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THE PROBLEM OF FUZZY RELATION DECOMPOSITION

by

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The Problem of Fuzzy Relation Decomposition*

Some preliminary results

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Table of Contents

List of Figures

List of Tables

1. INTRODUCTION

The results presented in this paper are the outcome of a preliminary study into an important problem that arises when we try to construct and analyze fuzzy models of complex or ill-defined systems. Whilst a fuzzy model can be defined in many ways, we use the term to refer to a set of fuzzy rules that together form an algorithmic description of system behavior. Any set of rules is easily transformed into a fuzzy relation (under some suitable assumptions about fuzzy implication), but the reverse transformation is considerably more difficult. Indeed, this problem of "Relation Decomposition" turns out to be surprisingly intractable in its more general forms. The bulk of this paper is concerned with a description of the principal issues, and the attempts we have made to deal with them.

The remainder of this introductory section includes a more detailed description of the problem, provides some basic definitions, and indicates two general strategies for uncovering rule-based decompositions of an arbitrary fuzzy relation. Section 2 describes our first solution technique, "Iterative Decomposition", and Section 3 describes our second technique, " α - Decomposition". Finally, Section 4 discusses the properties of the set of solutions to tne decomposition problem, considers future directions for this research and attempts to place our results in the framework of a more general fuzzy modeling methodology.

1.1 FUZZY MODELS

We view a fuzzy model as a collection of fuzzy rules that together form an algorithmic description of system behavior. Thus if a system has an input space $\mathcal{X}$ and an output space $\mathcal{Y}$, each rule is a statement about the relationship between $\mathcal{X}$ and $\mathcal{Y}$. More formally, a rule is denoted by $X_i \Rightarrow Y_i$, where X_i is a fuzzy subset of $\mathcal{X}$ and Y_i is a fuzzy subset of $\mathcal{Y}$. Every rule can be translated into a binary fuzzy relation, R_i on

$\mathcal{X} \times \mathcal{Y}$, under some suitable definition of implication. We have chosen the outer-product form described by Zadeh [5], partly because of its importance in most applications of fuzzy set theory to control system design (see Tong [3] for a review), and partly because we feel that in many contexts it is unreasonable for the modeler to infer anything at all if the antecedent to the implication is not true. Thus we have

$$R_i(x,y) \triangleq X_i(x) \wedge Y_i(y) \tag{1.1}$$

where $\wedge$ denotes the infimum (or minimum) operator.

The overall behavior of the system can be determined by firstly forming an aggregate of the rules and then secondly by defining a fuzzy rule of inference. The system relation R, is defined by

$$R(x,y) \triangleq \bigvee_{i=1}^{N} R_i(x,y) \tag{1.2}$$

where N is the number of rules and $\vee$ denotes the supremum (or maximum) operator. Then for an arbitrary input to the system, denoted X, we can compute the corresponding output Y from

$$Y(y) = \bigvee_{x} (X(x) \wedge R(x,y)) \tag{1.3}$$

We often write this more compactly as $Y = X \circ R$, and illustrate the concept in Figure 1-1.

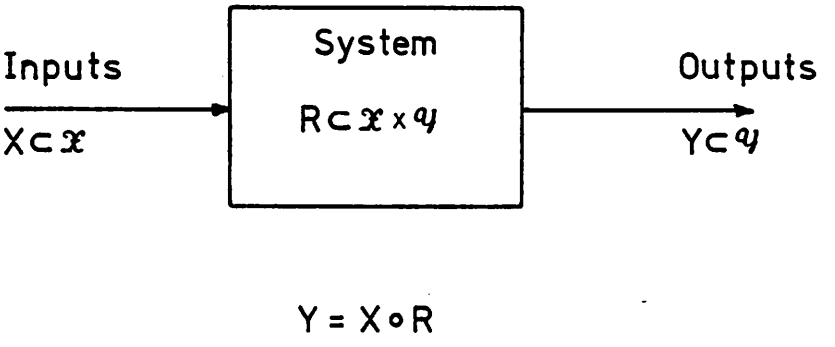


Figure 1-1: Fuzzy Relational Models

1.2 DECOMPOSITION

So far, we have shown that a system may be described by a relation, R, which is the union of several individual rules relating input and output conditions. It is clearly trivial to proceed from the set of R_i to R, but to reverse the process and move from R to a set of R_i is a much more complex and subtle problem. We call this reverse process Relation Decomposition. Before indicating two approaches to the solution, we will briefly discuss our motivation for attempting the problem.

Given a relation, R, there are a number of reasons why one might want to obtain a rule-based description. In the decision-making context, a relation may have been built up over a series of interviews, providing many rules. We could use Relation Decomposition to look for a more compact representation of the decision-maker's algorithm, possibly to check for any inconsistency. In the controller design context, we may have synthesized a

controller by solving a fuzzy relational equation and may desire a rule-based description rather than a relational one. A third possibility would be that we have constructed a fuzzy relation by observing the input-output behavior of the system and wish to transform it into a rule-based description of system behavior.

Relation Decomposition is thus a very powerful tool both for model assessment and model construction. We have divided the possible solutions to this problem into three classes: exact minimum rule solutions, term set matching solutions and approximate solutions. This is illustrated in Table 1-1. By exact minimum rule solutions, we mean those solutions with the smallest number of rules required to decompose R exactly. Term set matching solutions are those in which the antecedents and consequents of the rules are required to match elements of some predetermined term set. Approximate solutions are those which decompose R approximately.

We have confined our attention in this report to the first class of solutions. However, we believe our techniques to be applicable to all the problems although we have not fully explored the changes and generalizations that would be required. We are inclined to the view that approximate solutions might eventually be the most practical, but realize that an understanding of the exact minimum rule decomposition is a useful introduction to the complexity of the problem.

1.3 TWO APPROACHES TO THE PROBLEM

We recall that the relation R is the union of several rules, and that each rule is the cartesian product of a pair of input and output conditions. This suggests two approaches to the problem.

In the first approach, we observe that by combining Equations (1.1) and (1.2) we obtain

Classes of solution	#	Reasons for choosing	Comments
Exact minimum rule decomposition	N	Original rules exist, but alternative, possibly more compact, set required. Original rules "lost". Model altered through subsequent modification and is to be compared with original. No previous description of R exists.	Essentially mathematical.
Term set matching	>N	For training new process operators. For comparison with original linguistic rules.	Linguistic interpretation.
Approximate solutions	<N	Many rules required for an exact decomposition, several of which contribute very little to the overall model. Data of questionable accuracy.	Most useful in practice?

Table 1-1: Classes of Decomposition

$$R(x,y) = \bigvee_{i=1}^{N} X_i(x) \wedge Y_i(y) \tag{1.4}$$

Now, if we rewrite the i^{th} input condition, X_i, as the i^{th} column of a relation A, and the i^{th} output condition as the i^{th} row of a relation B, then

$$\begin{aligned} A(x,i) &\triangleq X_i(x) \\ B(i,y) &\triangleq Y_i(y) \end{aligned} \tag{1.5}$$

substituting into Equation (1.4) we obtain

$$R(x,y) = \bigvee_{i=1}^{N} A(x,i) \wedge B(i,y) \tag{1.6}$$

which is simply the composition of A and B. Therefore

$$R = A \circ B \tag{1.7}$$

The problem may thus be re-stated as follows: "Given a finite discrete relation R, find N, where N gives the number of rules in the solution, and a finite discrete relation pair (A,B), such that $A \circ B = R$".

We have developed an iterative technique for solving $A \circ B = R$, given only R. It will find N, where N is the minimum number of rules necessary to define R, and provide the complete set of solutions in this minimum rule space. It can also find a partial set of solutions in higher order spaces, involving more than N rules. This technique will be described in the next section.

Let us now turn to the second approach. In the first approach, we

proceed from R to an immediate listing of all the rules with their antecedents and consequents. The second approach involves two stages. Firstly, we break R down into its constituent R_i (Equation (1.2)) and then provide a range of solutions for each R_i (Equation (1.3)). This technique is much less easy to express mathematically than the iterative technique, because it requires the analyst to examine the α-levels which constitute R and we are not aware of a convenient and powerful mathematical notation to express the complexity involved. However, this technique is successful in revealing parts of the set of solutions which cannot be obtained under the iterative scheme. This α-level technique will be discussed in the third section of the paper.

In this section of the paper, we have introduced the problem of Relation Decomposition. We have indicated our motivation for attacking this problem, believing that a successful technique for solving it would be a widely useful tool in both control and decision-making contexts. There are several classes of solution to the decomposition problem, but we will be restricting ourselves to the problem of finding exact decompositions. We have developed two techniques for solving this problem which will be discussed in the next sections of the paper. The first is an iterative technique based on a direct approach to the problem and the second technique involves studying the α-levels of R , to produce decomposition indirectly.

2. THE ITERATIVE TECHNIQUE

In the first section, we showed that the decomposition problem could be restated as "Solve $A \circ B = R$ when only R is known and the number of rules is unknown". This is similar to some previous relational problems that have appeared in the literature and we have utilized these earlier results in our approach. (A preliminary discussion of the problem is given in Tong and Efstathiou [4]). Sanchez [2] has shown how given R and A(or B), one may find out if a B(or A) exists, which will solve $A \circ B = R$ and, furthermore, one may generate the maximum B(or A) such that $A \circ B = R$. Pappis and Sugeno [1] have shown mathematically how to find the set of minimal solutions, given R and A (or B). Observe that given R and A (or B), the set of possible solutions form an upper semi-lattice, i.e. one maximum and many mimina. For a statement of the relevant results from Sanchez, and Pappis and Sugeno, see Appendix A.

In this section, we will describe our iterative technique for solving the decomposition problem and illustrate it with an example. We shall indicate some general results as well as suggesting some conjectures which may or may not withstand analytic testing. We shall also demonstrate the shortcomings of this method.

2.1 GENERAL AIMS

It has been shown [2] that given R and A(or B), the set of solutions for B(or A) will form an upper semi-lattice. The dimensions of B(or A) will be uniquely determined.

Turning to the problem when R only is known, we seek for each N, a set of solutions for both A and B. Denote these as $\mathcal{A}_N$ and $\mathcal{B}_N$. We are seeking the set $\mathcal{A}_N$ so that any element of it, A, will be guaranteed a B such that $A \circ B = R$. . Similarly for $\mathcal{B}_N$. This may be stated:

$\forall A \in \mathcal{A}_N, \ \exists B : A \circ B = R$

and

$\forall B \in \mathcal{B}_N, \ \exists A : A \circ B = R$

Since N is not uniquely determined, we suppose that there is a minimum value for N for which $\mathcal{A}_N$ and $\mathcal{B}_N$ exist. Denote those by $\mathcal{A}_{min}$ and $\mathcal{B}_{min}$. There will be sets $\mathcal{A}_N$ and $\mathcal{B}_N$ with $N > min$. The iterative solution seeks $\mathcal{A}_{min}$ and $\mathcal{B}_{min}$, the complete set of solutions in the minimum rule space. It must also determine this minimum number of rules.

2.2 THE BASIC METHOD

The iterative method for finding $\mathcal{A}_{min}$ and $\mathcal{B}_{min}$ may be described in four steps. See Figure 2-1.

The crux of step one is, clearly, in picking the initial condition. These conditions must guarantee a solution, or the results of Sanchez, and Pappis and Sugeno, will not be useful. There are two obvious conditions which we may use

(i) set $A = R$ and $B = I_m$,
(ii) set $A = I_n$ and $B = R$

where I_1 is the 1×1 identity relation. In case (i), we are immediately in the $\mathcal{A}_m$ and $\mathcal{B}_m$ spaces. The free dimension has been set at m. Similarly, case (ii) puts us in the n-rule space. At this stage we have only one element in either $\mathcal{A}$ or $\mathcal{B}$. The next step allows us to expand this space.

Given an initial A, call it A_0, be it either R or I_n, we may use this with R to generate a B^1 and B_{1*}, where B^1 denotes the maximum and B_{1*} denotes the set of minima generated after one calculation. We now have a range of elements in $\mathcal{B}_{N^\circ}$, where N° is the dimension of the space. These B may be used to expand $\mathcal{A}$ from A_0. B^1 may be used to generate

1. Pick an initial A_0 and B_0 such that $A_0 \in \mathcal{A}_N$, and $B_0 \in \mathcal{B}_N$.

2. Iteratively expand $\mathcal{A}_N$ and $\mathcal{B}_N$.

3. Check for smaller rule solutions. If detected, restart iteration in lower rule space, $\mathcal{A}_{N-}$ and $\mathcal{B}_{N-}$.

4. Continue until no new elements are generated.

Figure 2-1: The Iterative Scheme

A_{2*} and B_{1*} will generate more A^{2*}. Repeating this cycle will extend the ranges of $\mathcal{A}$ and $\mathcal{B}$ until no new elements are generated. Recall that we may start from $A_0 = R$ or $B_0 = R$. Both halves of the problem must be investigated to obtain a complete picture of $\mathcal{A}$ and $\mathcal{B}$. See Figure 2-2.

We are seeking a minimum rule solution but do not know if the space in which we are working represents a minimum rule solution or not. We may move into a lower rule space if one or more of the rules generated by existing (A,B) pairs is redundant. Recall that each rule is given by the cartesian product of a column from A and the corresponding row from B.

