EXPONENTIAL LOWER BOUNDS FOR SOME NP-COMPLETE PROBLEMS IN RESTRICTED LINEAR DECISION TREE MODEL

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Exponential lower bounds for some NP-complete problems in restricted linear decision tree model*

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ABSTRACT

Let V be a set in R^n consisting of finitely many hyperplanes. The linear recognition problem given by V is to determine, using ternary comparisons of the form '$f(x)$:0' where $f:R^n \rightarrow R$ is a linear function, whether a point $x \in R^n$ is in V. We consider lower bounds on the number of comparisons when V corresponds to some NP-complete problems. A technique is proposed for proving such bounds. If the tests '$f(x)$:0' are restricted such that f always defines some hyperplane in V, then some NP-complete problems are shown to have exponential lower bounds in n. Examples of larger classes of linear test functions are found such that the exponential lower bounds are still valid.

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1. Introduction

The *linear recognition problem* investigated in this paper is to determine, given a point $x=(x_1, \cdots ,x_n)$ in the Euclidean n-space R^n, whether x lies in a fixed set V where V is a finite union of some hyperplanes. We study lower bounds on the complexity of *linear decision trees* for solving such problems. The complexity measure to be used is the height of the tree, that is, the number of three-valued linear comparisons of the form '$f(x)$:0' needed to solve the problem with a decision tree when each $f: R^n \to R$ is a linear function of the input x. This model was first studied by Rabin [5] and Reingold [6].

We are mainly interested in linear recognition problems obtained from NP-complete problems. Although discrete by nature, many NP-complete problems can be meaningfully analyzed also as linear recognition problems. For example, the NP-complete partition problem is to determine, given positive integers $x_1, \cdots ,x_n$, whether the index set $\{1, \cdots ,n\}$ can be partitioned into nonempty parts I and I' such that $\sum_I x_i = \sum_{I'} x_i$. If each x_i is thought to be a real variable, this equation defines a hyperplane in R^n. For each fixed n, the union V of all such hyperplanes defines a linear recognition problem in R^n. A linear recognition problem constructed in similar way from the NP-complete subset sum problem was previously analyzed in [1,2,8] (where it was called the knapsack problem; our terminology is from [3]). Similar problems are also studied in [7,9].

Each connected component in the complement of V is a polyhedron with faces defined by the hyperplanes in V. Therefore, if the number of $n-1$-dimensional faces of some component is exponential in n then the complexity of the linear recognition problem is at least exponential if only tests '$f(x)$:0' where f defines some hyperplane in V are allowed. Let $L(V)$ denote the set of all such functions f. We show that in all our example cases including the partition, subset sum, hamiltonian circuit and satisfiability problems, the complement of V

has components with exponentially many $n-1$-dimensional faces. This immediately leads to exponential lower bounds, given in Section 3, for decision trees with test functions limited to the set $L(V)$. Some of our examples even have the stronger property, called *face-completeness*, that the complement of V has a component which has as many $n-1$-dimensional faces as V has hyperplanes. For such problems we may give the exact value of the decision tree complexity when test functions are limited to the set $L(V)$.

It is natural to ask whether these results remain true if test functions outside $L(V)$ are allowed. For example, an exponential lower bound for some NP-complete problem, when we allow all tests '$\sum a_i x_i + a_0 : 0$' where the coefficients a_i are integers with lengths polynomially bounded in n, would be of considerable interest. In Section 4 we prove a result in this direction. For some V and V' corresponding to NP-complete problems it turns out, for example, that extending the class of test functions from $L(V)$ to $L(V) \cup L(V')$ does not essentially help in solving the problem given by V. For the partition problem we give also some other classes of test functions larger than $L(V)$ such that the problem is still of exponential decision tree complexity.

2. Recognition problems in R^n and the decision tree model

In this section we introduce the concepts necessary in formulating and proving our results. Some example problems are given for later use.

