EVALUATIONS OF POSSIBILITY IN A FUZZY ENVIRONMENT

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ABSTRACT

Sometimes problems considered as intrinsically vague, show their vagueness in an apparent form. A meticulous analysis of them could lead to a boolean modulation depending on the "logic" used in their treatment. Among them we can think about problems involving evaluation judgements as, for instance, preferences in competitions (etc. toral, etc...) a "subjective" evaluation of the candidates by several observers defines a collection of fuzzy sets on the set of candidates.

As it was already discussed in [18], in order to model the "fuzzy environment" in which these "experiments" take place it seems reasonable to accept that when we operate such fuzzy sets by previously adopted logic connectives or when transformations such that do not alter their "necessity" ([17]) operate in them, new fuzzy sets are obtained. Such reasonable families of fuzzy events are called fuzzy algebras and they are DeMorgan algebras closed by Watanabe transformations.

Once the logical reasonability of events connected with an evaluating experience has been established, measures to evaluate the possibility of occurrence of such events are defined. Such "evaluations of possibility" unify different definitions ([14],[28] and [8]) given with similar aims. Examples and basic properties of such measures are also studied.

Finally the concepts of "conditional fuzzy algebra" and "conditional evaluation of possibility" are introduced where the conditioning event is either in the initial algebra or does not belong to it. In the latter case a "wider" fuzzy environment, in which "old" and "new" elements could be considered together, is constructed, and a new evaluation of possibility which allows to evaluate the elements of such new environment is defined.

1. PRELIMINARIES

In this paper we accept the theory of Zadeh [22] for fuzzy subsets of a set S, in which $(A \cup B)(x) = \max\{A(x), B(x)\}$, $(A \cap B)(x) = \min\{A(x), B(x)\}$ [3] with the modification of defining fuzzy sets as functions $A: S \to J$ with $\{0,1\} \subset J \subset [0,1]$, and considering a negation function n which has as symmetry level [16] a significance level $\lambda \in (0,1) \cap J$ previously fixed. Then a "complementary" is obtained by $\overline{A} = n \cdot A$. For $J = [0,1]$ and $n = 1 - j$, being j the identity function in $[0,1]$ we obtain fuzzy sets in the classical sense.

So with $P_J(S) = J^S$ and $\phi(x) = 0$, $X(x) = 1$ for all $x \in S$, we have the DeMorgan algebra $(P_J(S); \cup, \cap, -, \phi, X)$ in which the set of its boolean elements is the set $P(S)$ of classical subsets of S, and $\underset{\sim}{\lambda} \in P_J(S)$ defined by $\underset{\sim}{\lambda}(x) = \lambda$ in any $x \in S$ satisfies $\underset{\sim}{\lambda} = \bigcup_{A \in P_J(S)} (A \cap \overline{A})$ which is essential for DeMorgan algebras to be embeddable [13]. This is a suitable hypothesis to the representation of questions by fuzzy sets.

From now on, when no mention is made we will assume $J = [0,1]$ and we will write $\underset{\sim}{P}(S)$ otherwise J will be specified.

Among admissible transformations of fuzzy sets, the so called Watanabe transformations [19], introduced by the author in [21], have a special importance. By one of such transformations a fuzzy set A is changed into another B which membership values are kept above, below or in the symmetry level λ as they were in the fuzzy set A. Functionally expressible morphisms [19] are examples of Watanabe transformations.

The relation $\sim$ defined in $\underset{\sim}{P}(S)$ by :"If $A, B \in \underset{\sim}{P}(S)$, it is $A \in B$ if and only if there exists a Watanabe transformation which transforms A into B" is an equivalence relation which is the same that the "ML-fuzzy at the level λ" relation defined in [19]. The equivalence class containing a fuzzy set A is denoted by $W^{\lambda}(A)$. We observe that given $A \in P_J(S)$, $J = \{0,1\}$ we have $W^{\lambda}(Z) = \{A\}$, if we consider such set with relation to $J \neq \{0,1\}$, then $W^{\lambda}(A)$ contains more elements than A.

Note that $B \in W^{\lambda}(A)$ implies $\underline{B} = \underline{A}$ (the nearest crisp to A and B with respect to λ, respectively). The converse of this assertion is not always true. It is $\underline{A} \in W^{\lambda}(A)$ if and only if $A^{-1}(\lambda) = \phi$.

If $A' \in W^{\lambda}(A)$ and $B' \in W^{\lambda}(B)$ then $A' \cup B' \in W^{\lambda}(A \cup B)$. Reciprocally, if $D \in W^{\lambda}(A \cup B)$ there exist $F \in W^{\lambda}(A)$ and $E \in W^{\lambda}(B)$ such that $D = F \cup E$. The smae result holds for $\cap$.

The definition of Watanabe transformation can be easily extended to consider Watanabe transformation of $A \in \underset{\sim}{P}(S)$, for any $c \in \underset{\sim}{P}(S)$ instead of only in the universe S: Let $c \in \underset{\sim}{P}(S)$, for any $A \in \underset{\sim}{P}(S)$ $A \in C$, $W_C^{\lambda}(A)$ is the set of Watanabe-transformed of A in C; that is $B \in W_C^{\lambda}(A)$ if and only if the following conditions hold:

(i) $C(x) \geq B(x) > \lambda$ if and only if $A(x) > \lambda$

(ii) $B(x) \leq (\underset{\sim}{\lambda} \cap C)(x)$, $B(x) \neq \lambda$ if and only if $A(x) < \lambda$, and

(iii) $B(x) = \lambda$ if and only if $A(x) = \lambda$

For $C = S$ we get the usual definition.

Fuzzy partitions are considered here in the general form defined in [18].

Let $P = \{P_1, \ldots, P_n\}$ be a finite fuzzy partition. We consider $\underline{P}_i = \{x \in S; P_i(x) > \lambda\}$ and $P_i^{\lambda} = \{x \in S; P_i(x) = \lambda\}$ for each $P_i \in P$, and $\hat{P} = \{x \in S; \max_{1 \leq i \leq n} P_i(x) < \lambda\}$. Let $I = \{1, \ldots, n\}$ be a set of integers. For each subset $K \subset I$ such that $K \neq \emptyset$ we define a subset of S in the following way: $T_K = \bigcap_{i \in K} P_i \cap \bigcap_{j \in K^c} \overline{P}_j$ being K^c the complementary set of K with respect to I.

It is immediately verified that $Q = \{\underline{P}_i\}_{i\in I} \cup \{T_K\}_{\substack{K\subset I \\ K\neq\emptyset}} \cup \{\hat{P}\}$ is a classical partition of S. We say that Q is the classical partition induced by the fuzzy partition P in S.

Let P and P' be fuzzy partitions. P is said to be finer than $P'(P > P')$ if and only if the classical partitions Q and Q' induced by P and P', respectively are Q finer than Q'. Such relation is a partial order relation in the set of all fuzzy partitions of S.

Given two finite fuzzy partitions $P = \{P_1,\ldots,P_n\}$ and $V = \{V_1,\ldots,V_m\}$, the family $P_x V = \{C_{1j};C_{ij} = P_i \cap V_j;\ i = 1,\ldots n,\ j = 1,\ldots m,\ C_{ij} \neq 0\}$ is a fuzzy partition of S, but such fuzzy partition is not always comparable with P_1 and P_2.

If P and V satisfy the condition that for each $x \in S$ there exist $P_i \in P$ and $V_j \in V$ such that $P_i(x) > \lambda$ and $V_j(x) > \lambda$ then $P \times V$ is a fuzzy partition finer than P and V. It is easy to verify that $P \times V = \sup\{P, V\}$.

When P is a fuzzy partition not satisfying the previous hypothesis, that is, such that $\{x \in S, \max_{1<i\leq n} P_i(x) \leq \lambda\} \neq S$ frequently one of the following hypothesis can be accepted.

(H-1) If, apart from not verifying the mentioned hypothesis P is such that $\{x \in S, \max_{1\leq i\leq n} P_i(x) = \lambda\} = \emptyset$, then a new fuzzy partition $P' \supset P$ can be considered instead of P with the convenient number of fuzzy set for the previous condition to be satisfied.

