TREATING A FUZZY SUBSET AS A PROJECTABLE RANDOM SUBSET

by

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Memorandum No. UCB/ERL M82/35

13 April 1982

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

TREATING A FUZZY SUBSET AS A PROJECTABLE RANDOM SUBSET

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The treatment of a fuzzy subset as a random subset has been studied by some authors. This paper contributes a unified framework to this approach. We call those random subsets of a space 'projectable' if they yield well-defined one point coverage functions and, hence, fuzzy subset membership functions. It is shown that this definition and related ones lend to the property that all projectable random subsets of a given space form a σ-algebra, called the random σ-algebra. A generalization, using measure theoretic results of Goodman's original construction is presented whereby any fuzzy subset of a given space is shown to correspond to some projectable random subset. Some characterizations are obtained for simple, i.e., finite-valued, projectable random sets and simple fuzzy sets. In addition, it is shown that a bijection exists between all projectable random intervals and all bivariate ordered (increasing) random vectors. This implies that the membership functions of the correspondong fuzzy subsets are computable via simple integration involving bivariate distributions. Finally, some specialization of these results to fuzzy numbers and corresponding projectable random intervals is presented.

Key Words: Random sets, measurability, random vectors, random intervals, fuzzy numbers

1. INTRODUCTION

Zhang Nan-lun presents a set of statistical data in [11] showing that there exists a stability in the observation of the frequency of a movable interval (expressing a fuzzy concept) that covers a fixed point in the real line $\mathbb{R}$. His paper supports L. A. Zadeh's theory in some respects, and his paper also supports treating a fuzzy subset as a random subset. Many authors, especially I. R. Goodman, have either already studied this or have been embroiled in some other way in the controversy between fuzzy set and probabilistic approaches to modeling uncertainties [1,9]. A lot of difficult problems is still awaiting our solutions.

2. PROJECTABLE RANDOM SUBSETS

Let U be a given base space and $\mathcal{B}$ a σ-algebra of subsets of U.

$\forall u \in U$, $C(u)$ = filter of $\mathcal{B}$ on U

$$C(u) \triangleq^{1} \{B \in \mathcal{B} \mid u \in B\}. \qquad (2.1)$$

Also, let $\check{\mathcal{B}}$ = σ-algebra generated by $\{C(u) \mid u \in U\}$

$$\check{\mathcal{B}} \triangleq \sigma(\{C(u) \mid u \in U\}), \qquad (2.2)$$

where $\sigma(C)$ denotes the smallest σ-algebra containing the class C.

<u>Definition 1</u>. Let (Ω,A,P) be a given probability space. We call $(\mathcal{B},\check{\mathcal{B}})$ a <u>projectable measurable space</u> on U, where $\check{\mathcal{B}}$ is defined in (2.2), and we call a mapping

$$S : \Omega \to \mathcal{B}, \qquad (2.3)$$

a <u>projectable random subset</u> of U, if it is $\check{\mathcal{B}}$-measurable, thus inducing the probability space $(\mathcal{B},\check{\mathcal{B}},\nu)$, where $\nu = P\circ S^{-1}$. See [4] for related notions (strong measurability, for example).

The collection of all projectable random subsets of U is denoted by $S(\Omega,A,P;U,\mathcal{B},\check{\mathcal{B}})$, or simply S.

Sometimes, the range of the mapping S is not the whole σ-algebra $\mathcal{B}$, but a subset of it. We can rewrite

$$S : \Omega \to L, \text{ where } L \subseteq \mathcal{B} \qquad (2.4)$$

$$C_L(u) \triangleq \{B \in L \mid u \in B\} \qquad (2.5)$$

$$\check{\mathcal{B}}_L \triangleq \sigma(\{C_L(u) \mid u \in U\}) \qquad (2.6)$$

<u>Lemma 1</u>. The mapping defined in (2.4) is $\check{\mathcal{B}}$-measurable iff it is $\check{\mathcal{B}}_L$-measurable.

The proof is trivial. ¤

<u>Definition 2</u>. Suppose that S is a projectable random subset of U from a given probability space(Ω,A,P), let $\underset{\sim}{S}$ be the fuzzy subset of U with membership function given by, for all $u \in U$,

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1The symbol "$\triangleq$" stands for "means" or "is equal by definition to".

$$\begin{aligned}\mu_{\check{S}}(u) &\triangleq P(\{\omega \mid u \in S(\omega)\}) \\ &= P(\{\omega \mid S(\omega) \in C(u)\}) \qquad (2.7) \\ &= \nu(C(u)) \text{ (one point coverage function of } S).\end{aligned}$$

We call $\check{S}$ the <u>fuzzy subset projected from</u> S, see [1] for related notions. We write S' ~ S" iff $\check{S}' = \check{S}''$. "~" is an equivalence relation on $\mathcal{S}$, so that $\mathcal{S}$ can be partitioned into different classes.