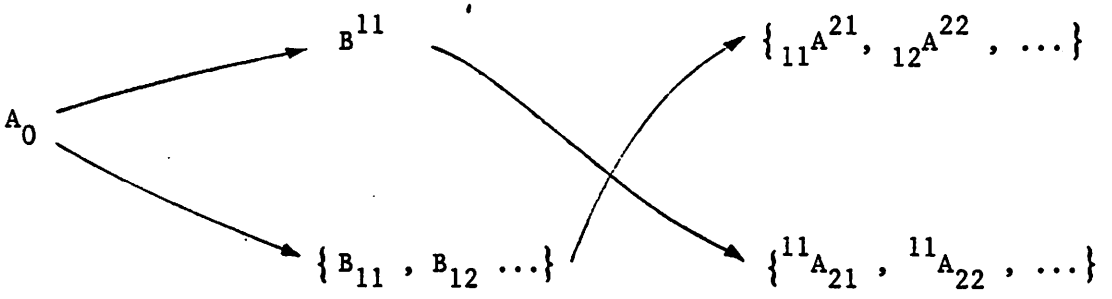
$$R_i = A_{*i} \wedge B_{i*}$$

A rule is redundant if it is a subset of another rule. This is obvious both because R is formed from the union of the R_i and because such a rule's input and output conditions would be supplied by another rule. Hence, we may say that a smaller rule space may be used if there exists a p and a q such that

$$a_{ip} \leq a_{iq} \ , \ \forall i \qquad \text{and} \qquad b_{pj} \leq b_{qj} \ , \ \forall j$$

Then column p may be deleted from A and row p may be deleted from B. Redundant rules are usually easy to spot in practice because a column consisting entirely of zeros appears in a minimum A or a row of zeros appears in a minimum B.

Once the smaller rule space has been detected we use the reduced A (or B) as an initial condition in $\mathcal{A}_{N-}$ (or $\mathcal{B}_{N-}$), which is then iteratively expanded, again watching for redundant rules. The process stops when no new elements are found, i.e., the maximum and minimum elements regenerate their parents.



Notation: right index in upper position means generated as a maximum,
right index in lower position means generated as a minimum,
first right digit indicates number of times operation has been performed,
second right digit is an identifier;

left index in upper position means parent was a maximum,
left index in lower position means parent was a minimum,
first left digit should be one less than the first right digit,
second left digit identifies the parent.

Figure 2-2: Iterative Expansion of $\mathcal{A}_N$ and $\mathcal{B}_N$

2.3 EXAMPLE

For notation, see Figure 2-2. Suppose R is given as

$$R = \begin{matrix} .3 & .5 & .6 & 1.0 \\ .5 & .7 & .7 & .4 \\ .8 & .8 & 1.0 & .3 \\ .9 & .7 & .2 & .2 \\ 1.0 & .7 & .2 & 0 \end{matrix}$$

2.3.1 A_0=R, find B^1

$$B^1 = \begin{matrix} 1.0 & .7 & .2 & 0 \\ .3 & 1.0 & .2 & 0 \\ .3 & .5 & 1.0 & 0 \\ .3 & .5 & .6 & 1.0 \end{matrix} \qquad A_2 = \begin{matrix} 0 & 0 & 0 & 1.0 \\ .5 & .7 & .7 & .4 \\ .8 & .8 & 1.0 & .3 \\ .9 & 0 & 0 & .2 \\ 1.0 & 0 & 0 & 0 \end{matrix}$$

$$B^3 = \begin{matrix} 1.0 & .7 & .2 & 0 \\ .5 & 1.0 & 1.0 & .3 \\ .5 & .8 & 1.0 & .3 \\ .3 & .5 & .6 & 1.0 \end{matrix} \qquad A_4 = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .7 & 0 & .4 \\ 0 & 0 & .7 & .4 \\ .8 & 0 & 1.0 & 0 \\ .9 & .2 & 0 & 0 \\ .9 & 0 & .2 & 0 \\ .9 & 0 & 0 & .2 \\ 1.0 & 0 & 0 & 0 \end{matrix}$$

where "}" denotes a choice of rows

Here we have followed our standard practice of alternating between generating a maximum or minima. Only one minimum was generated at A_2, but a total of 6 were found at A_4. These will now be traced separately.

$$A_{41} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .7 & 0 & .4 \\ .8 & 0 & 1.0 & 0 \\ .9 & .2 & 0 & 0 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad {}_{41}B^{51} = \begin{matrix} 1.0 & .7 & .2 & 0 \\ .5 & 1.0 & 1.0 & .4 \\ .8 & .8 & 1.0 & .3 \\ .3 & .5 & .6 & 1.0 \end{matrix}$$

$$
{}^{51}A_6 = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .7 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ .9 & .2 & 0 & 0 \\ .9 & 0 & .2 & 0 \\ .9 & 0 & 0 & .2 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad B^7 = B^{51}
$$

Here, three minima occur at ${}^{51}A_6$. They all generate B^7 as their maximum, but it equals B^{51} and so the iteration halts.

$$
A_{42} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .7 & 0 & .4 \\ .8 & 0 & 1.0 & 0 \\ .9 & 0 & .2 & 0 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad {}_{42}B^{52} = {}_{41}B^{51}
$$

$$
A_{43} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .7 & 0 & .4 \\ .8 & 0 & 1.0 & 0 \\ .9 & 0 & 0 & .2 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad {}_{43}B^{53} = {}_{41}B^{51}
$$

We have the result that each of A_{41}, A_{42} and A_{43} generates B^{51} and that this maximum generates three more minima which are all smaller by two elements than A_{41}, A_{42} and A_{43}.

$$
A_{44} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & 0 & .7 & .4 \\ .8 & 0 & 1.0 & 0 \\ .9 & .2 & 0 & 0 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad B^{54} = \begin{matrix} 1.0 & .7 & .2 & 0 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ .5 & .8 & 1.0 & .3 \\ .3 & .5 & .6 & 1.0 \end{matrix}
$$

$$
{}^{54}A_6 = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .4 & .7 & 0 \\ 0 & 0 & .7 & .4 \\ .8 & 0 & 1.0 & 0 \\ .9 & .2 & 0 & 0 \\ .9 & 0 & .2 & 0 \\ .9 & 0 & 0 & .2 \\ 1.0 & 0 & 0 & 0 \end{matrix}
$$

At ${}^{54}A_6$ we find six minima. Three of these are three of those found at A_4, but the other three, which have a new second row, must be examined again.

$$
{}^{54}A_{61} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .4 & .7 & 0 \\ .8 & 0 & 1.0 & 0 \\ .9 & .2 & 0 & 0 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad B^{71} = B^{54} = \begin{matrix} 1.0 & .7 & .2 & 0 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ .5 & .8 & 1.0 & .3 \\ .3 & .5 & .6 & 1.0 \end{matrix}
$$

$$
{}^{54}A_{62} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .4 & .7 & 0 \\ .8 & 0 & 1.0 & 0 \\ .9 & 0 & .2 & 0 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad B^{72} = B^{71}
$$

$$
{}^{54}A_{63} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .4 & .7 & 0 \\ .8 & 0 & 1.0 & 0 \\ .9 & 0 & 0 & .2 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad B^{73} = B^{71}
$$

This gives us no new leads and indicates that these are the best minima. Let us return to our investigation of A_4.

$$A_{45} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & 0 & .7 & .4 \\ .8 & 0 & 1.0 & 0 \\ .9 & 0 & .2 & 0 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad B^{55} = \begin{matrix} 1.0 & .7 & .2 & 0 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ .5 & .8 & 1.0 & .3 \\ .3 & .5 & .6 & 1.0 \end{matrix}$$

$$^{55}A_6 = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & 0 & .7 & .4 \\ .8 & 0 & 1.0 & 0 \\ .9 & 0 & .2 & 0 \\ .9 & 0 & 0 & .2 \\ 1.0 & 0 & 0 & 0 \end{matrix}$$

$$A_{46} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & 0 & .7 & .4 \\ .8 & 0 & 1.0 & 0 \\ .9 & 0 & 0 & .2 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad B^{56} = B^{55}$$

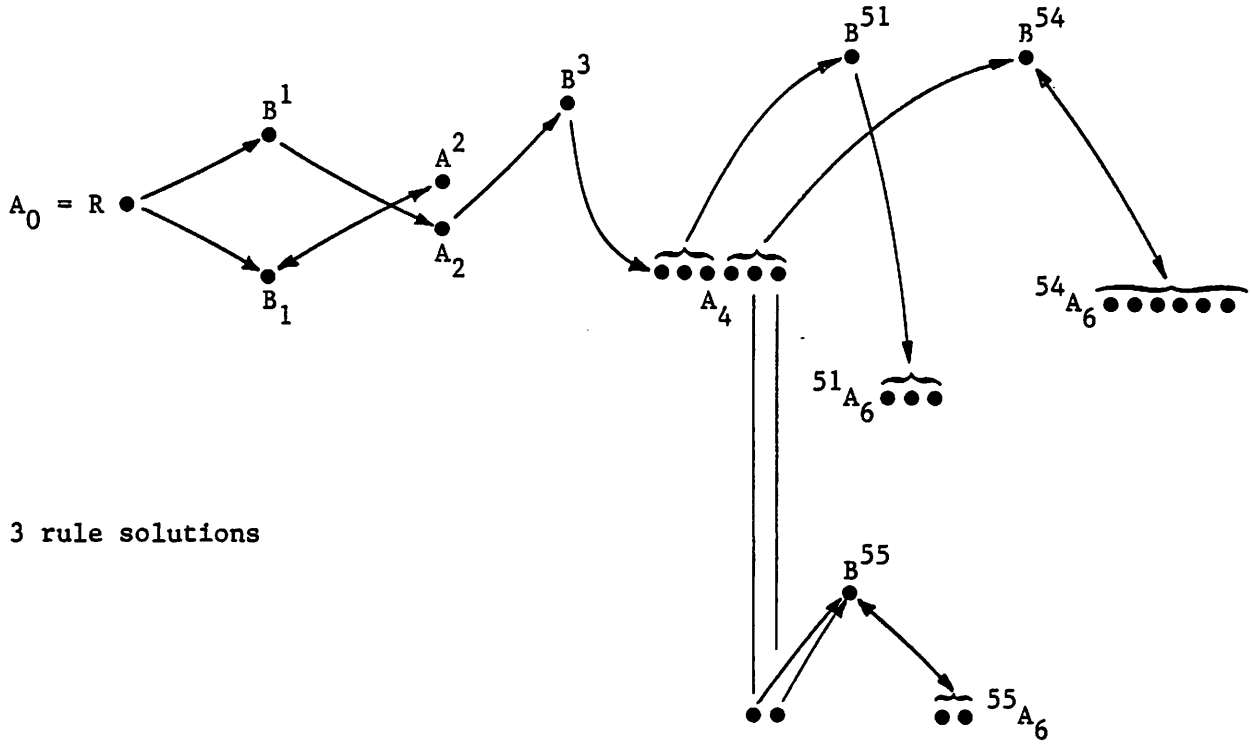
A_{45} and A_{46} both have a column of zeros which may be deleted, bringing us into $\mathcal{A}_3$ and $\mathcal{B}_3$; this cycle soon stops. For a complete map of this calculation, see Figure 2-3.

2.3.2 A_0=R, find B_1

$$B_1 = \begin{matrix} 1.0 & 0 & 0 & 0 \\ 0 & .8 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{matrix} \qquad A^2 = \begin{matrix} .3 & .5 & .6 & 1.0 \\ .5 & .7 & .7 & .4 \\ .8 & 1.0 & 1.0 & .3 \\ .9 & .7 & .2 & .2 \\ 1.0 & .7 & .2 & 0 \end{matrix}$$

$$B_3 = B_1$$

There is little of interest here. It is not surprising that seeking the minimum yields the so-called sub-identity, although other minima may sometimes be found. This has also been included in Figure 2-3.



Note: the relative heights of the spots indicates which are the greatest maxima or lowest minima.

Figure 2-3: Solution from $A_0 = R$

2.3.3 B_0=R, find A^1

$$A^1 = \begin{matrix} 1.0 & .3 & .3 & .3 & .3 \\ .4 & 1.0 & .5 & .5 & .5 \\ .3 & .3 & 1.0 & .8 & .8 \\ .2 & .2 & .2 & 1.0 & .9 \\ 0 & 0 & 0 & 0 & 1.0 \end{matrix} \qquad B_2 = \begin{matrix} 0 & .5 & .6 & 1.0 \\ 0 & .7 & .7 & 0 \\ 0 & .8 & 1.0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.0 & .7 & .2 & 0 \end{matrix}$$

$$A^3 = \begin{matrix} 1.0 & .5 & .5 & .3 \\ .4 & 1.0 & .7 & .5 \\ .3 & 1.0 & 1.0 & .8 \\ .2 & .2 & .2 & .9 \\ 0 & .2 & .2 & 1.0 \end{matrix} \qquad B_4 = \begin{matrix} 0 & 0 & .6 & 1.0 \\ 0 & 0 & 0 & 0 \\ 0 & .8 & 1.0 & 0 \\ 1.0 & .7 & 0 & 0 \end{matrix}$$

$$A^5 = \begin{matrix} 1.0 & .5 & .3 \\ .4 & .7 & .5 \\ .3 & 1.0 & .8 \\ .2 & .2 & .9 \\ 0 & .2 & 1.0 \end{matrix} \qquad B_6 = \begin{matrix} 0 & 0 & .6 & 1.0 \\ 0 & .8 & 1.0 & 0 \\ 1.0 & .7 & 0 & 0 \end{matrix}$$

This chain is most interesting. Each time a minimum B is calculated, only one exists and it has a redundant rule. Once the three rule solution is found, the cycle stops. We would not expect to find any new elements in the ranges of $\mathcal{A}_4$, $\mathcal{A}_5$, $\mathcal{B}_4$ or $\mathcal{B}_5$ had the column or row of 1's been retained. [Note that we have automatically deleted redundant rules from A(or B); we continue this practice in the remainder of the report].