(H-2) If $\{x \in S, \max_{1\leq i\leq n} P_i(x) = \lambda\} = 0$ then we substitute P by a new partition $P' = \{P'_1,\ldots,P'_n,P'_{n+1},\ldots,P'_m\}$ being $P'_1,\ldots,P'_n$ the fuzzy sets of P conveniently modified by "modifiers" [26] (such as very, highly, approximately, etc.) to get the condition of H-1): $\{x \in S, \max_{1\leq i\leq n} P_i(x) = \lambda\} = 0$. Then as in the previous case we add the convenient number of fuzzy sets to get the desired hypothesis.

For example, if S is the set of political parties of a democratic county whatsoever, let R and L be the fuzzy sets (given by an observer) associated to the concepts "rightist" and "leftist" respectively. If $p = \{R,L\}$ does not satisfy that for each $x \in S$ either $R(x) > \lambda$ or $L(x) > \lambda$, but $\{x \in S; \max\{R(x),L(x)\} = \lambda\} = \emptyset$ then, we make the hypothesis that considering, for instance, the fuzzy set C corresponding to the concept "centered" we may get a new partition $P' = \{R,L,C\}$ satisfying the required condition. In due case in which $\{x \in S, \max\{R(x),L(x)\} = \lambda\} \neq 0$ we make the hypothesis that changing R and L by fuzzy sets as for instance ER and CL corresponding to the concepts "extremely rightist" and "center leftist" and then proceeding as in the previous case we may get the desired hypothesis.

The possibility of admitting or not H-1 and H-2 will depend on the kind of problem we are faced. To admit there is any problem would mean that changing the initial partition into a proper one "everything becomes clear enough" and that is doubtful specially in problems with inherent vagueness.

Given fuzzy partitions P and V a proper fuzzy partition finer than P and V can be also constructed in the following way: Let Q and R be the classical partitions induced by P and V respectively, then the classical partition $Q \times R = \{Q_i \cap R_j, Q_i \in Q, R_j \in R\}$ is finer than Q and R. So, any fuzzy partition which has $Q \times R$ as induced classical partition will be finer than P and V. Particularly $Q \times R$ itself is finer than P and V. This is another way of treating the same problem which is specially useful in problems where H-1 and H-2 are difficult to accept.

All these concepts and results will be used in the next chapters.

2. A CONCEPT OF FUZZY ENVIRONMENT

2.1 Fuzzy Experiments and Related Questions

A random experiment is carried out any time that, by a random procedure which can be repeated, one and only one element of a universe S is selected. In many problems one or more of such basic hypothesis are not fulfilled, and to give a probabilistic modelization of them can be a matter of discussion. If, for example, the problem is "to choose a painting that you like" from an exhibition, it seems clear that subjective criteria are followed in the election and that the unicity of the result is also a questionable hypothesis.

Then it seems convenient to construct a new mathematical model which includes also situations as the previous one; that is, where problems involving other kinds of vagueness, different from randomness, could be reflected. This is the aim of this work.

The word "proposition" is considered in this paper in a primitive sense, so it is not going to be defined. Of course such propositions are considered as it is usual in multiple valued logic; that is, once there is argreement in what a proposition is, we accept that the concept of "truth" applied to them can be evaluated, and that such evaluation is much more a matter of degree than an absolute concept. In this sense we accept that given a universe of discourse S and a set P(s) of propositions on it, there exist mappings $v:p(s)\to[0,1]$ such that $v(p)\in[0,1]$ is the degree of truth of the proposition p. In this line $v(p)=1$ means "p is true" and $v(p)=0$ means "p is false" as far as v is concerned.

Then a fuzzy question on S is denoted by p? being $p\in P(S)$. We accept the English use of ? and we agree that we know what p? means. Of course once a linguistic question p? has been made we expect an answer: we agree that possible answers are not only yes or not but a graded sequence of them ordered from yes to no.

Next definitions and results are an attempt to formalize these ideas.

<u>Definition 2.1</u>. A <u>fuzzy experiment</u> is made when a real number of $[0,1]$ is assigned to each element of a classical set S (finite or infinite); that is, when a fuzzy set A on S is given. Element $A \in \underset{\sim}{P}(S)$ is called <u>result of the fuzzy experiment</u> and $\underset{\sim}{P}(S)$ is the set of all possible outcomes.

Random experiments are particular case of fuzzy experiments in which the result is the characteristic function of a classical singleton.

<u>Defintion 2.2</u>. Given a set S, a <u>fuzzy question</u> A? is any question with a graded sequence of possible answers. An <u>answer</u> to A? is a function r with $r(A?) \in [0,1]$.

<u>Example 2.1.1</u>.

a) Let S be a set of individuals and $x \in S$. We consider the fuzzy question "x is rich ?". The proposition "x is rich" can be expressed by its relational assignment equation R(amount money (x)) = "rich," being "rich" a fuzzy set which has an adequate membership function $\Gamma_{\text{"rich"}}: [0,10^8] \to [0,1]$, and R a fuzzy restriction on the attribute "amount of money" of x to which the value "rich" is assigned by this equation [26]. Then a fuzzy set $A: S \to [0,1]$ such that $A(x) = \Gamma_{\text{rich}}$ (amount money (x)) can be defined. An answer to the question can be given by r(is x rich ?) = r(A?) = A(x).

b) Let S be an exhibition of paintings considered as the set of all paintings: $S = \{x_i\}$ $i = 1,\dots,n$, and "is S nice ?" a fuzzy question.

In this case the proposition "S is nice" can be expressed by n equations: $R(\text{beauty}(x_i)) = \text{nice}$ $i = 1,\ldots,n$ being "nice" a fuzzy set with membership function: $\Gamma_{\text{"nice"}}: S \to [0,1]$ given by exemplification [26]. We define $A = \Gamma_{\text{nice}}$. The rule to give the answer $r(A?)$ will be defined depending on the characteristics of the problem.

Of course classical questions in the sense that we only admit for them two answers "yes" and "not" are particular case of fuzzy questions.

Let n be an strong negation function [16] with symmetry level λ and $A,B \in \underset{\sim}{P}(S)$.

If A? and B? are fuzzy questions we accept that they are also fuzzy questions:

1) $(A?)$, given by $(A?) = \overline{A}? = (\text{no}A)?$

2) $(A?) \vee (B?)$ and $(A?) \wedge (B?)$ defined by:

   $(A?) \vee (B?) = A \cup B?)$ and $(A?) \wedge (B?) = (A \cap B?)$

3) All fuzzy questions D? being $D \in W^{\lambda}(A)$ and H? being $H \in W^{\lambda}(b)$.

4) $\underline{A?}$ and $\underline{B?}$ defined by $\underline{A?} = \underline{A}?$ and $\underline{B?} = \underline{B}?$

5) $\underset{\sim}{\lambda}?$

<u>Definition 2.3</u>. A family of fuzzy questions F is said to be a <u>reasonable family</u> if and only if

(i) If $A? \in F$, then $(A?) \in F$

(ii) If $A?\ B? \in F$, then $(A?) \wedge (B?) \in F$. (and as a consequence $(A?) \vee (B?) \in F$)

(iii) If $A? \in F$, then $D? \in F$ for any $D \in W^{\lambda}(F)$

(iv) If $A? \in F$, then $\underline{A}? \in F$.

(v) $\lambda? \in F$.

Previous definitions generalize the classical case as the following scheme shows:

Fuzzy Case	Classical Case
$A?,\ A \in \underset{\sim}{P}(S),\ J = [0,1]$	$A?,\ A \in \underset{\sim}{P}(S),\ J = \{0,1\}$
$(A?) = (noA)?$	$(A?) = ((1-j)\quad A)?$
$(A?) \vee (B?) = (A \cup B)?$ $(A?) \wedge (B?) = (A \cap B)?$	$(A?) \vee (B?) = (A \cup B)?$ $(A?) \wedge (B?) = (A \cap B)?$
$\{D?; D \in W^{\lambda}(A)\}$	$\{D?; D \in W^{\lambda}(A)\} = \{A?\}$
$\underline{A}?$	$\underline{A}? = A?$

2.2. Family of Fuzzy Sets Associated to a Reasonable Family of Fuzzy Questions: Fuzzy Events

Let F be a family of fuzzy questions related to a universe S. Between F and $\underset{\sim}{P}(S)$ a morphism ϕ with respect to lattice operations V, and can be defined as follows:

$$\phi : F \longrightarrow \underset{\sim}{P}(S)$$
$$A? \longrightarrow A$$

Of course it is a morphism since:

$$\phi(\ (A?)) = \phi(\overline{A}?) = \overline{A}$$
$$\phi((A?) \wedge (B?)) = \phi((A \cap B)/) = A \cap B,$$

which implies $\phi((A?) \vee (B?)) = \phi((A \cup B)?) = A \cup B.$

Moreover, if $D \in W^{\lambda}(A)$ it is $(D?) = D \in W^{\lambda}(A).$

$$\phi(\underline{A?}) = \phi(\underline{A}?) = \underline{A}$$
$$\phi(\underset{\sim}{\lambda}?) = \underset{\sim}{\lambda}.$$

The image set $\phi(F)$ of ϕ is a family of fuzzy sets such that satisfies:

(FA1) If A is in the family, then $\overline{A}$ is in the family.