Let $(\mathcal{B}_t, \check{\mathcal{B}}_t)_{t\in T}$, be a family of projectable measurable spaces, where T is an arbitrary index set. When T is finite, supposing that $T = \{1,2,\ldots,n\}$, we set

$$\mathcal{B}_T \triangleq \bigtimes_{t\in T} \mathcal{B}_t = \{(B_1,\ldots,B_n) \mid B_i \in \mathcal{B}_i (i=1,\ldots,n)\} \qquad (2.8)$$

$$\check{\mathcal{B}}_T \triangleq \dot{\bigtimes_{t\in T}} \check{\mathcal{B}}_t \triangleq \sigma(\{\mathcal{D}_1 \times \ldots \times \mathcal{D}_n \mid \mathcal{D}_i \in \check{\mathcal{B}}_i (i = 1,\ldots,n)\}) \qquad (2.9)$$

When the index set T is infinite, set

$$\mathcal{B}_T \triangleq \bigtimes_{t\in T} \mathcal{B}_t = \{(B_t)_{t\in T} \mid \forall t \in T,\ B_t \in \mathcal{B}_t\} \qquad (2.10)$$

$$\check{\mathcal{B}}_T \triangleq \dot{\bigtimes_{t\in T}} \check{\mathcal{B}}_t \triangleq \sigma(\{\mathcal{D}_J \times \bigtimes_{i\in T-J} \check{\mathcal{B}}_i \mid \mathcal{D}_J \in \check{\mathcal{B}}_J,\ J(\text{finite}) \subseteq T\}) \qquad (2.11)$$

<u>Definition 3</u>. We call $(\mathcal{B}_T, \check{\mathcal{B}}_T)$ the <u>product projectable measurable space</u> of $(\mathcal{B}_t, \check{\mathcal{B}}_t)_{t\in T}$.

If $\mathcal{B}_t \equiv \mathcal{B}\ (t \in T)$, and $T = \{1,2,\ldots,n\}$, we can write

$$\mathcal{B}_T = \mathcal{B}^n; \quad \check{\mathcal{B}}_T = \check{\mathcal{B}}^n \qquad (2.12)$$

<u>Lemma 2</u>. Consider $(S_t)_{t\in T}$ where $\forall t \in T$, $S_t \in \mathcal{S}(\Omega,\mathcal{A},P;U,\mathcal{B}_t,\check{\mathcal{B}}_t)$ and set $S_T : \Omega \to \mathcal{B}_T$, defined by

$$\omega \mapsto S_T(\omega) \triangleq (S_t(\omega))_{t\in T}, \qquad (2.13)$$

then S_T is $\check{\mathcal{B}}_T$-measurable.

Proof. If $T = \{1,2,\ldots,n\}$, for any $\mathcal{D}_i \in \check{\mathcal{B}}_i$ $(i = 1,\ldots,n)$, we have

$$S_T^{-1}(\mathcal{D}_1 \times \mathcal{D}_2 \times \ldots \times \mathcal{D}_n) = S_1^{-1}(\mathcal{D}_1) \cap \ldots \cap S_n^{-1}(\mathcal{D}_n) \in \mathcal{A} \qquad (2.14)$$

so that

$$S_T^{-1}(\mathcal{B}_T) \subseteq \mathcal{A}\ . \qquad (2.15)$$

When T is infinite, we have

$$S_T^{-1}(\mathcal{D}_J \times \bigtimes_{i\in T-J} \check{\mathcal{B}}_i) = S_J^{-1}(\mathcal{D}_J) \in \mathcal{A}, \qquad (2.16)$$

where $\mathcal{D}_J \in \check{\mathcal{B}}_J$, and

$$S_J : \Omega \longrightarrow \mathcal{B}_J, \quad \omega \longmapsto S_J(\omega) = (S_t(\omega))_{t\in J}. \qquad (2.17)$$

Then we can get

$$S_T^{-1}(\check{\mathcal{B}}_T) \subseteq \mathcal{A} \qquad (2.18)$$

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Let $(\mathcal{B}_t, \check{\mathcal{B}}_t)_{t\in T}$ be a family of projectable measurable spaces on U, where T is finite or denumerable, with

$$\mathcal{B}_t \equiv \mathcal{B}; \quad \check{\mathcal{B}}_t \equiv \check{\mathcal{B}} \quad (t \in T).$$

We can define two mappings as follows.

$$\cup_T : \mathcal{B}_T \to \mathcal{B}, \quad (B_t)_{t\in T} \mapsto \bigcup_{t\in T} B_t; \qquad (2.19)$$

$$\cap_T : \mathcal{B}_T \to \mathcal{B}, \quad (B_t)_{t\in T} \mapsto \bigcap_{t\in T} B_t. \qquad (2.20)$$

<u>Lemma 3</u>. The mappings $\cup_T$ and $\cap_T$ are $\check{\mathcal{B}}$-measurable.