2.3.4 B_0=R, find A_1

$$A_1 = \begin{matrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & .7 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & .9 & 0 \\ .2 & 0 & 0 & 0 & .9 \\ 0 & .2 & 0 & 0 & .9 \\ 0 & 0 & .2 & 0 & .9 \\ 0 & 0 & 0 & .2 & .9 \\ 0 & 0 & 0 & 0 & 1.0 \end{matrix}$$

Five minima appear at A.
We shall study each in turn..

$$A_{11} = \begin{matrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & .7 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & .9 & 0 \\ 0 & 0 & 0 & 0 & 1.0 \end{matrix} \qquad B^{21} = \begin{matrix} .3 & .5 & .6 & 1.0 \\ .5 & 1.0 & 1.0 & .4 \\ .8 & .8 & 1.0 & .3 \\ 1.0 & .7 & .2 & .2 \\ 1.0 & .7 & .2 & 0 \end{matrix}$$

$$^{21}A_3 = A_1$$

$$A_{12} = \begin{matrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & .7 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 \\ .2 & 0 & 0 & 0 & .9 \\ 0 & 0 & 0 & 0 & 1.0 \end{matrix} \qquad B^{22} = \begin{matrix} .3 & .5 & .6 & 1.0 \\ .5 & 1.0 & 1.0 & .4 \\ .8 & .8 & 1.0 & .3 \\ 1.0 & .7 & .2 & 0 \end{matrix}$$

$$A_{32} = \begin{matrix} 1.0 & 0 & 0 & 0 \\ 0 & .7 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ .2 & 0 & 0 & .9 \\ 0 & .2 & 0 & .9 \\ 0 & 0 & .2 & .9 \\ 0 & 0 & 0 & 1.0 \end{matrix}$$

$$A_{13} = \begin{matrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & .7 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 \\ 0 & .2 & 0 & 0 & .9 \\ 0 & 0 & 0 & 0 & 1.0 \end{matrix} \qquad B^{23} = B^{22}$$

$$A_{14} = \begin{matrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & .7 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & .2 & 0 & .9 \\ 0 & 0 & 0 & 0 & 1.0 \end{matrix} \qquad B^{24} = B^{22}$$

$$A_{15} = \begin{matrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & .7 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & .2 & .9 \\ 0 & 0 & 0 & 0 & 1.0 \end{matrix} \qquad B^{25} = \begin{matrix} .3 & .5 & .6 & 1.0 \\ .5 & 1.0 & 1.0 & .4 \\ .8 & .8 & 1.0 & .3 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & .7 & .2 & 0 \end{matrix}$$

$$A_{26} = \begin{matrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & .7 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 \\ .2 & 0 & 0 & 0 & .9 \\ 0 & .2 & 0 & 0 & .9 \\ 0 & 0 & .2 & 0 & .9 \\ 0 & 0 & 0 & .2 & .9 \\ 0 & 0 & 0 & 0 & 1.0 \end{matrix}$$

The most interesting thing to observe from this exercise is that it does not provide any 3-rule solutions at all. See Figure 2-4 for the map to this problem. Observe that in the 3-rule space, there is only one maximum A and one maximum B. We have only one minimum B, but <u>two</u> minimum A.

2.4 COMMENTARY

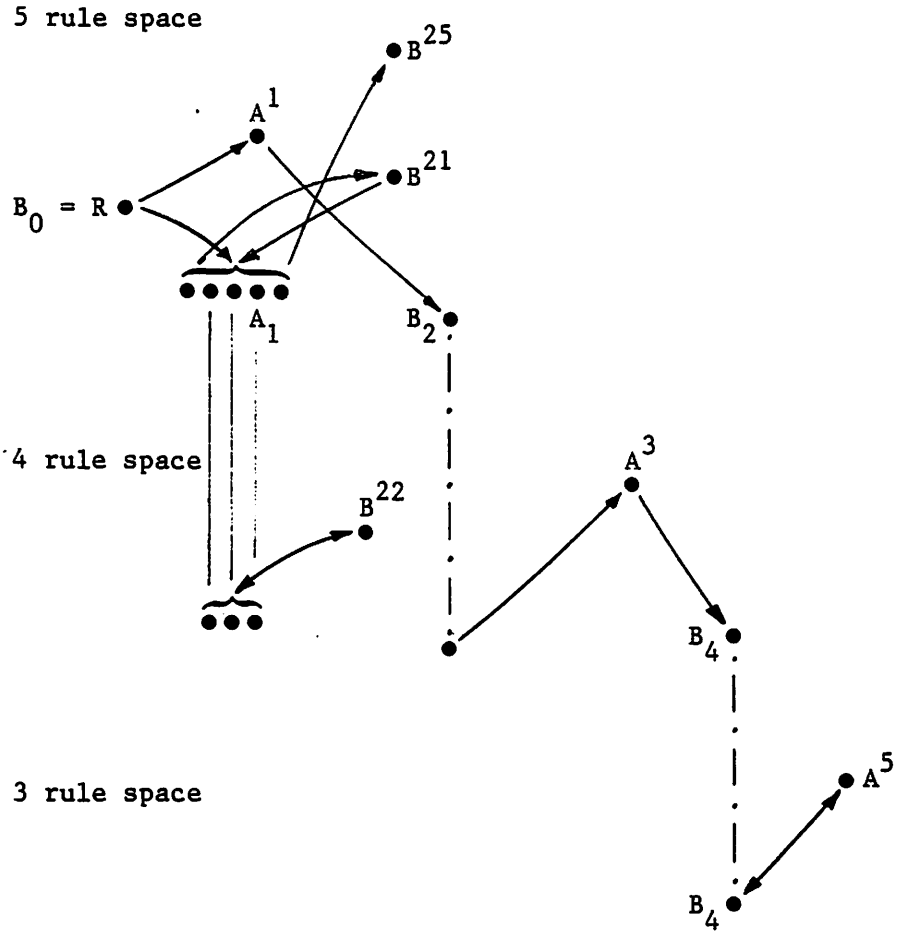
Despite our assertions about our iterative method, we have been unable to prove any significant global properties for the scheme. The things that we can prove are described in Appendix A, and in this section we will consider some questions that we hope will be the starting points for future research.

2.4.1 Shortcomings in Higher Rule Spaces

Our experimental experience leads us to believe that the iterative method can detect the complete set of solutions in the minimum rule space. It would clearly be useful if it could do the same in higher rule spaces. Unfortunately, this is not so. It seems that the initial conditions $A_0 = R$ or $B_0 = R$ impose some structure on the class of solutions which will be detected.

This may be illustrated using the example of Section 2.3. The range of $\mathcal{A}_4$ which was found, is as follows:

$A_0 = R$ yields one maximum A and nine minimum A, viz-



Note: in 5 rule space

$A_{11} \rightarrow B^{21} \rightarrow A_{11}$, A_{12} , A_{13} , A_{14} , A_{15}

A_{12} , A_{13} , $A_{14} \leftrightarrow B^{22}$

$A_5 \rightarrow B^{25} \rightarrow A_{12}$, A_{13} , A_{14} , A_{15}

Figure 2-4: Solution from $B_0 = R$

$$\hat{A}_a = \begin{matrix} .3 & .5 & .6 & 1.0 \\ .5 & .7 & .7 & .4 \\ .8 & 1.0 & 1.0 & .3 \\ .9 & .7 & .2 & .2 \\ 1.0 & .7 & .2 & 0 \end{matrix} \qquad \overset{\vee}{A}_a = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .7 & 0 & 0 \\ 0 & 0 & .7 & .4 \\ 0 & .4 & .7 & 0 \\ .8 & 0 & 1.0 & 0 \\ .9 & .2 & 0 & 0 \\ .9 & 0 & .2 & 0 \\ .9 & 0 & 0 & .2 \\ 1.0 & 0 & 0 & 0 \end{matrix}$$

$B_0 = R$ yields one maximum $\hat{A}$ and three minimum $\overset{\vee}{A}$, viz-

$$\hat{A}_b = \begin{matrix} 1.0 & 1.0 & .5 & .3 \\ .4 & 1.0 & .7 & .5 \\ .3 & 1.0 & 1.0 & .8 \\ .2 & 1.0 & .2 & .9 \\ 0 & 1.0 & .2 & 1.0 \end{matrix} \qquad \overset{\vee}{A}_b = \begin{matrix} 1.0 & 0 & 0 & 0 \\ 0 & .7 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ .2 & 0 & 0 & .9 \\ 0 & .2 & 0 & .9 \\ 0 & 0 & .2 & .9 \\ 0 & 0 & 0 & 1.0 \end{matrix}$$

An extra subscript has been added to indicate their source. Permuting the $\overset{\vee}{A}$ to make them easily comparable, we obtain

$$\overset{\vee}{A}_a = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .7 & 0 & 0 \\ 0 & 0 & .7 & .4 \\ 0 & .4 & .7 & 0 \\ .8 & 0 & 1.0 & 0 \\ .9 & .2 & 0 & 0 \\ .9 & 0 & .2 & 0 \\ .9 & 0 & 0 & .2 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad \overset{\vee}{A}_b = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .7 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ .9 & .2 & 0 & 0 \\ .9 & 0 & .2 & 0 \\ .9 & 0 & 0 & .2 \\ 1.0 & 0 & 0 & 0 \end{matrix}$$

Observe that all these minima have a .9 and 1.0 in the first column. Does a class of solutions exist for which this is not the case? Indeed so.

$$A^* = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & 0 & .7 & .4 \\ 0 & .8 & 1.0 & 0 \\ 0 & .9 & 0 & 0 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad B^* = \begin{matrix} 1.0 & .7 & .2 & 0 \\ .9 & .7 & .2 & .2 \\ .5 & .8 & 1.0 & .2 \\ .3 & .5 & .6 & 1.0 \end{matrix}$$

This is just one of the possible minimum A which would not be detected under the iterative scheme. To be sure, many more examples exist.

We can obtain some insight into the reasons why these undetected minima exist, by looking at the patterns of the rules which they produce. There are three main groups of minima as detected by the iterative method. An example of each is below, together with the minimum B such that $A \circ B = R$.

$$A_{01} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .7 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ .9 & .2 & 0 & 0 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad {}_{01}B_{11} = \begin{matrix} 1.0 & .7 & .2 & 0 \\ .5 & .7 & .7 & .4 \\ .8 & .8 & 1.0 & .3 \\ .3 & .5 & .6 & 1.0 \end{matrix}$$

$$A_{02} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & 0 & .7 & .4 \\ .8 & 0 & 1.0 & 0 \\ .9 & .2 & 0 & 0 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad {}_{02}B_{11} = \begin{matrix} 1.0 & .7 & .2 & 0 \\ 0 & 0 & 0 & .2 \\ .5 & .8 & 1.0 & .3 \\ .3 & .5 & .6 & 1.0 \end{matrix}$$

$$A_{03} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & .4 & .7 & 0 \\ .8 & 0 & 1.0 & 0 \\ .9 & .2 & 0 & 0 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad {}_{03}B_{11} = \begin{matrix} 1.0 & .7 & .2 & 0 \\ 0 & 0 & 0 & .4 \\ .5 & .8 & 1.0 & .3 \\ .3 & .5 & .6 & 1.0 \end{matrix}$$

Each column-row pair constitutes a rule and their product will form a relation. Now, because we have chosen minima, this will mean that there is the least amount of redundant information in the (A,B) pair. The rules will not overlap any more than is absolutely necessary, which makes the patterns of the rules very clear. See Figure 2-5 and 2-6. We may observe that whereas

$A_{01} \circ {}_{01}B_{11}$

.	.	.	.		.	.	.	.		.	.	.	.		.3	.5	.6	1.0
.	.	.	.		.5	.7	.7	.4		.	.	.	.		.	.	.	.
.	.	.	.		.	.	.	.		.8	.8	1.0	.3		.	.	.	.
.9	.7	.2	.		.2	.2	.2	.2		.	.	.	.		.	.	.	.
1.0	.7	.2	.		.	.	.	.		.	.	.	.		.	.	.	.

$A_{02} \circ {}_{02}B_{11}$

.	.	.	.		.	.	.	.		.	.	.	.		.3	.5	.6	1.0
.	.	.	.		.	.	.	.		.5	.7	.7	.3		.3	.4	.4	.4
.8	.7	.2	.		.	.	.	.		.5	.8	1.0	.3		.	.	.	.
.9	.7	.2	.		.	.	.	.2		.	.	.	.		.	.	.	.
1.0	.7	.2	.		.	.	.	.		.	.	.	.		.	.	.	.

$A_{03} \circ {}_{03}B_{11}$

.	.	.	.		.	.	.	.		.	.	.	.		.3	.5	.6	1.0
.	.	.	.		.	.	.	.4		.5	.7	.7	.3		.	.	.	.
.8	.7	.2	.		.	.	.	.		.5	.8	1.0	.3		.	.	.	.
.9	.7	.2	.		.	.	.	.2		.	.	.	.		.	.	.	.
1.0	.7	.2	.		.	.	.	.		.	.	.	.		.	.	.	.

Figure 2-5: Rule Patterns Under the Iterative Scheme

$\overset{v}{A}^* \circ B^*$																
.	.	.	.		.	.	.	.		.	.	.	.		.3	.5	.6	1.0
.	.	.	.		.	.	.	.		.5	.7	.7	.2		.3	.4	.4	.4
.	.	.	.		.8	.7	.2	.2		.5	.8	1.0	.2		.	.	.	.
.	.	.	.		.9	.7	.2	.2		.	.	.	.		.	.	.	.
1.0	.7	.2	.		.	.	.	.		.	.	.	.		.	.	.	.

Figure 2-6: Extra Non-Iteratively Discovered Decompositions

the iterative scheme always couples the fourth and fifth rows of R, the extra solution couples other rows of R and leaves the fifth row of R by itself. The reason for rows to dominate in these patterns is due to our selection of minimum A's. The reason for this will become clearer in the section on the α-level method.

Generally, in a higher rule space the iterative method is limited by the starting conditions. The $A_0 = R$, $B_0 = R$ starting points limit subsequent exploration of $\mathcal{A}$ and $\mathcal{B}$. We have not devised any method for decoupling rules or redistributing information from one rule to another or others. Given that we know how many rules are in the minimum rule space, it is likely that the construction of patterns of rules in a higher space directly from the

minimum rules is possible. Clearly, we require some means of studying the fundamental structure of R. The next section of this report describes a means of so doing.

2.4.2 Some General Questions

To conclude this section we consider, or rather merely state, some questions for which we would like an answer. As yet we do not have even an intuitive notion of what the answers might be, and we are sure the list is incomplete.

