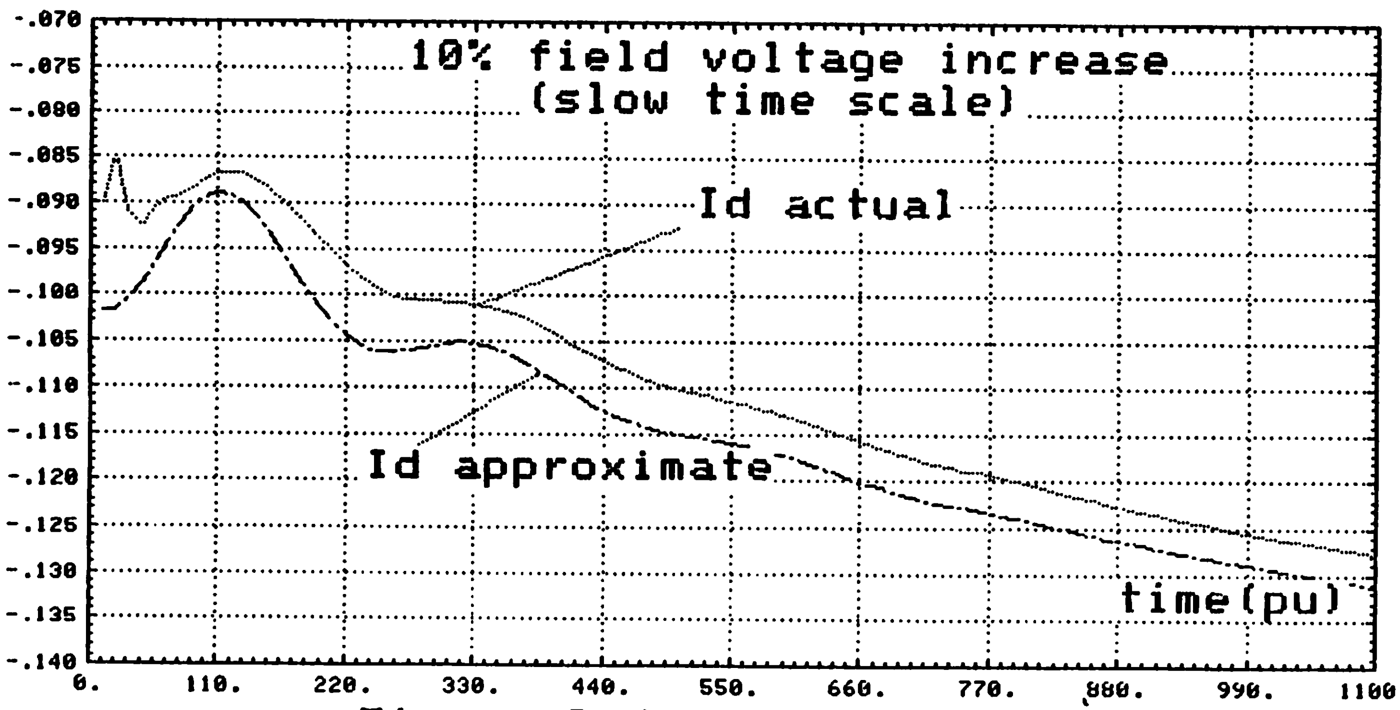

Figure 3 d-axis current