Proof.

$$\cup_T^{-1}(C(u)) = \bigcup_{t\in T} (C(u) \times \check{\mathcal{B}}_{T-\{t\}}) \in \check{\mathcal{B}}_T \qquad (2.21)$$

$$\cap_T^{-1}(C(u)) = \bigcap_{t\in T} C_t \in \check{\mathcal{B}}_T \quad (C_t \equiv C(u)) \qquad (2.22)$$

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<u>Definition 4</u>. The following operations are defined on $\mathcal{S}$.

$$\bigcup_{t\in T} S_t \triangleq \cup_T(S_T), \qquad (2.23)$$

$$(\bigcup_{t\in T} S_t)(\omega) = \cup_T(S_T(\omega)) = \bigcup_{t\in T} S_t(\omega);$$

$$\bigcap_{t\in T} S_t \triangleq \cap_T(S_T),$$

$$(\bigcap_{t\in T} S_t)(\omega) = \cap_T(S_T(\omega)) = \bigcap_{t\in T} S_t(\omega) \qquad (2.24)$$

where T is finite or denumerable.

From Lemma 2 and Lemma 3, we can see that class $\mathcal{S}$ is closed under the operations $\cup$ and $\cap$.

Complementation is now defined by

$$(S^c)(\omega) \triangleq (S(\omega))^c. \qquad (2.25)$$

Obviously, for any fixed element $\omega \in \Omega$, the class

$$\mathcal{S}(\omega) \triangleq \{S(\omega) \mid S \in \mathcal{S}\}$$

is a σ-algebra with respect to the operations (2.23)-(2.25). The operations discussed are special cases of compositional set operators [12].

<u>Definition 5</u>. We call $\mathcal{S}$ the <u>random</u> σ-<u>algebra related to operations (2.23)-(2.25)</u>.

3. A MEASURABILITY THEOREM.

If a mapping $f : X \to Y$ is a bijection, then every σ-algebra A on X can be carried by f to become a σ-algebra on Y:

$$f(A) \triangleq \{\{f(x) | x \in A\} | A \in A\}. \qquad (3.1)$$

Theorem 1. Let (Ω, A, P) be a probability space and, let $(\mathcal{B}, \check{\mathcal{B}})$ be a projectable measurable space on U and $(E, \mathcal{E})$ any measurable space. Assume that the mapping

$$\tau : E \to L$$

where $L \subseteq \mathcal{B}$, is a bijection. If,

$$T(\mathcal{E}) \triangleq \{\{T(e) | e \in \alpha\} | \alpha \in \mathcal{E}\} \supseteq \{C_L(u) | u \in U\}, \qquad (3.2)$$

then, for an $\mathcal{E}$-measurable mapping

$$\xi : \Omega \to E,$$

we have

$$S \triangleq T \circ \xi \in S \qquad (3.3)$$

Proof. $T \circ \xi$ is $T(\xi)$-measurable iff ξ is $\mathcal{E}$-measurable. From (3.2) we have $\check{\mathcal{B}}_L \subseteq T(\mathcal{E})$, so that S is $\check{\mathcal{B}}_L$-measurable and hence $\check{\mathcal{B}}$-measurable too. Thus, (3.3) is true. ¤

Theorem 1 is of many uses. For example, Goodman's theorem 1 in [1] can be easily proved using this theorem. We shall rewrite it with a few changes.

We say that a probability space (Ω, A, P) is sufficient for a given mesurable space $(X, \mathcal{B})$ if for any probability measure m defined on $\mathcal{B}$, there is a $\mathcal{B}$-measurable mapping $\xi : \Omega \to X$ such that m is the induced probability measured by ξ, i.e.

$$m(B) = P(\xi^{-1}(B)) \quad \forall B \in \mathcal{B} \qquad (3.4)$$

Theorem 2. Let (Ω, A, P) be a probability space which is sufficient for $(\mathbb{R}, \mathcal{B}_0)$, where $\mathcal{B}_0$ is the Borel field on $\mathbb{R}$. Let $(\mathcal{B}, \check{\mathcal{B}})$ be a projectable measurable space on U. Given a fuzzy subset A of U, it is always projected from some $S \in S(\Omega, A, P; U, \mathcal{B}, \check{\mathcal{B}})$ provided that μ_A is $(\mathcal{B}, \mathcal{B}_0)$-measurable.