(FA2) If A and B are in the family, then A B is in the family.

(FA3) If A is in the family, then all fuzzy sets of $W^{\lambda}(A)$ are in the family.

(FA4) If A is in the family, then $\underline{A}$ is in the family.

(FA5) $\underset{\sim}{\lambda}$ is in the family.

Definition 2.2.1. Elements of P(S) corresponding to fuzzy questions are called fuzzy events. A family A of fuzzy events satisfying (FA1), (FA2), (FA3), (FA4) and (FA5) is a fuzzy algebra [18].

Obviously from (FA1), (FA2) and Demorgan laws it follows that A B is also in the family. So, fuzzy algebras are DeMorgan algebras closed by Watanabe transformations.

It is clear that, for any fuzzy algebra A, it is $W^{\lambda}(\phi) \cup W^{\lambda}(S) \subset A$.

Definition 2.2.2. We call the coarse fuzzy algebra A_G the fuzzy algebra

$$A_G = \{\underset{\sim}{\lambda}\} \cup W^{\lambda}(\phi) \cup W^{\lambda}(S)$$

Obviously A_G is the smallest fuzzy algebra and it is contained in any other.

Definition 2.2.3. A Bernouilli fuzzy algebra is any fuzzy algebra having the following form:

$$A(A) = W^{\lambda}(A) \cup W^{\lambda}(\overline{A}) \cup W^{\lambda}(A \cup \overline{A}) \cup W^{\lambda}(A \cup \overline{A}) \cup \{\underset{\sim}{\lambda}\}$$

It corresponds to the idea of algebra associated to "only one fuzzy question."

In the classical case, the coarse algebra is $A_G = \{S, \phi\}$, and a

Bernouilli algebra is $A(A) = \{A,\bar{A},\phi,S\}$. Both are particular cases of previous definitions.

Obviously the finest fuzzy algebra is $\underset{\sim}{P}(S)$. In the classical case the finest algebra is $P(S)$.

Proposition 2.2.1. The intersection of fuzzy algebras is a fuzzy algebra.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be fuzzy algebras.

(i) If $A \in \mathcal{A} \cap \mathcal{B}$, then $A \in \mathcal{A}$ and $A \in \mathcal{B}$. As $\bar{A} \in \mathcal{A}$ and $\bar{A} \in \mathcal{B}$ it is $\bar{A} \in \mathcal{A} \cap \mathcal{B}$.

(ii) If $A,B \in \mathcal{A} \cap \mathcal{B}$, then $A,B \in \mathcal{A}$ and $A,B \in \mathcal{B}$. As $A \cup B \in \mathcal{A}$ and $A \cup B \in \mathcal{B}$, it is $A \cup B \in \mathcal{A} \cap \mathcal{B}$.

(iii) If $A \in \mathcal{A} \cap \mathcal{B}$ and $B \in W^{\lambda}(A)$, then $A \in \mathcal{A}$ and $A \in \mathcal{B}$. As $B \in \mathcal{A}$ and $B \in \mathcal{B}$ it is $B \in \mathcal{A} \cap \mathcal{B}$.

(iv) If $A \in \mathcal{A} \cap \mathcal{B}$, then $A \in \mathcal{A}$ and $A \in \mathcal{B}$ as $\underline{A} \in \mathcal{A}$ and $\underline{A} \in \mathcal{B}$ it is $\underline{A} \in \mathcal{A} \cap \mathcal{B}$.

(v) $\underset{\sim}{\lambda} \in \mathcal{A}$ and $\underset{\sim}{\lambda} \in \mathcal{B}$, then $\underset{\sim}{\lambda} \in \mathcal{A} \cap \mathcal{B}$. ¤

Definition 2.2.4. Let $\mathcal{P}$ be a collection of fuzzy subsets of S. The fuzzy algebra generated by $\mathcal{P}$, $A(\mathcal{P})$, is the smallest fuzzy algebra containing $\mathcal{P}$. Then $\mathcal{P}$ is said a generating system of $A(\mathcal{P})$.

The fuzzy algebra generated by $\mathcal{P}$ is the intersection of all fuzzy algebras that contain $\mathcal{P}$. If $\mathcal{P}$ is a fuzzy partition [18] then $A(\mathcal{P})$ is called the fuzzy algebra generated by the fuzzy partition $\mathcal{P}$.

It is easy to prove the following.

Proposition 2.2.2. If in a generating system $\mathcal{P} = \{P_i\}_{i \in I}$ a fuzzy algebra $A(\mathcal{P})$ a fuzzy set P_i is replaced by any other of $W^{\lambda}(P_i)$ then we obtain a new generating system of the same algebra.

The motion of fuzzy algebra can be easily extended to consider fuzzy algebras in any $c \in \underset{\sim}{P}(S)$ (instead of considering them only in S). In this general form a fuzzy algebra in c is defined as a fmaily A_c of fuzzy sets pointwise included in c such that:

(i) Each $A \in A_c$ has a complement in A_c,

(ii) A_c is closed by unions and intersections,

(iii) $W_c^{\lambda}(A) \in A_c$ for each $A \in A_c$ (see § I),

(iv) $\underline{A} \cap c \in A_c$ for each $A \in A_c$,

(v) $\underset{\sim}{\lambda} \cap c \in A_c$.

(Of course for $c = S$ we get the defintion of fuzzy algebra previously given and in this case we write A instead of A_s.)

A fuzzy algebra models the fuzzy environment in which our problem takes place in the sense that it contains all the fuzzy events that we can observe as far as our model is concerned.

Once the fuzzy environment has been constructed, special real functions (called variables) on the universe s can be defined (see [18]). Such functions behave with respect to the fuzzy environment as it is usual for measurable functions but according to the theory of Zadeh and they generalize the concept of random variable. An elementary calculus in the set of such functions is also established.

A characterization of such variables when the fuzzy algebra is generated by a finite fuzzy partition is also given in [18].

2.3 Fuzzy Questions Related to Fuzzy Experiments

Let $A \in \underset{\sim}{P}(S)$ be the result of a fuzzy experiment and $B \in \underset{\sim}{P}(S)$ a fuzzy event. It can be interesting to formulate fuzzy questions about the result A such as: Does A satisfy B? In order to answer this kind of question we proceed in the following way:

From fuzzy sets A and B we define a new fuzzy set $B^A: S \to [0,1]$ by:

$$B^A(x) = \begin{cases} 1 - |A(x)-B(x)| , & \text{if } A(x)\cdot B(x) \neq 0 \\ 0 , & \text{if } A(x)\cdot B(x) = 0 \end{cases}$$

Then, through such new fuzzy set, by some criterion (depending on the problem) an answer to old question can be given: A fuzzy even B <u>occurs with a certain degree</u> α whenever the actual outcome A satisfy such previously given criterion. We represent our question by B^A? as, actually we have moved our question on B to a question on B^A.

If $\{w\} \in P(S)$ is the result of a random experiment then classical questions related to it have the following form: $(\{w\} \in A)$? being $A \in P(S)$. Such questions can be represented by $A^{\{w\}}$? where $A^{\{w\}}$ is defined by

$$A^{\{w\}}(x) = \begin{cases} 1 , & \text{if } x = w \\ 0 , & \text{if } x \neq w \end{cases}$$

that is $A^{\{w\}} = A \quad \{w\}$.

Then to answer the question we give the following criterion:

$$r(A^{\{w\}}?) = \begin{cases} 1, & \text{if } A \cap \{w\} = \{w\} \quad \text{(answer: "yes")} \\ 0, & \text{if } A \cap \{w\} \neq 0 \quad \text{(answer: "no")} \end{cases}$$

which is a translation of the ordinary version "$w \in A$" and "$w \notin A$."