Set $A \subseteq R^n$ is *affine* if it is obtained by translating a linear subspace. The *dimension* of A is defined to be the dimension of that subspace. A *hyperplane* $H \subset R^n$ is an affine set of dimension $n-1$. Thus each H is the set of solutions $(x_1, \cdots, x_n)$ to a nontrivial linear equation $\sum a_i x_i = b$. The hyperplane H cuts R^n into two open half-spaces defined by $\sum a_i x_i > b$ and $\sum a_i x_i < b$. By allowing equality in these conditions the closed half-spaces are obtained. A *polyhedron* is an intersection of finitely many (open or closed) half-spaces. If such a set A is obtained by intersecting the half-spaces defined by hyperplanes $\{H_i \mid i \in I\}$ then for each subset $I' \subseteq I$, the set $F = \bar{A} \cap (\bigcap_{I'} H_i)$ is a *face* of A; here $\bar{A}$ denotes the closure of A. The dimension of F is s if the smallest affine set in R^n containing F has dimension s.

Let V be a union of finitely many hyperplanes $H_i \subset R^n$. Each such V defines a *linear recognition problem* in R^n. We call n the *dimension* of the problem. The problem is to determine for each input $x = (x_1, \cdots, x_n)$ whether or not x is in V. In other words, we must decide whether or not x is a root of at least one of the linear equations $\sum a_i x_i + b = 0$ defining hyperplanes H_i. The number of different hyperplanes in V is called the *degree* of V.

We are interested in the solution of linear recognition problems using *linear decision trees*. An algorithm in this form is a ternary tree with each internal node representing a test of the form '$\sum_{i=1}^{n} a_i x_i - b : 0$', and each leaf containing a 'yes' or 'no' answer. For any input $x = (x_1, \cdots, x_n)$ in R^n, the algorithm proceeds by moving down the tree, testing and branching according to the test results ($<, =$ or $>$) until a leaf is reached. At that point, the answer to the question 'Is $x \in V$?' is supplied by the leaf.

The *cost* of a decision tree is the height of the tree, i.e. the maximum number of tests made for any input. The (*linear decision tree*) *complexity* of the problem given by V is the minimum cost of any linear decision tree solving the problem, and is denoted by $C(V)$. If S is some subclass of linear functions, then the complexity of V over trees using test functions only from S, is denoted by $C_S(V)$. In particular, if W is some set of hyperplanes, the complexity of V over trees using linear test functions f such that f defines some hyperplane in W, is denoted by $C_W(V)$, too.

Ternary decision tree model has the convenient property that $C_V(V) \le m$ where m is the degree of V. In the *binary* decision tree model where each test can have only two-valued result ($>$, $\le$) this is not true.

Many well-known computational problems can easily be formulated as linear recognition problems. In particular, some important discrete combinatorial problems have more or less natural embeddings into R^n as linear recognition problems. We now list examples of such problems to be analyzed in more detail in Sections 3 and 4. The terminology is as in [3].

The NP-complete *partition problem* defined in the Introduction leads to a linear recognition problem in R^n given by the following union of 2^{n-1} hyperplanes

$$V_{P(n)} = \{(x_1, \cdots, x_n) \mid \sum_I x_i = \sum_{I'} x_i \text{ for disjoint } I, I' \text{ such that } I \cup I' = \{1, \cdots, n\}\}. \quad (1)$$

The *balanced partition problem* is the general partition problem restricted such that the index sets I and I' must be of equal size. Thus, given positive integers $x_1, \cdots, x_n$, where n is even, we must decide whether $\sum_I x_i = \sum_{I'} x_i$ for some index sets I and I' such that $I \cup I' = \{1, \cdots, n\}$ and $|I| = |I'|$. This problem clearly is in NP. Its NP-hardness is seen using a polynomial time transformation from the partition problem. If $(x_1, \cdots, x_n)$ is an instance of the partition problem then $(x_1+1, \cdots, x_n+1, y_1, \cdots, y_n)$ where $y_i=1$ for $i=1, \cdots, n$, is an instance of the balanced partition problem which has a solution if and only if $(x_1, \cdots, x_n)$ has a solution as an instance of the partition problem.

For each even n, the linear recognition problem in R^n which corresponds to the balanced partition problem is given by

$$V_{BP(n)} = \{(x_1, \cdots, x_n) \mid \begin{array}{l} \sum_I x_i = \sum_{I'} x_i \text{ for disjoint } I, I' \text{ such that} \\ I \cup I' = \{1, \cdots, n\} \text{ and } |I| = |I'|\}. \end{array} \quad (2)$$

Set $V_{BP(n)}$ consists of $\binom{n}{n/2}/2$ hyperplanes. Moreover, $V_{BP(n)} \subset V_{P(n)}$.