- How many maxima and minima are there in $\mathcal{A}_N$ and $\mathcal{B}_N$?
- Can we select any other viable initial conditions?
- How can we guarantee that all the minima have been found for any N-rule space? How can we tell?
- Why does the apparent structure of $\mathcal{A}_N$ and $\mathcal{B}_N$ differ?

3. α-LEVEL TECHNIQUES

In Section 1.3, we mentioned two techniques for decomposing fuzzy relations. The first of these, the iterative method, was described in Section 2, and now we move on to consider the second technique. This is based on studying the α-levels which define the relation R. From the α-levels, we may construct rules individually and then provide a range of solutions for each rule, indirectly solving $A \circ B = R$. This technique suffers from the lack of a convenient mathematical notation, but we shall describe it as well as possible. First, we require some definitions.

3.1 DEFINITIONS

In [6], Zadeh provides the following definition:

For α in [0,1], an α-level-set of a fuzzy relation R is denoted by R_α and is a non-fuzzy set in $\mathcal{X} \times \mathcal{Y}$ defined by

$$R_\alpha = \{(x,y) : R(x,y) \geq \alpha\}$$

Thus the R_α form a nested sequence of non-fuzzy relations, with

$$\alpha_1 \geq \alpha_2 \Rightarrow R_{\alpha_1} \subset R_{\alpha_2}$$

From this it follows that any fuzzy relation from $\mathcal{X}$ to $\mathcal{Y}$ admits of the resolution

$$R = \bigvee_\alpha \alpha R_\alpha \qquad 0 < \alpha \leq 1$$

where $\vee$ stands for the union and αR_α denotes a subnormal non-fuzzy set defined by

$$\alpha R_\alpha(x,y) = \alpha \qquad (x,y) \in R_\alpha$$
$$= 0 \qquad \text{otherwise}$$

3.2 α-LEVEL SETS

Any α-level-set may be considered as a binary relation, i.e., a relation from $\mathcal{X}$ to $\mathcal{Y}$, but consisting entirely of zeros and ones. For us, the most important feature of such a Boolean relation is that it may be decomposed into two relations, A and B. We decompose a Boolean relation by looking for patterns in the 0's and 1's which form a relation and then use these to make the rules.

Now, an important feature of a Boolean rule, i.e., a rule which consists of a column of 0's and 1's combined with a row of 0's and 1's, is that it is always arranged in blocks. Such a rule can never give a triangular pattern, for example. In this way

$$a_\alpha = [\,0 \;\; 1 \;\; 1 \;\; 0 \;\; 1\,]^T \qquad b_\alpha = [\,1 \;\; 1 \;\; 1 \;\; 0\,]$$

yields

$$R_\alpha = \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{matrix}$$

The blocks in this R_α are separated because of the zero in the 4^{th} place of a_α.

To decompose a relation which consists of a regular pattern like this, set

$$a_\alpha(i) = \text{row maximum} \qquad b_\alpha(j) = \text{column maximum}$$

$$= \bigvee_j R_\alpha(i,j) \qquad = \bigvee_i R_\alpha(i,j)$$

It will frequently be the case, however, that a relation consists of more than one rule. To decompose it successfully, one must separate out the patterns. For example,

$$R_\alpha = \begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{matrix} \qquad = R_{\alpha_1} \cup R_{\alpha_2}$$

where

$$R_{\alpha_1} = \begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \qquad R_{\alpha_2} = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{matrix}$$

Hence, $R_\alpha = A_\alpha \circ B_\alpha$, where

$$A_\alpha = \begin{matrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{matrix} \qquad B_\alpha = \begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

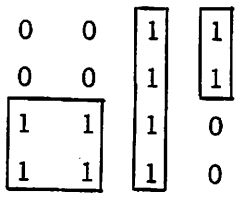
The rules are combined column by column for A and row by row for B as explained in 1.3.

Under the iterative scheme we observed several properties of the range of solutions. They also exist for Boolean relations and we may obtain some insight into why they arise.

3.2.1 Higher Rule Spaces

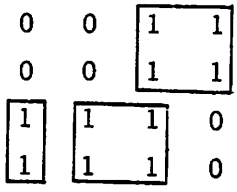
For the example R_α considered above, there is clearly only one decomposition under two rules, i.e., there is only one way in which R may be blocked into two patterns. However, there are several ways in which it may be blocked to give three rules. See Figure 3-1. It is obvious that some

(i)



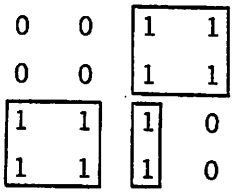
$$A_\alpha = \begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \qquad B_\alpha = \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

(ii)



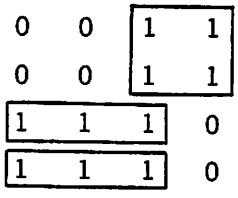
$$A_\alpha = \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \qquad B_\alpha = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$$

(iii)



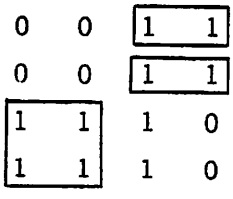
$$A_\alpha = \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \qquad B_\alpha = \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$$

(iv)



$$A_\alpha = \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \qquad B_\alpha = \begin{matrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$$

(v)



$$A_\alpha = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \qquad B_\alpha = \begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}$$

Figure 3-1: Three Rule Decomposition of R

blocks could easily be united to give one block, but the first pattern could not be transformed simply into two blocks, i.e., without redistributing one block between the other two.

It would be nice if we had some means of looking at R_α and computing immediately how many decompositions exist for a given number of rules. Such a procedure should be possible, but we do not know it.

3.2.2 The Range of Values

To continue with this example, we can see that there is only one two-rule solution. However, in the three-rule space where a choice is available to us, we have a range of solutions. The three-rule decompositions of Figure 3-1 contain no overlap of the rules. If we permit the rules to overlap, we can construct a range of solutions. See Figure 3-2. The rules as drawn in 3-2 have been extended as far as possible to overlap with other rules in 3-2. Note that the extension has been to lengthen the row dimension of the pattern. This means that only B'_α is changed.

This illustrates the important property that for a given A_α, there will be a range of B_α, depending on the degree to which the rule patterns overlap.

3.2.3 Several Minima

Looking again at Figure 3-1, we see that the A_α are the same for (ii) and (iii) although the corresponding B_α are different. This is the phenomenon of several minima. It is often the case that given A_α, there may be several ways of choosing the blocking patterns on R, without overlap. This leads to several minima, one minimum B_α for each unique pattern of blocking R. However, for this A_α, there is only one maximum B_α. See Figure 3-3. Observe that the B_α's are <u>not</u> homomorphic under row permutation.

(a)

0	0	1	1
0	0	1	1
1	1	1	0
1	1	1	0

A_α =

0	1	1
0	1	1
1	1	0
1	1	0

B_α =

1	1	0	0
0	0	1	0
0	0	0	1

(b)

0	0	1	1
0	0	1	1
1	1	1	0
1	1	1	0

A'_α =

0	1	1
0	1	1
1	1	0
1	1	0

B'_α =

1	1	1	0
0	0	1	0
0	0	1	1

(a) without overlap of rules

(b) with overlap of rules

Figure 3-2: Range of Solutions

$$A_\alpha = \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 0 & \boxed{1} & \boxed{1} \\ 0 & 0 & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{1} & \boxed{1} & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & 0 \end{matrix}$$

maximum B_α = $\begin{matrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$

$$\begin{matrix} 0 & 0 & \boxed{1} & \boxed{1} \\ 0 & 0 & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{1} & \boxed{1} & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & 0 \end{matrix}$$

first minimum B_α = $\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$

$$\begin{matrix} 0 & 0 & \boxed{1} & \boxed{1} \\ 0 & 0 & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{1} & \boxed{1} & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & 0 \end{matrix}$$

second minimum B_α = $\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$

Figure 3-3: Maximum and Minimum B_α's

3.2.4 Maximum A_α vs. Maximum B_α

We have just seen that to obtain a maximum B , we should seek to extend the rule patterns along the rows. Similarly, to obtain a maximum A , we should extend a column pattern along a column. This means that rules drawn without overlap to accommodate a maximum A will tend to be tall and thin, whereas those drawn for a maximum B will be broad. The particular R_α which we have been using does not illustrate this very well (although compare A_α (ii) and (iv) in Figure 3-1) so a different R_α has been used in Figure 3-4. Observe the different shapes of the rules. It is apparent that as much information as possible has been placed in B_α and A'_α somehow, so as to make them maximal. When a choice is available, extending rows will maximize B and extending columns will maximize A. Had we been seeking the two-rule decomposition here, no choice would have been available.

3.2.5 Comments

Some of the results discussed above might seem trivial and obvious, but they are very important when we come to consider the complete R, as formed from the nested R_α. These results will help us recognize which features of the nested R_α are important in determining the properties of $\mathcal{A}_N$ and $\mathcal{B}_N$ and the reasons for the minimum number of rules. One of the most important results is that several minima, ranges of results etc., come from the existence at some stage of a choice in the ways of drawing the blocks of rules on the α-level-set. Unfortunately, we lack a compact mathematical representation of this and any analytic means of identifying the number of choices available in general.

(a)

0	1	1	1
1	1	1	1
1	1	0	0
1	1	0	0

A_α =

0	0	1
0	1	0
1	0	0
1	0	0

B_α =

1	1	0	0
1	1	1	1
0	1	1	1

(b)

0	1	1	1
1	1	1	1
1	1	0	0
1	1	0	0

A'_α =

0	1	1
1	1	1
1	1	0
1	1	0

B'_α =

1	0	0	0
0	1	0	0
0	0	1	1

(a) seeking maximum B

(b) seeking maximum A

Figure 3-4: Seeking Maxima

3.3 NESTED α-LEVEL SETS

In Section 3.1, we quoted Zadeh's result that

$$R = \bigvee_{\alpha} \alpha R_{\alpha} \qquad 0<\alpha\leq 1$$

We have already looked at the decomposition of each α-level-set into an A_{α} and B_{α}. Now we must turn to the problems of decomposing complete relations. The most important features here are producing "consistent" decompositions and finding what is the minimum number of rules.

3.3.1 Consistent Decompositions

In Section 1.1 we wrote

$$R = \bigvee_{i=1}^{N} R_i$$

Each of the relations which describes a rule can also be written in terms of its α-levels.

$$R_i = \bigvee_{\alpha} R_{i\alpha} \qquad 0<\alpha\leq 1; \quad i=1,\ldots,N \tag{3.1}$$

This means that

$$R_{\alpha} = \bigvee_{i} R_{i\alpha}$$

i.e., any α-level-set of R is the union of the α-level-sets of all the R_i. But Equation (3.1) also implies that the α-levels for each rule are nested according to:

$$\alpha_1 \geq \alpha_2 \quad \Rightarrow \quad R_{\alpha_1} \subset R_{\alpha_2}$$

The problem is, therefore: "At each α-level, draw blocks to represent the rules, so that the blocks remain nested for $0 < \alpha \leq 1$ ".

When drawing rules on the α-levels, we start at $\alpha = 1.0$. Starting at $\alpha = 0.1$, or the lowest level of discretisation such that $\alpha > 0$, does not provide enough information on the shapes the rules are likely to take. At $\alpha = 1.0$, every normal rule will be represented and may be distinguished.

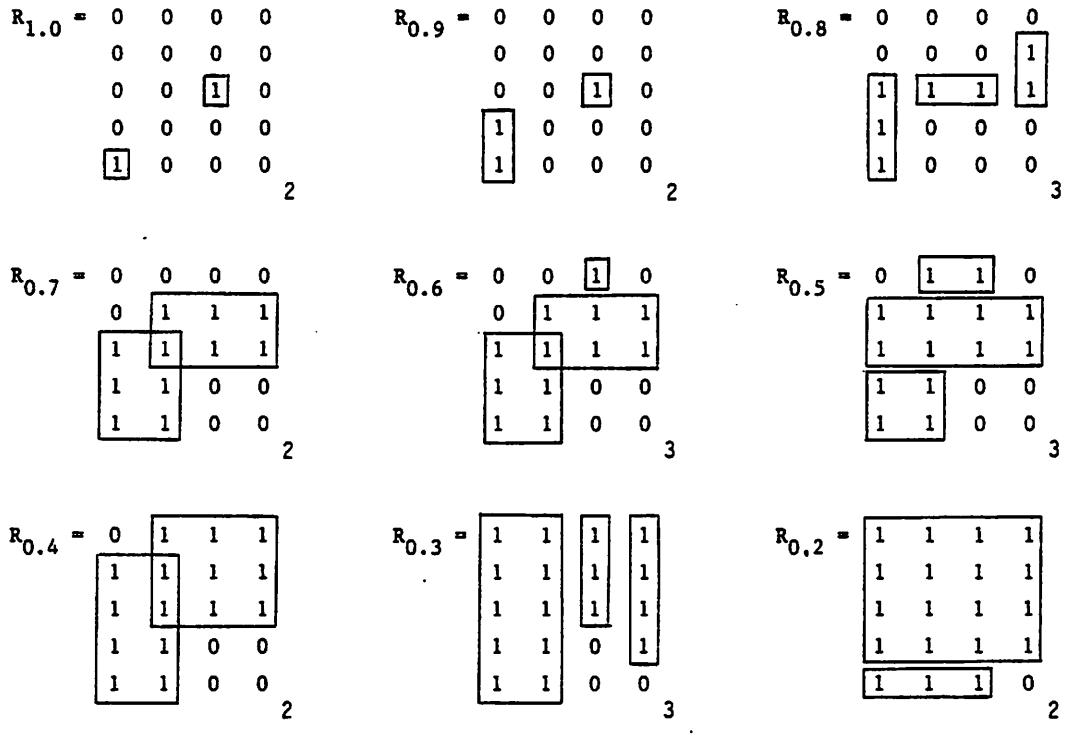
As we proceed down the α-levels, we will be faced with choices. These are:

1. assign information to an A,
2. assign information to a B,
3. create a new rule.