Then, the study of fuzzy questions related to fuzzy experiments has the study of questions related to random experiments as particular case. This is shown in the following scheme:

	Experiments	Questions	Answers
Random Case	$\{v_i\} \in S$ or $\{w\} \in S$ or $\{w\} \subset P(S)$	$\{w\} \in A?$ or $\{w\} \in A?,\ A \in P(S)$ $A^{\{w\}}?,\ A^{\{w\}} = A \cap \{w\}$	$r(A^{\{w\}}?) = \begin{cases} 0, \text{ "no," if } A \cap \{w\} = \emptyset \\ 1, \text{ "yes," if } A \cap \{w\} = \{w\} \end{cases}$
Fuzzy Case	$A \in \underset{\sim}{P}(S)$	$B\,?\ ,\ B \in \underset{\sim}{P}(S)$ $B^A?\ ,\ B^A \in \underset{\sim}{P}(S)$	$r(B^A?) \in J \subseteq [0,1]$

3. EVALUATIONS OF POSSIBILITY

3.1 Concept and Examples

Let A be a fuzzy algebra. In order to have a measure to evaluate the possibility of occurrence with a certain degree α of a fuzzy event of A, we give the following

Definition 3.1.1. An evaluation of possibility m defined in a fuzzy algebra A is a function $m: A \to \mathbb{R}^+$ such that:

(EP1) If $A, B \in A$, $A \subseteq B$ then $m(A) \leq m(B)$

(EP2) If $A_n \uparrow A$, $A_n \in A$ for all n and $A \in A$ then $\lim_{n \to \infty} m(A_n) = m(A)$.

(EP3) If $B \in W^\lambda(A)$, then $m(b) = m(A)$.

An evaluation of possibility is called θ-additive if:

(EP4) There exists an operation $\theta: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that, for any pair $A, B \in A$ of incompatible elements ($A \cap B \subset \underset{\sim}{\lambda}$), satisfies $m(A \cup B) = \theta(m(a), m(B))$.

Axiom (EP1) establishes that m is an increasing function with respect to the pointwise order. Particularly, as $\phi, S \in A$ it is $m(\phi) \leq m(A) \leq m(S)$ for any $A \in A$; that is $m(\phi)$ and $m(S)$ are the minimum and maximum values of m, respectively. Axiom (EP2) postulate the continuity of m with respect to order-increasing sequences of functions of A. Axiom (EP3) is characteristic of the study we are carrying out as in the classical case it is always $W^\lambda(A) = \{A\}$ and in this case no new information is given by $W^\lambda(A)$. The incompatibility relation $A \cap B \subset \underset{\sim}{\lambda}$ in the classical case, is equivalent to $A \cap B = \phi$, so axiom (EP4) generalizes the classical additivity.

We give now some examples of evaluations of possibility.

Example 3.1.1.

Any σ-additive probability is a θ-additive evaluation of possibility with respect to $\theta = +$.

Example 3.1.2.

Fuzzy measures of Sugeno [14] defined in a σ-algebra are evaluations of possibility and fuzzy-additivity is a particular case of θ-additivity ($\theta = \vee$).

Example 3.1.3.

Possibility measures of Zadeh [28] and Ngugen [10] are Max-additive evaluations of possibility.

Example 3.1.4.

Seaks of Namhias [8] are Max-additive evaluations of possibility.

Example 3.1.5.

If $f:\mathbb{R}^+ \to \mathbb{R}^+$ is non increasing and such that $f(m(\phi)) = +\infty$ and $f(m(s)) = 0$, and m is an evaluation of possibility, then $f \circ m$ is a generalized information in the sense of Kampe de Feriet-Bruno Forte [6] and the condition (EP4) is equivalent to the F-composibility of such information ($F = \theta$).

Example 3.1.6.

Let A be the minimum fuzzy algebra such that it contains the following set:

$$K = \{f:[0,1] \to [0,1] \ ; \ f \text{ continuous}, \ \lambda \in \text{Ran } f, \ f(0) = f(1) = 0\}$$

Let $m:A \to \mathbb{R}^+$ be a function defined by:

$$m(f) = \begin{cases} 0 & \text{, if } f \notin K \text{ and } f \subset \underset{\sim}{\lambda} \\ \sup\{x; f(x) = \lambda\} & \text{, if } f \in K \\ \sup\{x; f(x) \geq \lambda\} & \text{, otherwise.} \end{cases}$$

Then m is a Max-additive evaluation of possibility. Indeed,

(EP1) Let $A, B \in \mathcal{A}$ such that $A \subseteq B$, we have three possible cases:

a) If $m(B) = 0$, then $B \notin K$ and $B \subset \underset{\sim}{\lambda}$ so $A \subseteq B \subset \underset{\sim}{\lambda}$, $A \notin K$ and $m(A) = 0 \leq m(B) = 0$

b) Let $m(b) = \sup\{x; B(x) = \lambda\}$; that is $B \in K$. If $m(A) = 0$ we easily get the inequality. If $A \in K$, then $m(A) = \sup\{x; A(x) = \lambda\}$ and $m(a) \leq m(B)$ as A and B are continuous. Otherwise, if $m(A) = \sup\{x; A(x) \geq \lambda\}$, then $m(A) \leq m(B)$ as in any $x \in (m(B), 1)$ it is $A(x) \leq B(x) < \lambda$.

c) Otherwise, $m(B) = \sup\{x; B(x) \geq \lambda\}$. If $m(A) = 0$ there is nothing to prove. If $A \in K$, from the condition $A \subseteq B$ we get $\{x; A(x) = \lambda\} \subset \{x; B(x) \geq \lambda\}$ which implies $m(A) \leq m(B)$. If $m(A) = \sup\{x; A(x) \geq \lambda\}$, as $\{x; A(x) \geq \lambda\} \subset \{x; B(x) \geq \lambda\}$ we also have $m(A) \leq m(B)$.

(EP2) It can be proved from the usual behavior of supremums with respect to increasing sequences of sets.

(EP3) Let $B \in W^{\lambda}(A)$. If $m(A) = 0$, it is $A \notin K$ and $A \subset \underset{\sim}{\lambda}$, so $B \subseteq A \subset \underset{\sim}{\lambda}$ and $B \notin K$; that is $m(B) = 0 = m(A)$. If $A \in K$ finally we will have $m(B) = m(A)$. Otherwise, it is $\{x; A(x) \geq \lambda\} = \{x; B(x) \geq \lambda\}$ and $m(A) = m(B)$.

(EP4) Let $A, B \in \mathcal{A}$ be incompatible, that is $A \cap B \subset \underset{\sim}{\lambda}$. Then, if $m(A \cup B) = 0$ we have $m(A) = m(B) = 0$. If $m(A \cup B) = \sup\{x; A \cup B(x) = \lambda\}$ $= \sup[\{x; A(x) = \lambda] \cup \{x; B(x) = \lambda\}] = \max\ \{\sup\{x; A(x) = \lambda\},$ $\sup\{x; B(x) = \lambda\}\} = \max\{m(A), m(B)\}$. If $m(A \cup B) = \sup\{x; A(x) \quad B(x) \geq \lambda\}$,

then it is $\sup\{x;A(x)\cup B(x)\geq\lambda\} = \sup(\{x;A(x)\geq\lambda\}\cup\{x;B(x)\geq\lambda\})$:
$= \max(m(A),m(B))$.

Example 3.1.7.

Let (S,A,μ) be a classical measurable space and A the minimum fuzzy algebra containing all functions $f:S\to\mathbb{R}^+$ such that $\{x;f(x)\geq\lambda\}\in A$. We define

$$m(A) = \lambda\cdot\mu(\{x\in S;A(x)>\lambda\})$$

Then m is a θ-additive evaluation of possibility with respect to $\theta=\sum$.

Theorem 3.1.1. Axioms (EP1), (EP2), (EP3) and (EP4) are independent.

Proof. We will give four examples of mappings from A to $\mathbb{R}^+$, each one of them satisfying three of the four axioms but no the other one.

(i) Let $\{\Omega,A,P\}$ be a space of probability such that P is θ-additive. Considering $n(x)=1-j$ and $a=A$, and defining $m(A)=\frac{1}{P(A)}$, axiom (EP1) is not satisfied. But (EP2) is easily verified and so are (EP3), as $W^{1/2}(A)=\{A\}$, and (EP4) for $\theta(x,y) = \frac{xy}{x+y}$, as $\theta(m(A),m(B)) = \frac{m(A)\cdot m(B)}{m(A)+m(B)} = \frac{1}{P(A)+P(B)} = m(A\cup B)$ when $A\cap B = \emptyset$.

