A COMPUTATIONAL APPROACH TO FUZZY QUANTIFIERS IN NATURAL LANGUAGES

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ABSTRACT

The generic term *fuzzy quantifier* is employed in this paper to denote the collection of quantifiers in natural languages whose representative elements are: *several, most, much, not many, very many, not very many, few, quite a few, large number, small number, close to five, approximately ten, frequently,* etc. In our approach, such quantifiers are treated as fuzzy numbers which may be manipulated through the use of fuzzy arithmetic and, more generally, fuzzy logic.

A concept which plays an essential role in the treatment of fuzzy quantifiers is that of the cardinality of a fuzzy set. Through the use of this concept, the meaning of a proposition containing one or more fuzzy quantifiers may be represented as a system of elastic constraints whose domain is a collection of fuzzy relations in a relational database. This representation, then, provides a basis for inference from premises which contain fuzzy quantifiers. For example, from the propositions "Most U's are A's" and "Most A's are B's," it follows that " Most^2 U's are B's," where $most^2$ is the fuzzy product of the fuzzy proportion *most* with itself.

To Professor Kokichi Tanaka

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The computational approach to fuzzy quantifiers which is described in this paper may be viewed as a derivative of fuzzy logic and test-score semantics. In this semantics, the meaning of a semantic entity is represented as a procedure which tests, scores and aggregates the elastic constraints which are induced by the entity in question.

1. INTRODUCTION

During the past two decades, the work of Montague and others (Montague (1974), Partee (1976), Dowty (1981)) has contributed much to our understanding of the proper treatment of the quantifiers *all*, *some* and *any* when they occur singly or in combination in a proposition in a natural language.

Recently, Barwise and Cooper and others (Barwise and Cooper (1981), Peterson (1979)) have described methods for dealing with so-called *generalized quantifiers* exemplified by *most*, *many*, etc. In a different approach which we have described in a series of papers starting in 1975 (Zadeh (1975a, 1975b, 1977, 1978a, 1978b, 1981a)), the quantifiers in question – as well as other quantifiers with imprecise meaning such as *few*, *several*, *not very many*, etc. – are treated as fuzzy numbers and hence are referred to as *fuzzy quantifiers*. As an illustration, a fuzzy quantifier such as *most* in the proposition "Most big men are kind" is interpreted as a fuzzily defined proportion of the fuzzy set of kind men in the fuzzy set of big men. Then, the concept of the cardinality* of a fuzzy set is employed to compute the proportion in question and find the degree to which it is compatible with the meaning of *most*.

* Informally, the cardinality of a fuzzy set F is a real or fuzzy number which serves as a count of the number of elements "in" F. A more precise definition of cardinality will be given in Section 2.

We shall employ the class labels "fuzzy quantifiers of the first kind" and "fuzzy quantifiers of the second kind" to refer to absolute and relative counts, respectively, with the understanding that a particular quantifier, e.g., *many*, may be employed in either sense, depending on the context. Common examples of quantifiers of the first kind are: *several, few, many, not very many, approximately five, close to ten, much larger than ten, a large number*, etc., while those of the second kind are: *most, many, a large fraction, often, once in a while, much of*, etc. Where needed, ratios of fuzzy quantifiers of the second kind will be referred to as *fuzzy quantifiers of the third kind*. Examples of quantifiers of this type are the likelihood ratios and certainty factors which are encountered in the analysis of evidence, hypothesis testing and expert systems. (Shortliffe (1976), Duda and Hart (1978), Barr and Feigenbaum (1982).)

An important aspect of fuzzy quantifiers is that their occurrence in human discourse is, for the most part, implicit rather than explicit. For example, when we assert that "Basketball players are very tall," what we usually mean is that "Almost all basketball players are very tall." Likewise, the proposition, "Lynne is never late," would normally be interpreted as "Lynne is late very rarely." Similarly, by "Overeating causes obesity," one may mean that "Most of those who overeat are obese," while "Heavy smoking causes lung cancer," might be interpreted as "The incidence of lung cancer among heavy smokers is much higher than among nonsmokers."

An interesting observation that relates to this issue is that *property inheritance* -- which is exploited extensively in knowledge representation systems and high-level AI languages (Barr and Feigenbaum (1982)) -- is a *brittle* property with respect to the replacement of the nonfuzzy quantifier *all* with the fuzzy quantifier *almost all*.* What this means is that if in the inference rule**

* The brittleness of property inheritance is of relevance to nonmonotonic logic, default reasoning and exception handling.
** The symbol $\triangleq$ stands for "denotes" or "is defined to be."

$$p \triangleq all\ A\text{'s are } B\text{'s}$$

$$\underline{q \triangleq all\ B\text{'s are } C\text{'s}}$$

$$r \triangleq all\ A\text{'s are } C\text{'s}$$

the quantifier *all* in p and q is replaced by *almost all*, then the quantifier *all* in r should be replaced by *none-to-all*. Thus, a slight change in the quantifier *all* in the premises may result in a large change in the quantifier *all* in the conclusion.*

Another point which should be noted relates to the close connection between fuzzy quantifiers and fuzzy probabilities. Specifically, it can be shown (Zadeh (1975a, 1981b)) that a proposition of the form $p \triangleq Q$ A's are B's, where Q is a fuzzy quantifier (e.g., $p \triangleq$ most doctors are not very tall), implies that the conditional probability of the event B given the event A is a fuzzy probability which is equal to Q. What can be shown, in fact, is that most statements involving fuzzy probabilities may be replaced by semantically equivalent statements involving fuzzy quantifiers. This connection between fuzzy quantifiers and fuzzy probabilities plays an important role in expert systems and fuzzy temporal logic, but we shall not dwell on it in the present paper.

As was stated earlier, the main idea underlying our approach to fuzzy quantifiers is that the natural way of dealing with such quantifiers is to treat them as fuzzy numbers. However, this does not imply that the concept of a fuzzy quantifier is coextensive with that of a fuzzy number. Thus, in the proposition "Vickie is several years younger than Mary," the fuzzy number *several* does

* An example which relates to this phenomenon is: What is rare is expensive. A cheap apartment in Paris is rare. Therefore, a cheap apartment in Paris is expensive. This example was suggested to the author in a different connection by Professor O. Botta of the University of Lyon.

not play the role of a fuzzy quantifier, whereas in "Vickie has several good friends," it does. More generally, we shall view a fuzzy quantifier as a fuzzy number which provides a fuzzy characterization of the absolute or relative cardinality of one or more fuzzy or nonfuzzy sets. For example, in "Vickie has several credit cards," *several* is a fuzzy characterization of the cardinality of the nonfuzzy set of Vickie's credit cards; in "Vickie has several good friends," *several* is a fuzzy characterization of the cardinality of the fuzzy set of Vickie's good friends; and in "Most big men are kind," *most* is a fuzzy characterization of the relative cardinality of the fuzzy set of kind men in the fuzzy set of big men. There are propositions, however, in which the question of whether or not a constituent fuzzy number is a fuzzy quantifier does not have a clear cut answer.

A simple example may be of help at this point in providing an idea of how fuzzy quantifiers may be treated as fuzzy numbers. Specifically, consider the propositions

$$p \triangleq 80\% \textit{ of students are single}$$

$$\underline{q \triangleq 60\% \textit{ of single students are male}}$$

$$r \triangleq Q \textit{ of students are single} \text{ and } \textit{male}$$

in which r represents the answer to the question "What percentage of students are single males?" given the premises expressed by p and q.

Clearly, the answer is: 80% x 60% = 48%, and, more generally, we can assert that:

$$p \triangleq Q_1 \textit{ of } A\text{'s are } B\text{'s} \tag{1.1}$$

$$\underline{q \triangleq Q_2 \textit{ of } (A \text{ and } B)\text{'s are } C\text{'s}}$$

$$r \triangleq Q_1\, Q_2 \textit{ of } A\text{'s are } (B \text{ and } C)\text{'s}$$

where Q_1 and Q_2 are numerical percentages, and A, B and C are labels of non-fuzzy sets or, equivalently, names of their defining properties.

