ALGEBRAIC THOERY FOR ROBUST STABILITY OF INTERCONNECTED SYSTEMS: NECESSARY AND SUFFICIENT CONDITIONS

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Memorandum No. UCB/ERL M81/89

14 December 1981

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ALGEBRAIC THEORY FOR ROBUST STABILITY OF INTERCONNECTED SYSTEMS: NECESSARY AND SUFFICIENT CONDITIONS

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ABSTRACT

We consider an interconnected system S_o made of linear multivariable subsystems which are specified by matrix fractions with elements in a ring of stable scalar transfer functions H. Given that the k̲th subsystem is perturbed from $G_k = N_{rk}D_k^{-1}$ to $\tilde{G}_k = (N_{rk}+\Delta N_{rk})(D_k+\Delta D_k)^{-1}$ and that the system S_o is H-stable, we derive a computationally efficient necessary and sufficient condition for the H-stability of the perturbed system. These fractional perturbations are more general than the conventional additive and multiplicative perturbations. The result is generalized to handle simultaneous perturbations of two or more subsystems.

Research sponsored by the Air Force Office of Scientific Research (AFOSR) United States Air Force under Contract No. F49620-79-0178.

1. Introduction

Within the theory of large interconnected systems, the problem of determining whether the system remains stable after being subjected to perturbations is a very important one which has an abundant literature. The existing results impose restrictions on the nature of the perturbations (e.g., they must be "small", must be "stable", etc.). In this paper, we propose a general algebraic theory that allows large perturbations without any essential restrictions. We also consider carefully the computational aspects of the problem.

Most of the results on robust stability use the formulation of additive/multiplicative perturbations. For example, for linear time-invariant systems: Desoer et. al. [Des 1] considered coefficient perturbations of subsystem descriptions for lumped feedback systems. Singular perturbation considerations impose some rather unnatural restrictions on such perturbations (no numerator- or denominator-degree increase). Åstrom [Åst. 1] and Francis [Fra. 1] considered stable perturbations on single-input single-output stable plants: Åstrom discussed the robustness of a design method for lumped feedback systems with a two-input one-output controller while Francis examined various notions of perturbations for distributed unity-feedback systems. The key mathematical technique used in [Des. 1], [Åst. 1] and [Fra. 1] is Rouché's theorem and hence only sufficient conditions for robust stability are obtained. Still considering stable perturbations, [Cru. 1], [Pos. 1] and [Zam. 1] also included a number of sufficient conditions. Recently, Doyle and Stein [Doy. 1], considering lumped feedback systems, stated elegant necessary and sufficient conditions for robust stability over a prescribed class of possibly unstable perturbations. Later, Chen and Desoer [Chen 1] proved and generalized these conditions for distributed systems having more general feedback configurations. For nonlinear and time-varying systems, considering stable perturbations on open-loop I/O maps and using the small gain theorem, Zames [Zam. 2] and later Sandell [San. 1] gave sufficient conditions for robust stability of unity-feedback systems. Safonov [Saf. 1] gave sufficient conditions for the robust stability using general state-space models.

More recently, considering matrix fraction description of transfer functions, Vidyasagar et. al. [Vid. 1] introduce a novel formulation of perturbations. More precisely, using coprime facotorizations, they define a topology for unstable systems and show that it is the weakest topology such that the map from open-loop transfer functions to closed-loop transfer functions is continuous. Throughout this paper, we use this more general formulation of perturbations and call them the fractional perturbations.

In this paper, we consider an interconnected system S_o made of μ linear time-invariant multivariable subsystems each described by a matrix fraction with elements in the ring of stable scalar transfer functions H. Suppose the kth subsystem is the only one subjected to fractional perturbations; more precisely, let it be perturbed from a r.c.f. (right coprime factorization) $G_k = N_{rk}D_k^{-1}$ to a r.c.f. $\tilde{G}_k := (N_{rk}+\Delta N_{rk})(D_k+\Delta D_k)^{-1}$ where both ΔN_{rk} and ΔD_k have elements in H but are not assumed to be "small". Given that the nominal system S_o is H-stable, we derive an efficient necessary and sufficient condition for the H-stability of the fractionally perturbed system. The result is generalized to handle simultaneous perturbations of two or more subsystems. Finally, using

Nyquist type argument, we obtain graphic stability tests for the four special algebras of transfer functions commonly used in control problems. Computational considerations are also included.

This paper is organized as follows. In section 2, we define the algebras and the matrix fraction descriptions of transfer functions. In section 3, we describe the nominal and the fractionally perturbed interconnected systems. In section 4, we derive efficient necessary and sufficient conditions for the stability of the perturbed system. In section 5, we generalize the results to handle simultaneous perturbations. In section 6, considering the four commonly used algebras, we give Nyquist-type stability tests and discuss their computational aspects.

2. Preliminaries

2.1 Algebraic Framework

Throughout this paper, we assume the following general algebraic structure:

H : an entire ring, i.e., a commutative ring with no zero-divisor. Let 0 and 1 denote the additive and multiplicative neutral elements, respectively.

I : a multiplicative subset of H, i.e., $I \subset H$, $0 \notin I$, and $x, y \in I \Rightarrow x \cdot y \in I$. W.ℓ.o.g., let $1 \in I$.

G : = $[H][I]^{-1} := \{n/d : n \in H,\ d \in I\}$, i.e., G is the ring of fractions with denominators in I[Bou. 1][Lan. 1, p. 66]

$\mathbb{F}$: a field. Typically, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

We assume that both $(H, \mathbb{F})$ and $(G, \mathbb{F})$ form vector spaces over the field $\mathbb{F}$ (i.e., multiplication by scalars is defined on $\mathbb{F} \times H$ and on $\mathbb{F} \times G$, and the axioms of vector spaces are assumed satisfied).

Table I shows four special cases of the algebraic structure above: (see Sec. 6 for the definition of $\mathcal{U}$). These special cases have additional properties which will be used in Sec. 6 in order to obtain Nyquist-type test.

Comments:

(a) Since by assumption $1 \in \mathcal{I}$, we can identify $n \in \mathcal{H}$ and $n/1 \in \mathcal{G}$; hence we view $\mathcal{H}$ as a subring of $\mathcal{G}$.

(b) By construction of $\mathcal{G}$, every element of $\mathcal{I}$ has an inverse in $\mathcal{G}$.

(c) Since both $\mathcal{H}$ and $\mathcal{G}$ are commutative rings, both $(\mathcal{H},\mathbb{F})$ and $(\mathcal{G},\mathbb{F})$ are commutative algebras over the field $\mathbb{F}$. ¤

2.2 Coprime Factorizations

Definition 2.1

Let $H \in \mathcal{G}^{m\times n}$. We say that $N_r D^{-1}$ ($D^{-1}N_\ell$, resp.) is a right-coprime factorization (r.c.f.) (left-coprime factorization (l.c.f.), resp.) of H if and only if

(i) $H = N_r D^{-1}$ ($D^{-1}N_\ell$, resp.);

(ii) $N_r \in \mathcal{H}^{m\times n}$, $D \in \mathcal{H}^{n\times n}$ ($D \in \mathcal{H}^{m\times m}$, $N_\ell \in \mathcal{H}^{m\times n}$, resp.), and $\det D \in \mathcal{I}$;

(iii) (N_r,D) are right-coprime (r.c.), i.e., $\exists\ U_r \in \mathcal{H}^{n\times m}$ and

$$V_r \in \mathcal{H}^{n\times n} \text{ such that } U_r N_r + V_r D = I_n. \tag{2.5}$$