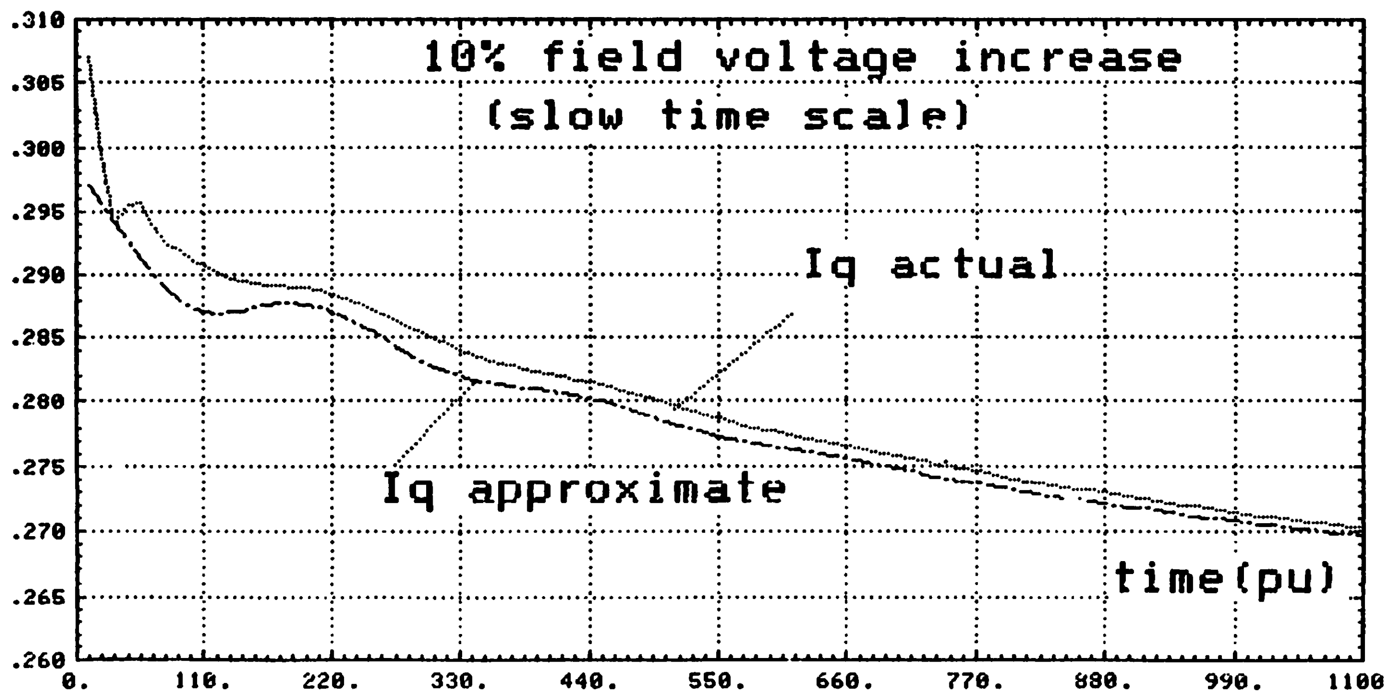

Figure 4 q-axis current

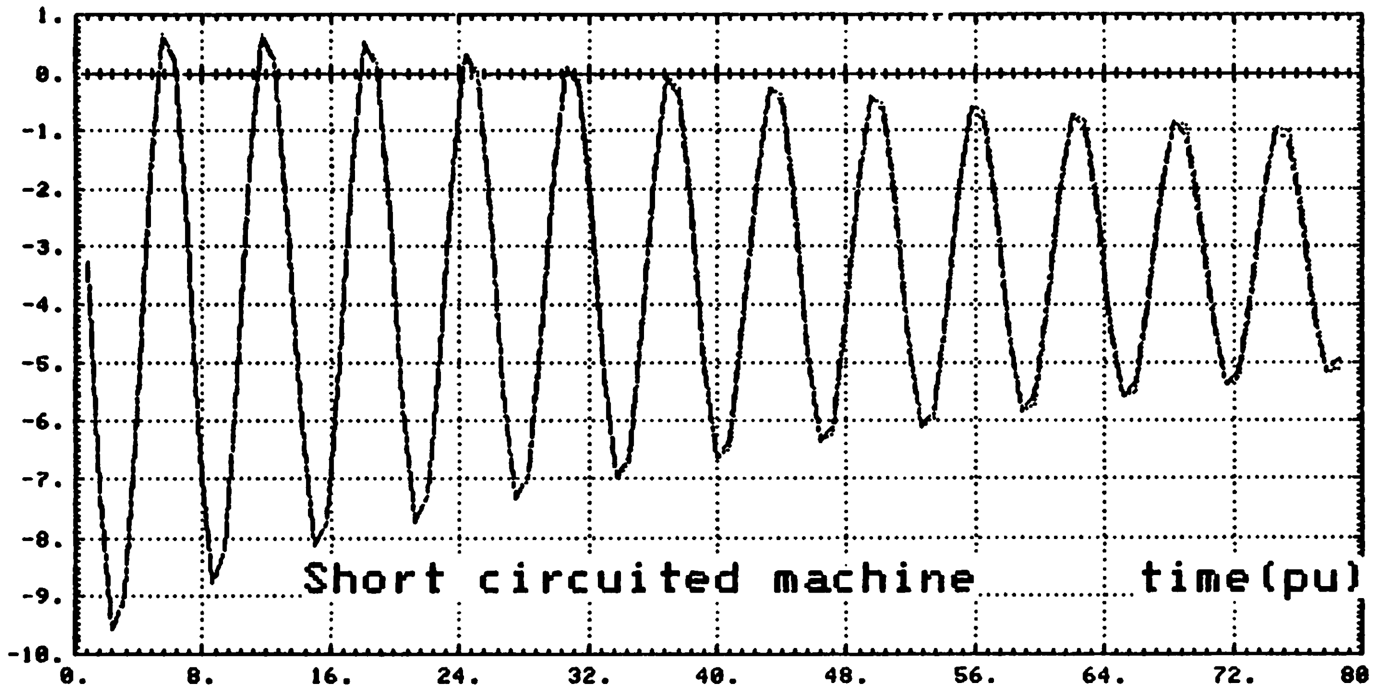