The first two choices refer to the decision on whether to extend an existing rule block lengthwise (assign to A) or breadthwise (assign to B). There will be cases where no rule may be extended to cover a new α-level, in which case we must create a new rule. Sometimes the possibility arises where new information may be assigned to a number of the existing rules. This gives the possibility of many minima. Many of these ideas, and their connection to our previous results on α-level-sets, will be made clearer by an example. (See Section 3.4).

3.3.2 The Minimum Number of Rules

The minimum number of rules is determined as α-level analysis is performed, but we may place a lower bound upon the number of rules by examining the α-level-sets themselves. See Figure 3-5. If we draw blocks upon each α-level as if to represent rules, but ignoring other α-levels and conditions for consistency for now, we will see that there is a minimum number of blocks required to cover each α-level. Denote the minimum number of rules at α by $\overset{v}{N}_{\alpha}$. We know that any α-level-set which requires $\overset{v}{N}_{\alpha}$ rules can



$$R = \begin{matrix} .3 & .5 & .6 & .4 \\ .5 & .7 & .7 & .8 \\ .8 & .8 & 1.0 & .8 \\ .9 & .7 & .2 & .3 \\ 1.0 & .9 & .2 & 0 \end{matrix}$$

Figure 3-5: How Many Rules?

be decomposed into N'_α rules, where $N'_\alpha > \overset{v}{N}_\alpha$. But by definition an α-level-set which requires at least $\overset{v}{N}_\alpha$ rules can never be decomposed using less than $\overset{v}{N}_\alpha$ rules. So, we must look for the α-level which requires most rules to place a lower bound on the minimum number of rules. Hence,

$$\overset{v}{N} \geq \bigvee_\alpha N_\alpha$$

In Figure 3-5, we see that some α-levels require only two rules, but some require three. [Note that $\overset{v}{N}_\alpha$ is written alongside each α-level-set]. The fact that $R_{1.0}$ requires less than the $\bigvee_\alpha \overset{v}{N}_\alpha$ rules indicates that at least one rule will be subnormal. In fact, the example of Figure 3-3 requires four rules, as we shall see when we return to it in Section 3.4.3.

The minimum number of rules is determined by the existence of a "critical" α-level. This α-level may appear to require only $\overset{v}{N}_\alpha$ rules for decomposition but it is impossible to construct a $\overset{v}{N}_\alpha$-rule map on this α-level which is consistent, in the sense that

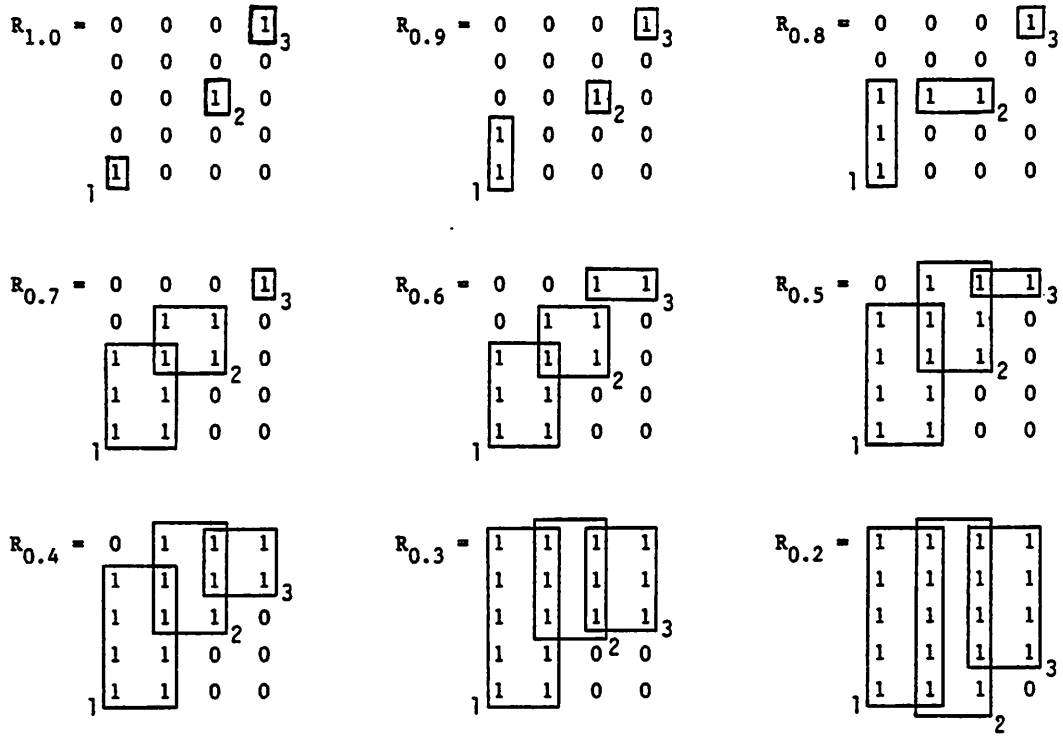
$$R_{\alpha'} \subset R_\alpha \subset R_{\alpha''} \quad \text{for} \quad \alpha' \geq \alpha \geq \alpha''$$

The number of rules which critical α-levels require, determine the minimum number of rules. If more than one critical α-level exists, then the maximum of their respective minimum number of rules is required.

3.4 EXAMPLES

3.4.1 Minimum Rule Solutions

We take the example from Section 2.3, see Figure 3-6. By inspection, we can see that at least three rules will be needed to cover R. To restrict our choices, we will seek to form the maximum A, i.e., assign to columns wherever possible.



$$R = \begin{matrix} .3 & .5 & .6 & 1.0 \\ .5 & .7 & .7 & .4 \\ .8 & .8 & 1.0 & .3 \\ .9 & .7 & .2 & .2 \\ 1.0 & .7 & .2 & 0 \end{matrix}$$

Figure 3-6: Seeking Maximum A

$R_{1.0}$ and $R_{0.9}$ are easy to decompose. The decisions to use only three rules and assign to columns, block the 0.9 with 1.0. At $R_{0.8}$, we can add to rule 1 columnwise, but must extend rule 2 row-wise because 0.8 is a critical α-level. The columnwise extension of rule 1 would not be consistent in three rules with $R_{0.7}$. $R_{0.7}$ requires rule 1 to be extended row-wise, causing some overlap with rule 2. Rule 2 will produce a 0.8 in this position, rule 1 will contribute only 0.7. At $R_{0.3}$, we had the choice of extending rule 2 row-wise or rule 3 columnwise. Here is an example of our decision to maximize A taking effect. The resulting A solution is:

$$A = \begin{matrix} .3 & .5 & 1.0 \\ .5 & .7 & .4 \\ .8 & 1.0 & .3 \\ .9 & .2 & .2 \\ 1.0 & .2 & 0 \end{matrix}$$

There were no choices in assigning the rows, so the minimum B is

$$B = \begin{matrix} 1.0 & .7 & 0 & 0 \\ 0 & .8 & 1.0 & 0 \\ 0 & 0 & .6 & 1.0 \end{matrix}$$

To find the maximum B solution, we assign to rows. See Figure 3-7. Given the three rule constraint, no choice is possible until $R_{0.8}$. The obvious row-extending pattern here would have been:

$$R = \begin{matrix} 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & 0 \\ \boxed{1} & 0 & 0 & 0 \\ \boxed{1} & 0 & 0 & 0 \end{matrix}$$

However, it is not possible to draw a pattern on $R_{0.7}$ which is consistent with decomposition of $R_{0.8}$ and so it must be discarded in favor

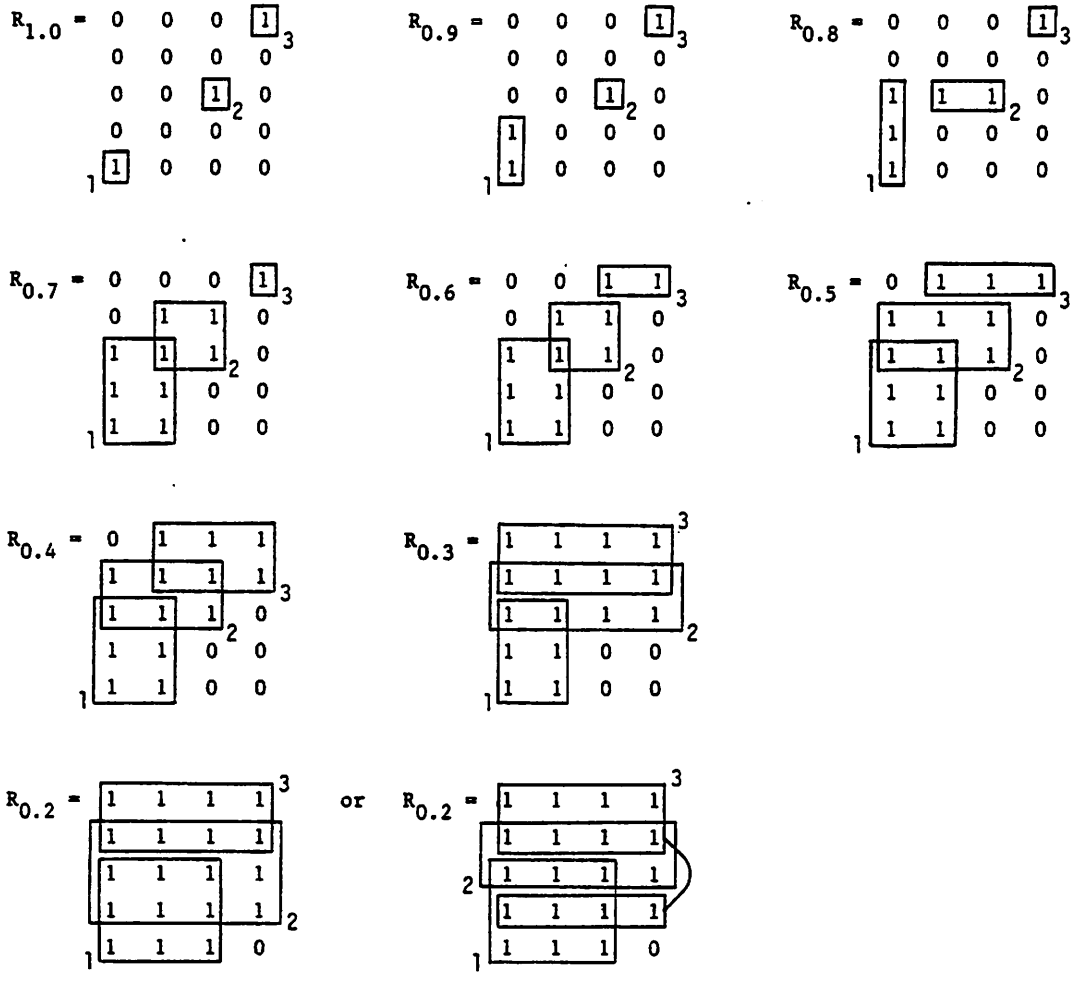


Figure 3-7: Seeking Maximum B

of the A maximum-like decomposition. Our first departure from the previous example comes at $R_{0.5}$ when rules 2 and 3 are both extended row-wise, rather than 1 and 2 columnwise. Thereafter, the decomposition proceeds smoothly until $R_{0.2}$, where rule 1 must be extended row-wise, and either rule 2 or 3 column-wise. Extending rule 3 to cover the fourth row is a little unnatural because it will give a bimodal rule, but it is a legitimate alternative. Our solutions are now:

$$B_{max} = \begin{matrix} 1.0 & .7 & .2 & 0 \\ .5 & .8 & 1.0 & .3 \\ .3 & .5 & .6 & .1 \end{matrix}$$

$$A_{min} = \begin{matrix} 0 & 0 & 1.0 \\ 0 & .7 & .4 \\ .8 & 1.0 & 0 \\ .9 & .2 & 0 \\ 1.0 & 0 & 0 \end{matrix} \quad \text{or} \quad \begin{matrix} 0 & 0 & 1.0 \\ 0 & .7 & .4 \\ .8 & 1.0 & 0 \\ .9 & 0 & .2 \\ 1.0 & 0 & 0 \end{matrix}$$

We may confidently assert that the A's and B which were obtained, together with the respective A or B maxima, are minima because of the overlap involved. To be sure, a fair amount of overlap does occur, but this is the minimum amount required. To find the maximum B which corresponds with the maximum A we may use Sanchez' result, or by taking the α-level-blocks and extending them row-wise wherever possible, so long as consistency with lower α-levels is maintained. This is not so simple as it might sound and Sanchez' method is reliable. Just taking α-level-sets and trying to maximise overlap is not a good method, because it can easily produce sub-maxima. Over-enthusiastic row-wise extension at one α-level can delete opportunities for column-wise extension elsewhere.

3.4.2 Non-Minimum-Rule Solutions

In the earlier discussion of this example, we indicated that not all of the four rule solutions available could be detected under the iterative scheme. The α-level patterns for these solutions are indicated in Figures 3-8, 3-9 and 3-10.

Here, we can see that the fourth rule is not introduced until $\alpha = 0.7$, Figure 3-8, $\alpha = 0.2$, Figure 3-9, or $\alpha = 0.4$, Figure 3-10. In fact, with the existence of four ones at $\alpha = 0.9$, the fourth rule could have been introduced there. See Figures 3-11 and 3-12 for two decompositions where the fourth rule was introduced at $\alpha = 0.9$.

It could also have been introduced at $\alpha = 0.8$, but we shall not give any more examples here.

From this, we confirm that the iterative scheme does not see all the possibilities for creating solutions in the higher than minimum rule spaces. It would seem that the iterative scheme does not make any choices about $\alpha = \alpha^*$ where

$$\alpha^* = \bigwedge_j \left(\bigvee_i R(i,j) \right)$$

i.e., the minimum column maximum. The details of this mechanism are still to be investigated.