Now suppose that Q_1 and Q_2 are fuzzy quantifiers of the second kind, as in the following example:

$$p \triangleq \textit{most students are single}$$

$$\underline{q \triangleq \textit{a little more than a half of single students are male}}$$

$$r \triangleq \, ?Q \textit{ of students are single } \text{and } \textit{male}$$

where the question mark indicates that the value of Q is to be inferred from p and q.

By interpreting the fuzzy quantifiers *most*, *a little more a than half*, and Q as fuzzy numbers which characterize, respectively, the proportions of single students among students, males among single students and single males among students, we can show that Q may be expressed as the product, in fuzzy arithmetic (see Appendix), of the fuzzy numbers *most* and *a little more than a half*. Thus, in symbols,

$$Q = \textit{most} \otimes \textit{a little more than a half} \tag{1.2}$$

and, more generally, for fuzzy Q's, A's, B's and C's, we can assert the syllogism:

$$p \triangleq Q_1 \textit{ of } A\text{'s } \textit{are } B\text{'s} \tag{1.3}$$

$$\underline{q \triangleq Q_2 \textit{ of } (A \text{and} B)\text{'s } \textit{are } C\text{'s}}$$

$$r \triangleq Q_1 \otimes Q_2 \textit{ of } A\text{'s } \textit{are } (B \text{and} C)\text{'s},$$

which will be referred to as the *intersection/product* syllogism. A pictorial representation of (1.2) is shown in Fig. 1.

The point of this example is that the syllogism (or the *inference schema*) expressed by (1.1) generalizes simply and naturally to fuzzy quantifiers when they are treated as fuzzy numbers. Furthermore, through the use of *linguistic approximation* (Zadeh (1975b), Mamdani and Gaines (1981)) – which is analogous to rounding to an integer in ordinary arithmetic – the expression for Q may be approximated to by a fuzzy quantifier which is an element of a specified context-free language. For example, in the case of (1.2), such a quantifier may be expressed as ***about a half***, or ***more or less close to a half***, etc., depending on how the fuzzy numbers ***most***, ***a little more than a half***, and ***close to a half*** are defined through their respective possibility distributions (see Appendix).

In our discussion so far, we have tacitly assumed that a fuzzy quantifier is a fuzzy number of type 1, i.e., a fuzzy set whose membership function takes values in the unit interval. More generally, however, a fuzzy quantifier may be a fuzzy set of type 2 (or higher), in which case we shall refer to it as an *ultrafuzzy quantifier* . The membership functions of such quantifiers take values in the space of fuzzy sets of type 1, which implies that the compatibility of an ultrafuzzy quantifier with a real number is a fuzzy number of type 1. For example, the fuzzy quantifier *not so many* would be regarded as an ultrafuzzy quantifier if the compatibility of *not so many* with 5, say, would be specified in a particular context as *rather high*, where *rather high* is interpreted as a fuzzy number in the unit interval.

Although the rule of inference expressed by (1.3) remains valid for ultrafuzzy quantifiers if $\otimes$ is interpreted as the product of ultrafuzzy numbers (see Fig. 2), we shall restrict our attention in the present paper to fuzzy quantifiers of type 1, with the understanding that most of the inference schemas derived on this assumption can readily be generalized to fuzzy quantifiers of higher type.

As will be seen in the sequel, a convenient framework for the treatment of fuzzy quantifiers as fuzzy numbers is provided by a recently developed meaning-representation system for natural languages termed *test-score semantics* (Zadeh (1981a)). Test-score semantics represents a break with the traditional approaches to semantics in that it is based on the premise that almost everything that relates to natural languages is a matter of degree. The acceptance of this premise necessitates an abandonment of bivalent logical systems as a basis for the analysis of natural languages and suggests the adoption of fuzzy logic (Zadeh (1975a, 1977), Bellman and Zadeh (1977)) as the basic conceptual framework for the representation of meaning, knowledge and strength of belief.

Viewed from the perspective of test-score semantics, a semantic entity such as a proposition, predicate, predicate-modifier, quantifier, qualifier, command, question, etc., may be regarded as a system of elastic constraints whose domain is a collection of fuzzy relations in a database -- a database which describes a state of affairs (Carnap (1952)) or a possible world (Lambert and van Fraassen (1970)) or, more generally, a set of objects or derived objects in a universe of discourse. The meaning of a semantic entity, then, is represented as a test which when applied to the database yields a collection of partial test scores. Upon aggregation, these test scores lead to an overall vector test score, τ, whose components are numbers in the unit interval, with τ serving as a measure of the compatibility of the semantic entity with the database. In this respect, test-score semantics subsumes both truth-conditional and possible-world semantics as limiting cases in which the partial and overall test scores are restricted to {pass, fail} or, equivalently, {true, false} or {1,0}.

In more specific terms, the process of meaning representation in test-score semantics involves three distinct phases. In Phase I, an *explanatory database* *frame* or EDF, for short, is constructed. EDF consists of a collection of relational frames, i.e., names of relations, names of attributes and attribute domains, whose meaning is assumed to be known. In consequence of this assumption, the choice of EDF is not unique and is strongly influenced by the knowledge profile of the addressee of the representation process as well as by the desideratum of explanatory effectiveness. For example, in the case of the proposition $p \triangleq$ Over the past few years Nick earned far more than most of his close friends, the EDF might consist of the following relations*: *INCOME* [*Name; Amount; Year*], which lists the income of each individual identified by his/her name as a function of the variable *Year; FRIEND* [*Name;* μ], where μ is the degree to which *Name* is a friend of Nick; *FEW* [*Number;* μ], where μ is the degree to which *Number* is compatible with the fuzzy quantifier *FEW; MOST* [*Proportion;* μ], in which μ is the degree to which *Proportion* is compatible with the fuzzy quantifier *MOST;* and *FAR.MORE* [*Income1; Income2;* μ], where μ is the degree to which *Income1* fits the fuzzy predicate *FAR.MORE* in relation to *Income2*. Each of these relations is interpreted as an elastic constraint on the variables which are associated with it.

In Phase 2, a test procedure is constructed which acts on relations in the explanatory database and yields the test scores which represent the degrees to which the elastic constraints induced by the constituents of the semantic entity are satisfied. For example, in the case of p, the test procedure would yield the test scores for the constraints induced by the relations *FRIEND, FEW, MOST* and *FAR.MORE.*

In Phase 3, the partial test scores are aggregated into an overall test score, τ, which, in general, is a vector which serves as a measure of the compatibility

* Generally, we follow the practice of writing the names of fuzzy subsets and fuzzy relations in uppercase symbols.

of the semantic entity with an instantiation of EDF. As was stated earlier, the components of this vector are numbers in the unit interval, or, more generally, possibility/probability distributions over this interval. In particular, in the case of a proposition, p, for which the overall test score is a scalar, τ may be interpreted -- in the spirit of truth-conditional semantics -- as the degree of truth of the proposition with respect to the explanatory database ED (i.e., an instantiation of EDF). Equivalently, τ may be interpreted as the possibility of ED given p, in which case we may say that p ***induces a possibility distribution.*** More concretely, we shall say that p ***translates into a possibility assignment equation*** (Zadeh 1978a)):

$$p \to \Pi_{(X_1, \ldots, X_n)} = F, \tag{1.4}$$

where F is a fuzzy subset of a universe of discourse U, $X_1, \ldots, X_n$ are variables which are explicit or implicit in p, and $\Pi_{(X_1, \ldots, X_n)}$ is their joint possibility distribution. For example, in the case of the proposition $p \triangleq$ Danielle is tall, we have

$$\textit{Danielle is tall} \to \Pi_{Height(Danielle)} = TALL \quad , \tag{1.5}$$

where $TALL$ is a fuzzy subset of the real-line, $Height(Danielle)$ is a variable which is implicit in p, and $\Pi_{Height(Danielle)}$ is the possibility distribution of the variable $Height(Danielle)$. Equation (1.5) implies that

$$Poss\{Height(Danielle) = u\} = \mu_{TALL}(u),$$

where u is a specified value of the variable $Height(Danielle)$, $\mu_{TALL}(u)$ is the grade of membership of u in the fuzzy set $TALL$, and $Poss\{X = u\}$ should be read as "the possibility that X is u." In effect, (1.5) signifies that the proposition "Danielle is tall," may be interpreted as an elastic constraint on the variable $Height(Danielle)$, with the elasticity of the constraint characterized by the unary relation *TALL* which is defined as a fuzzy subset of the real line.