Figure 5 d-axis current

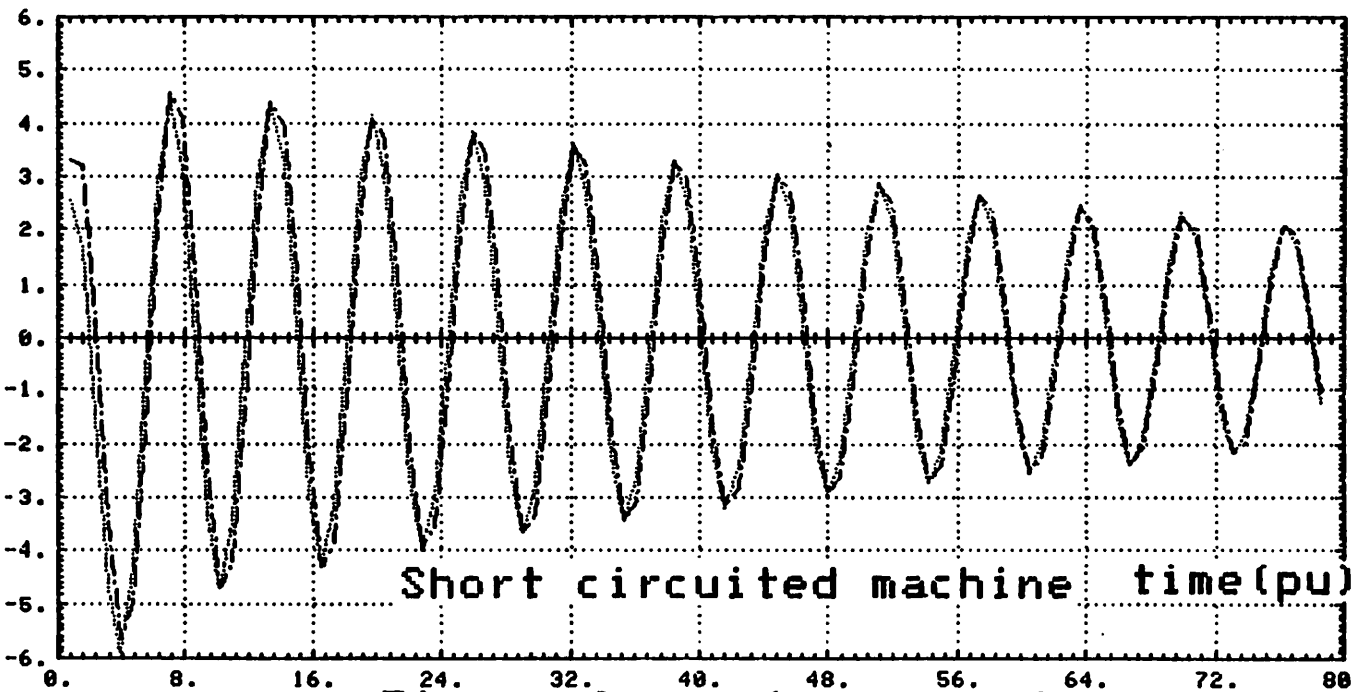