3.4.3 $\overset{\vee}{N} \neq \bigvee_\alpha \overset{\vee}{N}_\alpha$

In Section 3.3.2, we described how bounds were placed on the minimum number of rules and mentioned the existence of critical α-levels. Let us return to the example of Figure 3-11, which has $\bigvee_\alpha \overset{\vee}{N}_\alpha = 3$, although we indicated in the text that four rules would in fact be needed. See Figure 3-13.

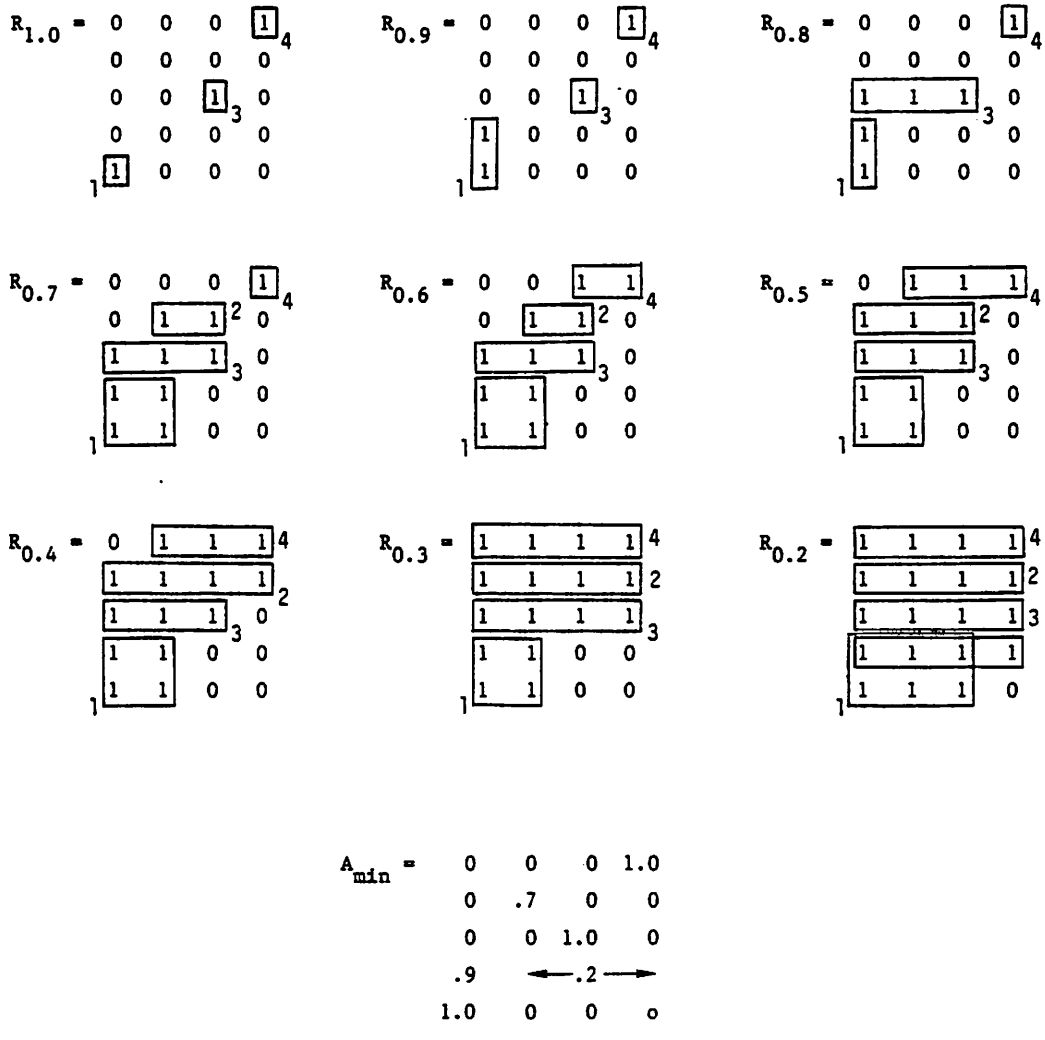
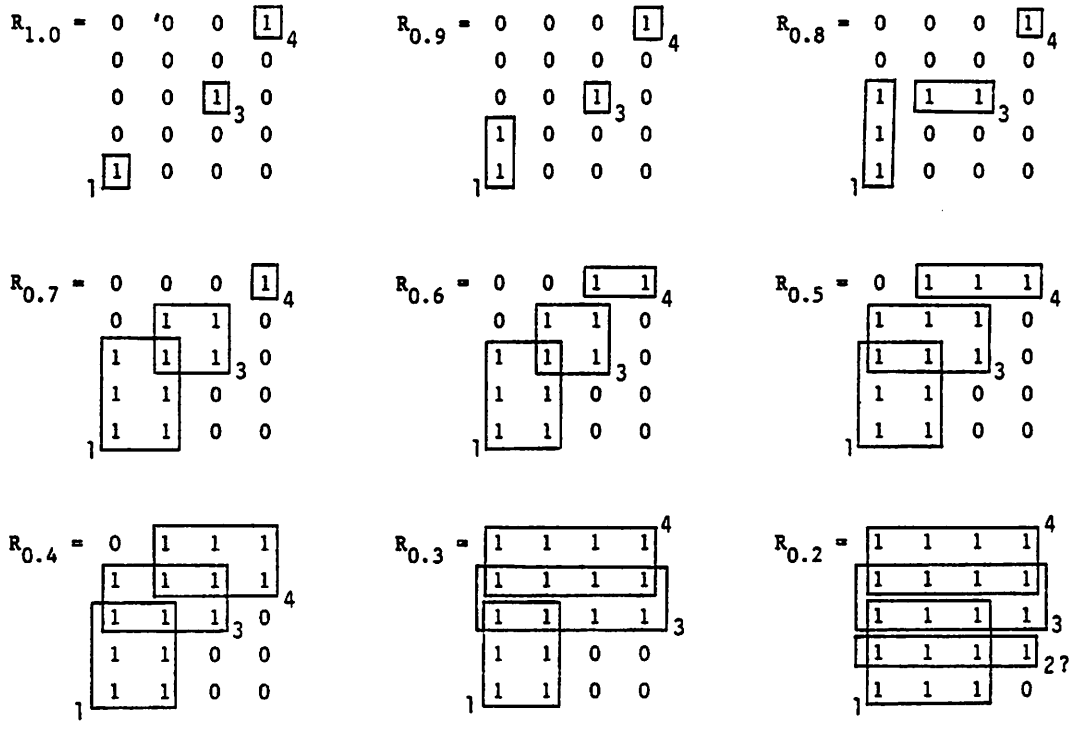


Figure 3-8: A Four Rule Solution: Part 1



$$A_{min} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & 0 & .7 & .4 \\ .8 & 0 & 1.0 & 0 \\ .9 & \leftarrow & .2 & \rightarrow \\ 1.0 & 0 & 0 & 0 \end{matrix}$$

Figure 3-9: A Four Rule Solution: Part 2

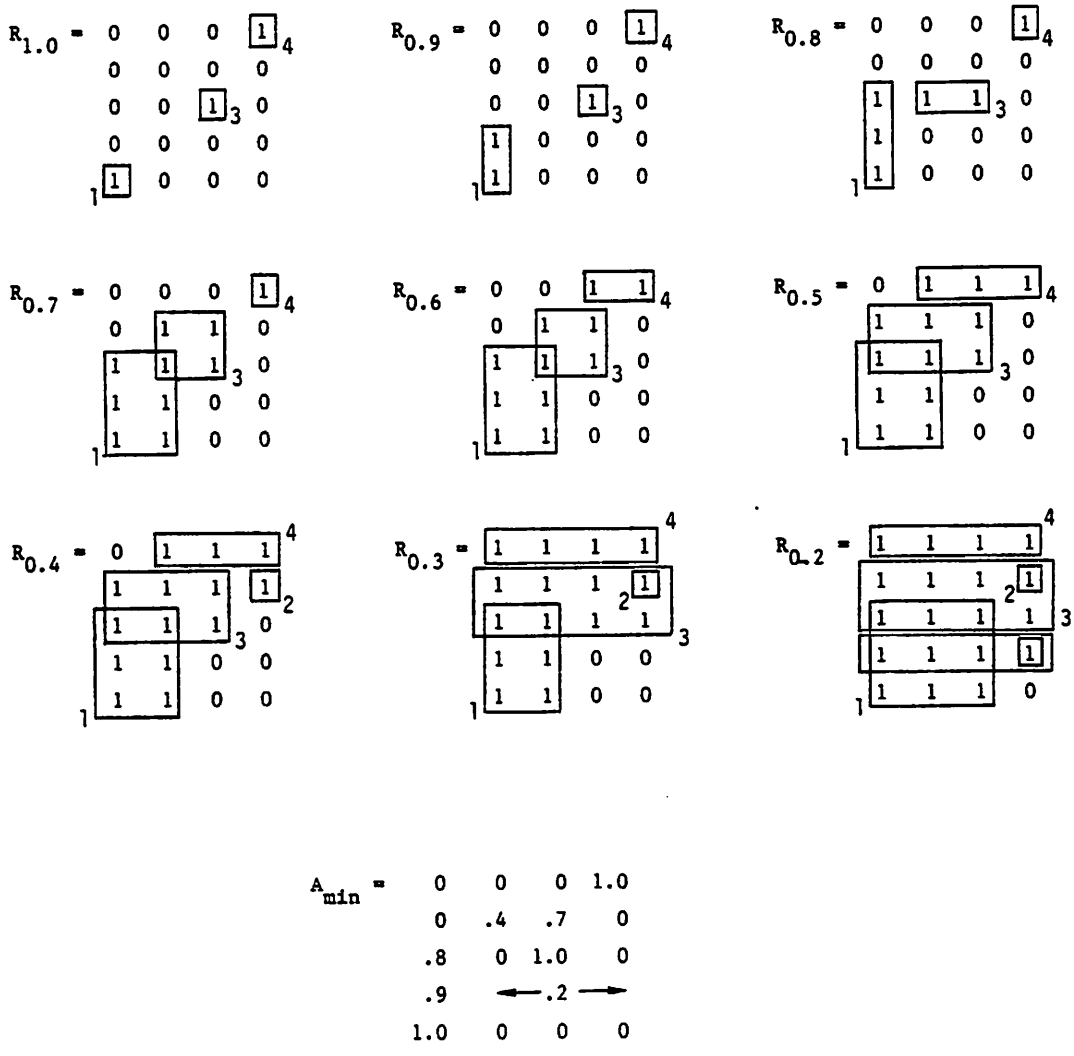
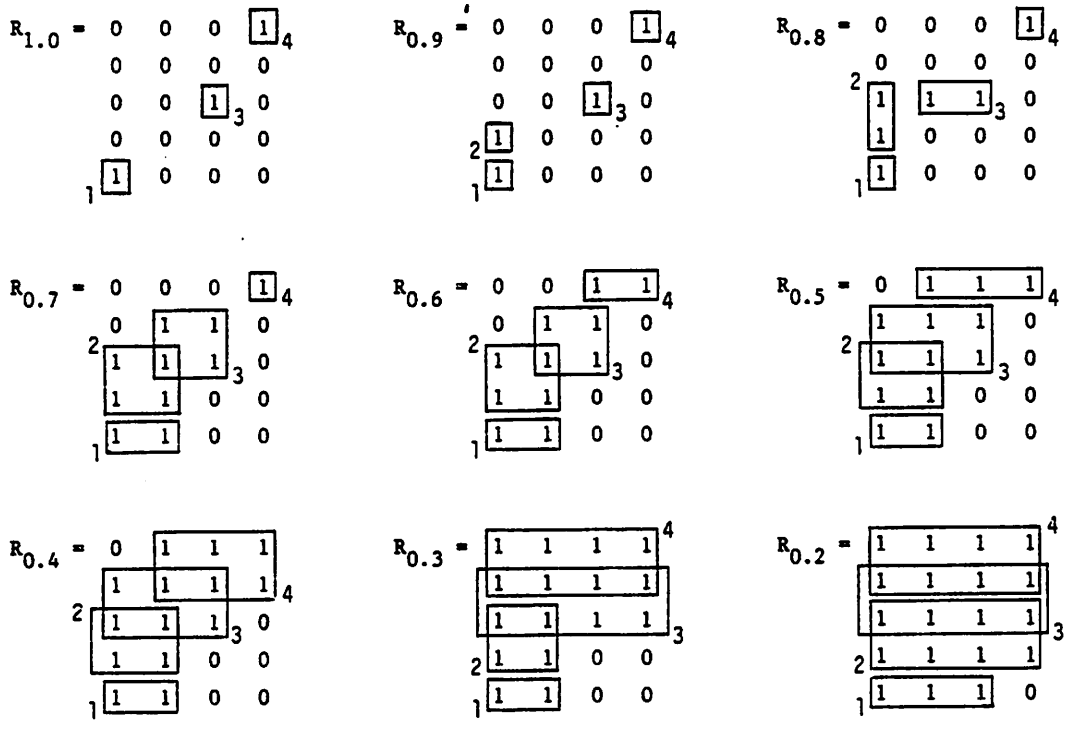
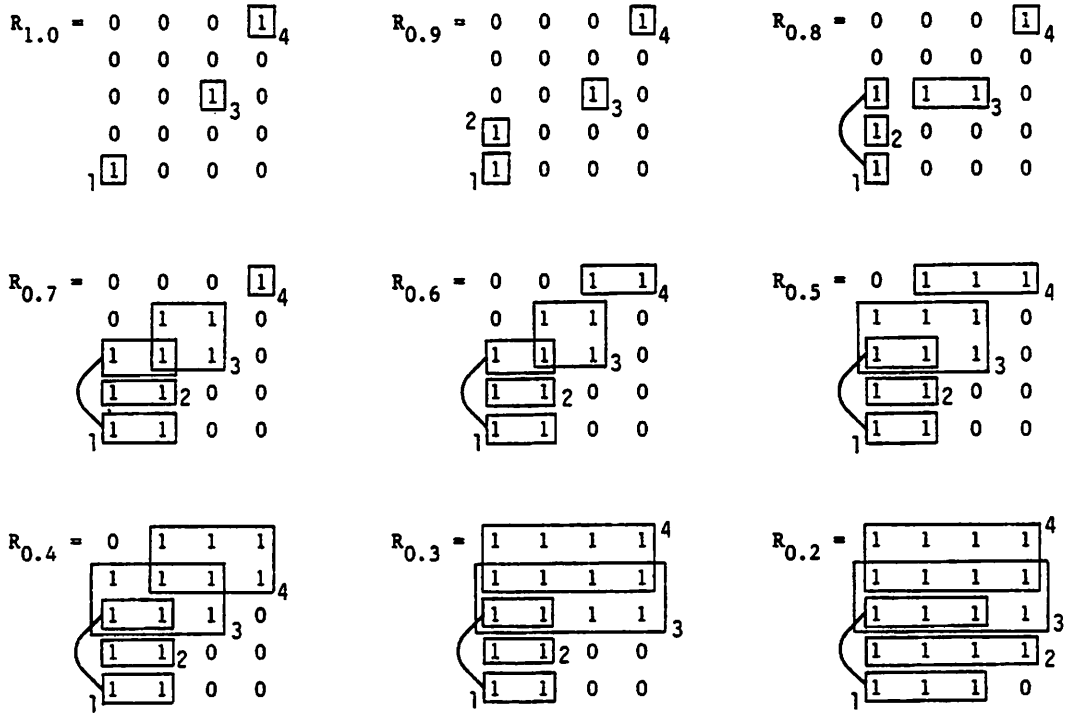