Proof. Since μ_A is $(\mathcal{B}, \mathcal{B}_0)$-measurable,

$$L \triangleq \{A_\lambda | \lambda \in [0,1]\} \subseteq \mathcal{B}, \qquad (3.5)$$

where

$$A_\lambda \triangleq \{u \in U | \mu_A(u) \geq \lambda\} \text{ is the } \lambda\text{-cut of } A. \qquad (3.6)$$

Clearly, if $\lambda_1 > \lambda_2$, then $A_{\lambda_1} \subseteq A_{\lambda_2}$. For any $\lambda, \mu \in [0,1]$,

$$\lambda \sim \mu \text{ iff } A_\lambda = A_\mu \qquad (3.7)$$

"~" is an equivalence relation on $[0,1]$. Defining $E = [0,1]/\sim$, the elements of E are either singleton sets or intervals in $[0,1]$. Thus, E has a linear order $\leq$:

$$e_1 \leq e_2 \Leftrightarrow \exists \lambda_1 \in e_1, \lambda_2 \in e_2 \text{ such that } \lambda_1 \leq \lambda_2. \qquad (3.8)$$

Hence, we can define the interval in E:

$$[e_1, e_2] \triangleq \{e \in E | e_1 \leq e \leq e_2\}.$$

Set

$$\mathcal{E} \triangleq \sigma(\{[e_1, e_2] | e_1, e_2 \in E\}). \qquad (3.9)$$

Set

$$T : E \to L, \quad e \mapsto A_e \triangleq A_\lambda \ (\forall \lambda \in e), \qquad (3.10)$$

T is a bijection.

Moreover, we have

$$T^{-1}(C_L(u)) = [\dot{0}, \dot{\mu_A}(u)] \in \mathcal{E}, \qquad (3.11)$$

where $\dot{\lambda}$ is the class containing λ in E. Therefore (3.2) is satisfied.

Because (Ω, A, P) is sufficient for $(\mathbb{R}, \mathcal{B}_0)$, there is a $(\mathcal{B}, \mathcal{B}_0)$-measurable mapping $\xi_0 : \Omega \to [0,1]$ such that its distribution is uniform in $[0,1]$, i.e. $m_0 = P \circ \xi_0^{-1}$, m_0 Lebesgue measure over $[0,1]$. Set, for each $\omega \in \Omega$, $\xi(\omega) \triangleq (\dot{\xi_0}(\omega))$. Clearly, ξ is $\mathcal{E}$-measurable. According to theorem 1, we have

$$S \triangleq \tau \circ \xi \in S(\Omega, A, P; U, \mathcal{B}, \check{\mathcal{B}}).$$

But

$$\begin{aligned} P\{\omega | S(\omega) \in C(u)\} &= P\{\omega | S(\omega) \in C_L(u)\} \\ &= P(\xi \in [\dot{0}, \dot{\mu_A}(u)]) \\ &= P(\xi_0 \in [0, \mu_A(u)]) \\ &= \mu_A(u). \end{aligned}$$ ¤

Does the converse proposition of Theorem 2 hold? We cannot answer this yet. But we could obtain a partial result here.

Definition 6. A projectable random subset is called a simple projectable random subset if its range only contains a finite subclass of $\mathcal{B}$. The collection of all simple projectable random subsets is denoted by S^*.

Lemma 4. Suppose that (Ω, A, P) is sufficient for $(\mathbb{R}, \mathcal{B}_0)$, then each simple projectable random subset of $\mathbb{R}$ can be projected into a simple fuzzy set membership function on $(U, \mathcal{B})$. Conversely, each simple fuzzy set membership function on $(U, \mathcal{B})$ is projected from some $S^* \in S^*$.

Proof. For any $S^* \in S^*$, there is a partition of Ω : $A_i \in A$ $(i = 1, \ldots, n)$

and $S^*(\omega) = B_i$ $(\forall\omega \in A_i)$ $(B_i \in \mathcal{B})$ $(i = 1,\ldots,n)$. Then, the membership function of $\underset{\vee}{S}^*$ is a simple function on $(U,\mathcal{B})$:

$$\mu_{\underset{\vee}{S}^*}(u) = \sum_{i=1}^{n} P(A_i)\chi_{B_i}(u). \tag{3.12}$$

the converse conclusion is clear from theorem 2. ¤

Definition 7. We write $S_1 \subseteq S_2$ iff $S_1(\omega) \subseteq S_2(\omega)$ $(\forall\omega \in \Omega)$, and we write $S_n \uparrow(\downarrow) S$ iff $S_n(\omega) \uparrow(\downarrow) S(\omega) = \bigcup_{n=1}^{\infty} S_n(\omega)$ $\left(\bigcap_{n=1}^{\infty} S_n(\omega)\right)$ $(\forall\omega \in \Omega)$.