Figure 6 q-axis current

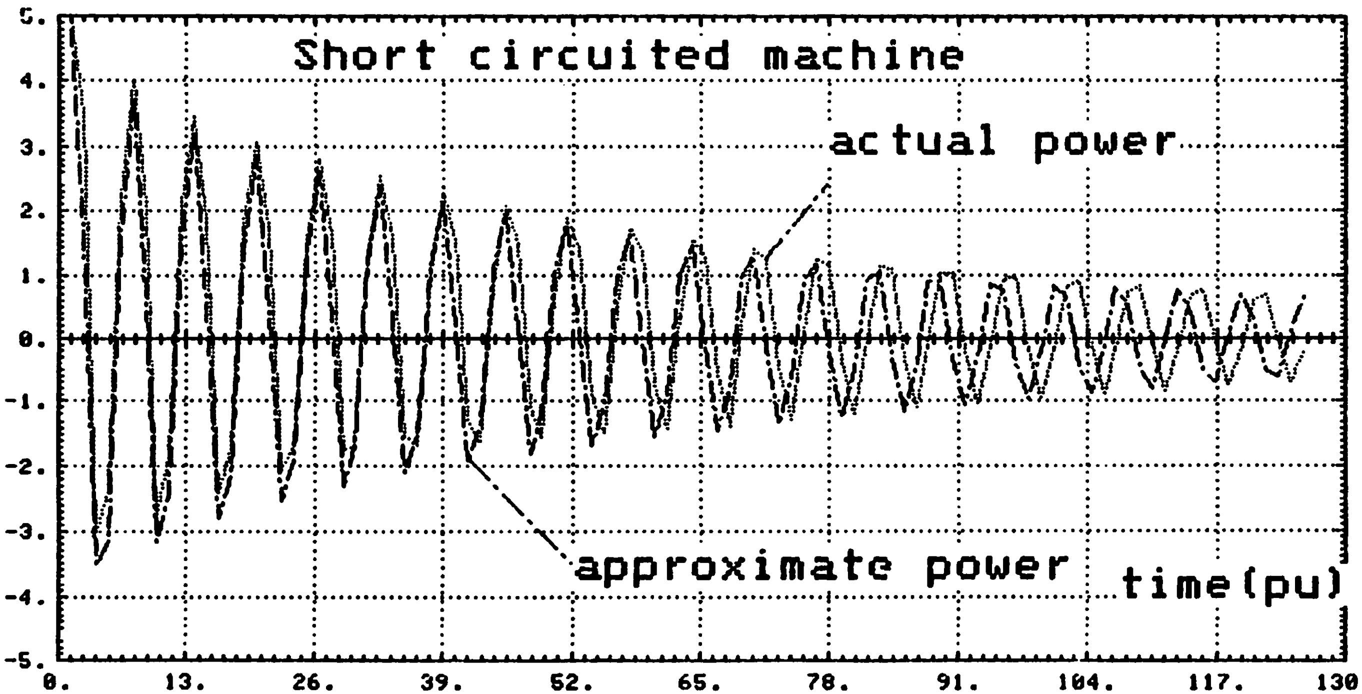

Figure 5' Power Swing Curves