The same basic idea may be applied to propositions containing one or more fuzzy quantifiers. As a simple illustration, let us consider the proposition

$$p \triangleq \textit{Vickie has several credit cards},$$

in which *several* is regarded as a fuzzy quantifier which induces an elastic constraint on the number of credit cards possessed by Vickie. In this case, X may be taken to be the count of Vickie's cards, and the possibility assignment equation becomes

$$\textit{Vickie has several credit cards} \rightarrow \Pi_{Count(Cards(Vickie))} = SEVERAL, \tag{1.6}$$

in which *SEVERAL* plays the role of a specified fuzzy subset of the integers 1, 2, ..., 10. Thus, if the integer 4, say, is assumed to be compatible with the meaning of *several* to the degree 0.8, then (1.6) implies that, given p and the definition of *several*, the possibility that Vickie has four credit cards is expressed by

$$Poss\{Count(Cards(Vickie))=4\}=0.8 \quad .$$

In the above example, the class of Vickie's credit cards is a nonfuzzy set and hence there is no problem in counting their number. By contrast, in the proposition

$$p \triangleq \textit{Vickie has several close friends}$$

the class of close friends is a fuzzy set and thus we must first resolve the question of how to count the number of elements in a fuzzy set or, equivalently, how to determine its cardinality. This issue is addressed in the following section.

2. CARDINALITY OF FUZZY SETS

In the case of a crisp (nonfuzzy) subset, A, of a universe of discourse, U, the proposition "u is an element of A," is either true or false, and hence there is just one way in which the cardinality of A, i.e., the count of elements of A, may be defined. However, even though the count may be defined uniquely, there may be some uncertainty about its value if there is an uncertainty regarding the membership status of points of U in A.

By contrast, in the case of a fuzzy subset, F, of U, the proposition "u is an element of F," is generally true to degree, with the result that the concept of cardinality admits of a variety of definitions. Among them, some associate with a fuzzy set F a real number, in which case the cardinality of a fuzzy set is nonfuzzy. Others associate with F a fuzzy number, since it may be argued that the cardinality of a fuzzy set should be a fuzzy number. A brief discussion of these viewpoints is presented in the following. For simplicity, we shall restrict our attention to finite universes of discourse, in which case a fuzzy subset, F, of $U=\{u_1, \ldots, u_n\}$ may be expressed symbolically as

$$F = \mu_1/u_1 + \ldots + \mu_n/u_n$$

or, more simply, as

$$F = \mu_1 u_1 + \ldots + \mu_n u_n ,$$

in which the term μ_i/u_i, i = 1,..., n, signifies that μ_i is the grade of membership of u_i in F, and the plus sign represents the union.*

* For the most part we shall rely on the context to disambiguate the meaning of +.

Nonfuzzy Cardinality

A simple way of extending the concept of cardinality to fuzzy sets is to form the *sigma-count* (DeLuca and Termini (1972), Zadeh (1972)), which is the arithmetic sum of the grades of membership in F. Thus

$$\Sigma Count(F) \triangleq \Sigma_i \mu_i, \quad i=1,\ldots,n, \tag{2.1}$$

with the understanding that the sum may be rounded, if need be, to the nearest integer. Furthermore, one may stipulate that the terms whose grade of membership falls below a specified threshold be excluded from the summation. The purpose of such an exclusion is to avoid a situation in which a large number of terms with low grades of membership become count-equivalent to a small number of terms with high membership.

As a simple illustration of the concept of sigma-count, assume that the fuzzy set of close friends of Teresa is expressed as

$$F = 1/\mathit{Enrique} + 0.8/\mathit{Ramon} + 0.7/\mathit{Elie} + 0.9/\mathit{Sergei} + 0.8/\mathit{Ron}.$$

In this case,

$$\Sigma Count(F) = 1 + 0.8 + 0.7 + 0.9 + 0.8$$

$$= 4.2 \quad .$$

A sigma-count may be *weighted*, in the sense that if $w=(w_1, \ldots, w_n)$ is an n-tuple of nonnegative real numbers, then the *weighted sigma-count of F with respect to w* is defined by

$$\Sigma Count(F;w) \triangleq \Sigma_i w_i \mu_i, \quad i=1, \ldots, n.$$

This definition implies that $\Sigma Count(F;w)$ may be interpreted as the sigma-count of a fuzzy multiset* $'F$ in which the grade of membership and the multiplicity of u_i, $i=1, \ldots, n$, are, respectively, μ_i and w_i. The concept of a weighted sigma-count is closely related to that of the *measure* of a fuzzy set (Zadeh (1968), Klement (1981abc)).

Whether weighted or not, the sigma-count of a fuzzy set is a real number. As was stated earlier, it may be argued that the cardinality of a fuzzy set should be a fuzzy number. If one accepts this argument, then a natural way of defining fuzzy cardinality is the following (Zadeh (1977)).

Fuzzy Cardinality**

In this case, the point of departure is a stratified representation of F in terms of its *level sets* (Zadeh (1971)), i.e.,

$$F = \Sigma_\alpha \alpha F_\alpha,$$

in which the α-level-sets F_α are nonfuzzy sets defined by

$$F_\alpha \triangleq \{u \mid \mu_F(u) \geq \alpha\}, \quad 0 < \alpha \leq 1,$$

and

$$\mu_{\alpha F_\alpha}(u) = \alpha \mu_F(u), \quad u \in U.$$

*A fuzzy multiset, $'F$, may be represented as $'F = \Sigma_i \mu_i / m_i \times u_i$, in which m_i is the *multiplicity* of u_i and μ_i is the grade of membership of u_i in the fuzzy set $F = \Sigma_i \mu_i / u_i$. The multiplicity, m_i, is a nonnegative real number which is usually, but not necessarily, an integer. Thus, a fuzzy multiset may have identical elements, or elements which differ only in their grade of membership.

** Although it is perhaps a more natural extension of the concept of cardinality than the sigma-count, fuzzy cardinality is a more complex concept and is more difficult to manipulate. The exposition of fuzzy cardinality in this section may be omitted on first reading.

In terms of this representation, there are three fuzzy counts, *FCounts*, that may be associated with F. First, the *FGCount* is defined as the conjunctive fuzzy integer* (Zadeh (1981a))

$$FGCount(F) = 1/0 + \Sigma_\alpha \alpha / Count(F_\alpha)\ , \qquad \alpha > 0\ .$$

Second, the *FLCount* is defined as

$$FLCount(F) = (FGCount(F))' \ominus 1$$

where ' denotes the complement and $\ominus 1$ means that 1 is subtracted from the fuzzy number *FGCount(F)*. And finally, the *FECount(F)* is defined as the intersection of *FGCount(F)* and *FLCount(F)*, i.e.,

$$FECount(F) = FGCount(F) \cap FLCount(F).$$

Equivalently -- and more precisely -- we may define the counts in question via the membership function of F, i.e.,

$$\mu_{FECount(F)}(i) \triangleq sup_\alpha \{\alpha \mid Count(F_\alpha) \geq i\}, \qquad i = 0, 1, \ldots, n, \tag{2.2}$$

$$\mu_{FLCount(F)}(i) \triangleq sup_\alpha \{\alpha \mid Count(F_\alpha) \geq n - i\} \tag{2.3}$$

$$\mu_{FECount(F)}(i) \triangleq \mu_{FGCount(F)}(i) \wedge \mu_{FLCount(F)}(i), \tag{2.4}$$

where $\wedge$ stands for min in infix position.