Figure 3-10: A Four Rule Solution: Part 3



$$A_{min} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & 0 & .7 & .4 \\ 0 & .8 & 1.0 & 0 \\ 0 & .9 & 0 & 0 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad B = \begin{matrix} 1.0 & .7 & .2 & 0 \\ .9 & .7 & .2 & .2 \\ .5 & .8 & 1.0 & .3 \\ .3 & .5 & .6 & 1.0 \end{matrix}$$

Figure 3-11: A Four Rule Solution: Part 4



$$A_{min} = \begin{matrix} 0 & 0 & 0 & 1.0 \\ 0 & 0 & .7 & 0 \\ .8 & 0 & 1.0 & 0 \\ 0 & .9 & 0 & 0 \\ 1.0 & 0 & 0 & 0 \end{matrix} \qquad B = \begin{matrix} 1.0 & .7 & .2 & 0 \\ .9 & .7 & .2 & .2 \\ .5 & .8 & 1.0 & .3 \\ .3 & .5 & .6 & 1.0 \end{matrix}$$

Figure 3-12: A Four Rule Solution: Part 5

$$R = \begin{matrix} .3 & .5 & .6 & .4 \\ .5 & .7 & .7 & .8 \\ .8 & .8 & 1.0 & .8 \\ .9 & .7 & .2 & .3 \\ 1.0 & .7 & .2 & 0 \end{matrix}$$

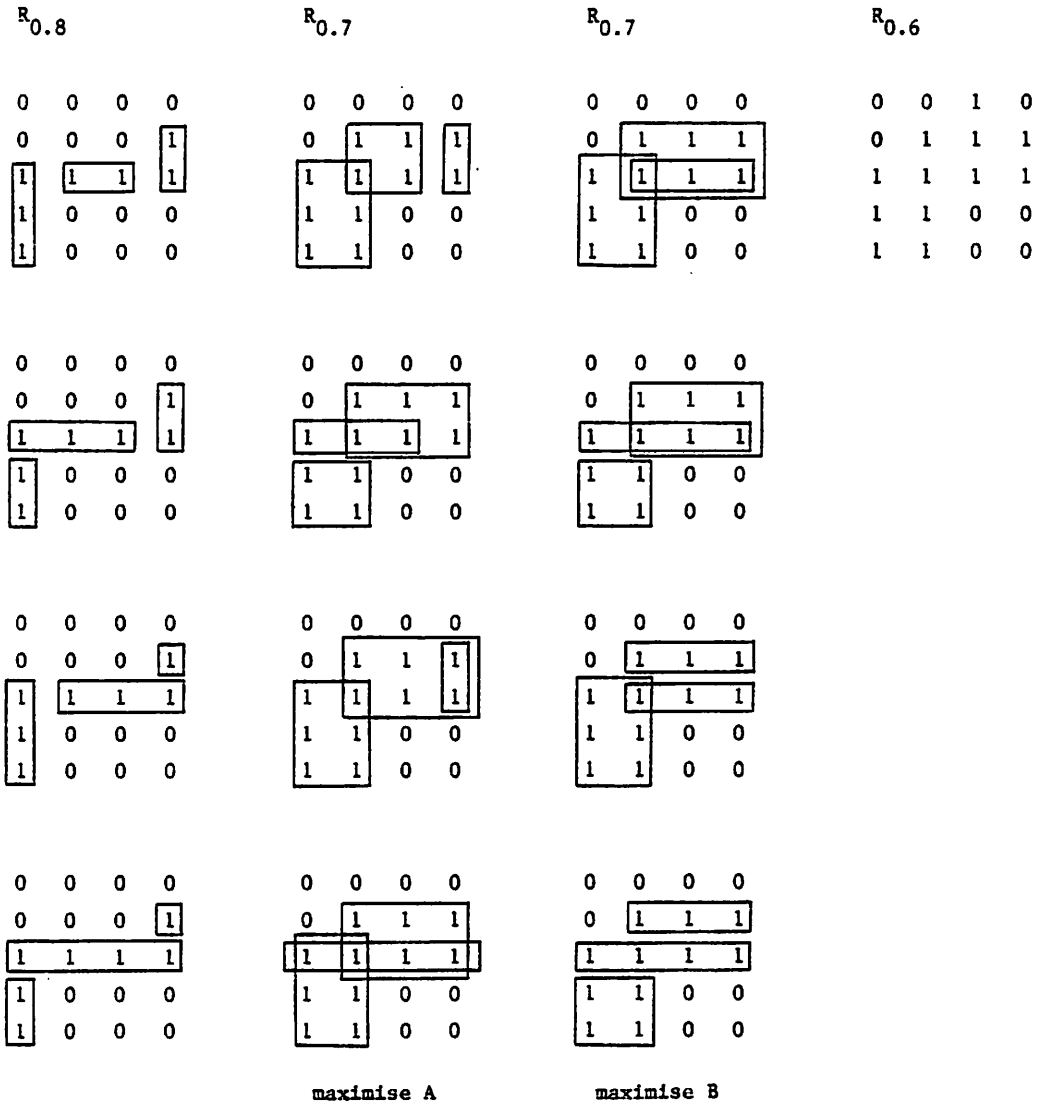


Figure 3-13: Complete Four Rule Solution

At $\alpha = 0.8$, there are four ways of drawing three rules, consistent with $R_{0.9}$ and $R_{0.7}$. However, none of these can be extended to cover $R_{0.6}$. This α-level introduces the row pattern [0 0 1 0] and none of the existing rules can cope with it. Hence an extra rule needs to be introduced. This could be done either at $\alpha = 0.8$ or $\alpha = 0.6$. The critical α-levels are $R_{0.6}$ and $R_{0.8}$. Although both can be covered by three rules, four rules are required for consistency.

3.5 CONCLUDING REMARKS

The α-level technique is very useful, particularly because of the insight it gives us on the structure of R. The patterns which the rules combine to form on the α-level-sets shows how many minima may arise and is especially useful in choosing maximum A or maximum B. We can find parts of the solution space which are invisible to the iterative technique. Indeed, so many new solutions become available that we immediately encounter the main problem of this technique: the lack of an adequate notation.

It soon becomes very difficult to express an α-level-set conveniently. Generally we require information on both the row and column patterns of R, which needs to be compared with that from the α-level above. We need to be able to automate the assignment of portions of the α-level-set to particular rules and we do not have a mathematically acceptable way of doing this, although it is a very easy task for a human to perform.

The most important insight which this technique gave us, trivial as it may seem, was that assigning to columns produced maximum A's and assigning to rows produced maximum B's. By using these rules as our guide when faced with a choice at any α-level, we were able to structure our study of $\mathcal{A}$ and $\mathcal{B}$. The choices are also limited by remaining in the minimum rule space. However, once we move above that space, the choices become bewildering very soon, because one can choose at which level to introduce the extra rule.

We will have no way of easily counting the number of minima, short of analyzing all the α-levels. This is a problem particularly in higher spaces where we would like to know the effects of introducing an extra rule at various α-levels. The number of choices seem to grow geometrically. But, it is the fundamental nature of this approach which leads us to believe that it would supply answers to many of our questions.

4. REVIEW AND COMMENTARY

Relation decomposition has proved to be a frustrating and fascinating problem. We have gained some insights and appear to be close to a complete solution. This final section of the report will review our work so far, point out our problems and suggest directions for this work in the future.

4.1 THE ITERATIVE TECHNIQUE

The iterative technique has been fairly successful, but our understanding of it is still incomplete. The main problem is in the so-called starting conditions. Our use of A = R and B = R as the first positions in the bootstrapping procedure seem to severely limit the range of solutions which can be detected. Taking A = R and seeking B seem to have the effect of giving us no choice on the α-levels above the sub-identity. We need some other intelligently chosen starting conditions which should have the effect of coupling other rows and columns in a way that the identity cannot do. We also need to know how many sets of starting conditions should be sought in order to reveal all of $\mathcal{A}$ and $\mathcal{B}$.

The other persistent problem with the iterative approach is in knowing exactly how many minima exist in either $\mathcal{A}$ or $\mathcal{B}$. As we move up towards the maximum $B \in \mathcal{B}$, the number of minimum A which it can generate seems to decline. At first we thought there might be some rather magical B, less than B_{max}, which would generate all possible A_{min}. This is not the case, but it would be nice to know how to choose sufficient B to generate all the minimum A's, and we would need to be able to count these B's as well. It would also be nice to be able to examine R alone and determine directly how many minima exist, but we suspect this will not be possible.

Our stated aim with the iterative technique is to explore the minimum rule space. However, we have no proof that the iterative technique is guaranteed to find the minimum rule space, although it seems never to fail to

do so in practice. There would seem to be some rather special constraints in effect in this space, so it might be a little easier to prove things. For example, how many maxima exist in the minimum rule space? Would it be easier to count minima in the minimum rule space, before actually searching for them all? We have never proven anything about convergence of the iterative technique, nor the number of iterations required to reach convergence. Would it be possible to solve this, given minimum rule space conditions?

The range of solutions, $\mathcal{A}$ and $\mathcal{B}$, represent a transfer of information between A and B. A minimum has, in some sense, the minimum amount of information possible, with the rest of the information contained in its partner. But, there is a sense in which some minima are "better" than others. The minima with most zeros are "better". A "good" minimum would be one, such that the maximum and minimum generated from it are the same. In this sense, the identity is a very good minimum, but it needs too many rules. The good minimum produces very little overlap of its rules. Overlapping rules imply redundant information, but this is sometimes unavoidable in the minimum rule space.

We have not looked very closely at the influence of starting conditions on the rules which are guaranteed. We know that taking A = R and seeking B_{min} means that B_1, looks like the identity relation. This will make A look like columns of R. Seeking B_{max} would make B look like rows of R. Indeed, rows or columns of R often appear unchanged in the solutions. With a square R, it would be particularly interesting to observe how information is balanced between A and B depending on the starting point. The starting point, A or B = R, also affects the range of solutions which are found and we have not examined whether it is best (if it is best) to seek maxima or minima first.

4.2 THE α-LEVEL TECHNIQUE

The α-level technique extended our knowledge of the decomposition problem considerably. We realized the importance of certain critical α-levels which determined the minimum number of rules required to decompose R. Choices which could be made at α-levels with $\alpha < \alpha_{crit}$ determined the number of minima which could be generated. The number of minima is closely dependent upon choices made at higher α-levels.

The main problem in applying this approach, has been in finding a convenient notation. The A and B matrices into which a particular R may be decomposed are probably as convenient a method as any, but what we really require is a reliable technique for finding all possible decompositions of a given R. It is a simple task for a human to perform, since it consists of enclosing all the 1's in an α-level set in rectangular boxes. This is quite difficult to put into a firm algorithm because of the number of choices which must be made. Since boxes may overlap one must be careful not to eliminate valid choices because they might seem to have plenty of redundancy. The columns of A can be combined under union to form columns of R, so it is no easy matter to ensure all possible combinations have been tried.

Studying column patterns leads us to suspect the existence of disjoint sets of solutions. This result was completely unexpected and led us to look at many more properties of the solution space. We have since realized that the behavior of a given α-level within a complete relation does <u>not</u> seem to match the behavior of that α-level if it is analyzed separately. This does seem a most surprising result and could well yield the final solution to this problem.

4.3 STRUCTURE OF THE SOLUTION SPACE

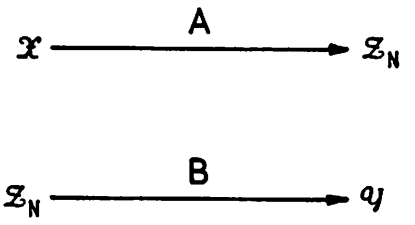
A difficulty we have had throughout this work is a less than perfect understanding of the structure of the solution spaces for A and B. Although we know that the range of solutions for each step in the iterative scheme is an upper semi-lattice, it is clear that the union of these is not. Furthermore, our investigation of the α-level scheme has raised the possibility of partitioned solution spaces. We are left with a incoherent picture of the the inter-relationships between the elements in the solution space.

Our approach to the decomposition problem has been to search for algorithmic solution procedures. Unfortunately, this tends to obscure the structural issues. We feel that an understanding of solution space structure will only be gained by examining some of the basic algebraic issues that our problem raises. We have conducted no research on this, but what follows is a very rudimentary outline of some algebraic concepts.