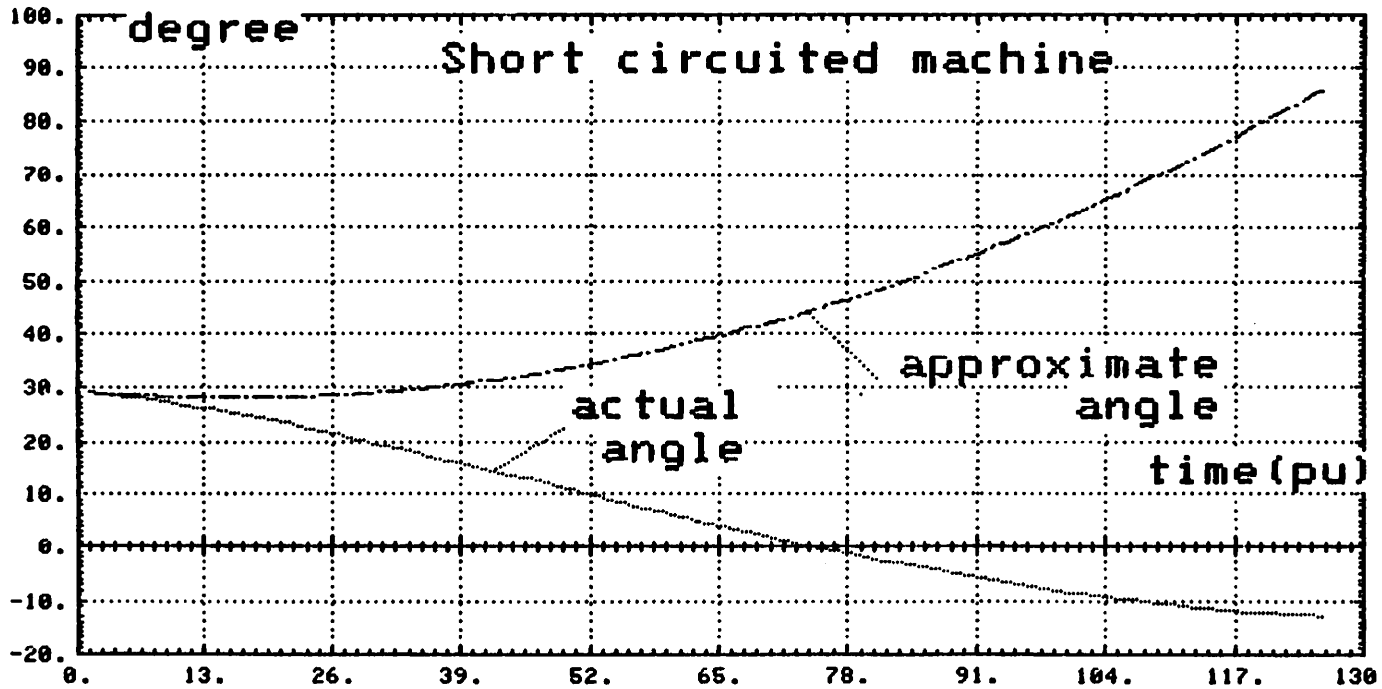

Figure 6' Angle Deviation