(ii) As it is very well known, with any probability P defined in a classical algebra A, being not θ-additive, with $a=A$ and with $m(A)=P(A)$, axioms (EP1) (EP3) and (EP4) with $\theta=+$ are satisfied whereas (EP2) may fail.

(iii) Let $a=\underset{\sim}{P}([0,1])$ and $n=1-j$. We define a function $m:\underset{\sim}{P}[0,1]\to\mathbb{R}^+$ by $m(f)=\sup\{f(x);x\in[0,1]\}$. As $m(f)\leq m(g)$ it is $f\subseteq g$ for any $f,g\in a$, so axiom (EP1) is satisfied. Given $f,g\in\underset{\sim}{P}([0,1])$ and $\varepsilon>0$, because of the definition of $m(f\vee g)$ there exist an $x_0\in[0,1]$ such that $m(f\vee g)-(f\vee g)(x_0)<\varepsilon$. The, $m(f\vee g) -< (f\vee g)(x_0)$ $\leq\max(m(f),m(g))$, which implies $0\leq m(f\vee g)-\max\{m(f),m(g)\}<\varepsilon$

that is $m(f \vee g) = \max(m(f), m(g))$. So, m satisfied (EP2) and (EP4) with respect to max operation. To establish that (EP3) is not satisfied, let us consider $f(x) = \frac{1}{2}x + \frac{1}{4}$ and $g(x) = x$. It is $g \in W^{1/2}(f)$ but $m(g) = 1 \neq m(f) = \frac{3}{4}$.

(iv) Let (Ω, A, P) be a space of probability being P θ-additive. Then $m(A) = P(A) + 1$ with $a = A$ satisfying the first three axioms. If there existed $\theta: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $m(A \cup B) = \theta(m(A), m(B))$ for any pair A,B such that $A \cap B = \emptyset$ it would be $(x = P(A),\ y = P(B))$ $x + y + 1 = \theta(x+1, y+1)$; that is for any pair $u, v, 2 \geq u,\ v \geq 1$ it would be $\theta(u,v) = u+v-2$. But, as $m(A \cup B) \geq \max(m(A), m(B))$ it is $\theta(u,v) \geq \max(u,v)$. So $\theta(u,v) = u+v-2 \geq \max(u,v)$ which implies $\min(u,v) \geq 2$ which is absurd. Then m is not θ-additive for any operation θ. ⌑

If θ-additivity is satisfied, we will establish now which properties should θ necessarily satisfy.

Theorem 3.1.2. Let $m: a \to \mathbb{R}^+$ be an onto evaluation of possibility such that for each $(x,y,z) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ there exist $A, B, C \in a$ mutually incompatible with $m(A) = x$, $m(B) = y$, $m(C) = z$ in a way that if $x \leq y$ then there exist $A, B \in a$, $A \subseteq B$, such that $m(A) = x$ and $m(B) = y$. Then if m is θ-additive, θ should necessarily be associative, commutative, non-decreasing, with $m(\phi)$ as null element, and such that $\theta \geq \max$.

Proof. Given $x, y, z \in \mathbb{R}^+$, let $A, B, C \in a$ such that $m(A) = x$, $m(B) = y$, $m(C) = z$, being all of them mutually incompatible fuzzy events. Then each one of them will be incompatible with the supremum of the other and we will have: $\theta(x, \theta(y,z)) = \theta(m(A), \theta(m(B), m(C))) = \theta(m(A), m(B \cup C))$ $= m(A \cup (B \cup C)) = m((A \cup B) \cup C) = \theta(m(A \cup B), m(C)) = \theta(\theta(m(A), m(B)), m(C))$ $= \theta(\theta(x,y), z)$, then θ is associative. (Notice we use the fact that if

C is incompatible with A and B then C is also incompatible with $A \cup B$.) Analogously $\theta(x,y) = \theta(m(A),m(B)) = m(A \cup B) = m(B \cup A) = \theta(m(B),m(A)) = \theta(y,x)$ leads to the commutativity of θ.

As ϕ is incompatible with any $A \in \mathcal{a}$, we have $\theta(m(\phi),x) = \theta(m(\phi),m(A)) = m(A \cup \phi) = m(A) = x$, this is, $m(\phi)$ is the null element of θ. θ is non-decreasing because given $x \leq z$; $x,z \in \mathbb{R}^+$ and being A,B such that $A \subseteq B$, $m(A) = x \leq m(B) = z$, then taking C incompatible with A and B and such that $m(C) = y$ we have $\theta(x,y) = \theta(m(A),m(C)) = m(A \cup C) \leq m(B \cup C) = \theta(m(B),m(C)) = \theta(z,y)$. ¤

Notice that, if to the fact of θ being associative, commutative, non-decreasing and with $m(\phi)$ as null element, we add the hypothesis of θ being continuous, strictly increasing and Archimedean ($\theta(x,x) > x$ for any $x > 0$), we have that the restriction of θ to $[m(\phi),+\infty]^2$ admit the Aczel-Ling [7] representation: $\theta(x,y) = \phi^{-1}(\phi(x) + \phi(y))$, being $\phi:[m(\phi),+\infty] \to \mathbb{R}^+$ strictly increasing and satisfying $\phi(m(\phi)) = 0$, $\phi(+\infty) = +\infty$. Function ϕ is called the additive generator of θ. If $\theta = +$, it is $\phi = j$ (the identity).

Through this functional representation, from the θ-addivity condition we have: $m(A \cup B) = \theta(m(A),m(B)) = \phi^{-1}(\phi(m(A)) + \phi(m(B)))$; that is, $(\phi o m)(A \cup B) = (\phi o m)(A) + (\phi o m)(B)$. Then, in this case $\phi o m$ comes to be a measure, in the traditional sense, associated to m.

3.2 Properties

In order to analyze the basic properties of an evaluation of possibility we will define some auxiliary functions.

Given $a,b \in \mathbb{R}^+$, let T_{ab} be the ordering symbol defined by:

$$T_{ab} = \begin{cases} 1 \ , & \text{if } a \geq b \\ 0 \ , & \text{if } a < b \end{cases}$$

For any $A,B \in \underset{\sim}{P}(S)$ we define:

$$N_{AB}(x) = A(x)T_{A(x),B(x)} \quad , \quad \hat{N}_{AB}(x) = A(x)(1-T_{A(x),B(x)}),$$

$$N_{BA}(x) = B(x)T_{B(x),A(x)} \quad , \quad \hat{N}_{BA}(x) = B(x)(1-T_{B(x),A(x)}).$$

Then by simple computation we can prove the following

Proposition 3.2.1. For any $A,B \in \underset{\sim}{P}(S)$ the following conditions are satisfied:

(i) $N_{AB} \subseteq A$ and $\hat{N}_{AB} \subseteq A$.

(ii) $N_{AB} \cup N_{BA} = A \cup B$ and $\hat{N}_{AB} \cup \hat{N}_{BA} = A \cap B$.

(iii) If $N_{AB}(x) > \lambda$ it is $N_{BA}(x) = 0$, $N_{AB} \cap N_{BA} = \phi < \underset{\sim}{\lambda}$; that is N_{AB} is incompatible with N_{BA}. The same holds for $\hat{N}_{AB}$ and $\hat{N}_{BA}$.

(iv) $\hat{N}_{AB}$ and $\hat{N}_{BA}$ are the unique elements in the set $[\phi,A] \in \underset{\sim}{P}(S)$ such that $N_{AB} \cup \hat{N}_{AB} = A$, $N_{BA} \cup \hat{N}_{BA} = B$ and $N_{AB} \cap \hat{N}_{AB} = N_{BA} \cap \hat{N}_{BA} = \phi$; so, N_{AB} is incompatible with $\hat{N}_{AB}$.

(v) If $A \in W^{\lambda}(P_i)$ and $B \in W^{\lambda}(P_j)$ being $P_i,P_j \in \mathcal{P}$, $i \neq j$, and $\mathcal{P}$ a fuzzy partition then $N_{AB},N_{BA} \in \mathcal{a}(\mathcal{P})$ being $\mathcal{a}(\mathcal{P})$ the fuzzy algebra generated by $\mathcal{P}$.

Now we will analyze some basic properties of an evaluation of possibility.