As a simple illustration, consider the fuzzy set expressed as

$$F = 0.6/u_1 + 0.9/u_2 + 1/u_3 + 0.7/u_4 + 0.3/u_5. \tag{2.5}$$

In this case,

*It should be noted that the membership function of a conjunctive fuzzy number is not a possibility distribution.

$$F_1 = u_3$$

$$F_{0.9} = u_2 + u_3$$

$$F_{0.7} = u_2 + u_3 + u_4$$

$$F_{0.6} = u_1 + u_2 + u_3 + u_4$$

$$F_{0.3} = u_1 + u_2 + u_3 + u_4 + u_5,$$

which implies that, in stratified form, F may be expressed as

$$F = 1(u_3) + 0.9(u_2 + u_3) + 0.7(u_2 + u_3 + u_4) + 0.6(u_1 + u_2 + u_3 + u_4) + 0.3(u_1 + u_2 + u_3 + u_4 + u_5),$$

and hence that

$$FGCount(F) = 1/0 + 1/1 + 0.9/2 + 0.7/3 + 0.6/4 + 0.3/5$$

$$FLCount(F) = 0.1/2 + 0.3/3 + 0.4/4 + 0.7/5 + 1/6 + \cdots \ominus 1$$

$$= 0.1/1 + 0.3/2 + 0.4/3 + 0.7/4 + 1/5 + \cdots$$

$$FECount(F) = 0.1/1 + 0.3/2 + 0.4/3 + 0.6/4 + 0.3/5$$

while, by comparison,

$$\Sigma Count(F) = 0.6 + 0.9 + 1.0 + 0.7 + 0.3$$

$$= 3.5.$$

A useful interpretation of the defining relations (2.2), (2.3) and (2.4) may be stated as follows:

(a) $\mu_{FGCount}(i)$ is the truth value of the proposition "F contains at least i elements."

(b) $\mu_{FLCount}(i)$ if the truth value of the proposition "F contains at most i elements."

(c) $\mu_{FECount}(i)$ is the truth value of the proposition "F contains i and only i elements."

From (a), it follows that *FGCount(F)* may readily be obtained from F by first sorting F in the order of decreasing grades of membership and then replacing u_i with i and adding the term 1/0. For example, for F defined by (2.5), we have

$$F\downarrow = 1/u_3+0.9/u_2+0.7/u_4+0.6/u_1+0.3/u_5 \tag{2.6}$$

$$NF\downarrow = 1/1+0.9/2+0.7/3+0.6/4+0.3/5$$

and

$$FGCount(F) = 1/0+1/1+0.9/2+0.7/3+0.6/4+0.3/5,$$

where $F\downarrow$ denotes F sorted in descending order, and $NF\downarrow$ is $F\downarrow$ with *ith* u replaced by i. An immediate consequence of this relation between $\Sigma Count(F)$ and $FGCount(F)$ is the identity

$$\Sigma Count(F) = \Sigma_i \mu_{FGCount}(i)-1,$$

which shows that, as a real number, $\Sigma Count(F)$ may be regarded as a "summary" of the fuzzy number $FGCount(F)$.

Relative Count

A type of count which plays an important role in meaning representation is that of ***relative count*** (or ***relative cardinality***) (Zadeh (1975b)). Specifically, if F and G are fuzzy sets, then the relative sigma-count of F in G is defined as the ratio:

$$\Sigma Count(F/G) = \frac{\Sigma Count(F \cap G)}{\Sigma Count(G)}, \tag{2.7}$$

which represents the proportion of elements of F which are in G, with the intersection $F \cap G$ defined by

$$\mu_{F \cap G}(u) = \mu_F(u) \wedge \mu_G(u). \tag{2.8}$$

The corresponding definition for the *FGCount* is

$$FGCount(F/G) = \Sigma_\alpha \alpha / \frac{Count(F_\alpha \cap G_\alpha)}{Count(G_\alpha)}, \tag{2.9}$$

where the F_α and G_α represent the α - sets of F and G, respectively. It should be noted that the right-hand member of (2.9) should be treated as a fuzzy multiset, which implies that terms of the form α_1/u and α_2/u should not be combined into a single term $(\alpha_1 \vee \alpha_2)/u$, as they would be in the case of a fuzzy set.

The $\Sigma Count$ and *FCounts* of fuzzy sets have a number of basic properties of which only a few will be stated here. Specifically, if F and G are fuzzy sets, then from the identity

$$a \vee b + a \wedge b = a + b$$

which holds for any real numbers, it follows at once that

$$\Sigma Count(F \cap G) + \Sigma Count(F \cup G) = \Sigma Count(F) + \Sigma Count(G) \tag{2.10}$$

since

$$\mu_{F \cap G}(u) = \mu_F(u) \wedge \mu_G(u) , \qquad u \varepsilon U$$

and

$$\mu_{F \cup G}(u) = \mu_F(u) \vee \mu_G(u).$$

Thus, if F and G are disjoint (i.e., $F \cap G = \vartheta$), then

$$\Sigma Count(F \cup G) = \Sigma Count(F) + \Sigma Count(G) \tag{2.11}$$

and, more generally,

$$\Sigma Count(F) \vee \Sigma Count(G) \leq \Sigma Count(F \cup G) \leq \Sigma Count(F) + \Sigma Count(G) \tag{2.12}$$

and

$$(\Sigma Count(F) + \Sigma Count(G) - Count(U)) \leq \Sigma Count(F \cap G) \leq \tag{2.13}$$

$$\Sigma Count(F) \wedge \Sigma Count(G).$$

These inequalities follow at once from (2.10) and

$$\Sigma Count(F \cap G) \leq \Sigma Count(F)$$

$$\Sigma Count(F \cap G) \leq \Sigma Count(G)$$

$$\Sigma Count(F \cup G) \leq \Sigma Count(U).$$

In the case of *FCounts* and, more specifically, the *FGCount*, the identity corresponding to (2.10) reads (Zadeh (1981ab), Dubois (1981)),

$$FGCount(F \cap G) \oplus FGCount(F \cup G) = FGCount(F) \oplus FGCount(G). \tag{2.14}$$

where $\oplus$ denotes the addition of fuzzy numbers, which is defined by (see Appendix)

$$\mu_{A \oplus B}(u) = sup_v(\mu_A(v) \wedge \mu_B(u-v)), \qquad u, v \,\varepsilon\, (-\infty, \infty), \tag{2.15}$$

where A and B are fuzzy numbers, and μ_A and μ_B are their respective membership functions.

A basic identity which holds for relative counts may be expressed as:

$$\Sigma Count(F \cap G) = \Sigma Count(G) \Sigma Count(F/G) \tag{2.16}$$

for sigma-counts, and as

$$FGCount(F \cap G) = FCount(G) \otimes FGCount(F/G) \tag{2.17}$$

for *FGCounts*, where $\otimes$ denotes the multiplication of fuzzy numbers, which is defined by (see Appendix)

$$\mu_{A \otimes B}(u) = sup_v(\mu_A(v) \wedge \mu_B(\frac{u}{v})), \qquad u, v \,\varepsilon\, (-\infty, \infty), \quad v \neq 0 \ . \tag{2.18}$$

An inequality involving relative sigma-counts which is of relevance to the analysis of evidence in expert systems is the following:

$$\Sigma Count(F/G) + \Sigma Count(\neg F/G) \geq 1 \tag{2.19}$$

$$= 1 \text{ if } G \text{ is nonfuzzy},$$

where $\neg F$ denotes the complement of F, i.e.,

$$\mu_{\neg F}(u) = 1 - \mu_F(u), \qquad u \,\varepsilon\, U \ . \tag{2.20}$$

Note that (2.19) implies that if the relative sigma-count $\Sigma Count(F/G)$ is identified with the conditional probability $Prob(F/G)$ (Zadeh (1981b)), then

$$Prob(\neg F/G) \geq 1 - Prob(F/G) \tag{2.21}$$

rather than

$$Prob(\neg F/G) = 1 - Prob(F/G), \tag{2.22}$$

which holds if G is nonfuzzy.

The inequality in question follows at once from

$$\Sigma Count(\neg F/G) = \frac{\Sigma_i(1-\mu_F(u_i)) \wedge \mu_G(u_i)}{\Sigma_i\mu_G(u_i)} \tag{2.23}$$

$$\geq \frac{\Sigma_i(1-\mu_F(u_i))\mu_G(u_i)}{\Sigma_i\mu_G(u_i)}$$

$$\geq 1 - \frac{\Sigma_i\mu_F(u_i)\mu_G(u_i)}{\Sigma_i\mu_G(u_i)}$$

$$\geq 1 - \frac{\Sigma_i\mu_F(u_i) \wedge \mu_G(u_i)}{\Sigma_i\mu_G(u_i)}$$

since

$$\Sigma Count(F/G) = \frac{\Sigma_i\mu_F(u_i) \wedge \mu_G(u_i)}{\Sigma_i\mu_G(u_i)}.$$

This concludes our brief exposition of some of the basic aspects of the concept of cardinality of fuzzy sets. As was stated earlier, the concept of cardinality plays an essential role in representing the meaning of fuzzy quantifiers. In the following sections, this connection will be made more concrete and a basis for inference from propositions containing fuzzy quantifiers will be established.