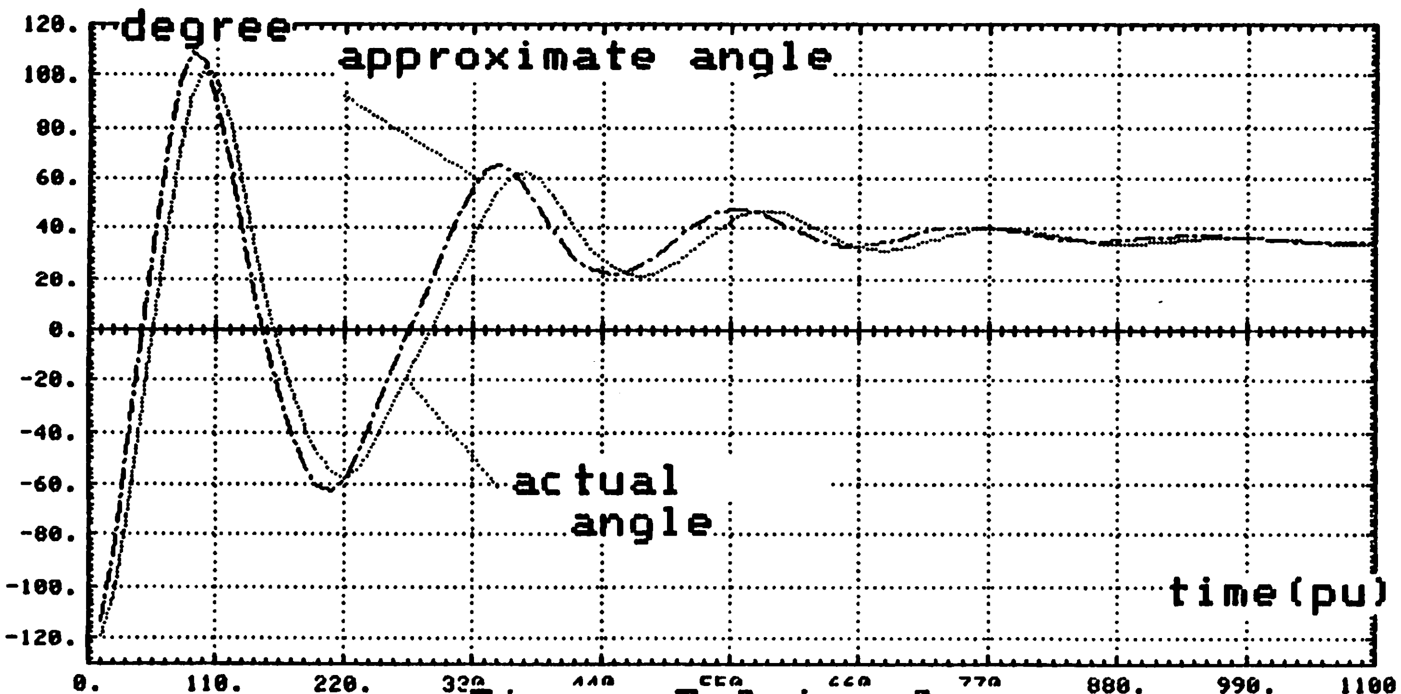

Figure 7 Swing Curves

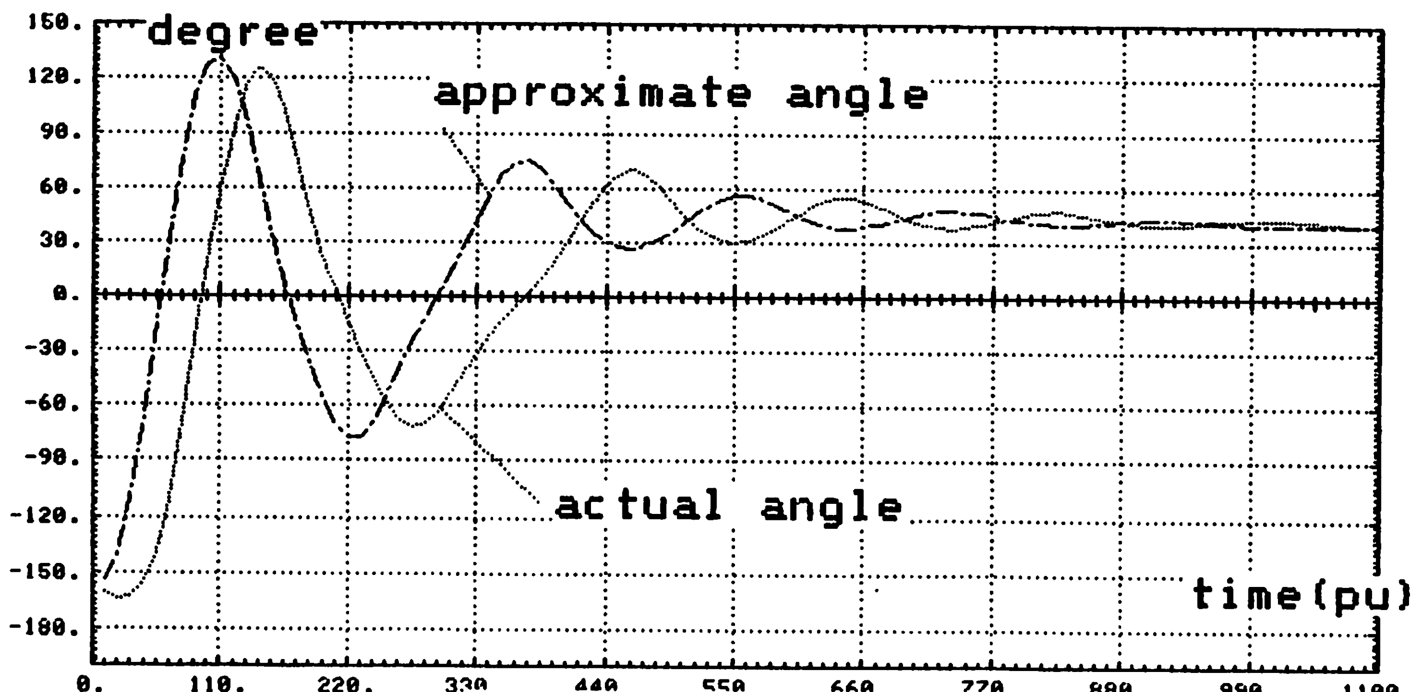