((iii') (D,N_ℓ) are left coprime (l.c.), i.e., $\exists\ U_\ell \in \mathcal{H}^{n\times m}$ and

$$V_\ell \in \mathcal{H}^{m\times m} \text{ such that } N_\ell U_\ell + D V_\ell = I_m, \text{ resp.}) \tag{2.6}$$

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Definition 2.2

Let $H \in \mathcal{G}^{m\times n}$. We say that $N_r D^{-1} N_\ell$ is a right-left-coprime factorization (r.l.c.f.) of H if and only if (i) $H = N_r D^{-1} N_\ell$, (ii) N_r, D and N_ℓ all have their elements in H with det $D \in I$, (iii) conditions (2.5) and (2.6) hold. ¤

Comment:

Recently, Vidyasagar et. al. give a set of sufficient conditions for the existence of coprime-factorizations [Vid. 1, Thm. 3.34]; it is easily seen that all the examples in Table I satisfy those conditions. In this paper, we assume the existence of coprime-factorizations throughout (see assumptions (3.4) and (3.15) in Sec. 3). ¤

3. System Descriptions

3.1 The Nominal System S_o

Given μ subsystems, each one described by its transfer function matrix $G_j \in G^{n_{oj} \times n_{ij}}$ $(j=1,\ldots,\mu)$, we consider the interconnected system S_o obtained as follows:

(i) We assign a summing-node to each subsystem input; (3.1a)

(ii) We associate an additive exogenous input with each summing node; (3.1b)

(iii) We feed each subsystem-output through gain-matrices with elements in $\mathbb{F}$ to all the summing nodes. (Some of these gain-matrices may be zero). (3.1c)

More precisely, as shown in Fig. 1, the subsystems are interconnected according to :

$$\left.\begin{aligned} e_j &= u_j + \sum_{\alpha=1}^{\mu} F_{j\alpha} y_\alpha \qquad (3.2) \\ y_j &= G_j e_j \qquad (3.3) \end{aligned}\right\} j = 1,\ldots,\mu,$$

where e_j is the jth subsystem-input, y_j is the jth subsystem-output, u_j is the jth exogenous input at the jth summing node, and $F_{j\alpha} \in \mathbb{F}^{n_{ij} \times n_{o\alpha}}$ represents the gain-matrix from the output y_α to the jth summing node. We assume that

$$\text{for } j = 1,\ldots,\mu,\ G_j \text{ has a r.l.c.f. } G_j = N_{rj} D_j^{-1} N_{\ell j} \ . \tag{3.4}$$

Now, let $u := [u_1^T \vdots \ldots \vdots u_\mu^T]^T$, $y := [y_1^T \vdots \ldots \vdots y_\mu^T]^T$, $\xi := [\xi_1^T \vdots \ldots \vdots \xi_\mu^T]^T$, (where ξ_j is defined in Fig. 1); let $D := \mathrm{diag}[D_1,\ldots,D_\mu]$, $N_r := \mathrm{diag}[N_{r1},\ldots,N_{r\mu}]$, and $N_\ell := \mathrm{diag}[N_{\ell 1},\ldots,N_{\ell\mu}]$; let $n_o := \sum_{\alpha=1}^{\mu} n_{o\alpha}$, $n_i := \sum_{\alpha=1}^{\mu} n_{i\alpha}$, and denote by F the $n_o \times n_i$ matrix with its (α,β)th block equal to $F_{\alpha\beta}$, for $\alpha, \beta = 1,\ldots,\mu$. Thus, the nominal system S_o is described by

$$D_c \xi = N_\ell u, \qquad N_r \xi = y, \tag{3.5}$$

where

$$D_c := D - N_\ell F N_r \ . \tag{3.6}$$

From (3.5), $H_{yu} : u \mapsto y$, the I/O map of the system S_o, is given by

$$H_{yu} = N_r D_c^{-1} N_\ell \ . \tag{3.7}$$

Comments:

(a) The G_j's are assumed in (3.4) to be specified by a r.l.c.f. in order to have a flexible general theory : the resulting framework allows some G_j's to be specified by l.c.f. while others may be specified by a r.c.f. or a r.l.c.f.

(b) Assumption (3.4) implies that

$$(N_r, D_c) \text{ are r.c.; } (D_c, N_\ell) \text{ are l.c.} \tag{3.9}$$

Indeed, by definition, $\exists$ U_r, V_r, U_ℓ, V_ℓ, all with elements in H, such that $U_r N_r + V_r D = I_{n_\xi}$ and $N_\ell U_\ell + D V_\ell = I_{n_\xi}$, where $n_\xi := \sum_{\alpha=1}^{\mu} n_{\xi_\alpha}$ and n_{ξ_α} is the dimension of ξ_α. Consequently,

$$(U_r+V_r N_\ell F)N_r + V_r(D-N_\ell F N_r) = I_{n_\xi}, \tag{3.11}$$

$$N_\ell(U_\ell+FN_r V_\ell) + (D-N_\ell F N_r)V_\ell = I_{n_\xi}, \tag{3.12}$$

and hence (3.9) follows.

(c) Note that none of the G_j's are assumed to be H-stable.[†]

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3.2 The Perturbed System $S(\Delta N_{rk},\Delta D_k)$

Suppose that one subsystem, say G_k, is perturbed into the subsystem $\tilde{G}_k$. We assume that

(i) $G_k = N_{rk} D_k^{-1}$, (i.e., $N_{\ell k} = I_{n_{ik}}$); (3.15a)

(ii) $\tilde{G}_k := (N_{rk}+\Delta N_{rk})(D_k+\Delta D_k)^{-1} \in G^{n_{ok} \times n_{ik}}$; and (3.15b)

(iii) ΔN_{rk} and ΔD_k, all with elements in H, are such that (3.15b) is a r.c.f. of $\tilde{G}_k$. (3.15c)

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Perturbations of this type are called the fractional perturbations. We denote by $S(\Delta N_{rk},\Delta D_k)$ the resulting fractionally perturbed interconnected system.

Comments:

(a) Note that the fractional perturbations ΔN_{rk} and ΔD_k are H-stable. Compared to H-stable additive or multiplicative perturbations (i.e.,

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[†] A transfer function matrix is said to be H-stable iff it has all its elements in H.

$G_k \leftarrow G_k + \Delta G_k$, $G_k \leftarrow (I_{n_{ok}} + M_k)G_k$), fractional perturbations are much more flexible : in the context of stable proper rational functions $H = R(0)$, fractional perturbations allow us to change the number and the locations of poles and zeros anywhere in $\mathbb{C}$. In contrast, both stable ΔG_k and stable M_k cannot move $\mathbb{C}_+$-poles; furthermore, stable ΔG_k cannot change the number of $\mathbb{C}_+$-poles while stable M_k may delete some $\mathbb{C}_+$-poles with the consequent difficulties of unstable pole-zero cancellations.

(b) The fractionally perturbed subsystem can be obtained by applying (see Fig. 2) (i) an H-stable feed-forward perturbation ΔN_{rk} on N_{rk}, and (ii) an H-stable feedback perturbation $-\Delta D_k$ on D_k^{-1}.