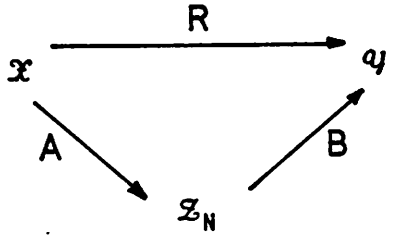
From our basic definitions (see Section 1.1) we have the result that $R \subset \mathcal{X} \times \mathcal{Y}$. An alternative view would be that R is a mapping from $\mathcal{X}$ to $\mathcal{Y}$ which we can write as

$$\mathcal{X} \xrightarrow{\quad R \quad} \mathcal{Y}$$

Now let us define a finite discrete space $\mathcal{Z}_N$ of cardinality N, then the unknown relation A is a mapping from $\mathcal{X}$ to $\mathcal{Z}_N$, and the unknown relation B is a mapping from $\mathcal{Z}_N$ to $\mathcal{Y}$. Or, using the notation defined above



Algebraically, we are looking for a decomposition of R such that the following diagram is satisfied



We can now ask some very general questions of this structure. For example, does such a $\mathcal{Z}_N$ exist, what is the least cardinality of $\mathcal{Z}_N$, what properties must the mappings A and B have, etc.? Our previous investigations tell us that a $\mathcal{Z}_N$ does always exist, and we can place some bounds on its cardinality. However, the other questions are unresolved, and must await further investigation.

4.4 PRACTICAL CONSIDERATIONS

In Section 1.2, we mentioned some aspects of approximate solutions, although we have confined ourselves to seeking exact solutions to the problem. In considering an approximate solution, one must decide firstly why an

approximate solution is desired and then choose the definition of "equal", or "nearly equal", which is most appropriate.

For example, if we decide to look for a solution with a smaller number of rules than that stated by the minimum rule (exact) solution, one must accept that the resulting R° will not match R in all the contours. The gradients of R will be smoothed out. But if some low level detail is all that is lost, then it is probably adequate. In this case, we could choose α and β such that

$$\begin{array}{ll} R(i,j) \geq \alpha & R_\circ^\circ(i,j) = R(i,j) \\ R(i,j) < \alpha & R_\circ(i,j) = R(i,j) \pm \beta \end{array}$$

or $\quad R^\circ(i,j) \leq \alpha$

An β-approximation such as

$$R^\circ(i,j) = R(i,j) \pm \beta$$

could cause displacement of the peaks unless some scaling factor were introduced, but the approximation rules above would avoid that problem. However, there may well be some cases in which displacement of the peaks is quite acceptable.

We could take an opposite rule such as:

$$\begin{array}{ll} R(i,j) \leq \alpha & R_\circ^\circ(i,j) = R(i,j) \\ R(i,j) > \alpha & R^\circ(i,j) \geq \alpha \end{array}$$

This rule might be better for large, sparse relations when it is more important to have zeros and small elements properly positioned.

We could explore the possibilities of regarding "approximately equal" as an operator. If $A \subset \mathcal{U}$ and $B \subset \mathcal{V}$ and $R \subset \mathcal{G}$, then we could regard

"approximately equal" as an operation in $\mathcal{U} \times \mathcal{V} \times \mathcal{G}$. This operation returns the identity when $A \circ B = R$. Otherwise, the returned value gives some idea of the mis-match between $A \circ B$ and R. An element-by-element comparison would be just such an operation, but it could be useful to have something a little more subtle.

We are inclined to feel that approximate solutions are likely to be of the most practical benefit, although it is a necessary first step to study exact solutions. Since many practical relations are based upon not wholly reliable data, there would not seem to be strong arguments for single-mindedly demanding an exact decomposition. When modeling an ill-understood process, one wants to be able to construct a model, describe it, and then submit it to further observation and test. If a crude approximation does not suffice then more elaborate descriptions can be found and tested instead. This seems to be in keeping with a pragmatic, fuzzy approach.

This work has been done in the hope that it would be practical and useful, so we must consider the problems of implementing a rule-based model and of calculating decompositions for relations of the sort which one would expect to meet. So far, we lack firm analytic procedures and the short-cuts which good theorems would provide. So, it would still present a problem should we try to analyze a many-element, multi-dimensional relation. The relations we have been using to test our ideas are only toys in real terms. Clearly, we still have a very long way to go before realizing practical results.

4.5 POSSIBLE DIRECTIONS FOR THE FUTURE

We have often been struck by the concept of information, and the way in which information is shared between the A and B. So far, we have not been able to devise a suitable measure for the information content of A or B which reflect the number of rules, the amount of necessary, yet redundant,

information and the number of non-zero elements. If we could assess the information content of R directly, we might be able to gauge better the amount of information required in the A and B, and thereby how to achieve the "best" minima. A definition of "best" here is also pending, awaiting a good definition of information. We must be aware of the information content of each rule and how the combination of rules affects this. We know in some sense that as an A or B lose information, they somehow lose their capacity to produce minima. We know that a rule which can take a larger range of possible values is somehow less useful. This is observed when a row or column of R features in a rule. Generally, these take only very restricted ranges, implying a great importance. On the other extreme, rules which have a range anywhere between 0 and 1 are completely useless and may be deleted. Our measure of information must involve some sort of comparison between A, B and R, and possibly require rule-by-rule calculations.

The work on α-levels has brought us many new insights. We should look again at row and column patterns and how an α-level set can be reliably broken down into the column or row patterns, looking out for choices which might lead to disjoint sets of solutions. We need to look at the interactions between α-levels and how this affects the number of minima which may be found. Once we have chosen a convenient notation, this could prove to be very fruitful.

We could also think about changing the problem slightly. For example, had we chosen other definitions of implication or composition, would our results be affected drastically? Why and under what circumstances would other definitions be a better model of the processes of reasoning? We have been concentrating upon the minimum rule solution, but when in a higher space, how easy is it to prevent the solution for dropping back to a lower rule space and how easy is it, in fact, to match to a predetermined term set? We have held all these notions in reserve, awaiting better progress.

4.6 A FINAL COMMENT

The results we have reported are an attempt to define some procedures for fuzzy relation decomposition. As a final comment on the work, we would like to show its relevance in the wider context of a complex system modeling methodology.

A complex system is characterized by the large amount of information that flows between its component parts. The difficulty of manipulating and understanding this information is a major problem when attempting to analyze, and make decisions in, complex environments. One way of addressing this issue is to ask if there are ways of aggregating the information so that its essential meaning and structure are preserved. We believe that fuzzy set theory is an ideal tool with which to develop aggregation techniques, and that fuzzy relational models are very powerful basic concepts.

Thus we view fuzzy models as a way of representing low-level highly-detailed information in a more compact and meaningful form. Given this basic premise, we need to develop a set of tools that allow us to manipulate fuzzy relations. Relation decomposition is one such tool. It has, when fully developed, the ability to generate linguistic descriptions of complex information structures, and may help resolve some very important questions.

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Appendix A. DEFINITIONS AND LEMMAS

This appendix contains the elementary definitions and lemmas that we have used in the development of our iterative scheme for fuzzy relation decomposition. We start with some results from Sanchez [2] and Pappis and Sugeno [1], and conclude with some of our own.

Definition 1: For any given pair of elements a and b in [0,1], we define the α-operator such that $c = a \alpha b$ and

$$c = 1 \quad \text{if} \quad a \leq b$$
$$c = b \quad \text{if} \quad a > b$$

Definition 2: A fuzzy relation R is contained in a fuzzy relation R^* (written $R \subseteq R^*$) whenever

$$R(i,j) \leq R^*(i,j) \quad \forall\, i,j$$

Definition 3: Let R be a fuzzy relation, then the fuzzy relation R^T is the transpose, or inverses, of R and is defined by

$$R^T(i,j) = R(j,i) \quad \forall\, i,j$$

Definition 4: Given two relations P an Q, then the $\circ$-composition of P and Q (written $P \circ Q$) is given by

$$P \circ Q(i,j) = \bigvee_k P(i,k) \wedge Q(k,j)$$

Definition 5: Given two relations P an Q, then the α-composition of P and Q (written $P @ Q$) is given by

$$P @ Q(i,j) = \bigwedge_k P(i,k) \,\alpha\, Q(k,j)$$

Lemma 6: For every pair of fuzzy relations A and B, we have (assuming dimensional consitency)

$$B \subseteq A^T @ (A \circ B)$$

$$A \subseteq (B @ (A \circ B)^T)^T$$

Proof: see Sanchez [2]

Lemma 7: For every set of fuzzy relations A, B, R we have (assuming dimensional consistency)

$$A \circ (A^T @ R) \subseteq R$$
$$(B @ R^T)^T \circ B \subseteq R$$

Proof: see Sanchez [2]

Lemma 8: Given fuzzy relations A and R, let $\mathcal{B}$ be the set of fuzzy relations B such that $A \circ B = R$. Then

(i) $\mathcal{B} \neq 0$ iff $\hat{B} \triangleq A^T @ R \in \mathcal{B}$

(ii) if $\mathcal{B} \neq 0$ then $\hat{B}$ is the largest element in $\mathcal{B}$

Proof: see Sanchez [2]

Lemma 9: Given fuzzy relations B and R, let $\mathcal{A}$ be the set of fuzzy relations A such that $A \circ B = R$. Then

(i) $\mathcal{A} \neq 0$ iff $\hat{A} \triangleq (B @ R^T)^T \in \mathcal{A}$

(ii) if $\mathcal{A} \neq 0$ then $\hat{A}$ is the largest element in $\mathcal{A}$

Proof: see Sanchez [2]

Definition 10: For any given pair of elements a and b in [0,1], then we define the <u>β-operator</u> such that $c = a \beta b$ and

$c = 0$ if $a < b$
$c = b$ if $a \geq b$

Definition 11: Given a column vector $\underline{a} = [a_1, \ldots, a_n]^T$ such that $a_i = \hat{a}$ (the maximum element in $\underline{a}$) or 0 $(i=1,\ldots,n)$, then the set $\Phi(\underline{a})$ of ζ-vectors is given by

$$\Phi(\underline{a}) \triangleq \{ \zeta(\underline{a}) \}$$

where $\zeta(\underline{a}) = (\zeta_1, \ldots, \zeta_n)^T$

$$\zeta_i = 0 \text{ or } \hat{a}$$

$$\sum_i \zeta_i = \hat{a}$$

Note: (i) that there is only one non-zero element in each ζ-vector , corresponding to the maximum element in each $\underline{a}$; (ii) that $\underline{a}$ contains only $\hat{a}$ and zeros, and that the number of ζ-vectors is equal to the number of non-zero elements in $\underline{a}$.

Definition 12: Given a fuzzy relation A with m columns $\underline{a}_j$, assume that $\Phi(\underline{a}_j)$ is defined for $j=1,\ldots,m$. Then the set $\Phi(A)$ of matrices $\zeta(A)$ is given by

$$\Phi(A) \triangleq \{ \zeta(A) \}$$

where $\zeta(A) = [\ \zeta(\underline{a}_1), \ldots, \zeta(\underline{a}_n)\ ]$

Definition 13: Given a fuzzy relation B and a row vector $\underline{r}$, then the <u>δ-composition</u> of B with $\underline{r}$ (written $B \textcircled{\delta} \underline{r}$) is given by

$$B \textcircled{\delta} \underline{r}(i,j) = \left(\bigwedge_k B(i,k) \alpha \underline{r}(k) \right) \beta \left(B(i,j) \beta \underline{r}(j) \right)$$

Lemma 14: Given the fuzzy relation B and the fuzzy set $\underline{r}$, all the fuzzy sets $\underline{a}$ such that $\underline{a} \circ B = \underline{r}$ are given by

$$\bigvee \left(\zeta (B \textcircled{\delta} \underline{r}) \right)^T \leqslant \underline{a} \leqslant (B \textcircled{@} \underline{r}^T)$$

$$\forall\ \zeta(B \textcircled{\delta} \underline{r}) \in \Phi(B \textcircled{\delta} \underline{r})$$

Proof: see Pappis and Sugeno [1].

Remark: the set $\bigvee \left(\zeta_i (B \textcircled{\delta} \underline{r}) \right)^T$ contains all the minimum solutions.

Lemma 15: Given the fuzzy relation A and the fuzzy set $\underline{r}$, all the fuzzy sets $\underline{b}$ such that $A \circ \underline{b} = \underline{r}$ are given by

$$\bigvee \left(\zeta (A^T \textcircled{\delta} \underline{r}^T) \right) \leqslant \underline{b} \leqslant A^T \textcircled{@} \underline{r}$$

Proof: similar to previous lemma.

Remark: the set $\bigvee \left(\zeta_i (A^T \textcircled{\delta} \underline{r}^T) \right)^T$ contains all the minimum solutions.

These definitions and lemmas provide all the necessary theory for determining the complete set of solutions at every step in the iterative scheme. What follows are a few results that allow us understand the structure of the solution space (but see section 4.3 in the main body of the report).

Lemma 16: Given A, $A^* \in \mathcal{A}$ and $A \subseteq A^*$ then

$$(A^{*T} \circledcirc R) \subseteq (A^T \circledcirc R)$$

Proof: by Definition 5

$$LHS(i,j) = \bigwedge_k A^*(k,i) \alpha R(k,j)$$
$$RHS(i,j) = \bigwedge_k A(k,i) \alpha R(k,j)$$

But $A(k,i) \leq A^*(k,i)$, therefore by definition 1

$$A^*(k,i) \alpha R(k,j) \leq A(k,i) \alpha R(k,j) \qquad \forall i,j,k$$

So that $LHS(i,j) \leq RHS(i,j) \ \forall i,j$

Lemma 17: Given B, $B^* \in \mathcal{B}$ and $B \subseteq B^*$ then

$$(B^* \circledcirc R^T)^T \subseteq (B \circledcirc R^T)^T$$

Proof: same form as proof for Lemma 16.

Lemma 18: The maxima of $\mathcal{A}$ (or $\mathcal{B}$) must be generated by the minima of $\mathcal{B}$ (or $\mathcal{A}$)

Proof: follows from the ordering results of Lemmas 16 and 17.