3. FUZZY QUANTIFIERS AND CARDINALITY OF FUZZY SETS

As was stated earlier, a fuzzy quantifier may be viewed as a fuzzy characterization of absolute or relative cardinality. Thus, in the proposition $p \triangleq Q$ *A's are B's* , where Q is a fuzzy quantifier and A and B are labels of fuzzy or nonfuzzy sets, Q may be interpreted as a fuzzy characterization of the relative cardinality of B in A. The fuzzy set A will be referred to as the *base set*.

When both A and B are nonfuzzy sets, the relative cardinality of B in A is a real number and Q is its possibility distribution. The same is true if A and/or B are fuzzy sets and the sigma-count is employed to define the relative cardinality. The situation becomes more complicated, however, if an *FCount* is employed for this purpose, since Q, then, is the possibility distribution of a conjunctive fuzzy number.

To encompass these cases, we shall assume that the following propositions are semantically equivalent (Zadeh (1978b)):

$$\textit{There are } Q \textit{ A's} \leftrightarrow \textit{Count}(A) \textit{ is } Q \tag{3.1}$$

$$Q \textit{ A's are B's} \leftrightarrow \textit{Prop}(B/A) \textit{ is } Q, \tag{3.2}$$

where the more specific term *Proportion* or *Prop*, for short, is used in place of *Count* in (3.2) to underscore that *Prop(B/A)* is the relative cardinality of B in A, with the understanding that both *Count* in (3.1) and *Prop* in (3.2) may be fuzzy or nonfuzzy counts. In the sequel, we shall assume for simplicity that, except where stated to the contrary, both absolute and relative cardinalities are defined via the sigma-count.

The right-hand members of (3.1) and (3.2) may be translated into possibility assignment equations (see (1.1)). Thus we have

$$\textit{Count}(A) \textit{ is } Q \rightarrow \Pi_{\textit{Count}(A)} = Q \tag{3.3}$$

and

$$Prop(B/A)\ is\ Q \rightarrow \Pi_{Prop(B/A)} = Q, \tag{3.4}$$

in which $\Pi_{Count(A)}$ and $\Pi_{Prop(B/A)}$ represent the possibility distributions of $Count(A)$ and $Prop(B/A)$, respectively. Furthermore, in view of (3.1) and (3.2), we have

$$There\ are\ Q\ A\text{'s} \rightarrow \Pi_{Count(A)} = Q \tag{3.5}$$

$$Q\ A\text{'s}\ are\ B\text{'s} \rightarrow \Pi_{Prop(B/A)} = Q. \tag{3.6}$$

These translation rules in combination with the results established in Section 2, provide a basis for deriving a variety of syllogisms for propositions containing fuzzy quantifiers, an instance of which is the *intersection/product* syllogism described by (1.3), namely,

$$\begin{array}{c} Q_1\ A\text{'s}\ are\ B\text{'s} \\ \underline{Q_2\ (A \text{ and } B)\text{'s}\ are\ C\text{'s}} \\ Q_1 \otimes Q_2\ A\text{'s}\ are\ (B \text{ and } C)\text{'s} \end{array} \tag{3.7}$$

in which Q_1, Q_2, A, B and C are assumed to be fuzzy, as in

$$\begin{array}{c} \textit{most tall men are fat} \\ \underline{\textit{many tall} \text{ and } \textit{fat men are bald}} \\ \textit{most} \otimes \textit{many tall men are fat} \text{ and } \textit{bald}. \end{array} \tag{3.8}$$

To establish the validity of syllogisms of this form, we shall rely, in the main, on the semantic entailment principle (Zadeh (1977), (1978b)), and on a special case of this principle which will be referred to as the *quantifier* *extension principle*.

Stated in brief, the semantic entailment principle asserts that a proposition p entails proposition q, which we shall express as $p \rightarrow q$ or

$$\frac{p}{q} \quad ,$$

if and only if the possibility distribution which is induced by p, $\Pi^p(X_1, \ldots, X_n)$, is contained in the possibility distribution induced by q, $\Pi^q(X_1, \ldots, X_n)$ (see (1.4)). Thus, stated in terms of the possibility distribution functions of Π^p and Π^q, we have

$$\frac{p}{q} \text{ if and } \textit{only} \text{ if } \quad \pi^p(X_1, \ldots, X_n) \leq \pi^q(X_1, \ldots, X_n) \tag{3.9}$$

for all points in the domain of π^p and π^q.

Informally, (3.9) means that p entails q if and only if q is less specific than p. For example, the proposition $p \triangleq$ Diana is 28 years old, entails the proposition $q \triangleq$ Diana is in her late twenties, because p is less specific than q, which in turn is a consequence of the containment of the nonfuzzy set "28" in the fuzzy set "late twenties."

It should be noted that, in the context of test-score semantics, the inequality of possibilities in (3.9) may be expressed as a corresponding inequality of overall test scores. Thus, if τ^p and τ^q are the overall test scores associated with p and q, respectively, then

$$\frac{p}{q} \text{ if and } \textit{only} \text{ if } \quad \tau^p \leq \tau^q , \tag{3.10}$$

with the understanding that the tests yielding τ^p and τ^q are applied to the same explanatory database and that the inequality holds for all instantiations of EDF.

In our applications of the entailment principle, we shall be concerned, for the most part, with an entailment relation between a collection of propositions $p_1, \ldots, p_n$ and a proposition q which is entailed by the collection. Under the assumption that the propositions which constitute the premises are noninteractive (Zadeh (1978b)), the statement of the entailment principle (3.9) becomes:

$$\begin{array}{ll} p_1 & \text{if and } \textit{only} \text{ if } \quad \pi^{p_1} \wedge \cdots \wedge \pi^{p_n} \le \pi^q \\ \vdots & \\ \underline{p_n} & \\ q & \end{array} \tag{3.11}$$

where $\pi^{p_1}, \ldots, \pi^{p_n}, \pi^q$, are the possibility distribution functions induced by $p_1, \ldots, p_n, q$, respectively, and likewise for (3.10).

We are now in a position to formulate an important special case of the entailment principle which will be referred to as the ***quantifier extenson principle***. This principle may also be viewed as an inference rule which is related to the transformational rule of inference described in Zadeh (1980).

Specifically, assume that each of the propositions $p_1, \ldots, p_n$ is a fuzzy characterization of an absolute or relative cardinality which may be expressed as $p_i \triangleq C_i \textit{ is } Q_i$, $i = 1, \ldots, n$, in which C_i is a count and Q_i is a fuzzy quantifier, e.g.,

$$p_i \triangleq \Sigma Count(B/A) \textit{ is } Q_i$$

or, more concretely,

$$p_i \triangleq \textit{most } A\text{'s } \textit{are } B\text{'s}.$$

Now, in general, a syllogism involving fuzzy quantifiers has the form of a collection of premises of the form $p_i \triangleq C_i$ *is* Q_i , $i = 1, \ldots, n$, followed by a conclusion of the same form, i.e., $q \triangleq C$ *is* Q , where C is a count that is related to $C_1, \ldots, C_n$, and Q is the fuzzy quantifier which is related to $Q_1, \ldots, Q_n$. The quantifier extension principle makes these relations explicit, as represented in the following inference schema:

$$C_1 \textit{ is } Q_1 \tag{3.12}$$

$$\underline{C_n \textit{ is } Q_n}$$

$$C \textit{ is } Q,$$

where Q is given by

$$\textit{If } C = g(C_1, \ldots, C_n) \textit{ then } Q = g(Q_1, \ldots, Q_n),$$

in which g is a function which expresses the relation between C and the C_i , and the meaning of $Q = g(Q_1, \ldots, Q_n)$ is defined by the extension principle (see Appendix). A somewhat more general version of the quantifier extension principle which can also be readily deduced from the extension principle is the following:

$$C_1 \textit{ is } Q_1 \tag{3.13}$$

$$\underline{C_n \textit{ is } Q_n}$$

$$C \textit{ is } Q,$$

where Q is given by

$$\textit{If } f(C_1, \ldots, C_n) \le C \le g(C_1, \ldots, C_n) \textit{ then } f(Q_1, \ldots, Q_n) \le Q \le g(Q_1, \ldots, Q_n).$$