Also the *satisfiability problem* of boolean formulas can be embedded into a linear recognition problem in R^n. Let U be a finite set of boolean variables and C a collection of clauses (i.e., sets containing some variables or their negations) over U. The problem is to determine whether there is a satisfying truth assignment in U for C. Let $|U|=m$ and $|C|=k$, and assume that no clause contains both a variable and its negation. Then this instance of the problem is encoded as an $k \times m$ -matrix $x=(x_{ij})$ over values -1, 0, 1. The rows of the matrix correspond to the clauses. The elements of the ith row correspond to the elements of the ith clause c_i such that

$$x_{ij} = \begin{cases} 1, \text{ if clause } c_i \text{ contains the } j\text{th variable } u_j, \\ -1, \text{ if clause } c_i \text{ contains the negation } \bar{u}_j \text{ of } u_j \\ 0, \text{ otherwise.} \end{cases}$$

Note that this transformation is a part of the transformation used in [4] to establish the NP-hardness of 0-1 integer programming. If x is interpreted as a point in R^{km}, formula C is satisfiable if and only if x belongs to some of hyperplanes defined by equations

$$a_{j_1} x_{1j_1} + a_{j_2} x_{2j_2} + \cdots + a_{j_k} x_{kj_k} = k \quad (3)$$

where $j_1, j_2, \cdots, j_k$ are in $\{1, \cdots, m\}$ and $a_{j_1}, \cdots, a_{j_k}$ (truth values) are in $\{-1, 1\}$. So we could associate with the satisfiability problem the linear recognition problem that is given by hyperplanes in R^{km} defined by equations (3). To get a simpler formulation we prefer to replace the constant k on the right hand side of (3) by a new free variable x_{km+1}. In this way we obtain for the satisfiability problem of k clauses over m variables an embedding into a linear recognition problem in R^{km+1} which is given by the set

$$V_{SAT(m,k)} = \{(x_1, \cdots, x_{km+1}) \mid a_{j_1}x_{1j_1} + a_{j_2}x_{2j_2} + \cdots + a_{j_k}x_{kj_k} = x_{km+1}\} \quad (4)$$

where $j_1, j_2, \cdots, j_k$ are in $\{1, \cdots, m\}$ and $a_{j_1}, \cdots, a_{j_k}$ are in $\{-1,1\}$. The degree of $V_{SAT(m,k)}$ is between $2m^k$ and $2^m m^k$.

The *subset sum problem* is, given positive integers $x_1, \cdots, x_n$, to decide whether there is index set $I \subseteq \{1, \cdots, n-1\}$ such that $\sum_I x_i = x_n$. Also this problem is NP-complete. For each n, the corresponding linear recognition problem in R^n is given by (c.f. [1,2,8])

$$V_{S(n)} = \{(x_1, \cdots, x_n) \mid \sum_I x_i = x_n \text{ for some } I \subseteq \{1, \cdots, n-1\}\}. \quad (5)$$

Set $V_{S(n)}$ consists of 2^{n-1} hyperplanes.

Some NP-complete problems on graphs can be formulated as linear recognition problems. For example, the *Hamiltonian circuit problem* is to decide, given an undirected graph, whether it contains a Hamiltonian circuit. A graph over n vertices has a natural encoding x as an upper triangular $n \times n$ matrix over elements 0,1. Thus x can be interpreted as a point in R^m where $m = n(n-1)/2$. The graph represented by x has a hamiltonian circuit if and only if $\sum_I x_i = n$ for some index set I such that the edges specified by I form a Hamiltonian circuit. We again replace the constant n by a new variable. So we obtain, for each $m = n(n-1)/2$ and $n = 1,2, \cdots$, a linear recognition problem in R^{m+1} defined by

$$V_{H(n)} = \{(x_1, \cdots, x_{m+1}) \mid \sum_I x_i = x_{m+1}\} \quad (6)$$

where $I \subset \{1, \cdots, m\}$ ranges over all index sets that specify a Hamiltonian circuit in the complete graph over n vertices. The degree of $V_{H(n)}$ is $(n-1)!$. Moreover, $V_{H(n)} \subset V_{S(n(n-1)/2+1)}$. Also note that the problem given by $V_{H(n)}$ can be understood as the linear recognition problem associated with the (undirected) *traveling salesman problem*. Further examples of subsets of $V_{S(n(n-1)/2+1)}$ that correspond to NP-complete graph problems can be obtained in a similar way from, say, the *clique problem* and the *degree constrained spanning tree problem* (definitions of these problems can be found in [3]).