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Let $H_{yu}(\Delta N_{rk}, \Delta D_k)$ denote the I/O map of the perturbed system $S(\Delta N_{rk}, \Delta D_k)$; it is given by

$$H_{yu}(\Delta N_{rk}, \Delta D_k) = N_r(\Delta N_{rk}) D_c(\Delta N_{rk}, \Delta D_k)^{-1} N_\ell, \tag{3.18}$$

where $D_c(\Delta N_{rk}, \Delta D_k)$, $N_r(\Delta N_{rk})$ are obtained from D_c and N_r, respectively, by the substitutions: $N_{rk} \leftarrow N_{rk} + \Delta N_{rk}$, $D_k \leftarrow D_k + \Delta D_k$. Using assumptions (3.4) and (3.15), we can easily prove that, (similar derivation to (3.9) above):

$$(N_r(\Delta N_{rk}),\ D_c(\Delta N_{rk}, \Delta D_k)) \text{ are r.c.; } (D_c(\Delta N_{rk}, \Delta D_k),\ N_\ell) \text{ are l.c.} \tag{3.20}$$

4. System Stability

In this section, we define stability and derive necessary and sufficient conditions for the stability of interconnected systems.

Definition 4.1

An interconnected system such as S_o (specified by (3.1), (3.2) and

(3.3)) is said to be *H-stable* iff all the (closed-loop) transfer function matrices from any exogenous input u_j to any subsystem-output y_α have all their elements in *H*. In the case these transfer functions have all their elements in *G*, the interconnected system is said to be well-posed.

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Lemma 4.2

(I) Consider S_o specified by (3.1) - (3.4). U.t.c.,

the system S_o is *H*-stable (4.6a)

(by def., $H_{yu} \in H^{n_o x n_i}$)

$\Leftrightarrow$

det D_c has an inverse in *H*. (4.7a)

(II) Consider $S(\Delta N_{rk}, \Delta D_k)$ defined as S_o except for $\tilde{G}_k$ specified in (3.15b,c). U.t.c.,

the system $S(\Delta N_{rk}, \Delta D_k)$ is *H*-stable (4.6b)

(by def., $H_{yu}(\Delta N_{rk}, \Delta D_k) \in H^{n_o x n_i}$)

$\Leftrightarrow$

det $D_c(\Delta N_{rk}, \Delta D_k)$ has an inverse in *H*. (4.7b)

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Comments:

(a) The conditions (4.7) are necessary and sufficient conditions for the *H*-stability of the systems S_o and $S(\Delta N_{rk}, \Delta D_k)$, respectively.

(b) Lemma 4.2 remains valid when we replace *H* by *G* in (4.6) and (4.7). The conditions (4.7), with *H* replaced by *G*, are then necessary and sufficient conditions for the systems S_o and $S(\Delta N_{rk}, \Delta D_k)$, respectively, to be well-posed.

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Proof of Lemma 4.2:

(I) $\Leftarrow$. Since H is a commutative ring, (4.7a) implies, by Cramer's rule, that $D_c^{-1} \in H^{n_\xi \times n_\xi}$ [Mac.1, p.303]. Consequently, $H_{yu} = N_r D_c^{-1} N_\ell \in H^{n_o \times n_i}$ by the closure properties of H.

$\Rightarrow$. Let $\overline{U}_r := U_r + V_r N_\ell F$, $\overline{U}_\ell := U_\ell + F N_r V_\ell$. Postmultiply (3.11) by $D_c^{-1} N_\ell \overline{U}_\ell$, premultiply (3.12) by D_c^{-1}, and add:

$$D_c^{-1} = \overline{U}_r H_{yu} \overline{U}_\ell + V_r N_\ell \overline{U}_\ell + V_\ell \tag{4.10}$$

Equation (4.10), the closure properties of H, and assumption (4.6a) give $D_c^{-1} \in H^{n_\xi \times n_\xi}$. Hence, conclusion (4.7a) follows.

(II) Same as above.

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Theorem 4.3

Consider the systems S_o and $S(\Delta N_{rk}, \Delta D_k)$ defined in (3.1) - (3.4) and (3.15). Assume that

the nominal system S_o is H-stable (4.13)

(by def., $H_{yu} \in H^{n_o \times n_i}$).

U.t.c., the following statements are equivalent:

(I) The perturbed system $S(\Delta N_{rk}, \Delta D_k)$ is H-stable

(by def., $H_{yu}(\Delta N_{rk}, \Delta D_k) \in H^{n_o \times n_i}$); (4.14)

(II) $\det[I_{n_{ik}} + H_{\xi_k u_k} \Delta D_k - (H_{\xi u} F)_{kk} \Delta N_{rk}]$ has an inverse in H (4.15)

where $H_{\xi_k u_k}$ and $H_{\xi u}$ are the transfer function matrices of the nominal system S_o mapping u_k into ξ_k and u into ξ, respectively, and

$$(H_{\xi u}F)_{kk} := \sum_{\alpha=1}^{\mu} H_{\xi_k u_\alpha} F_{\alpha k} \tag{4.16}$$

is the (k,k)th block of $(H_{\xi u}F)$;

$$\text{(III)} \quad \tilde{H}(\Delta N_{rk}, \Delta D_k) \in H^{n_{ik} \times (n_{ik}+n_{ok})} \tag{4.17}$$

where $\tilde{H}(\Delta N_{rk}, \Delta D_k)$ is the transfer function matrix of the system $\tilde{S}(\Delta N_{rk}, \Delta D_k)$ shown in Fig. 4 mapping $[\tilde{u}_k^T \vdots \tilde{d}_k^T]^T$ into $\tilde{\xi}_k$.

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Comments:

(a) There is no restrictions on (i) the "size" of the perturbations ΔN_{rk} and ΔD_k, and (ii) the number of unstable poles of the perturbed subsystem $\tilde{G}_k$ when considering any of the algebras of Table I.

(b) The transfer functions $H_{\xi_k u_k}$ and $(H_{\xi u}F)_{kk}$ have the following interpretations: Consider Fig. 3 which shows the fractionally perturbed system $S(\Delta N_{rk}, \Delta D_k)$ with interconnections cut at (A_k), (B_k) and (C_k). $H_{\xi_k u_k}$ is the transfer function mapping the input injected at (A_k) into the "output" measured at (C_k); $(H_{\xi u}F)_{kk}$ is the transfer function mapping the "input" injected at (B_k) into the "output" measured at (C_k). These transfer functions describe the behavior of S_o at the site of the perturbation.

(c) Theorem 4.3 shows that, in order to test the stability of the fractionally perturbed system $S(\Delta N_{rk}, \Delta D_k)$, we need only know the transfer functions $H_{\xi_k u_k}$ and $(H_{\xi u}F)_{kk}$ of the H-stable nominal system S_o. This is illustrated by Fig. 4: When we examine the stability of the perturbed system $\tilde{S}(\Delta N_{rk}, \Delta D_k)$, the nominal system S_o is reduced to a equivalent system with <u>two inputs u_k and d_k</u>, and <u>one output ξ_k</u>. Furthermore, the stability of $S(\Delta N_{rk}, \Delta D_k)$ is equivalent to that of the system $\tilde{S}(\Delta N_{rk}, \Delta D_k)$

with <u>two inputs $\tilde{u}_k$ and $\tilde{d}_k$,</u> and <u>one output $\tilde{\xi}_k$.</u>

(d) Since both $H_{\xi_k u_k}$ and $(H_{\xi u}F)_{kk}$ do <u>not</u> depend on the perturbations $(\Delta N_{rk}, \Delta D_k)$, the stability test (4.15) is very efficient when one has to examine the effects of a number of specified perturbations.