Lemma 5.

$$S_1 \subseteq S_2 \Rightarrow \mu_{\underset{\vee}{S}_1} \leq \mu_{\underset{\vee}{S}_2} \tag{3.13}$$

$$S_n \uparrow(\downarrow) S \Rightarrow \mu_{\underset{\vee}{S}_n} \uparrow(\downarrow) \mu_{\underset{\vee}{S}} \tag{3.14}$$

The proof is clear. ¤

Theorem 3. Suppose that $\mathcal{S} = \mathcal{S}(\Omega,\mathcal{A},P;U,\mathcal{B},\check{\mathcal{B}})$ is the smallest closure, containing $\mathcal{S}^*$, under the operations of denumerable union, denumerable intersection and complement, then for every $S \in \mathcal{S}$, the projected fuzzy subset $\underset{\vee}{S}$ is $(\mathcal{B},\mathcal{B}_0)$-measurable.

Proof. Clearly, $\mathcal{S}^*$ is a random algebra, i.e., $\mathcal{S}^*$ is a closed class under the operations of finite union, finite intersection and complement. As an analogy to measure theory, because $\mathcal{S}$ is smallest σ-algebra containing $\mathcal{S}^*$, so is the closure containing $\mathcal{S}^*$ under the monotonic limit operations defined in Definition 7.

$$\text{Set } \phi : \mathcal{S} \to [0,1]^U$$
$$S \mapsto \mu_{\underset{\vee}{S}} \tag{3.15}$$

and let $\mathcal{M} \subseteq [0,1]^U$ be the collection of all $(\mathcal{B},\mathcal{B}_0)$-measurable functions which are the membership functions of fuzzy subsets of U, i.e.,

$$\mathcal{M} \triangleq \{\mu \mid \mu : U \to [0,1] \text{ \& } \mu \text{ is } (\mathcal{B},\mathcal{B}_0)\text{-measurable}\}. \tag{3.16}$$

Let $\mathcal{M}^*$ be the simple functions subclass of $\mathcal{M}$, i.e.,

$$\mathcal{M}^* \triangleq \{\mu \mid \mu \in \mathcal{M} \text{ \& } \mu \text{ is a simple function}\}. \tag{3.17}$$

The mapping ϕ conserves the operations of monotonic limits so that $\phi(\mathcal{S})$ is the smallest closure containing $\mathcal{M}^*$ under monotonic convergence. But $\mathcal{M} \supseteq \mathcal{M}^*$ and $\mathcal{M}$ is a closed class under monotonic convergence, too, so that $\phi(\mathcal{S}) \subseteq \mathcal{M}$. ¤

4. PROJECTABLE RANDOM INTERVALS

Let $\mathcal{B}_0$ be the Borel field on $\mathbb{R}$, and let $(\mathcal{B}_0,\check{\mathcal{B}}_0)$ be the projectable measurable space on $\mathbb{R}$, i.e.,

$$\check{\mathcal{B}}_0 = \sigma(\{C(x) \mid x \in \mathbb{R} \text{ \& } C(x) \triangleq \{B \in \mathcal{B}_0 \mid x \in B\}\}). \tag{4.1}$$

Let

$$\delta \triangleq \{[x_1,x_2] \mid x_1,x_2 \in \mathbb{R} \text{ \& } x_1 \leq x_2\}. \tag{4.2}$$

Definition 8. A projectable random subset $S \in \mathcal{S}(\Omega,\mathcal{A},P;U,\mathcal{B},\check{\mathcal{B}})$ is called a projectable random interval if it is interval-valued, i.e.,

$$\text{range}(S) \subseteq \delta \tag{4.3}$$

the class of all of them being denoted by $\mathcal{I}$.

What is most convenient in our definition of a projectable measurable space is that we can transform a projectable random interval into a bivariate random vector.

Set

$$\mathbb{R}^2_{/} \triangleq \{(x,y) \mid x,y \in \mathbb{R} \text{ \& } x \leq y\} \tag{4.4}$$

Let $\mathcal{B}^2_0$ be the Borel field on $\mathbb{R}^2$. Set

$$\mathcal{B}^2_{/} \triangleq \{D \mid D \in \mathcal{B}^2_0 \text{ \& } D \subseteq \mathbb{R}^2_{/}\} \tag{4.5}$$

$\mathcal{B}^2_{/}$ is the restriction of $\mathcal{B}^2_0$ to $\mathbb{R}^2_{/}$, so it is a σ-algebra too. Let $(\Omega,\mathcal{A},P)$ be a given probability space.

Let

$$\mathcal{V} \triangleq \{\xi \mid \xi : \Omega \to \mathbb{R}^2_{/} \text{ \& } \xi \text{ is } (\mathcal{A},\mathcal{B}^2_{/})\text{-measurable}\}, \tag{4.6}$$

i.e., $\mathcal{V}$ is the set of all bivariate random vectors over the upper triangular real plane.

Set

$$\tau : \mathbb{R}^2_{/} \to \delta,$$

where

$$\text{ordered pair } (x,y) \mapsto \text{interval } [x,y]. \tag{4.7}$$

Obviously, τ is a bijection.