As in (3.12), the meaning of the inequalities which bound Q is defined by the extension principle. In more concrete terms, these inequalities imply that Q is a fuzzy interval which may be expressed as

$$Q = (\geq f(Q_1, \ldots, Q_n)) \cap (\leq g(Q_1, \ldots, Q_n)), \tag{3.14}$$

where the fuzzy s-number $\geq f(Q_1, \ldots, Q_n)$ and the fuzzy z-number $\leq g(Q_1, \ldots, Q_n)$ (see Appendix) should be read as "at least $f(Q_1, \ldots, Q_n)$" and "at most $g(Q_1, \ldots, Q_n)$," respectively, and are the compositions* of the binary relations $\geq$ and $\leq$ with $f(Q_1,\ldots,Q_n)$ and $g(Q_1,\ldots,Q_n)$. In terms of (3.14), then, the relation between C and Q may be expressed as:

$$\textit{If } f(C_1, \ldots, C_n) \leq C \leq g(C_1, \ldots, C_n) \textit{ then} \tag{3.15}$$

$$Q = (\geq f(Q_1, \ldots, Q_n)) \cap (\leq g(Q_1, \ldots, Q_n)).$$

An important special case of (3.12) and (3.15) is one where f and g are arithmetic or boolean expressions, as in

$$C = C_1 C_2 + C_3$$

and

$$C_1 + C_2 - 1 \leq C \leq C_1 \wedge C_2 \ .$$

For these cases, the quantifier extension principle yields

$$Q = Q_1 \otimes Q_2 \oplus Q_3$$

and

* The composition, RoS, of a binary relation R with a unary relation S is defined by $\mu_{RoS}(v) = \vee_u(\mu_R(v,u) \wedge \mu_S(u))$ $, u \in U, v \in V$, where μ_R, μ_S, and μ_{RoS} are the membership functions of R, S and RoS, respectively, and $\vee_u$ denotes the supremum over U. Where no confusion can result, the symbol o may be suppressed.

$$Q = (\geq (Q_1 \oplus Q_2 \ominus 1)) \cap \leq (Q_1 \oslash Q_2) \ ,$$

where Q, Q_1, Q_2 and Q_3 are fuzzy numbers, and $\otimes$, $\oplus$ and $\oslash$ are the product, sum and min in fuzzy arithmetic.*

We are now in a position to apply the quantifier extension principle to the derivation of the intersection/product syllogism expressed by (3.7). Specifically, we note that

$$Q_1 \ A\text{'s } are \ B\text{'s} \Leftrightarrow Prop(B/A) \ is \ Q_1 \tag{3.16}$$

$$Q_2 \ (A \text{ and } B)\text{'s } are \ C\text{'s} \Leftrightarrow Prop(C/A \cap B) \ is \ Q_2 \tag{3.17}$$

and

$$Q \ A\text{'s } are \ (B \text{ and } C)\text{'s} \Leftrightarrow Prop(B \cap C/A) \ is \ Q. \tag{3.18}$$

where

$$Prop(B/A) = \frac{\Sigma Count(B \cap A)}{\Sigma Count(A)} \tag{3.19}$$

$$Prop(C/A \cap B) = \frac{\Sigma Count(A \cap B \cap C)}{\Sigma Count(A \cap B)} \tag{3.20}$$

$$Prop(B \cap C/A) = \frac{\Sigma Count(A \cap B \cap C)}{\Sigma Count(A)} \ . \tag{3.21}$$

From (3.19), (3.20) and (3.21), it follows that the relative counts $C_1 \triangleq Prop(B/A)$, $C_2 \triangleq Prop(C/A \cap B)$ and $C \triangleq Prop(B \cap C/A)$ satisfy the identity

* Where typographical convenience is a significant consideration, a fuzzy version of an arithmetic operation * may be expressed more simply as (*).

$$Prop(B \cap C / A) = Prop(B / A)\, Prop(C / A \cap B). \tag{3.22}$$

and hence

$$C = C_1\, C_2 \quad . \tag{3.23}$$

On the other hand, from (3.16), (3.17) and (3.18), we see that Q_1, Q_2 and Q are the respective possibility distributions of C_1, C_2 and C. Consequently, from the quantifier extension principle applied to arithmetic expressions, it follows that the fuzzy quantifier Q is the fuzzy product of the fuzzy quantifiers Q_1 and Q_2, i.e.,

$$Q = Q_1 \otimes Q_2 \quad , \tag{3.24}$$

which is what we wanted to establish.

As a corollary of (3.7), we can deduce at once the following syllogism:

$$\begin{array}{c} Q_1\ A\text{'s are } B\text{'s} \\ \underline{Q_2\ (A \text{ and } B)\text{'s are } C\text{'s}} \\ (\geq (Q_1 \otimes Q_2))\ A\text{'s are } C\text{'s}, \end{array} \tag{3.25}$$

where the quantifier $(\geq (Q_1 \otimes Q_2))$, which represents the composition of the binary relation $\geq$ with the unary relation $Q_1 \otimes Q_2$, should be read as *at least* $(Q_1 \otimes Q_2)$. This syllogism is a consequence of (3.7) by virtue of the inequality

$$\Sigma Count(B \cap C) \leq \Sigma Count(C), \tag{3.26}$$

which holds for all fuzzy or nonfuzzy B and C. For, if we rewrite (3.7) in terms of proportions,

$$Prop(B/A) \text{ is } Q_1 \tag{3.27}$$

$$\underline{Prop(C/A \cap B) \text{ is } Q_2}$$

$$Prop(B \cap C/A) \text{ is } (Q_1 \otimes Q_2),$$

then from (3.26) it follows that

$$Prop(B \cap C/A) \text{ is } (Q_1 \otimes Q_2) \Longrightarrow Prop(C/A) \text{ is } (\geq (Q_1 \otimes Q_2)). \tag{3.28}$$

Thus, based on (3.28), the syllogism (3.7) and its corollary (3.25) may be represented compactly in the form:

$$Q_1 \; A\text{'s are } B\text{'s} \tag{3.29}$$

$$\underline{Q_2 (A \cap B)\text{'s are } C\text{'s}}$$

$$\underline{(Q_1 \otimes Q_2) \; A\text{'s are } (B \text{ and } C)\text{'s}}$$

$$(\geq (Q_1 \otimes Q_2)) \; A\text{'s are } C\text{'s}.$$

As an additional illustration of the quantifier extension principle, consider the inequality established in Section 2, namely,

$$0 \vee (\Sigma Count(A) + \Sigma Count(B) - Count(U)) \leq \Sigma Count(A \cap B) \tag{3.30}$$

$$\leq \Sigma Count(A) \wedge \Sigma Count(B).$$

Let Q, Q_1 and Q_2 be the fuzzy quantifiers which characterize $C \triangleq \Sigma Count(A \cap B)$, $C_1 \triangleq \Sigma Count(A)$, and $C_2 \triangleq \Sigma Count(B)$, respectively. Then

$$0 \otimes (Q_1 \oplus Q_2 \ominus 1) \leq Q \leq Q_1 \otimes Q_2. \tag{3.31}$$

where, as stated earlier, $\oplus$, $\otimes$, $\oslash$ and $\odot$ are the operations of sum, product, min and max in fuzzy arithmetic. Consequently, as a special case of (3.31), we can assert that in the inference schema

$$\begin{array}{c} \textit{most students are single} \\ \underline{\textit{many students are male}} \\ Q \textit{ students are single } \text{and } \textit{male} \end{array} \tag{3.32}$$

Q is a fuzzy interval given by

$$Q = (\geq (0 \odot (\, most \oplus many \ominus 1))) \cap (\leq (\, most \oslash many)) \; . \tag{3.33}$$

In more general terms, the inference schema of (3.32) may be stated as the *conjunction schema:*

$$\begin{array}{c} Q_1 \, A\text{'s are } B\text{'s} \\ \underline{Q_2 \, A\text{'s are } C\text{'s}} \\ Q \, A\text{'s are } (B \text{ and} C)\text{'s} \end{array}$$

where

$$Q = (\geq (0 \odot (Q_1 \oplus Q_2 \ominus 1))) \cap (\leq (Q_1 \oslash Q_2))$$

Monotonicity

In the theory of generalized quantifiers (Barwise and Cooper (1981)), a generalized quantifier Q is said to be *monotonic* if a true proposition of the form $p \triangleq Q$ *A's are B's*, where A and B are nonfuzzy sets, remains true when B is replaced by any superset (or any subset) of B. In this sense, *most* is a monotonic generalized quantifier under the assumption that B is replaced by a superset of B.