3. Lower bounds on $C_V(V)$

Let T be a decision tree that solves the linear recognition problem given by $V \subset R^n$. For each leaf v of T, we denote by $I(v) \subseteq R^n$ the set of those inputs that reach the leaf v. Sets $I(v)$ constitute a partition of R^n. In particular, sets $I(v)$ where v is a 'yes' leaf partition set V and sets $I(v)$ where v is a 'no' leaf partition set $-V$, the complement of V in R^n. Because each $I(v)$ is a polyhedron we get the following simple observation.

Lemma 1. *If T has a leaf v such that the set of inputs $I(v) \subseteq R^n$ has d different $n-1$-dimensional faces then the height of T is at least d.*

Proof. Immediate by contradiction. ∎

Thus each $I(v)$ may have only a polynomial number of $n-1$-dimensional faces if the complexity of T is a polynomial in n. Assume, in particular, that T contains only tests '$f(x)>0$' where f defines some hyperplane in V. For such a tree T, if v is a 'no' leaf, set $I(v)$ must be equal to some component of $-V$. So we arrive at the following principle of proving lower bounds on $C_V(V)$, that is, on the decision tree complexity restricted to trees that can test only those linear functions which define the problem.

Theorem 2. *If some component of $-V$ has m different $n-1$-dimensional faces,* *then* $C_V(V) \geq m$. ∎

We will now use this theorem in proving lower bounds when V is any of our example sets defined by (1), (2), (4)-(6) in Section 2. All bounds will be larger than polynomial in the dimension of V. These results can be established using surprisingly simple proof techniques. Assume that a point y in $-V$ is given. If we can find a path in $-V$ from y to a point x such that x belongs to exactly one hyperplane H of V, then we know that the component of $-V$ that contains y must have an $n-1$-dimensional face $\subseteq H$. If m different hyperplanes H can be found, then the number of faces must be $\geq m$.

First we study the structure of $-V_{P(n)}$ where $V_{P(n)}$ was defined by (1).

Theorem 3. (a) *Set* $-V_{P(n)} \subset R^n$ *has a component with at least* $\binom{n}{\lfloor n/2 \rfloor}/2$ *different* $n-1$-*dimensional faces.*

(b) *Set* $-V_{P(n)}$ *has at least* $\binom{n}{\lfloor n/2 \rfloor -1}$ *components such that each of them has at least* $\binom{\lfloor n/2 \rfloor -1}{\lfloor n/4 \rfloor -1}$ *different* $n-1$-*dimensional faces.*

Proof. (a) Let $y=(1/2,1,\cdots,1) \in R^n$. The first coordinate of y equals $1/2$ and the others equal 1. Point y is in $-V_{P(n)}$ because only one coordinate value of y is not an integer. Choose a set $I \subset \{1,\cdots,n\}$ such that $1 \in I$ and $|I| = \lfloor n/2 \rfloor$. We show that for each such I, point y can be continuously transformed in $-V_{P(n)}$ to a point $x=(x_1,\cdots,x_n)$ satisfying equation

$$\sum_I x_i = \sum_{-I} x_i. \tag{7}$$

At y, difference $\sum_I y_i - \sum_{-I} y_i$ equals $1/2$ when n is odd and $-1/2$ when n is even. Clearly, the absolute value $1/2$ is the the smallest possible for any choice of I.

To find a path from y to the hyperplane defined by (7), replace in y every coordinate y_i where $i \in I$ by value $y_i - t$ if n is odd and by $y_i + t$ if n is even. The other coordinates remain unchanged. Denote by $y(t)$ the vector obtained. Thus $y(0)=y$. A simple case analysis shows that as long as $0 \le t < 1/(2\lceil n/2 \rceil)$, point $y(t)$ is not in $V_{P(n)}$. Point $x = y(1/(2\lceil n/2 \rceil))$, however, satisfies (7) but no other equations defining some hyperplane in $V_{P(n)}$. Hence x belongs to exactly one hyperplane in $V_{P(n)}$.