Theorem 3.2.1. Let m be a θ-additive evaluation of possibility defined in a fuzzy algebra $\mathcal{a}$ such that for any pair $A,B \in \mathcal{a}$ it is N_{AB}, N_{BA}, $\hat{N}_{AB}$, $\hat{N}_{BA} \in \mathcal{a}$. Then:

(a) For any $A,B \in \underset{\sim}{P}(S)$ it is $m(A \cup B) \leq \theta(m(A),m(B))$. If $\theta = \max$ then $m(A \cup B) = \max(m(A),m(B))$.

(b) $\theta(m(A \cup B) \cdot m(A \cap B)) \leq \theta(\theta(m(A),m(B)), \min(m(A),m(B)))$.

(c) If $A_1,\ldots,A_n$ are mutually incompatible, each one of them is incompatible with the union of the other and $m(\bigcup_{i=1}^{n} A_i) = \underset{i=1}{\overset{n}{\theta}} m(A_i)$.

(d) $\theta(m(A\cup B),m(A\cap B)) = \theta(m(A),m(B))$.

Proof. Property (a) can be proved by simple computation using the previous proposition and the fact of θ being non-decreasing: $m(A\cup B) = m(N_{AB}\cup N_{BA}) = \theta(m(N_{AB}),m(N_{BA})) \leq \theta(m(A),m(B))$. (b) is obtained using (a) and $m(A\cap B) \leq \min(m(A),m(B))$. To prove (c) it is basic to consider the associativity of θ. (d) comes in the following way: $\theta(m(A),m(B)) = \theta(m(N_{AB}\cup \hat{N}_{AB}), m(N_{BA}\cup \hat{N}_{BA})) = \theta(\theta(m(N_{AB}),m(\hat{N}_{AB})), \theta(m(N_{BA}),m(\hat{N}_{BA}))) = \theta(\theta(m(N_{AB}),m(N_{BA})), \theta(m(\hat{N}_{AB}),m(\hat{N}_{BA}))) = \theta(m(A\cup B),m(A\cap B))$. ¤

It is necessary to remark that Theorem 3.2.2 is not true in any fuzzy algebra, though the fuzzy algebra is generated by a fuzzy partition, as not always N_{AB}, $\hat{N}_{AB}$, N_{BA} and $\hat{N}_{BA}$ are in the algebra. However, when property (v) of Proposition 3.2.1 holds, Theorem 3.2.1 can be established.

4. CONDITIONING IN FUZZY ENVIRONMENT

The model introduced in §2 represents the result of a fuzzy experiment as one of the elements A of the set of fuzzy sets $\underset{\sim}{P}(S)$ of a predetermined universe S. Fuzzy events are also fuzzy subsets of S.

In §3 a positive real number is assigned to each fuzzy event in a manner to evaluate the possibility of the occurrence with degree α of the fuzzy event.

But, it occurs frequently that before the actual outcome of the trial is known, partial information is received about the character of the outcome. Such information may lead to revise what the possibility of occurrence with certain degree of any given fuzzy event should be. We wish to examine how the original evaluation of possibility should be modified when partial information is obtained about the character of the outcome.

We begin with an example which illustrates the essential question.

Let S be the set of all paintings of an exhibition and $A_i \in \underset{\sim}{P}(S)$ the fuzzy subset of S obtained when a qualitative opinion about this painting is given qualitatively by an individual i. Such assignment is a fuzzy experiment and A_i its result.

Let B be the fuzzy set on S corresponding to the concept "nice." On basis to the opinion of art experts we could assign a real number to the fuzzy event B which evaluates the possibility of occurrence with certain degree, of such event when the fuzzy experiment is carried out.

Suppose information is received that the opinion of i with respect some other related concept (for instance "visual harmony") is given by the fuzzy set G. Of course, this new knowledge will make experts to modify the possibility assigned to B.

This example suggests that when partial information is received concerning the nature of the outcome of a fuzzy experiment, the result may be expressed as knowledge that a specific conditioning fuzzy event has occurred. Our aim now is to establish how, given knowledge that a conditioning event G has occurred a possibility of occurrence, with a certain degree, to the fuzzy event B could be assined.

It seens clear that when such situation happens what we are really looking for is the possibility of occurrence of $B \cap C$, so in a certain sense a new basic space C has been identified as all new fuzzy events are obtained by intersection of the previous fuzzy events with the fuzzy set C and, as a consequence, new events are pointwise included in C.

4.1 Construction of the Conditional Environment

Let S be the universe of discourse, $a \in \underset{\sim}{P}(S)$ a fuzzy algebra in S, and $C \in \underset{\sim}{P}(S)$ a fuzzy event. In order to construct the corresponding conditional fuzzy algebra we will distinguish two cases:

(a) $C \in a$. In this case let a' be the subset of a defined by $a' = \{A'; A' = A \cap C \text{ for every } A \in a\}$. The fuzzy set $\hat{A}' = \overline{A} \cap C$ (being $\overline{A} = \text{no}A$ the complement of A in $\underset{\sim}{P}(S)$ defined by $\hat{A}'(x) = \overline{A}(x) \wedge C(x)$ for any $x \in S$ comes to be a complement of $A' = A \cap C$ in the sense that it satisfies the involution property and DeMorgan laws. Clearly $\hat{A}' = \overline{A} \cap C \in a'$, then we given the following:

<u>Definition 4.1.1</u>. Complement $\hat{A}' = \overline{A} \cap C$ is called the <u>complement</u> of $A' = A \cap C$ with respect to C.

Then we can prove the following.

<u>Theorem 4.1.1</u>. If a is a fuzzy algebra in S and $C \in a$ then the set $a' = \{A'; A' = A \cap C, \text{ for each } A \in a\}$ is a fuzzy algebra in C.

Proof. (i) Taking $\hat{A}' = \bar{A} \cap C$ as complement of A', we have already mentioned that $\hat{A}' \in \mathcal{a}'$.

(ii) If $A', B' \in \mathcal{a}'$ then $A' \cap B' \in \mathcal{a}'$ as $A' \cap B' = (A \cap C) \cap (B \cap C) = (A \cap B) \cap C = (A \cap B)' \in \mathcal{a}'$. In the same way $A' \cup B' \in \mathcal{a}'$.

(iii) $W_C^\lambda(A') \subset \mathcal{a}'$ for any $A' \in \mathcal{a}'$. Indeed, if $B' \in W_C^\lambda(A')$ then $B' \in W^\lambda(A')$, as $A' \in \mathcal{a}$ it is $B' \in \mathcal{a}$. Then $B' \cap C = B' \in \mathcal{a}'$.

(iv) If $A' \in \mathcal{a}'$ then $\underline{A}' \cap C \in \mathcal{a}'$ as $A' \in \mathcal{a}$ and so is A'. Of course.

(v) $\underset{\sim}{\lambda} \cap C \in \mathcal{a}$. ¤

Definition 4.1.2. $\mathcal{a}'$ is called the conditional fuzzy algebra relative to C.

(b) $C \notin \mathcal{a}$. Let $\mathcal{a}(C)$ be the Bernouilli fuzzy algebra (Def. 4.2.3) generated by C: $\mathcal{a}(C) = W^\lambda(C) \cup W^\lambda(\bar{C}) \cup W^\lambda(C \cup \bar{C}) \cup W^\lambda(C \cap \bar{C}) \cup$ ($\mathcal{a}(C)$ can also be regarded as the fuzzy algebra generated by the finite fuzzy partition $C = \{C, \bar{C}\}$).

Definition 4.1.3. The smallest fuzzy algebra which contains $\mathcal{a}$ and $\mathcal{a}(C)$ is called the conditional fuzzy algebra relative to C.

If $\mathcal{a}$ is the fuzzy algebra generated by a finite fuzzy partition $P = \{P_i\}_{i \in I}$, $I = \{1, \ldots, n\}$, satisfying the condition that for any $x \in S$ there exists a $P_i \in P$ such that $P_i(x) > \lambda$, and if the finite fuzzy partition $C = \{C, \bar{C}\}$ satisfy the same hypothesis, then (§I) $P \times C = \{P_1 \cap C, P_1 \cap \bar{C}, \ldots, P_n \cap C, P_n \cap \bar{C}\}$ is a finite fuzzy partition finer than P and C with respect to the order relation given in §I, satisfying also the previous hypothesis required to P and C.