In the case of fuzzy quantifiers of the first or second kinds, a similar but more general definition which is valid for fuzzy sets may be formulated in terms of the membership function or, equivalently, the possibility distribution function of Q. More specifically:

A fuzzy quantifier Q is *monotone nondecreasing (nonincreasing)* if and only if the membership function of Q, μ_Q, is monotone nondecreasing (nonincreasing) over the domain of Q. From this definition, it follows at once that

$$Q \textit{ is monotone nondecreasing} \Leftrightarrow \geq Q = Q \tag{3.34}$$

$$Q \textit{ is monotone nonincreasing} \Leftrightarrow \leq Q = Q, \tag{3.35}$$

where, as stated earlier, $\geq Q$ and $\leq Q$ should be read as "at least Q" and "at most Q," respectively. Furthermore, from (2.7) it follows that, if $B \subset C$, then

$$Q \textit{ is monotone nondecreasing} \Leftrightarrow \tag{3.36}$$

$$Prop(B/A) \textit{ is } Q \Longrightarrow Prop(C/A) \textit{ is } Q$$

and

$$Q \textit{ is monotone nonincreasing} \Leftrightarrow \tag{3.37}$$

$$Prop(C/A) \textit{ is } Q \Longrightarrow Prop(B/A) \textit{ is } Q.$$

If Q is a fuzzy quantifier of the second kind, the ***antonym*** of Q, $antQ$, is defined by (Zadeh (1978b))

$$\mu_{antQ}(u) = \mu_Q(1-u), \quad u \,\varepsilon [0,1]. \tag{3.38}$$

Thus, if *few* is interpreted as the antonym of *most*, we have

$$\mu_{FEW}(u) = \mu_{MOST}(1-u)\,, \qquad u \varepsilon [0,1]. \tag{3.39}$$

A graphic illustration of (3.39) is shown in Fig. 3.

An immediate consequence of (3.38) is the following:

If Q is monotone nondecreasing (e.g., most), then its antonym (e.g., few) is monotone nonincreasing.

We are now in a position to derive additional syllogisms for fuzzily-quantified propositions and, inter alia, establish the validity of the example given in the abstract, namely,

$$\begin{array}{c} \textit{most } U\text{'s are } A\text{'s} \\ \underline{\textit{most } A\text{'s are } B\text{'s}} \\ \textit{most}^2\ U\text{'s are } B\text{'s}, \end{array} \tag{3.40}$$

where by U's we mean the elements of the universe of discourse U, and *most* is assumed to be monotone nondecreasing.

Specifically, by identifying A in (3.25) with U in (3.40), B in (3.25) with A in (3.40), C in (3.25) with B in (3.40), and noting that

$$U \cap A = A,$$

we obtain as a special case of (3.25) the inference schema

$$\begin{array}{c} \textit{most } U\text{'s are } A\text{'s} \\ \underline{\textit{most } A\text{'s are } B\text{'s}} \\ \underline{\geq 0\ (\textit{most} \otimes \textit{most})\ U\text{'s are } B\text{'s}} \end{array} \tag{3.41}$$

$$most^2 \ U\text{'s are } B\text{'s} \quad ,$$

where $most^2$ denotes $most \otimes most$. More generally, for any monotone nondecreasing fuzzy quantifiers Q_1 and Q_2, we can assert that

$$\begin{array}{c} Q_1 \ U\text{'s are } A\text{'s} \\ \underline{Q_2 \ A\text{'s are } B\text{'s}} \\ (Q_1 \otimes Q_2) \ U\text{'s are } B\text{'s}. \end{array} \tag{3.42}$$

If one starts with a rule of inference in predicate calculus, a natural question which arises is: How does the rule in question generalize to fuzzy quantifiers? An elementary example of an answer to a question of this kind is the following inference schema:

$$\frac{Q_1 \ A\text{'s are } B\text{'s}}{(\geq Q_2) \ A\text{'s are } B\text{'s}} \quad \text{if } Q_2 \leq Q_1 \quad , \tag{3.43}$$

which is a generalization of the basic rule:

$$\frac{(\forall x) \ P(x)}{(\exists x) \ P(x)} ,$$

where P is a predicate. In (3.43), the inequality $Q_2 \leq Q_1$ signifies that, as a fuzzy number, Q_2 is less than or equal to the fuzzy number Q_1 (see Fig. 4).

To establish the validity of (3.43), we start with the inference rule

$$\frac{Q_1 \ A\text{'s are } B\text{'s}}{Q_2 \ A\text{'s are } B\text{'s}} \quad \text{if } Q_1 \subset Q_2, \tag{3.44}$$

which is an immediate consequence of the entailment principle (3.9), since the conclusion in (3.44) is less specific than the premise. Then, (3.43) follows at once from (3.44) and the containment relation

$$Q_1 \leq Q_2 \Rightarrow Q_2 \subset (\geq Q_1) \tag{3.45}$$

which, in words, means that, if a fuzzy number Q_1 is less than or equal to Q_2, then, as a fuzzy set, Q_2 is contained in the fuzzy set which corresponds to the fuzzy number "at least Q_1."

In inferring from fuzzily-quantified propositions with negations, it is useful to have rules which concern the semantic equivalence or semantic entailment of such propositions. In what follows, we shall derive a few basic rules of this type.

The first rule, which applies to fuzzy quantifiers of the first kind, and to fuzzy quantifiers of the second kind when the base set, A, is nonfuzzy, is the following:

$$Q\ A\text{'s are } B\text{'s} \Leftrightarrow (antQ)\ A\text{'s are not } B\text{'s}, \tag{3.46}$$

where $antQ$ denotes the antonym of Q (see (3.38)). For example,

$$\textit{most men are tall} \Leftrightarrow (\textit{ant most})\ \textit{men are not tall}, \tag{3.47}$$

and

$$\textit{most men are tall} \Leftrightarrow \textit{few men are not tall}$$

if *few* is interpreted as the antonym of *most*.

To establish (3.46), we note that, in consequence of (3.6), we have

$$Q\ A\text{'s are } B\text{'s} \rightarrow \Pi_{\Sigma Count(B/A)} = Q. \tag{3.48}$$

The possibility assignment equation in (3.48) implies that the test score, τ_1, associated with the proposition "Q A's are B's," is given by

$$\tau_1 = \mu_Q\,(\Sigma Count\,(B/A)) \quad . \tag{3.49}$$

where μ_Q is the membership function of Q.

Similarly, the test score associated with the proposition "*(ant Q) A's are not B's*," is given by

$$\tau_2 = \mu_{antQ}(\Sigma Count(\neg B/A)) \quad . \tag{3.50}$$

Thus, to demonstrate that the two propositions are semantically equivalent, it will suffice to show that $\tau_1 = \tau_2$.