The proof is complete because set I in (7) can be chosen in at least $\binom{n}{n/2}/2$ different ways and each selection corresponds to a different hyperplane. Each of these hyperplanes forms an $n-1$-dimensional face of the component of $-V_{P(n)}$ which contains point y. Finally note that when n is even, $|I| = |-I| = n/2$. Therefore the hyperplanes selected in (7) constitute $V_{BP(n)}$, the set corresponding to the balanced partition problem.

(b) Choose $y = (y_1, \cdots, y_n)$ such that $y_i = 1$ for $2k+1$ different indexes i and $y_i = 0$ for the rest of i. Integer k is such that $2k+1$ is the largest odd integer which is $\le n/2$. Then y must be in $-V_{P(n)}$ since the sum of the y_i's is an odd number $2k+1$. There are $\ge \binom{n}{n/2-1}$ different points y. They all belong to different components of $-V_{P(n)}$ because the line segment joining any pair of such points necessarily intersects $V_{P(n)}$. We partition the 1's in y into two disjoint groups of sizes k and $k+1$. Let I_1 and I_2 be the corresponding sets of indexes. There are $\binom{2k+1}{k} \ge \binom{n/2-1}{n/4-1}$ different partitions. We will see that each of them corresponds to an $n-1$-dimensional face of the component containing y. This face is contained in the hyperplane defined by equation $\sum_I x_i = \sum_{-I} x_i$ where I contains I_1 and all indexes i such that $y_i = 0$. The hyperplane is accessed from y along the path where all 0's as well as the k 1's whose indexes are in I_1 are simultaneously increased by t. The path intersects the chosen hyperplane (but no other hyperplanes) when $t=1/(k+k')$ where $k'=n-(2k+1)$ is the number of 0's. ■

Corollary 4. $C_{V_{P(n)}}(V_{P(n)}) \geq \binom{n}{n/2}/2$. ■

Next we consider the balanced partition problem and the associated set $V_{BP(n)}$ as defined by (2). As noted in the proof of Theorem 3 (a), if n is even, all the $\binom{n}{n/2}$ different $n-1$-dimensional faces of a component of $-V_{P(n)}$ which were used in proving the lower bound are also contained in $V_{BP(n)}$. But this means, since $V_{BP(n)} \subset V_{P(n)}$, that $-V_{BP(n)}$ must have a component whose faces contain all these faces. Then we have shown:

Theorem 5. *Set $-V_{BP(n)} \subset R^n$ has a component with $\binom{n}{n/2}/2$ different $n-1$-dimensional faces.* ■

Set $-V_{BP(n)}$ has a component which has as many $n-1$-dimensional faces as $V_{BP(n)}$ has hyperplanes. This suggests the following definition.

Let V be a union of some hyperplanes. If $-V$ has a component A such that $\bar{A} \cap H$ is an $n-l$-dimensional face of A for every hyperplane $H \subset V$, then we call V as well as the associated linear recognition problem *face-complete*.

Clearly, if V is a face-complete set of degree m, then $C_V(V) \geq m$. On the other hand, $C_V(V)$ is always $\leq m$. Hence we have for face-complete V the stronger result that $C_V(V)=m$. Note that this conclusion is true only for ternary decision trees. In the binary tree model we can only conclude that m is a lower bound on $C_V(V)$.

Since $V_{BP(n)}$ is face-complete, we obtain:

Corollary 6. $C_{V_{BP(n)}}(V_{BP(n)}) = \binom{n}{n/2}/2$. ■

We also immediately obtain the following lemma:

Lemma 7. *Let V and V' be unions of hyperplanes such that V is face-complete, $V' \subseteq V$, and the degree of V' is m'.*

(a) *V' is face-complete.*

(b) $C_{V'}(V') = m'$. ■

The next set to be considered is $V_{S(n)}$ which was defined in (5) on the basis of the subset sum problem.