Lemma 6.

$$\tau(\mathcal{B}^2_{/}) \supseteq \{C_\delta(x) \mid x \in \mathbb{R}\}. \tag{4.8}$$

Proof. For any fixed $a \in \mathbb{R}$, we have

$$\begin{aligned} \tau^{-1}(C_\delta(a)) &= \{(x,y) \mid (x,y) \in \mathbb{R}^2_{/} \text{ \& } x \leq a \leq y\} \\ &= (-\infty,a] \times [a,+\infty) \\ &= \lrcorner_a \in \mathcal{B}^2_{/}, \end{aligned} \tag{4.9}$$

where

$$\lrcorner_x \triangleq (-\infty,x] \times [x,+\infty), \text{ for all } x \text{ in } \mathbb{R}. \tag{4.10}$$

¤

Theorem 4. There is a bijection

$$\psi : \mathcal{V} \to \mathcal{I} \tag{4.11}$$

Proof. For any $\xi \in \mathcal{V}$, define the mapping

$$\psi(\xi) : \Omega \to \delta, \tag{4.12}$$

where for all $\omega \in \Omega$, $(\psi(\xi))(\omega) \triangleq \tau(\xi(\omega))$.

Clearly, the mapping

$$\psi : \mathcal{V} \to \delta^{\Omega} \tag{4.13}$$

is an injection.

From Theorem 1 (take $L = \delta$, $E = \mathbb{R}^2_/$, $\mathcal{E} = \mathcal{B}^2_/$ and by Lemma 6), we can see that for any $\xi \in \mathcal{V}$, $\psi(\xi)$ is $(A, \check{\mathcal{B}}_0)$-measurable, i.e.,

$$\psi(\xi) \in \mathcal{I} \tag{4.14}$$

Hence, mapping ψ is an injection from $\mathcal{V}$ to $\mathcal{T}$ and we can rewrite ψ in the form:

$$\psi : \mathcal{V} \to \mathcal{I}. \tag{4.15}$$

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We want to prove that ψ is a bijection. This conclusion will be provided by Lemma 7.

Lemma 7.

$$\tau(\mathcal{B}^2_/) \subseteq \mathcal{B}_0 \tag{4.16}$$

Proof. It is not difficult to prove that

$$\mathcal{B}^2_/ = \sigma(\{(\Delta_1 \times \Delta_2) \cap \mathbb{R}^2_/ \mid \Delta_1, \Delta_2 \in \delta\})$$

$$= \sigma(\{\Delta_1 \times \Delta_2 \mid \Delta_1, \Delta_2 \in \delta \ \& \ \Delta_1 \times \Delta_2 \subseteq \mathbb{R}^2_/\}). \tag{4.17}$$

For $[a_1,b_1] \times [a_2,b_2] \subseteq \mathbb{R}^2_/$, we have

$$\begin{aligned}
[a_1,b_1] \times [a_2,b_2] &= (-\infty,b_1] \times [a_2,+\infty) - (-\infty,a_1] \\
&\quad \times [a_2,+\infty) - (-\infty,b_1] \times [b_2,+\infty) \\
&= ((-\infty,b_1] \times [b_1,+\infty) \cap (-\infty,a_2] \times [a_2,+\infty)) \\
&\quad - ((-\infty,a_1] \times [a_1,+\infty) \cap (-\infty,a_2] \times [a_2,+\infty)) \\
&\quad - ((-\infty,b_1] \times [b_1,+\infty) \cap (-\infty,b_2] \times [b_2,+\infty)) \\
&= (J_{b_1} \cap J_{a_2}) - (J_{a_1} \cap J_{a_2}) - (J_{b_1} \cap J_{b_2}).
\end{aligned} \tag{4.18}$$

From (4.9) and (4.18) we have

$$\begin{aligned}
[a_1,b_1] \times [a_2,b_2] &= \tau^{-1}(C_\delta(b_1) \cap C_\delta(a_2) \\
&\quad - C_\delta(a_1) \cap C_\delta(a_2) - C_\delta(b_1) \cap C_\delta(b_2)) \\
&\in \tau^{-1}(\check{\mathcal{B}}_0),
\end{aligned}$$

so that $\tau([a_1,b_1] \times [a_2,b_2]) \in \check{\mathcal{B}}_0$. (4.19)

From (4.17), the Lemma can be proved. ¤

The next thoerem will give us a relationship between $\underset{\smile}{S}$ and $\Psi^{-1}(S)$.

Theorem 5. Suppose that $\xi = (\xi_1, \xi_2) \in \mathcal{V}$ and $S = \psi(\xi)$, then

$$\mu_{\underset{\smile}{S}}(x) = P(\{\omega \mid \xi(\omega) \in J_x\}). \tag{4.20}$$

Suppose that ξ has density $p(x_1,x_2)$, then for any $x \in \mathbb{R}$,

$$\mu_{\underset{\smile}{S}}(x) = \iint_{J_x} p(x_1,x_2)\,dx_1 dx_2$$

$$= \int_x^{+\infty} \left(\int_{-\infty}^{x} p(x_1,x_2)\,dx_1\right) dx_2. \tag{4.21}$$

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The proof is trivial.