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<u>Proof of Theorem 4.3:</u>

<u>(4.14) $\Leftrightarrow$ (4.15)</u>. By assumption (4.13), $H_{\xi u} \in \mathcal{H}^{n_\xi \times n_i}$, and hence both $H_{\xi_k u_k}$ and $(H_{\xi u}F)_{kk}$ have all their elements in $\mathcal{H}$. Thus, the determinant in (4.15) is in $\mathcal{H}$. We claim that

$$\det D_c(\Delta N_{rk}, \Delta D_k) = \det D_c \cdot \det[I_{n_{ik}} + H_{\xi_k u_k}\Delta D_k - (H_{\xi u}F)_{kk}\Delta N_{rk}] \tag{4.22}$$

Indeed, by direct calculations†,

$$\det D_c(\Delta N_{rk}, \Delta D_k) = \det\{\mathrm{diag}[D_1, \ldots, D_k + \Delta D_k, \ldots, D_\mu] - N_\ell F\, \mathrm{diag}[N_{r1}, \ldots, N_{rk} + \Delta N_{rk}, \ldots, N_{r\mu}]\}$$

$$= \det\left\{(D - N_\ell F N_r) + \left[\begin{array}{c|c|c} & & \\ \hline & \Delta D_k & \\ \hline & & \end{array}\right] - N_\ell F \left[\begin{array}{c|c|c} & & \\ \hline & \Delta N_{rk} & \\ \hline & & \end{array}\right]\right\}$$

$$= \det\left\{D_c + \left(\left[\begin{array}{c} \vdots \\ \hline \Delta D_k \\ \hline \vdots \end{array}\right] - N_\ell F \left[\begin{array}{c} \vdots \\ \hline \Delta N_{rk} \\ \hline \vdots \end{array}\right]\right) \cdot \left[\begin{array}{c|c|c} \cdots & I_{n_{ik}} & \cdots \end{array}\right]\right\} \tag{4.23}$$

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†Throughout all unfilled blocks in a matrix have all their elements equal to zero.

$$= \det D_c \cdot \det\{I_{n_\xi} + D_c^{-1}\left(\begin{bmatrix} \vdots \\ \hline \Delta D_k \\ \hline \vdots \end{bmatrix} - N_\ell F \begin{bmatrix} \vdots \\ \hline \Delta N_{rk} \\ \hline \vdots \end{bmatrix}\right) \cdot \begin{bmatrix} \cdots & I_{n_{ik}} & \cdots \end{bmatrix}\}$$

$$= \det D_c \cdot \det\{I_{n_{ik}} + \begin{bmatrix} \cdots & I_{n_{ik}} & \cdots \end{bmatrix} D_c^{-1}\left(\begin{bmatrix} \vdots \\ \hline \Delta D_k \\ \hline \vdots \end{bmatrix} - N_\ell F \begin{bmatrix} \vdots \\ \hline \Delta N_{rk} \\ \hline \vdots \end{bmatrix}\right)\}$$

(4.24)

$$= \det D_c \cdot \det[I_{n_{ik}} + (D_c^{-1})_{kk}\, \Delta D_k - (D_c^{-1} N_\ell F)_{kk} \Delta N_{rk}] \tag{4.25}$$

where (i) we use the equality

$$\det(I+MN) = \det(I'+NM) \tag{4.26}$$

to obtain (4.24); and (ii) in (4.25), $(\cdot)_{kk}$ denotes the (k,k)th block of the matrix in the argument. Now, by (3.5), $H_{\xi u} = D_c^{-1} N_\ell$; hence $H_{\xi_k u_k} = (D_c^{-1})_{kk}$ since in (3.15a) we assumed that $N_{\ell k} = I_{n_{ik}}$. Consequently, (4.22) follows.

The equivalence of (4.14) and (4.15) follows immediately by (4.13), (4.22), and Lemma 4.2.

(4.15) ⇔ (4.17).

From Fig. 4, it is easy to show that

$$\tilde{H}(\Delta N_{rk}, \Delta D_k) = \tilde{N}_r \tilde{D}(\Delta N_{rk}, \Delta D_k)^{-1} \tag{4.31}$$

where

$$\tilde{N}_r := [H_{\xi_k u_k} \,|\, (H_{\xi u}F)_{kk}] \in H^{n_{ik} \times (n_{ik}+n_{ok})}, \tag{4.32}$$

$$\tilde{D}(\Delta N_{rk}, \Delta D_k) := \{I_{n_{ik}+n_{ok}} - \begin{bmatrix} -\Delta D_k \\ \Delta N_{rk} \end{bmatrix} \tilde{N}_r\} \in H^{(n_{ik}+n_{ok}) \times (n_{ik}+n_{ok})} \tag{4.33}$$

Direct calculation using (4.26) gives

$$\det \tilde{D}(\Delta N_{rk}, \Delta D_k) = \det\{I_{n_{ik}} - \tilde{N}_r \begin{bmatrix} -\Delta D_k \\ \Delta N_{rk} \end{bmatrix}\}$$

$$= \det[I_{n_{ik}} + H_{\xi_k u_k} \Delta D_k - (H_{\xi u}F)_{kk} \Delta N_{rk}] \tag{4.34}$$

From (4.31), (4.34) and the closure properties of H, the implication "(4.15) ⇒ (4.17)" follows immediately.

To prove "(4.17) ⇒ (4.15)" : observe that (4.32) and (4.33) give

$$\begin{bmatrix} -\Delta D_k \\ \Delta N_{rk} \end{bmatrix} \tilde{N}_r + \tilde{D}(\Delta N_{rk}, \Delta D_k) = I_{n_{ik}+n_{ok}},$$

(i.e., $(\tilde{N}_r, \tilde{D}(\Delta N_{rk}, \Delta D_k))$ are r.c.) and hence

$$\tilde{D}(\Delta N_{rk}, \Delta D_k)^{-1} = \begin{bmatrix} -\Delta D_k \\ \Delta N_{rk} \end{bmatrix} \tilde{H}(\Delta N_{rk}, \Delta D_k) + I_{n_{ik}+n_{ok}} . \tag{4.36}$$

(4.17) and (4.36) imply that $\tilde{D}(\Delta N_{rk}, \Delta D_k)^{-1} \in H^{(n_{ik}+n_{ok}) \times (n_{ik}+n_{ok})}$

Hence, conclusion (4.15) follows.

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If we assume the existence of a norm on the algebra H of stable transfer functions (and this holds for the four examples of H in Table I), then we can state a robust stability result:

Corollary 4.4 (Robust Stability)

Let the conditions of Theorem 4.3 hold. Let $(H,\|\cdot\|)$ be a Banach algebra and let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\|H_{\xi_k u_k}\|\rho_1 + \|(H_{\xi u}F)_{kk}\|\rho_2 < 1 . \tag{4.46}$$

U.t.c., if †

$$\Delta N_{rk} \in B(N_{rk};\rho_2) \text{ and } \Delta D_k \in B(D_k;\rho_1), \tag{4.47}$$

then

$$S(\Delta N_{rk},\Delta D_k) \text{ is } H\text{-stable.} \tag{4.48}$$

¤

Comment: Corollary 4.4 shows that the H-stability of the system S_o is robust with respect to fractional perturbation $(\Delta N_{rk},\Delta D_k)$.

¤

Proof of Corollary 4.4:

Assumptions (4.46) and (4.47) imply

$\|H_{\xi_k u_k}\Delta D_k - (H_{\xi u}F)_{kk}\Delta N_{rk}\| < 1$; thus [Die. 1, (8.3.2.1)],

$$[I_{n_{ik}} + H_{\xi_k u_k}\Delta D_k - (H_{\xi u}F)_{kk}\Delta N_{rk}]^{-1} \in H^{n_{ik}\times n_{ik}} .$$

Consequently, (4.15), or equivalently (by Theorem 4.3), (4.48) follow.

¤

† $\forall H \in H,\ \forall\rho > 0,\ B(H;\rho) := \{H' : \|H'-H\| < \rho\}$.