Theorem 8. *Set $-V_{S(n)} \subset R^n$ has a component with 2^{n-1} different $n-1$-dimensional faces. Set $V_{S(n)}$ is therefore face-complete.*

Proof. Consider the faces of the component of $-V_{S(n)}$ that contains point $y=(-1,\cdots,-1,1)$. The last coordinate of y equals 1 while the others equal -1. Choose a non-empty set $I \subseteq \{1,\cdots,n-1\}$. To find a path to a point $x=(x_1,\cdots,x_n)$ satisfying

$$\sum_I x_i = x_n \tag{8}$$

we let $y(t)$ denote a point obtained from y by replacing all elements y_i, where $i \in I$, by y_i+t. Then $y(t)$ is not in $V_{S(n)}$ as long as $0 \le t < 1+1/|I|$. However, $x=y(1+1/|I|)$ satisfies exactly (8) of the equations for $V_{S(n)}$.

The hyperplane corresponding to empty I is reached from y along the line on which the last coordinate decreases from 1 to 0 (thus we define the empty sum to be equal to 0).

So each of the 2^{n-1} hyperplanes in $V_{S(n)}$ borders the component of y. ■

In (6) we defined sets $V_{H(n)}$ on the basis of the Hamiltonian circuit problem. Because $V_{H(n)} \subset V_{S(n(n-1)/2+1)}$, Lemma 7 and Theorem 8 imply that also $V_{H(n)}$ is face-complete.

Corollary 9. (a) $C_{V_{S(n)}}(V_{S(n)}) = 2^{n-1}$.

(b) $C_{V_{H(n)}}(V_{H(n)}) = (n-1)!$ ■

Corollary 9 (a) improves a result in [2] where it is proved, using an adversary argument, that $\binom{n-1}{(n-1)/2}$ is a lower bound on $C_{V_{S(n)}}(V_{S(n)})$. Our proof technique, besides giving the exact value of the complexity, also clearly reveals the reasons leading to it.

Set $-V_{SAT(m,k)}$, defined in (4), also has components with many faces.

Theorem 10. *Set $-V_{SAT(m,k)} \subset R^{km+1}$ has a component with at least m^k different $km+1$-dimensional faces.*

Proof. Consider the faces of the component of $-V_{SAT(m,k)}$ that contains point $y = (-1/(k+1), -1/(k+1), \cdots, -1/(k+1), 1)$. According to (4), every hyperplane in $V_{SAT(m,k)}$ can be represented by an equation of the form

$$\sum_I x_i - \sum_{I'} x_i = x_{mk+1} \tag{9}$$

where I, I' are disjoint subsets of $\{1, \cdots, mk\}$ such that $|I|+|I'| = k$. Point y cannot belong to any of such hyperplanes. In particular, $V_{SAT(m,k)}$ contains m^k hyperplanes for which I' is empty. Note that these hyperplanes form a subset of $V_{S(mk+1)}$. For every such hyperplane, replace by t the coordinates of y whose indices are in I. Denote by $y(t)$ the point obtained in this way. Thus

$y=y(-1/(k+1))$. Then $x=y(1/k)$ satisfies (9) but does not belong to any other hyperplanes in $V_{SAT(m,k)}$. When $-1/(k+1) \le t < 1/k$, point $y(t)$ is not in $V_{SAT(m,k)}$ (or in $V_{S(mk+1)}$). ∎

Corollary 11. $C_{V_{SAT(m,k)}}(V_{SAT(m,k)}) \ge m^k$. ∎

4. More general lower bounds

So far we have considered the decision tree complexity of linear recognition problems under the strong restriction that only testing of functions used in the definition of the problem is allowed in the tree. Now we shall try to relax this restriction. Most of our results will be based on the following theorem.

Theorem 12. *Let V, $V' \subset R^n$ be unions of some hyperplanes. If $-(V \cup V')$ has a component A with at least m different $n-1$-dimensional faces such that each face is contained in V, then $C_{V \cup V'}(V) \ge m$.*

Proof. Suppose that a decision tree T solves the problem given by V using tests functions that define only hyperplanes in $V \cup V'$. Then T must have a 'no' leaf v such that the associated set of inputs $I(v)$ contains A. Let $H \subset V$ be a hyperplane that contains some $n-1$-dimensional face of A. If H does not contain also an $n-1$-dimensional face of $I(v)$ then $I(v) \cap V$ is not empty which means that T cannot solve the problem given by V. Thus $I(v)$ must have at least m different $n-1$-dimensional faces, which proves the theorem. ∎

Corollary 13. *Let V, $V' \subset R^n$ be unions of hyperplanes such that V is face-complete, $V' \subset V$, and V' is of degree m. Then $C_V(V') = m$.* ∎

Hence increasing the set of test functions in such a way that the increased set of hyperplanes is face-complete, does not help in solving the original problem.