Example 1. Suppose that η_1, η_2 are independent random variables which are both distributed uniformly in $[a,b]$, and set

$$\xi = (\xi_1,\xi_2) = (\min(\eta_1,\eta_2), \max(\eta_1,\eta_2)) \tag{4.22}$$

Clearly, $\xi \in \mathcal{V}$ and has the density function:

$$p_\xi(x_1,x_2) = \begin{cases} \dfrac{2}{(b-a)^2} & \text{if } a \le x_1, x_2 \le b \\ 0 & \text{otherwise} \end{cases} \tag{4.23}$$

According to Theorem 5, we have

$$\mu_{\underset{\smile}{\psi(\xi)}}(x) = \frac{2}{(b-a)^2} \int_{-\infty}^{x} \int_{x}^{+\infty} \chi_{[a \le x_1 \le x_2 \le b]}(x_1,x_2)\,dx_1 dx_2$$

$$= \begin{cases} \dfrac{2(x-a)(b-x)}{(b-a)^2} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases} \tag{4.24}$$

(see figure 1)

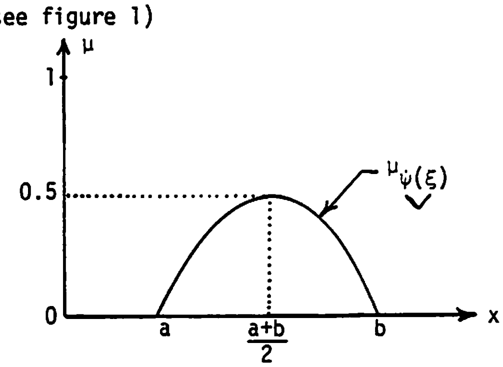

Figure 1

Example 2. If we wish to describe the following concepts on $[\alpha,\beta]$:

{SHORT, MEDIUM, TALL}

we can point two points ξ_1 and ξ_2 such that

SHORT $= [\alpha, \xi_1)$

MEDIUM $= [\xi_1, \xi_2)$

TALL $= [\xi_2, \beta)$.

Suppose that ξ_1, ξ_2 are independent random variables having density functions $p_{\xi_1}(x)$ and $p_{\xi_2}(x)$ that satisfy

$$p_{\xi_1}(x) = 0 \quad \text{if } x \notin (a_1, b_1)$$

$$p_{\xi_2}(x) = 0 \quad \text{if } x \notin (a_2, b_2),$$

here $\alpha < a_1 < b_1 < a_2 < b_2 < \beta$; then we have

$$\mu_{MEDIUM}(x) = \begin{cases} 0 & \text{if } \alpha \leq x \leq a_1, \\ \int_{a_1}^{x} p_{\xi_1}(u)du & \text{if } a_1 < x \leq b_1, \\ 1 & \text{if } b_1 < x \leq a_2, \\ \int_{x}^{b_2} p_{\xi_2}(u)du & \text{if } a_2 < x \leq b_2, \\ 0 & \text{if } b_2 < x \leq \beta. \end{cases} \tag{4.25}$$

$$\mu_{SHORT}(x) = \begin{cases} 1 & \text{if } \alpha \leq x \leq a_1; \\ \int_{x}^{b_1} p_{\xi_1}(u)du & \text{if } a_1 < x \leq b_1; \\ 0 & \text{if } b_1 < x \leq \beta. \end{cases} \tag{4.26}$$

$$\mu_{TALL}(x) = \begin{cases} 0 & \text{if } \alpha \leq x \leq a_2; \\ \int_{a_2}^{x} p_{\xi_2}(u)du & \text{if } a_2 < x \leq b_2; \\ 1 & \text{if } b_2 < x \leq \beta. \end{cases} \tag{4.27}$$

(see figure 2)

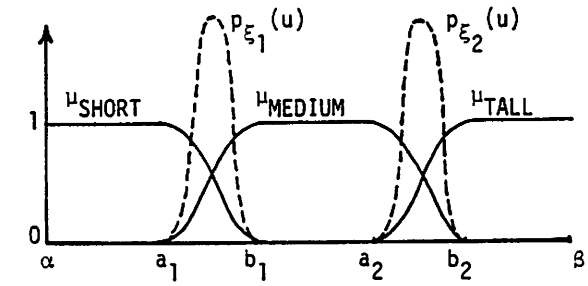

Figure 2

The method for constructing a membership function like in this example is called the <u>Three Phase Method</u>. It is given by Qu Yin-Sheng [7].