Let H be the fuzzy algebra generated by the fuzzy partition $P \times C$. Then

Proposition 4.1.1. For each $P_i \in \mathcal{P}$, it is $P_i \in \mathcal{H}$.

Proof. We will prove that $P_i \in W^{\lambda}[(P_i \cap C) \cup (P_i \cap \overline{C})]$ for any $i \in I$. Indeed, if $[(P_i \cap C) \cup (P_i \cap \overline{C})](x) > \lambda$ it is either $(P_i \cap C)(x) > \lambda$ or $(P_i \cap \overline{C})(x) > \lambda$ which implies both $P_i(x) > \lambda$ and $(C \cup \overline{C})(x) > \lambda$. If $[(P_i \cap C) \cup (P_i \cap \overline{C})](x) < \lambda$ then $(P_i \cap C)(x) < \lambda$ and $(P_i \cap \overline{C})(x) < \lambda$. If $C(x) < \lambda$ then $\overline{C}(x) \nless \lambda$ so the previous result implies $P_i(x) < \lambda$. $[(P_i \cap C) \cup (P_i \cap \overline{C})](x) = \lambda$ is only possible if $P_i(x) = \lambda$ as $(C \cup \overline{C})(x) > \lambda$ for any $x \in S$ because of the hypothesis assumed for C. ¤

Proposition 4.1.2. Fuzzy sets C and $\overline{C}$ satisfy $C \in \mathcal{H}$ and $\overline{C} \in \mathcal{H}$.

Proof. As in the previous proposition we can prove that $C \in W^{\lambda}[(P_1 \cap C) \cup (P_2 \cap C) \cup \ldots \cup (P_n \cap C)]$ and $\overline{C} \in W^{\lambda}\ [(P_1 \cap \overline{C}) \cup \ldots \cup (P_n \cap \overline{C})]$.

From the last two propositions it is easy to prove the following result.

Proposition 4.1.3. $\mathcal{H}$ is the smallest fuzzy algebra that contains $a(\mathcal{P})$ and $a(C)$.

In both cases the conditional fuzzy algebra modelizes the new fuzzy environment within which one problem can be more accurately reformulated, taking into consideration the occurrence, in a certain degree, of the fuzzy event C.

To study how a previously defined evaluation of possibility for the fuzzy events of a is affected by the occurrence of C, is the aim of next section.

4.2 Conditional Evaluation of Possibility

Let $\mathcal{A}$ be a fuzzy algebra, $C \in \mathcal{A}$ a fuzzy event and $m:\mathcal{A} \longrightarrow \mathbb{R}^+$ an evaluation of possibility in A. Let $\mathcal{A}' = \{A'; A' = A \cap C, \text{ for each } A \subset \mathcal{A}\}$ be the conditional fuzzy algebra relative to C.

As $\mathcal{A}' \subset \mathcal{A}$, events of $\mathcal{A}'$ can be evaluated by the same m. The new evaluation of possibility of an event $A \in \mathcal{A}$ given knowledge of the occurrence (in a certain degree) of the fuzzy event C is given by the following.

Definition 4.2.1. If C is a fuzzy event, the conditional evaluation of possibility given C, which is denoted by m_C is defined, for each fuzzy event A, by $m_C(A) = k_C * m(A \cap C)$ being k_C a positive constant depending on C and * any isotone operation in $\mathbb{R}^+$ satisfying $a*b = 0$ if and only if $a = 0$ or $b = 0$.

In fact m_C is an evaluation of possibility which most of the properties of the original evaluation on possibility m. This assignment is derived from the evaluation of $A \cap C$, and the consideration of a fixed constant related to C corresponds to the convenience of keeping in some problems levels or boundary conditions. This is due case when $\mathcal{A}$ is a σ-algebra and m a probability, then we define $k_C = m(C)$.

This problem becomes a little more complicated when we consider the conditional fuzzy algebra H related to a fuzzy event C which is not in $\mathcal{A}$. We will consider here only the case in which $\mathcal{A}$ is generated by a finite fuzzy partition $\mathcal{P}$ satisfying the condition required in Section 4.1 and already mentioned in §I.

So, let $\mathcal{A}(P)$ be a fuzzy algebra, $C \in \underset{\sim}{P}(S)$ such that $C \notin \mathcal{A}(P)$ and $m:\mathcal{A}(P) \to \mathbb{R}^+$ a θ-additive evaluation of possibility defined in $\mathcal{A}(P)$. Let $\mathcal{A}(C)$ be the Bernouilli fuzzy algebra generated by C, that is the fuzzy partition $C = \{C, \overline{C}\}$ (we make for C the hypothesis already made in

Section 4.1.) Suppose also that a θ-additive evaluation of possibility r has been defined in $\mathcal{a}(C)$ with-the only condition of being $r(\phi) = m(\phi)$ and $r(S) = m(S)$. Then in the conditional fuzzy algebra $H = \mathcal{a}(P \times C)$ evaluation of possibility can be defined such that if m_C is one of them, then $m_C(A) = m(A)$ for any $A \in \mathcal{a}(P)$ and $m_C(C') = r(C')$ for any $C' \in \mathcal{a}(C)$. In order to get it, let $*$ be an operation in $\mathbb{R}^+$ satisfying the conditions required in Definition 4.2.1, distributive with respect to θ, with $m(S) = r(S)$ as neutral element, and $m(\phi) = r(\phi)$ as absorbent. Clearly such operations exist.

Theorem 4.2.1. An evaluation of possibility defined in the elements of $P \times C$ by $m_C{*}P_i \cap C_j) = m(P_i){*}r(C_j)$ with $m_C(\phi) = m(\phi) = r(\phi)$ and $m_C(S) = m(S) = r(s)$ is such that $m_C(A) = m(A)$ for each $A \in \mathcal{a}(P)$ and $m_C(B) = r(B)$ for each $B \in \mathcal{a}(C)$.

Proof. It is $m_C(P_i) = m_C[(P_i \cap C) \cup (P_i \cap \overline{C})]$ as $P_i \in W^\lambda[(P_i \cap C) \cup (P_i \cap \overline{C})]$. Then $m_C[(P_i \cap C) \cup (P_i \cap \overline{C})] = \theta[m_C(P_i \cap C), m_C(P_i \cap \overline{C})]$ as $P_i \cap C$ and $P_i \cap \overline{C}$ are elements of a fuzzy partition and, as a consequence, they are incompatible. Taking account of the definition of m_C it is $\theta[m_C(P_i \cap C), m_C(P_i \cap \overline{C})]$ $= [r(C)\theta r(\overline{C})]{*}m(P_i) = r(C \cup \overline{C}){*}m(P_i) = r(S){*}m(P_i) = m(P_i)$ as, because of the hypothesis required to C, C and $\overline{C}$ are incompatible and $C \cup \bar{C} \in W^\lambda(S)$. Analogously we can prove that $m_C(C_j) = r(C_j)$.

As pairs of elements of P are incompatible, for any $A \in \mathcal{a}(P)$ such that $A \in W^\lambda(\bigcup_{i \in J} P_i)$, $J \subset I$, $I = \{1,\dots,n\}$ it is $m_C(A) = \underset{i \in J}{\theta}\, m_C(P_i) = \underset{i \in J}{\theta}\, m(P_i) = m(A)$ and for any A $W^\lambda(\bigcup_{i \notin J} P_i)$ it is $m_C(A) = m_C(\phi) = m(\phi) = m(A)$. In a similar way we will prove that $m_C(B) = r(B)$ for any $B \in \mathcal{a}(C)$. ¤

A general characterization of θ-additive evaluations of possibility is given by the following result.

Theorem 4.2.2. Let $H = A(P\times C)$ and $H_{ij} = P_i \cap C_j \in P \times C$. A function $m_i : H \to \mathbb{R}^+$ is a θ-additive evaluation of possibility if and only if

$$m_c(A) = \begin{cases} \overset{\ell}{\underset{ij}{\theta}} m_c(H_{ij}), & \text{when } A = \bigcup_{ij=1}^{\ell} H'_{ij},\ H'_{ij} \in W^\lambda(H_{ij}) \\ m_c(\phi) & , \text{when } A = \bigcap_{ij=1}^{\ell} H'_{ij},\ H'_{ij} \in W^\lambda(H_{ij}) \\ m_c(S) & , \text{when } A \in W^\lambda(S) \end{cases}$$

for any $\ell < n$, being $n = \#P \times C$.