5. Simultaneous Perturbations

The analysis above can easily be extended to handle simultaneous perturbations of two or more subsystems. For example, suppose that the jth and the kth subsystem are simultaneously subjected to perturbations. We assume that

$$\begin{aligned}&G_j = N_{rj}D_j^{-1} \text{ and } G_k = N_{rk}D_k^{-1}, \text{ both r.c.f.'s,}\\ &\text{are perturbed to r.c.f.'s } \tilde{G}_j := (N_{rj}+\Delta N_{rj})(D_j+\Delta D_j)^{-1}\\ &\text{and } \tilde{G}_k := (N_{rk}+\Delta N_{rk})(D_k+\Delta D_k)^{-1} \text{ where } \Delta N_{rj},\ \Delta D_j,\\ &\Delta N_{rk} \text{ and } \Delta D_k \text{ are all } H\text{-stable.}\end{aligned} \tag{5.5}$$

Let $S(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$ denote the resulting perturbed system and let the corresponding I/O map be given by

$$H_{yu}(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k) = N_r(\Delta N_{rj},\Delta N_{rk}) \cdot D_c(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)^{-1} \cdot N_\ell \tag{5.6}$$

where $N_r(\Delta N_{rj},\Delta N_{rk}) := \mathrm{diag}[N_{r1},\ldots,N_{rj}+\Delta N_{rj},\ldots,N_{rk}+\Delta N_{rk},\ldots,N_{r\mu}]$,
and

$$\begin{aligned}&D_c(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)\\ &\qquad := \mathrm{diag}\ [D_1,\ldots,D_j + \Delta D_j,\ldots,D_k + \Delta D_k,\ldots,D_\mu] - N_\ell\cdot F\cdot N_r(\Delta N_{rj},\Delta N_{rk}).\end{aligned} \tag{5.7}$$

As before, it is easy to see that coprimeness conditions similar to (3.20) hold.

Theorem 4.3 for one fractional perturbation can now be generalized to

Theorem 5.1

Consider the nominal system S_o and the perturbations (5.5). The perturbed system $S(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$, defined as in (3.1) - (3.4), has the I/O map defined in (5.6). Let

(a) $H_{\xi_a u_b}$, for a, b $\in$ {j,k}, and $H_{\xi u}$ denote the transfer function matrices of the nominal system S_o mapping u_b into ξ_a and u into ξ, respectively; (5.11)

(b) $(H_{\xi u}F)_{ab}$, for a,b $\in$ {j,k}, denote the (a,b)th block of $(H_{\xi u}F)$; (5.12)

(c)

$$X(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k) := \left[\begin{array}{c|c} H_{\xi_j u_j}\Delta D_j-(H_{\xi u}F)_{jj}\cdot\Delta N_{rj} & H_{\xi_j u_k}\Delta D_k-(H_{\xi u}F)_{jk}\cdot\Delta N_{rk} \\ \hline H_{\xi_k u_j}\cdot\Delta D_j-(H_{\xi u}F)_{kj}\cdot\Delta N_{rj} & H_{\xi_k u_k}\Delta D_k-(H_{\xi u}F)_{kk}\cdot\Delta N_{rk} \end{array}\right]. \tag{5.13}$$

U.t.c., if

S_o is $\mathcal{H}$-stable, (5.14)

then

$S(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$ is $\mathcal{H}$-stable (5.15)

$\Leftrightarrow$

$\det[I_{n_{ij}+n_{ik}} + X(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)]$ has an inverse in $\mathcal{H}$. (5.16)

¤

Proof of Theorem 5.1:

First, using (5.7) and calculating det $D_c(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$ (as in Sec. 4 to obtain (4.23) et seq.), we obtain

$$\det D_c(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k) = \det D_c \cdot \det[I_{n_{ij}+n_{ik}}+X(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)] \tag{5.21}$$

with $X(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$ defined in (5.11)-(5.13).

With (5.21), the rest of the proof is similar to that of Theorem 4.3.

¤

Remarks:

(a) We can also derive (5.21), and hence prove Theorem 5.1, by considering one perturbation at a time. More precisely, consider $S(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$ as the result of perturbing first S_o into $S(\Delta N_{rk},\Delta D_k)$ and second $S(\Delta N_{rk},\Delta D_k)$ into $S(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$: calculating directly (as in the derivation for (4.22)), we obtain for the second step

$$\det D_c(\Delta N_{rk},\Delta D_j;\Delta N_{rk},\Delta D_k)$$
$$= \det D_c(\Delta N_{rk},\Delta D_k) \cdot \det\{I_{n_{ij}}+H_{\xi_j u_j}(\Delta N_{rk},\Delta D_k)\Delta D_j-[H_{\xi u}(\Delta N_{rk},\Delta D_k)F]_{jj}\Delta N_{rj}\} \tag{5.31}$$

where $H_{\xi_j u_j}(\Delta N_{rk},\Delta D_k)$ and $[H_{\xi u}(\Delta N_{rk},\Delta D_k)F]_{jj}$ are the transfer function matrices of $S(\Delta N_{rk},\Delta D_k)$ defined as $H_{\xi_j u_j}$ and $(H_{\xi u}F)_{jj}$ of S_o. Substitution of (4.22) for the first step in (5.31) then gives

$$\det D_c(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k) = \det D_c \cdot \tilde{X}(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k) \tag{5.32}$$

where

$$\tilde{X}(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$$
$$:= \det[I_{n_{ik}}+H_{\xi_k u_k}\Delta D_k-(H_{\xi u}F)_{jj}\Delta N_{rk}] \cdot \det\{I_{n_{ij}}+H_{\xi_j u_j}(\Delta N_{rk},\Delta D_k)$$
$$\cdot \Delta D_j - [H_{\xi u}(\Delta N_{rk},\Delta D_k)F]_{jj} \cdot \Delta N_{rj}\} \tag{5.33}$$

Now, direct calculation shows that

$$H_{\xi_j u_j}(\Delta N_{rk},\Delta D_k)$$

$$= H_{\xi_j u_j} - [H_{\xi_{jk}}\Delta D_k - (H_{\xi u}F)_{jk}\Delta N_{rk}]\cdot[I_{n_{ik}} + H_{\xi_k u_k}\Delta D_k - (H_{\xi u}F)_{kk}\Delta N_{rk}]^{-1}\cdot H_{\xi_k u_j}, \tag{5.34}$$

$$[H_{\xi u}(\Delta N_{rk},\Delta D_k)F]_{jj}$$

$$= (H_{\xi u}F)_{jj} - [H_{\xi_{jk}}\Delta D_k - (H_{\xi u}F)_{jk}\Delta N_{rk}]\cdot[I_{n_{ik}} + H_{\xi_k u_k}\Delta D_k - (H_{\xi u}F)_{kk}\Delta N_{rk}]^{-1} \cdot (H_{\xi u}F)_{kj}. \tag{5.35}$$

Using (5.34) and (5.35), we can easily show that

$$\det[I_{n_{ij}+n_{ik}} + X(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)] = \tilde{\chi}(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k). \tag{5.36}$$

Therefore, we obtain (5.21) by substituting (5.36) in (5.32).

(b) By (5.36), the stability test (5.16) is equivalent to

$\tilde{\chi}(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$ has an inverse in H .