Figure 8 Swing Curves

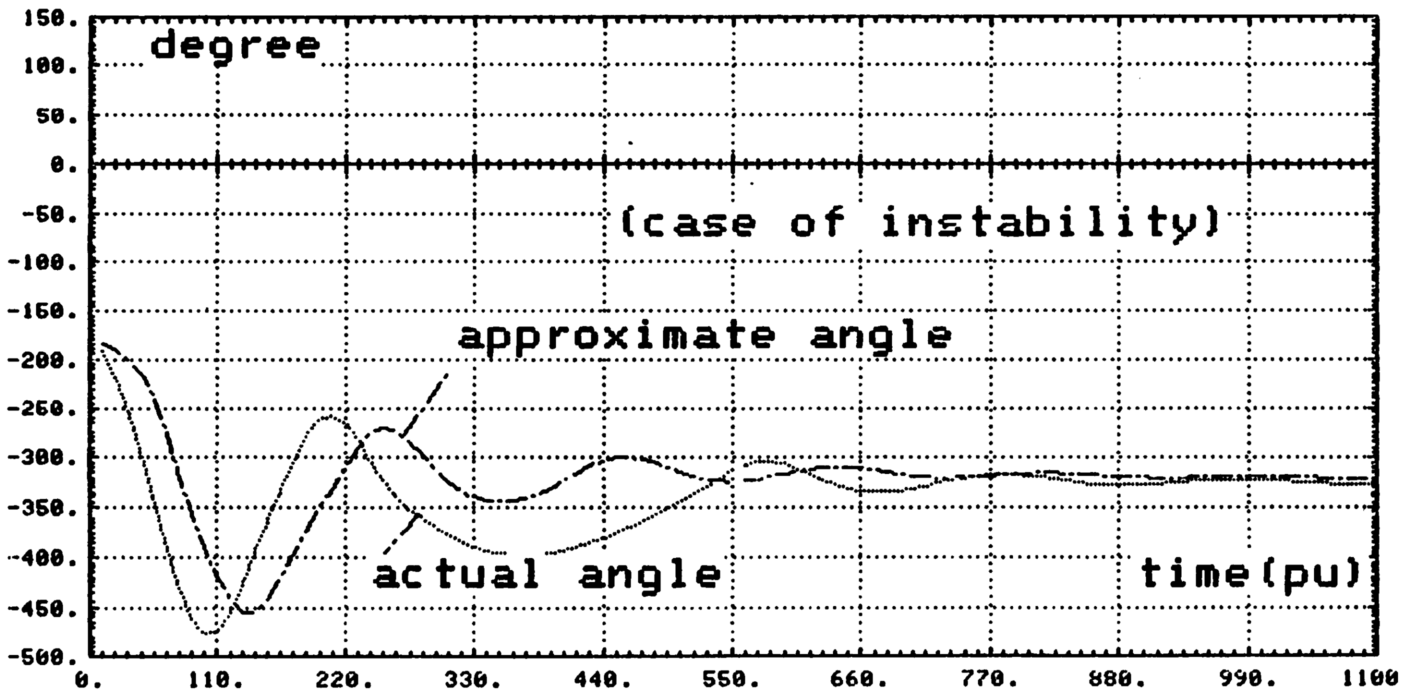