Proof. (EP4) If A and B are incompatible and such that $A = \bigcup_{si=1}^{k} H'_{si}$, $B = \bigcup_{tj=1}^{\ell} H'_{tj}$, then $m_c(A\cup B) = m_c(\bigcup_{si=1}^{k} H'_{si} \cup \bigcup_{tj=1}^{\ell} H'_{tj})$, as partition $P \times C$ satisfies the hypothesis mentioned in 4.1 all fuzzy sets H'_{si} are different from H'_{tj}. Then, $m_c(\bigcup_{si=1}^{k} H'_{si} \cup \bigcup_{tj=1}^{\ell} H'_{tj}) = m_c(\bigcup_{r=1}^{k+} H'_r)$ all H'_r different. Taking into account the definition of m_c it is

$$m_c(\bigcup_{r=1}^{k+\ell} H'_r) = \overset{k+\ell}{\underset{r=1}{\theta}} m_c(H'_r) = \overset{k}{\underset{si=1}{\theta}} m_c(H'_{si})\ \theta\ \overset{\ell}{\underset{tj=1}{\theta}} m_c(H'_{tj}) = m_c(A)\theta\ m_c(B).$$

If A and B are incompatible and such that $A = \bigcup_{\ell i=1}^{t} H'_{\ell i}$, $B = \bigcap_{js=1}^{m} H'_{js}$ (or vice versa), then $m_c(A\ \ B) = m_c(\bigcup_{i=1}^{t} H'_i \cup \bigcap_{js=1}^{m} H'_{js}) = m_c(\bigcup_{i=1}^{t} H'_i)$ $= \overset{t}{\underset{i=1}{\theta}} m_c(H_{\ell i}) = m_c(A)$. As $m_c(\phi)$ is the null element of θ in $m_c(H)$, then $m_c(A) = m_c(A)\theta\ m_c(\phi) = m_c(A)\theta\ m_c(B)$.

If A and B are incompatible and such that $A = \bigcap_{it=1}^{s} H'_{it}$, $B = \bigcap_{j\ell=1}^{t} H'_{j\ell}$, then it is $m_c(A\cup B) = m_c(\bigcap_{it=1}^{s} \cup \bigcap_{j\ell=1}^{t} H'_{j\ell}) = m_c(\phi) = m_c(\phi)\theta\ m_c(\phi)$ $= m_c(A)\ \ m_c(B)$.

(EP3) If $A' \in W^\lambda(A)$ and $A = \bigcup_{ij=1}^{\ell} H'_{ij}$. then it is $A' \in W^\lambda(\bigcup_{ij=1}^{\ell} H'_{ij})$ and there exist $V_{ij} \in W^\lambda(H'_{ij})$ for each ij such that $A' = \bigcup_{ij=1}^{\ell} V_{ij}$ (§I).

Because of the definition of m_c it is $m_c(A) = \underset{ij=1}{\overset{\ell}{\theta}} m_c(H_{ij})$, and because of being $V_{ij} \in W^\lambda(H'_{ij})$ it is $V_{ij} \in W^\lambda(H_{ij})$ and $m_c(A') = \underset{ij=1}{\overset{\ell}{\theta}} m_c(H_{ij})$.

This is $m_c(A) = m_c(A')$. Other cases can easily be proved

(EP1) If $A \subset B$ and $A = \bigcup_{ij=1}^{\ell} H'_{ij}$, $B = \bigcup_{rs=1}^{t} H'_{rs}$ it should be $\ell \leq t$, and because of the definition of m_c it is $m_c(A) = \underset{ij=1}{\overset{\ell}{\theta}} m_c(H_{ij})$ and $m_c(B) = \underset{rs=1}{\overset{t}{\theta}} m_c(G_{rs})$. As m_c has its values in $\mathbb{R}^+$ it is $m_c(A) = \underset{ij=1}{\overset{\ell}{\theta}} m_c(H_{ij})$ $\leq \underset{rs=1}{\overset{t}{\theta}} m_c(H_{rs}) = m_c(B)$. Other cases can easily be proved.

(EP2) can be stated from (EP1) and the definition of m_c. Easily we can prove that the condition is also necessary. ¤

4.3. Example

An exhibition of paintings is opened to the public during one month in an out gallery. The admittance is free but visitors are requested to fulfil forms in which they have to assign to each painting a quantitative value between 0 and 1 according to each one of k criteria $c_1,\ldots,c_k$ given to them in a list. For instance c_1 could be the degree of "visual pleasauce," c_2 due "harmony of sizes"---etc.---.

Let $S = \{p_1,\ldots,p_n\}$ be the set of paintings. Each visitor v_i establishes k fuzzy sets on S according with the criteria c_j, j = 1...k:

$$v_i^{c_j} : s \to [0,1]$$

Let V be the set of visitors and #V = m. With the assignments corresponding to each criterion we construct k fuzzy sets in the following way:

$$c_j : S \to [0,1]$$

$$Pi \to C_j(p_i) = (v_1^{c_j}(p_i),\ldots,v_m^{c_j}(p_i)$$

$j = 1,\ldots,k$; $i = 1,\ldots,n$ and being ϕ an adequate function of the values $v_r^{c_j}(p_i)$, $r = 1,\ldots,m$ (ϕ can be the arithmetic mean of values, due maximum, the weighted mean, etc....).

The set of all fuzzy sets c_j, $j = 1,\ldots,k$ is a λ-fuzzy partition P ([18]) of S for a certain $\lambda \in [0,1]$.

If the only possible value for λ is $\lambda = 1$, this means that more than one criterion c_j attains in a $p_i \in S$ the value 1. Then we consider that such criteria do not bring information different enough to be considered together and we omit, in the first list, the convenient criteria to keep $\lambda \in [0,1)$.

If the λ-fuzzy partition P obtained after the first score does not satisfy the hypothesis that for each $p_i \in S$ there exists a $c_j \in P$ such that $c_j(p_i) > \lambda$, we add some convenient criteria in the list to obtain a new partition $P' = \{c_1,\ldots,c_k,c_{k+1},\ldots,c_{k+t}\}$ which satisfies the required hypothesis. If such hypothesis is still not satisfied, new criteria would be added again, etc... (we assume that the hypothesis are reached after a finite number of criteria).

Let A(P) be the fuzzy algebra generated by P. Supposing that, following the rules of an evoluation of possibility, a real number $m(A_\ell) \geq 0$ can be assigned to each fuzzy set $A_\ell \in A(P)$ representing the "possibility" that a new visitor v following his own criterion c_α, defines a fuzzy set $v_0^{c_\alpha}$ satisfying A_ℓ in a certain degree (Particularly $m(A_\ell)$ can be frequentially deduced from early visitors). The obtained evoluation of possibility is a function

$$m : A(P) \to \mathbb{R}^+$$

$$A_\ell \to m(A_\ell).$$

If about visitor v_0 we know that he usually considers the criterion c_j, being c_j in the list and that when it occurs "$v_0^{c_j}$ satisfies a fuzzy set $c_j' \in A(P)$" (for instance $c_j' \in W\lambda(c_j)$), then the "possibility" of each $A_\ell \in A(P)$ is modified in the following form

$$m_{c_j'}(A_\ell) = k_{c_j'} * m(A_\ell \cap c_j')$$

If about v_0 we know that he usually considers the criterion c_ω, being c_ω not in the list, and that when this occurs $v_0^{c_\omega}$ does not satisfy any element of $A(P)$, i.e. $S_0^{c_\omega} \notin A(P)$. Then we have to consider the algebra H generated by $P \times C$ being $C = \{v_0^{c_\omega}, \bar{v}_0^{c_\omega}\}$.

Let denote by $H_{ij} = c_i \cap v_j$, $c_i \in P$, $v_j \in C$ each element of $P \times C$. Function $\hat{m}$ defined in H by

$$\hat{m}(A) = \begin{cases} \theta[m(c_i) * r(v_j)], \text{ if } A = \cup H_{ij}', \ H_{ij}' \in W^\lambda(H_{ij}), \\ m(\phi) = r(\phi), \text{ if } A \in W^\lambda(\phi); \\ m(s) = r(s), \text{ if } A \in W^\lambda(S); \end{cases}$$

is a -additive evaluation of possibility.

In this case $m_{c_\omega}(A) = k_{c_\omega} * \hat{m}(A \cap c_\omega)$. The election of proper operations $*$ and θ will depend on the things we are looking for in our problem.

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