We now apply these results on our example problems.

Corollary 14. (a) $C_{V_{P(n)}}(V_{BP(n)}) = \binom{n}{n/2}/2$.

(b) $C_{V_{S(n)}}(V_{H(n)}) = (n-1)!$.

(c) *Let* $V_0 = V_{SAT(m,k)} \cup V_{S(km+1)}$. *Then* $C_{V_0}(V_{S(km+1)}) \geq m^k$ *and* $C_{V_0}(V_{SAT(m,k)}) \geq m^k$.

Proof. (a) Noting the proof of Theorem 3, this follows from Theorem 12.

(b) This follows from Corollary 13 because $V_{S(n)}$ is face-complete by Theorem 8.

(c) From the proof of Theorem 10 we see that $-(V_{SAT(m,k)} \cup V_{S(mk+1)})$ has a component that has at least m^k different $mk+1$-dimensional faces such that each face is contained in $V_{SAT(m,k)} \cap V_{S(mk+1)}$. The result then follows from Theorem 12.

■

Let $V_1 \subset R^n$ be the union of all hyperplanes that can be define using linear homogeneous linear functions with coefficients -1, 0, 1, i.e.,

$$V_1 = \{x = (x_1, \cdots, x_n) \mid \sum a_i x_i = 0 \text{ for some } a_i \in \{-1,0,1\}\}.$$

Note that in all our example problems the hyperplanes to be recognized are in V_1.

Since we already have Corollaries 4 and 14 (a), it is natural further to ask whether, say, $C_{V_1}(V_{P(n)})$ or $C_{V_1}(V_{BP(n)})$ has nonpolynomial lower bounds in n. We leave these questions open. Instead, we prove such results for some sets V' such that $V_{P(n)} \subset V' \subset V_1$.

Our main tool is the following lemma which generalizes the proof of Theorem 3. If $H \subset V_1$ is a hyperplane, then there is exactly two linear functions f

that define H and are of the form $f(x)=\sum a_i x_i$ with coefficients a_i in $\{-1,0,1\}$. Those functions are negations of each other. Hence we may uniquely define the *distance* between a point $y \in R^n$ and H as $\mathrm{dist}(y,H) = |f(y)|$.

Lemma 15. *Let V', $V_{P(n)} \subset V' \subset V_1$, be an union of hyperplanes and $y \in R^n$ a point such that for some $c>0$ and for all hyperplanes $H \subset V'$, $\mathrm{dist}(y,H) \geq c$. If there exist m different hyperplanes $H \subset V_{P(n)}$ such that $\mathrm{dist}(y,H)=c$, then $C_{V'}(V_{P(n)}) \geq m$.*

Proof. For each hyperplane $H \subset V_{P(n)}$ such that $\mathrm{dist}(y,H)=c$ we describe a path from y to a point x such that $x \in H$ but the path does not intersect any other hyperplanes in V'. Condition $\mathrm{dist}(y,H)=c$ means that there is set $I \subset \{1, \cdots, n\}$ such that

$$\sum_I y_i - \sum_{-I} y_i = c.$$

Denote by $y(t)$ the point obtained by replacing every y_i by $y_i - t$ if $i \in I$ and by $y_i + t$ otherwise. Hence $y = y(0)$. As long as $0 \leq t < c/n$, point $y(t)$ is not in V' because $\sum |y_i(t) - y_i| = nt < c$ and $\mathrm{dist}(y,H') \geq c$ for all hyperplanes $H' \subset V'$. However, point $y(c/n)$ is in H. A simple case analysis shows that $y(c/n)$ cannot belong to any other hyperplanes in V'. The theorem now follows from Theorem 12. ■

It can be immediately seen that Lemma 15 is still true if we replace $V_{P(n)}$ by $V_{BP(n)}$.