Note that

$$\mu_{SHORT}(x) + \mu_{MEDIUM}(x) + \mu_{TALL}(x) \equiv 1$$
$$(\alpha \leq x \leq \beta) \tag{4.28}$$

5. FUZZY NUMBERS AND THEIR CHARACTERISTIC CURVES

Let σ be a mapping

$$\sigma : [0,1] \to \mathbb{R}^2_/$$
$$\lambda \mapsto \sigma(\lambda) = (\sigma_1(\lambda), \sigma_2(\lambda))$$
$$\text{with } \sigma_1(\lambda) \leq \sigma_2(\lambda) \tag{5.1}$$

<u>Definition 9</u>. We say that the mapping σ is a <u>characteristic curve</u> if σ_1 is monotonic increasing and σ_2 is monotonic decreasing. The collection of all of them is denoted by Σ. If $\sigma \in \Sigma$, then σ is $\mathcal{B}^2_/$-measurable. Let m_0 be the Lebesgue measure on $\mathbb{R}$, set $P_\sigma = m_0 \circ \sigma^{-1}$, i.e.,

$$P_\sigma(B) = m_0(\{\lambda | \sigma(\lambda) \in B\}) \ (\forall B \in \mathcal{B}^2_/), \tag{5.2}$$

P_σ is a probability measure on $\mathcal{B}^2_/$ which is induced by σ from m_0.

<u>Definition 10</u>. A mapping $\sigma \in \Sigma$ is called the <u>characteristic curve of a given projectable random interval</u> S (or of $\psi^{-1}(S)$ or of the <u>projected fuzzy</u> subset $\underset{\sim}{S}$) if

$$P_\sigma(\lrcorner_x) = P(S \in C(x)) = P(\psi^{-1}(S) \in \lrcorner_x) = \mu_{\underset{\sim}{S}}(x). \tag{5.3}$$

Obviously, if the characteristic curve exists, it is the characterization of a class of projectable random intervals having a same projected fuzzy subset.

When α is a fuzzy subset of $\mathbb{R}$, we call it a fuzzy number if for any $\lambda \in (0,1]$, its λ-cut is always a closed interval:

$$\alpha_\lambda \triangleq \{x \in \mathbb{R} | \mu_\alpha(x) \geq \lambda\} = [\alpha^1_\lambda, \alpha^2_\lambda]. \tag{5.4}$$

We say that α is normal if there exists $x_0 \in \mathbb{R}$ such that $\mu_\alpha(x_0) = 1$.

<u>Theorem 6</u>. Suppose that (Ω, A, P) is sufficient for $(\mathbb{R}, \mathcal{B}_0)$. For any normal fuzzy number α, there exists one and only one $\sigma \in \Sigma$ such that σ is the characteristic curve of α. Conversely, for any $\sigma \in \Sigma$, there exists one and only one normal fuzzy number α such that its characteristic curve is σ.

Proof. Given a normal fuzzy number α, set

$$\sigma : [0,1] \to \mathbb{R}^2_/$$
$$\lambda \mapsto (\alpha^1_\lambda, \alpha^2_\lambda), \tag{5.5}$$

where α^1_λ and α^2_λ are defined in (5.4). From

$$\lambda_1 < \lambda_2 \Rightarrow \alpha_{\lambda_1} \supseteq \alpha_{\lambda_2},$$

we have $\alpha_\lambda^1 \nearrow$ and $\alpha_\lambda^2 \searrow$. Moreover,

$$P_\sigma(\lrcorner_x) = m_0(\{x|\sigma(\lambda) \in \lrcorner_x\}) = m_0(\{\lambda|x \in \alpha_\lambda\})$$
$$= \mu_\alpha(x) \tag{5.6}$$

so that σ is the characteristic curve of α.

Conversely, given $\sigma \in \Sigma$, set

$$\mu_\alpha(x) \triangleq P_\sigma(\lrcorner_x) \tag{5.7}$$

so that $\mu_\alpha(x) \geq \lambda \Leftrightarrow P_\sigma(\lrcorner_x) \geq \lambda$

$$\Leftrightarrow \sigma(\lambda) \in \lrcorner_x$$
$$\Leftrightarrow \sigma_1(\lambda) \leq x \leq \sigma_2(\lambda) \tag{5.8}$$

hence,

$$\alpha_\lambda = [\sigma_1(\lambda),\sigma_2(\lambda)], \tag{5.9}$$

α is a fuzzy number. Moreover, for

$$x_0 = \max(\sigma_1(1),\sigma_2(1)) \tag{5.10}$$

we have

$$P_\sigma(\lrcorner_{x_0}) = 1, \tag{5.11}$$

so that α is a normal fuzzy number. ¤

ACKNOWLEDGEMENTS

The authors are grateful to Dr. I. R. Goodman for his valuable comments and criticisms that allowed to improve an earlier draft of this paper, and to Dr. H. T. Nguyen for his competent remarks.

Research sponsored in part by National Science Foundation Grants MCS79-06543 and IST-801896.

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