¤

6. Nyquist Tests for Special Cases

The results so far obtained invoke only algebraic properties of the transfer functions. In order to obtain Nyquist-type stability tests, we have to use their analytic properties. In this subsection, we consider the four algebraic structures listed in Table I, namely, the following four algebras of (scalar) transfer functions for single-input single-output linear time-invariant systems: (i) $\mathbb{R}_p(s)$ (continuous-time lumped case); (ii) $\hat{\mathcal{B}}(\sigma_0)$ (continuous-time distributed case [Cal. 1-2]);

(iii) $\mathbb{R}_p(z)$ (discrete-time lumped case); and (iv) $\tilde{b}(\rho_o)$ (discrete-time distributed case [Che. 1]).

6.1 <u>Nyquist Tests</u>

Referring to Table I, note that $\mathcal{U} \subset \mathbb{C}$ is the "region of instability" in the sense that (a) every element of $\mathcal{H}$ is analytic in $\mathcal{U}$; (b) for any $h \in \mathcal{H}$ that has an inverse in $\mathcal{G}$,

$$h \text{ has an inverse in } \mathcal{H} \Leftrightarrow h \text{ has no zeros in } \mathcal{U}; \qquad (6.1)$$

and (c) whenever $G \in \mathcal{G}^{mxn}$ is not in $\mathcal{H}^{mxn}$, G is analytic in $\mathcal{U}$ except for a <u>finite</u> number of poles [Cal. 1-2], [Che. 1]. Hence using the "argument principle" [Die. 1, p. 246-247] to determine whether (6.1) holds or not, we obtain the following

<u>Corollary 6.1</u> (Nyquist Test for Special Cases)

Let all the conditions of Theorem 5.1 hold with all transfer function matrices have elements in one of the four algebraic structures of Table I. For simplicity, let the transfer function matrix $X(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$, defined by (5.11)-(5.13), be <u>strictly proper</u> (i.e., goes to zero as s, (or z), goes to ∞ in $\mathbb{C}_+$).

the system $S(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$ is $\mathcal{H}$-stable

$\Leftrightarrow$

the Nyquist diagram of $\det[I_{n_{ij}+n_{ik}}+X(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)]$ neither goes through nor encircles the origin. (6.5)

¤

<u>Comments</u>:

(a) For given perturbations $(\Delta N_{rj},\Delta D_j)$ and $(\Delta N_{rk},\Delta D_k)$, Corollary 6.1

provides a graphic stability test (6.5) for the system $S(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$.

(b) Setting ΔN_{rj} and ΔD_j equal to zero matrices reduces (6.5) to a graphic stability test for the system $S(\Delta N_{rk},\Delta D_k)$.

¤

6.2 Computational Aspects

The stability test (6.5) is very convenient for computations when system studies require us to check the stability of the perturbed system for a prescribed finite set of perturbations $\mathcal{D}_j$ for G_j and a similar set $\mathcal{D}_k$ for G_k. More precisely, let $j = \mu - 1$ and $k = \mu$, then given a suitable finite set of frequencies Ω, we propose to sketch the corresponding Nyquist diagrams using the following

Algorithm 6.2 (Stability Test for $S(\Delta N_{r(\mu-1)},\Delta D_{\mu-1};\Delta N_{r\mu},\Delta D_\mu)$ over $\mathcal{D}_{\mu-1}$ and $\mathcal{D}_\mu$)

Data $\Omega := \{\omega_\alpha : \alpha = 1,\ldots,m_\Omega\}$;

$$\mathcal{D}_{\mu-1} := \{(\Delta N^{(\beta)}_{r(\mu-1)},\Delta D^{(\beta)}_{\mu-1}) : \beta = 1,\ldots,m_{\mu-1}\};$$

$$\mathcal{D}_\mu := \{(\Delta N^{(\gamma)}_{r\mu},\Delta D^{(\gamma)}_{\mu}) : \gamma = 1,\ldots,m_\mu\};$$

for $(\alpha=1,\ldots,m_\Omega)$

obtain $H_{\xi_a u_b}(j\omega_\alpha)$ and $[H_{\xi u}(j\omega_\alpha)F]_{ab}$, for $a,b \in \{\mu-1, \mu\}$, by solving appropriate sets of linear equations;

for $(\beta=1,\ldots,m_{\mu-1})$

compute and store $V^{(\beta)} := H_{\xi_{\mu-1}u_{\mu-1}}(j\omega_\alpha)\cdot\Delta D^{(\beta)}_{\mu-1}(j\omega_\alpha)$, $\overline{V}^{(\beta)} := H_{\xi_\mu u_{\mu-1}}(j\omega_\alpha)\cdot\Delta D^{(\beta)}_{\mu-1}(j\omega_\alpha)$, $W^{(\beta)} := [H_{\xi u}(j\omega_\alpha)F]_{(\mu-1)(\mu-1)} \cdot \Delta N^{(\beta)}_{r(\mu-1)}(j\omega_\alpha)$, and $\overline{W}^{(\beta)} := [H_{\xi u}(j\omega_\alpha)F]_{\mu(\mu-1)}\cdot\Delta N^{(\beta)}_{r(\mu-1)}(j\omega_\alpha)$;

for $(\gamma=1,\ldots,m_\mu)$

compute and store $\overline{Y}^{(\gamma)} := H_{\xi_{\mu-1}u_\mu}(j\omega_\alpha)\cdot\Delta D_\mu^{(\gamma)}(j\omega_\alpha)$,

$Y^{(\gamma)} := H_{\xi_\mu u_\mu}(j\omega_\alpha)\cdot\Delta D_\mu^{(\gamma)}(j\omega_\alpha)$, $\overline{Z}^{(\gamma)} := [H_{\xi u}(j\omega_\alpha)F]_{(\mu-1)\mu}\cdot\Delta N_{r\mu}^{(\gamma)}(j\omega_\alpha)$,

$Z^{(\gamma)} := [H_{\xi u}(j\omega_\alpha)F]_{\mu\mu}\cdot\Delta N_{r\mu}^{(\gamma)}(j\omega_\alpha)$;

for $(\beta=1,\ldots,m_{\mu-1})$

for $(\gamma=1,\ldots,m_\mu)$

compute and store

$$N_{\beta\gamma}(j\omega_\alpha) := \det\left[\begin{array}{c|c} I_{n_{i(\mu-1)}}+V^{(\beta)}-W^{(\beta)} & \overline{Y}^{(\gamma)}-\overline{Z}^{(\gamma)} \\ \hline \overline{V}^{(\beta)}-\overline{W}^{(\beta)} & I_{n_{i\mu}}+Y^{(\gamma)}-Z^{(\gamma)} \end{array}\right];$$

for $(\beta=1,\ldots,m_{\mu-1})$

for $(\gamma=1,\ldots,m_\mu)$

use the points $N_{\beta\gamma}(j\omega_\alpha)$, $\alpha = 1,\ldots,m_\Omega$, to plot the Nyquist diagram;

use the Nyquist test (6.5) to determine the stability of $S(\Delta N_{r(\mu-1)}^{(\beta)},\Delta D_{\mu-1}^{(\beta)};\Delta N_{r\mu}^{(\gamma)},\Delta D_\mu^{(\gamma)})$;

¤

Remark:

The algo above determines the stability of the perturbed system $S(\Delta N_{r(\mu-1)},\Delta D_{\mu-1};\Delta N_{r\mu},\Delta D_\mu)$ over the sets $\mathcal{D}_{\mu-1}$ and $\mathcal{D}_\mu$ by applying the Nyquist test (6.5) to the Nyquist diagram of $\det[I_{n_{i(\mu-1)}+n_{i\mu}}+X(\Delta N_{r(\mu-1)},\Delta D_{\mu-1};\Delta N_{r\mu},\Delta D_\mu)]$. Alternately, by (5.36), we can also determine the stability by checking the Nyquist diagram of $\tilde{\chi}(\Delta N_{rj},\Delta D_j;\Delta N_{rk},\Delta D_k)$ as prescribed by (5.33). In this case the labels $j = \mu$ and $k = \mu - 1$ are chosen so that $m_{\mu-1} \leq m_\mu$ [Bra. 1] and the complex matrices $H_{\xi_\mu u_\mu}(\Delta N_{r(\mu-1)},\Delta D_{\mu-1})(j\omega_\alpha)$ and $[H_{\xi u}(\Delta N_{r(\mu-1)},\Delta D_{\mu-1}) \cdot F]_{\mu\mu}(j\omega_\alpha)$ are obtained by first updating the LU-factors of $D_c(j\omega_\alpha)$ to obtain those of $D_c(\Delta N_{r(\mu-1)},\Delta D_{\mu-1})(j\omega_\alpha)$ [Haj. 1], and then using the resulting LU-factors to solve appropriate sets of linear equations. A careful study of operations count shows that the reduction of computational cost by using (5.33) is insignificant. Indeed, calculating at each frequency the second determininant on the right-hand side of (5.33) requires $(2n^3+n^3/3)$ multiplications while calculating the determinant in (6.16) requires $(2n)^3/3$ multiplications; furthermore, these calculations are repeated $(m_{\mu-1}\cdot m_\mu\cdot m_\Omega)$ times. In other words, the benefits of calculating the determinant of a smaller size matrix in (5.33) is almost wiped out by the cost of calculating 2 matrix products $(2n^3)$.

¤

6.3 Lumped Systems

For lumped systems whose rational transfer function matrices have the usual polynomial matrix fractions, we can perform all the calculations in the ring of polynomials $\mathbb{R}[s]$. For example, considering only one perturbed subsystem, we can easily prove the following

Corollary 6.3 (Continuous-Time Lumped Systems)

Consider the continuous-time lumped nominal system S_o defined in (3.1) - (3.4) and (3.15a) where all the N's and the D's are polynomial matrices. Suppose that the polynomial fractional perturbations ΔN_{rk} and ΔD_k are such that

(i) the perturbed kth subsystem is described by a (polynomial) r.c.f. $\tilde{G}_k := (N_{rk}+\Delta N_{rk})(D_k+\Delta D_k)^{-1}$,

(ii) the perturbed system $S(\Delta N_{rk},\Delta D_k)$ is well-posed

$$(\text{i.e., } H_{yu} \in \mathbb{R}_p(s)^{n_o \times n_i}) . \tag{6.21}$$

U.t.c., if

S_o is exp. (exponentially) stable

$$(\text{by def., } H_{yu} \in \mathcal{R}(0)^{n_o \times n_i}), \tag{6.22}$$

then

$S(\Delta N_{rk}, \Delta D_k)$ is exp. stable

$$(\text{by def., } H_{yu}(\Delta N_{rk}, \Delta D_k) \in \mathcal{R}(0)^{n_o \times n_i}) \tag{6.23}$$

$\Leftrightarrow$

$$\mathcal{Z}\{\det[I_{n_{ik}} + H_{\xi_k u_k} \Delta D_k - (H_{\xi u} F)_{kk} \Delta N_{rk}]\} \subset \mathring{\mathbb{C}}_- \tag{6.24}$$

$\Leftrightarrow$

for some convenient $\sigma > 0$, the Nyquist diagram of

$$\frac{1}{(s+\sigma)^\rho} \cdot \det[I_{n_{ik}} + H_{\xi_k u_k} \Delta D_k - (H_{\xi u} F)_{kk} \Delta N_{rk}]$$

neither goes through nor encircles the origin (6.25)

where†

$$\rho := \partial[\det D_c(\Delta N_{rk}, \Delta D_k)] - \partial[\det D_c]. \tag{6.26}$$

¤

Comments:

(a) It is easy to see that

$$\mathcal{P}[H_{yu}(\Delta N_{rk}, \Delta D_k)] \cap \mathbb{C}_+ \subset \mathcal{Z}\{\det[I_{n_{ik}} + H_{\xi_k u_k} \Delta D_k - (H_{\xi u} F)_{kk} \Delta N_{rk}]\}$$

---

† $\forall\, p \in \mathbb{R}[s]$, $\partial[p]$:= degree of p

(b) ρ in (6.26) is such that the Nyquist diagram goes to a nonzero constant at ∞. Indeed, from (4.22),

$$\frac{\det D_c(\Delta N_{rk},\Delta D_k)(s)}{(s+\sigma)^{\rho}\cdot \det D_c(s)} = \frac{1}{(s+\sigma)^{\rho}} \cdot \det[I_{n_{ik}} + H_{\xi_k u_k}\Delta D_k - (H_{\xi u}F)_{kk}\Delta N_{rk}](s). \tag{6.27}$$

By (6.21) and (6.22), $\det D_c(\Delta N_{rk},\Delta D_k) \not\equiv 0$ and $\det D_c \not\equiv 0$; hence from (6.26), both sides of (6.27) approach some nonzero constant as $s \to \infty$.

¤

7. Conclusions

The algebraic theory of robust stability developed in this paper shows that a single algebraic theory covers all the important classes of systems used in engineering (see Table I): the cost is small: "think in terms of commutative rings and define strictly proper as "tends to zero as s, (or z), goes to infinity in $\mathbb{C}_+$."

The formulation presented above is particularly efficient if one has to test the stability of a given interconnected system for a specified class of perturbations: at the cost of some overhead, the test cost per perturbation is considerably reduced by the consideration of the simple system shown Fig. 4.

The fractional perturbations used in this paper are the most general perturbations possible (while remaining within the class of systems under consideration): they do not suffer from the restrictions of the well known additive and multiplicative perturbations.

Table I. Examples of H, I, G, and U. (Note: $\sigma_0 \leq 0$ and $0 < \rho_0 \leq 1$)

H	$\mathcal{R}(0)$	$\hat{A}_-(\sigma_0)$	$\mathscr{r}(0)$	$\tilde{\ell}_{1-}(\rho_0)$
I	$\mathcal{R}^\infty(0)$	$\hat{A}_-^\infty(\sigma_0)$	$\mathscr{r}^\infty(0)$	$\tilde{\ell}_{1-}^\infty(\rho_0)$
G	$\mathbb{R}_p(s)$	$\hat{B}(\sigma_0)$	$\mathbb{R}_p(z)$	$\tilde{b}(\rho_0)$
U	$\mathbb{C}_+$	$\mathbb{C}_{\sigma_0+}$	$D(1)^c$	$D(\rho_0)^c$
Ref.	[Cal. 1-2]		[Che. 1]	

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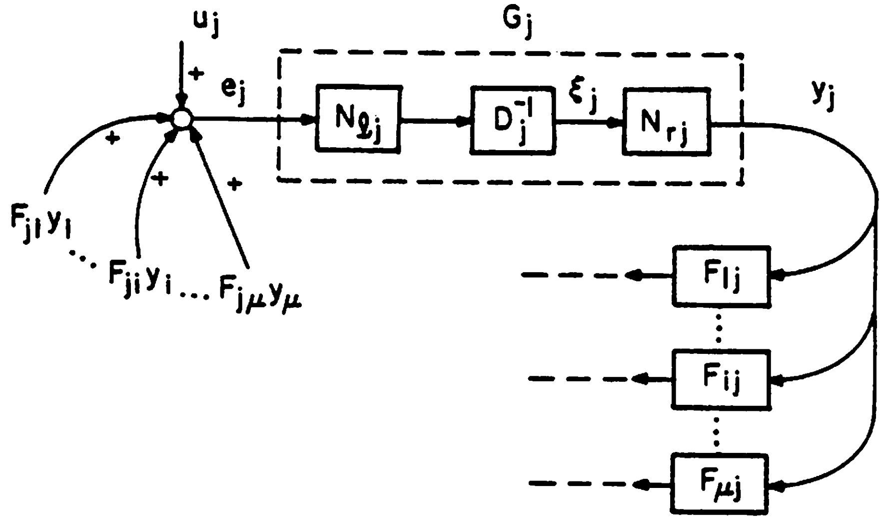
Figure Captions

Fig. 1. The $\underline{j}$th subsystem G_j with its interconnections.