Let now V' be a subset of V_1 consisting of hyperplanes $\{x \mid \sum a_i x_i = 0, a_1 = 1\} \cap V_1$. Hence each hyperplane in V' has a linear function f representing it such that f has coefficients in $\{-1,0,1\}$ and the first coefficient is 1.

Theorem 16. $C_{V'}(V_{P(n)}) \geq \binom{n}{n/2}/2$.

Proof. Let $y=(1/2,1,\cdots,1)\in R^n$. Since $y_1=1/2$ is the only coordinate of y that is not an integer, the definition of V' ensures that $\mathrm{dist}(x,H)\geq 1/2$ for every $H\subset V'$. On the other hand, $\mathrm{dist}(x,H)=1/2$ for all $H\subset V_{P(n)}$ such that $H=\{x\,|\sum_I x_i=\sum_{-I} x_i\}$ where $I\subset\{1,\cdots,n\}$ and $1\in I$. By Lemma 15, this completes the proof because I can be chosen in at least $\binom{n}{n/2}/2$ different ways. ∎

The result of Theorem 16 can be generalized as follows. Let $k\geq 2$ be a fixed integer and let $A\subset\{1,\cdots,n\}$ be a set such that $|A|=\lfloor(1+1/k)n\rfloor$. Furthermore, let V'', $V''\subset V_1$, denote the set of hyperplanes $\{x\,|\sum a_i x_i=0,\ a_i=1 \text{ if } i\in A\}$.

Theorem 17. $C_{V''}(V_{P(n)}) \geq \binom{n/k}{n/3k}$.

Proof. Assume for simplicity that $n=3km$ for some integer m. We prove the theorem for such n; the remaining n are left to the reader. Since $n=3km$, we have $|A|=3(k-1)m$. Because of symmetry, we may assume that $A=\{1,\cdots,r\}$ where $r=3(k-1)m$.

Let $y=(y_1,\cdots,y_n)$ where $y_i=1+2^{-i}$ for $i=1,\cdots,r-1$, $y_r=2^{-r}$, and $y^i=3(k-1)$ for $i>r$. Then $\mathrm{dist}(y,H)\neq 0$ for all H in V''. In fact, $\mathrm{dist}(y,H)$ is always $\geq 2^{-r}$. This can be seen from the binary representation of the value $f(y)=\sum a_i y_i$ where f defines H and each a_i is in $\{-1,0,1\}$. Let i be the largest index such that $i\leq r$ and $a_i\neq 0$. The definition of V'' ensures that such an i always exists. Then the contribution of $y_i=1+2^{-i}$ (or $y_i=2^{-i}$, if $i=r$) to the value $f(y)$ cannot be cancelled, that is, the position $-i$ has digit 1 in the binary representation of $\mathrm{dist}(y,H)$.

Let $I \subset \{1, \cdots, n\}$ be such that $|I| = m$ and $I \cap A$ is empty. Then a straightforward calculation shows that

$$\sum_{I \cup A} y_i - \sum_{-(I \cup A)} y_i = -2^{-r}.$$

Thus the hyperplane H in $V_{P(n)}$ defined by function f such that $f(n) = \sum_{I \cup A} x_i - \sum_{-(I \cup A)} x_i$ has the property $\text{dist}(y, H) = 2^{-r}$. By Lemma 15, this completes the proof because I can ce chosen in $\binom{3m}{m} = \binom{n/k}{n/3k}$ different ways. ∎

5. Conclusion

We developed in this paper a technique for proving lower bounds on the decision tree complexity of linear recognition problems. Tihs gave non-polynomial lower bounds for some np-complete problems in the linear decision tree model when the decisions are restricted to testing functions that are used in defining the problem. Similar bounds for some larger classes of test functions were also obtained although the problem of proving non-polynomial lower bounds when unrestricted linear tests are allowed, still remains open.

It is also noteworthy that our results were obtained without using adversary arguments. One easily sees, in fact, that all our lower bounds for problems given by different sets V are also lower bounds on the *width of complete proofs* of V as defined by Rabin [5], if the complete proofs may contain test functions from the same classes as our restricted linear decision trees.

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