Figure 9 Swing Curves

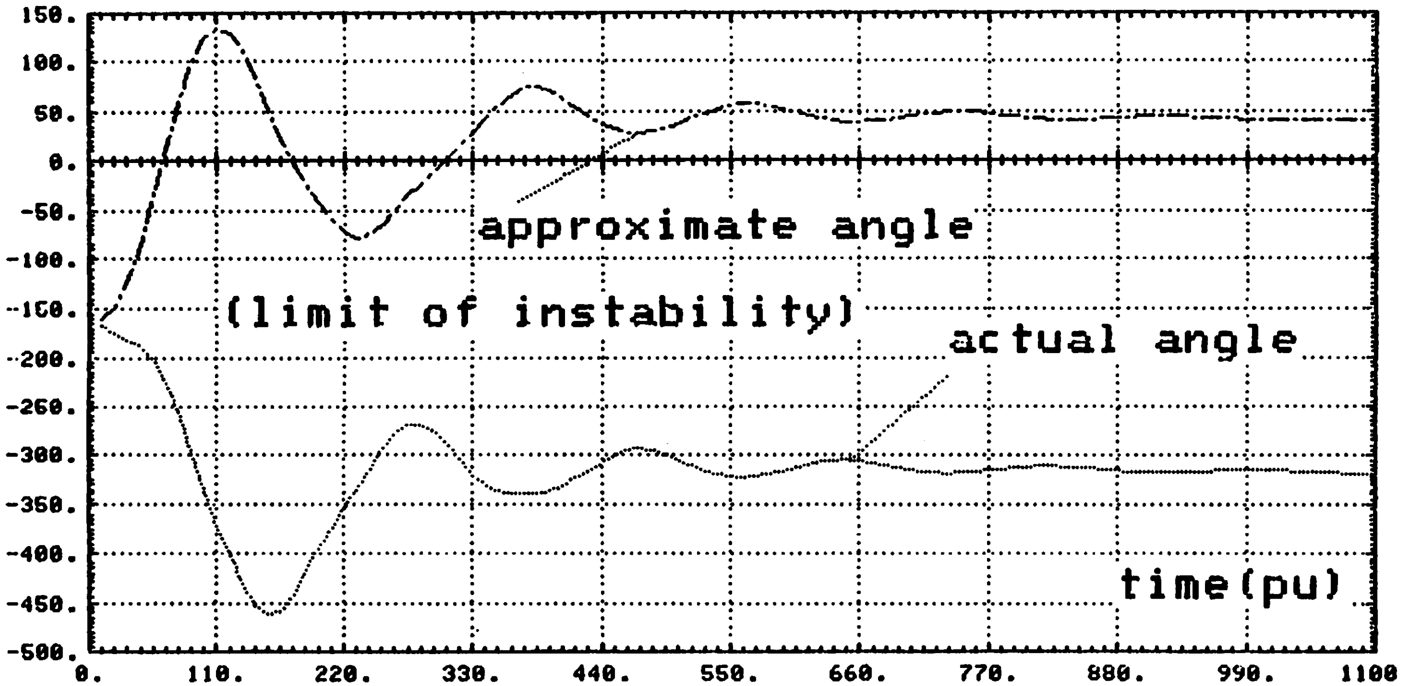

Figure 10 Swing Curves