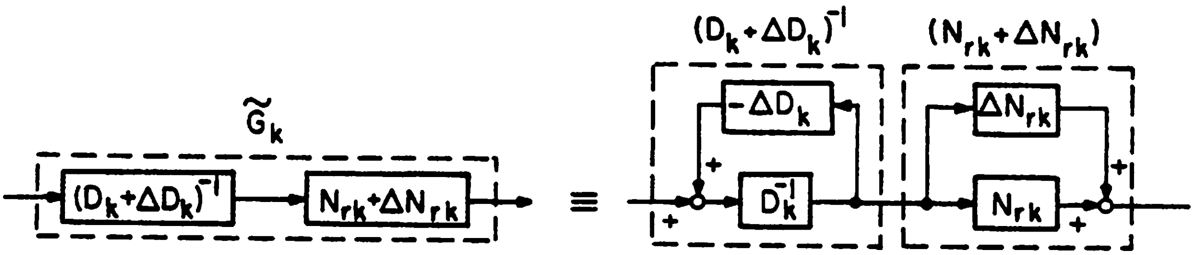
Fig. 2. The fractionally perturbed $\underline{k}$th subsystem $\tilde{G}_k$.

Fig. 3. The system $S(\Delta N_{rk},\Delta D_k)$ with interconnections cut at $\textcircled{A_k}$, $\textcircled{B_k}$ and $\textcircled{C_k}$.

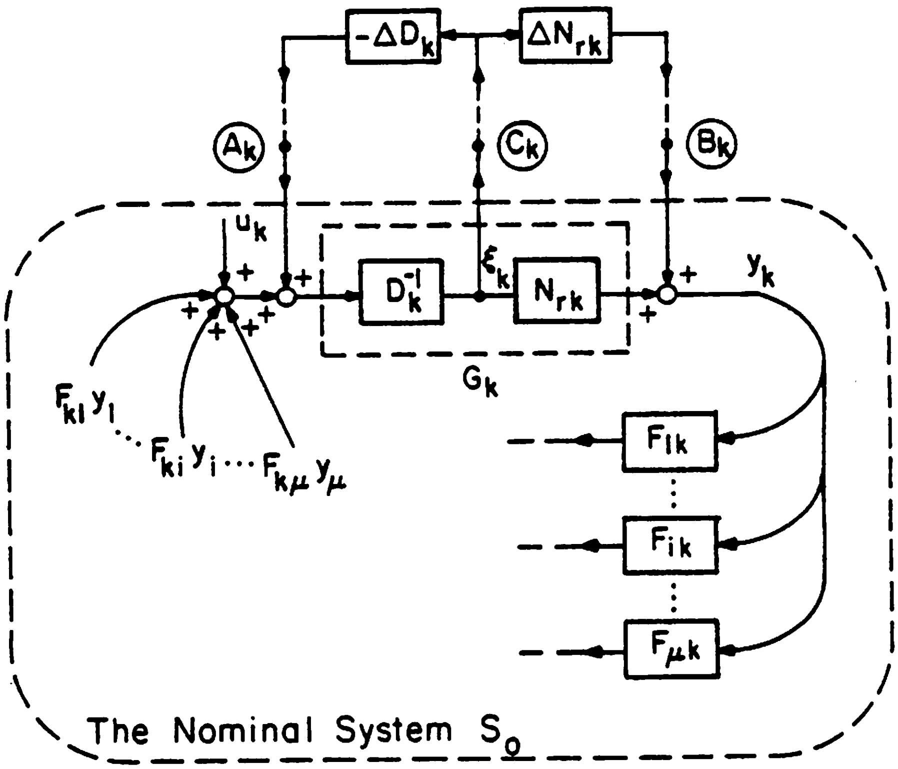
Fig. 4. The two-input ($\tilde{u}_k$ and $\tilde{d}_k$) one-output ($\tilde{\xi}_k$) system $\tilde{S}(\Delta N_{rk},\Delta D_k)$.



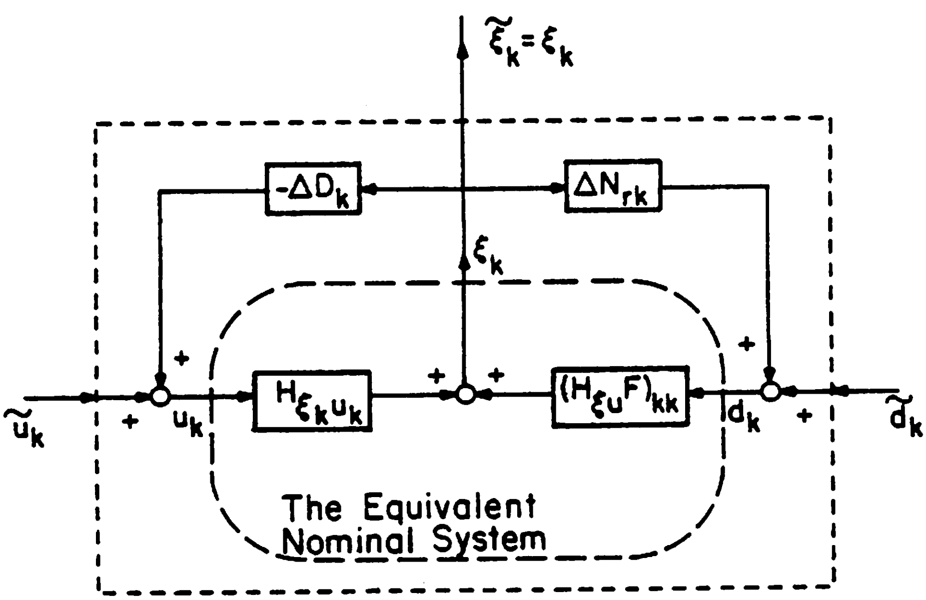
u_j
G_j
e_j
$N_{\ell j}$
D_j^{-1}
ξ_j
N_{rj}
y_j
$F_{j1}y_1$
$F_{ji}y_i$
$F_{j\mu}y_\mu$
F_{1j}
F_{ij}
$F_{\mu j}$



$\tilde{G}_k$
$(D_k+\Delta D_k)^{-1}$
$N_{rk}+\Delta N_{rk}$
$\equiv$
$(D_k+\Delta D_k)^{-1}$
$(N_{rk}+\Delta N_{rk})$
$-\Delta D_k$
D_k^{-1}
ΔN_{rk}
N_{rk}



-ΔD_k
ΔN_rk
A_k
C_k
B_k
u_k
ξ_k
y_k
G_k
F_k1 y_1 ··· F_ki y_i ··· F_kμ y_μ
F_1k
F_ik
F_μk
The Nominal System S_0



ξ̃_k = ξ_k
-ΔD_k
ΔN_rk
ξ_k
H_ξkuk
(H_ξuF)_kk
ũ_k
u_k
d_k
d̃_k
The Equivalent Nominal System