To this end, we note that

$$\Sigma Count(\neg B/A) = \frac{\Sigma Count(A \cap (\neg B))}{\Sigma Count(A)} \tag{3.51}$$

$$= \frac{\Sigma_i \mu_A(u_i) \wedge (1 - \mu_B(u_i))}{\Sigma_i \mu_A(u_i)}$$

and, if A is nonfuzzy, the right-hand member of (3.51) may be written as:

$$\frac{\Sigma_i \mu_A(u_i) \wedge (1 - \mu_B(u_i))}{\Sigma_i \mu_A(u_i)} = 1 - \Sigma Count(B/A). \tag{3.52}$$

Now, from the definition of the antonym (3.38), it follows that

$$\mu_{antQ}(1 - \Sigma Count(B/A)) = \mu_Q\,(\Sigma Count(B/A)), \tag{3.53}$$

and hence that $\tau_1 = \tau_2$, which is what we had to establish.

In the more general case where A is fuzzy, the semantic equivalence (3.46) does not hold. Instead, the following semantic entailment may be asserted:

If Q is monotone nonincreasing, then

$$\frac{Q \; A\text{'s are } B\text{'s}}{(antQ) \; A\text{'s are not } B\text{'s}} \tag{3.54}$$

To validate (3.54), we note that in Section 2 we have established the inequality (see (2.19))

$$1 - \Sigma Count(\neg B / A) \le \Sigma Count(B / A). \tag{3.55}$$

Now, if Q is monotone nonincreasing, then on application of μ_Q to both sides of (3.55) the inequality is reversed, yielding

$$\mu_Q(1 - \Sigma Count(\neg B / A)) \ge \mu_Q(\Sigma Count(B / A))$$

or, equivalently,

$$\mu_{antQ}(\Sigma Count(\neg B / A)) \ge \mu_Q(\Sigma Count(B / A)) \quad , \tag{3.56}$$

which establishes that the consequence in (3.54) is less specific than the premise and thus, by the entailment principle, is entailed by the premise.

In general, an application of the entailment principle for the purpose of demonstrating the validity of an inference rule reduces the computation of a fuzzy quantifier to the solution of a variational problem or, in discrete cases, to the solution of a nonlinear program. As an illustration, we shall consider the following inference schema

$$\frac{Q_1 \text{ } A\text{'s are } B\text{'s}}{?Q \text{ } A\text{'s are } (very \text{ } B)\text{'s}} \quad , \tag{3.57}$$

where $?Q$ is the quantifier to be computed; the base set A is nonfuzzy and the modifier *very* is an intensifier whose effect is assumed to be defined by (Zadeh (1972))

$$very \text{ } B = {}^2B \quad , \tag{3.58}$$

where the left exponent 2 signifies that the membership function of ^2B is the square of that of B.* Since A is nonfuzzy, we can assume, without loss of generality, that $A = U$.

With this assumption, the translation of the premise in (3.57) is given by

$$Q_1 \; U's \; are \; B's \rightarrow \Pi_{\Sigma Count(B/U)} = Q_1 \tag{3.59}$$

while that of the consequent is

$$Q \; U's \; are \; ^2B's \rightarrow \Pi_{\Sigma Count(^2B/U)} = Q \quad . \tag{3.60}$$

Let $\mu_1, \ldots, \mu_n$ be the grades of membership of the points $u_1, \ldots, u_n$ in B. Then, (3.59) and (3.60) imply that the overall test scores for the premise and the consequent are, respectively,

$$\tau_1 = \mu_{Q_1}(\frac{1}{N}\Sigma_i\mu_i) \tag{3.61}$$

$$\tau_2 = \mu_Q(\frac{1}{N}\Sigma_i\mu_i^2) \quad , \tag{3.62}$$

where $N = \Sigma Count(U)$.

The problem we are faced with at this point is the following. The premise, Q_1 $U's$ are $B's$, defines via (3.61) a fuzzy set, P_1, in the unit cube $C^N = \{\mu_1, \ldots, \mu_N\}$ such that the grade of membership of the point $\mu = (\mu_1, \ldots, \mu_N)$ in P_1 is τ_1. The mapping $C^N \rightarrow [0,1]$ which is defined by the sigma-count

$$\Sigma Count(very \; B/U) = \frac{1}{N}\Sigma_i\mu_i^2 \quad , \tag{3.63}$$

* In earlier papers, the meanings of B^2 and ^2B were interchanged.

induces the fuzzy set, Q, in [0,1] whose membership function, μ_Q, is what we wish to determine. For this purpose, we can invoke the extension principle, which reduces the determination of μ_Q to the solution of the following nonlinear program:

$$\mu_Q(v) = \max_\mu(\mu_{Q_1}(\frac{1}{N}\Sigma_i\mu_i)), \quad v\,\varepsilon[0,1] \quad , \tag{3.64}$$

subject to the constraint

$$v = \frac{1}{N}\Sigma_i\mu_i^2 \quad .$$

As shown in Zadeh (1977), this nonlinear program has an explicit solution given by

$$\mu_Q(v) = \mu_{Q_1}(\sqrt{v}), \quad v\,\varepsilon[0,1],$$

which implies that

$$Q = Q_1^2 = Q_1 \otimes Q_1. \tag{3.65}$$

We are thus led to the inference schema

$$\frac{Q_1\ A\text{'s are } B\text{'s}}{Q_1^2\ A\text{'s are (very } B\text{)'s}} \tag{3.66}$$

and, more generally, for any positive m and nonfuzzy A,

$$\frac{Q_1\ A\text{'s are } B\text{'s}}{Q_1^m\ A\text{'s are } (^mB)\text{'s}}, \tag{3.67}$$

and

$$\frac{Q_1\ A\text{'s are } {}^mB\text{'s}}{Q_1^{\frac{1}{m}}\ A\text{'s are } B\text{'s}} \tag{3.68}$$

where

$$\mu_{Q_1{}^m}(v) = \mu_{Q_1}(v^{\frac{1}{m}}), \quad v \varepsilon [0,1] \tag{3.69}$$

$$\mu_{Q_1{}^{1/m}}(v) = \mu_{Q_1}(v^m), \tag{3.70}$$

and

$$\mu_{m_B}(u) = (\mu_B(u))^m, \qquad u \varepsilon U. \tag{3.71}$$

As a simple example, assume that the premise in (3.66) is the proposition "Most men over sixty are bald." Then, the inference schema represented by (3.66) yields the syllogism:

$$\frac{\textit{most men over sixty are bald}}{\textit{most}^2 \textit{ men over sixty are very bald}} \quad . \tag{3.72}$$

It should be noted that an inference schema may be formed by a composition of two or more other inference schemas. For example, by combining (3.46) and (3.68), we are led to the following schema:

$$\frac{Q_1 \textit{ A's are (not very B)'s}}{(\textit{ant } Q_1)^{0.5} \textit{ A's are B's}} \quad , \tag{3.73}$$

in which the base set A is assumed to be nonfuzzy. Thus, the syllogism

$$\frac{\textit{most Frenchmen are not very tall}}{(\textit{ant most})^{0.5} \textit{ Frenchmen are tall}} \tag{3.74}$$

may be viewed as an instance of this schema (see Fig. 5).

In the foregoing discussion, we have attempted to show how the treatment of fuzzy quantifiers as fuzzy numbers makes it possible to derive a wide variety of inference schema for fuzzily-quantified propositions. These propositions were assumed to have a simple structure like "*Q A's are B's*," which made it unnecessary to employ the full power of test-score semantics for representing their meaning. We shall turn our attention to more complex propositions in the following section and will illustrate by examples the application of test-score semantics to the representation of meaning of various types of fuzzily-quantified semantic entities.

4. Meaning Representation by Test-Score Semantics

As was stated in the Introduction, the process of meaning representation in test-score semantics involves three distinct phases: Phase I, in which an explanatory database frame, EDF, is constructed; Phase II, in which the constraints induced by the semantic entity are tested and scored; and Phase III, in which the partial test scores are aggregated into an overall test score which is a real number in the interval $[0,1]$ or, more generally, a vector of such numbers.

In what follows, the process is illustrated by several examples in which Phase I and Phase II are merged into a single test which yields the overall test score. This test represents the meaning of the semantic entity and may be viewed as a description of the process by which the meaning of the semantic entity is composed from the meanings of the constituent relations in EDF.

In some cases, the test which represents the meaning of a given semantic entity may be expressed in a higher level language of logical forms. The use of such forms is illustrated in Examples 4 and 5.

When a semantic entity contains one or more fuzzy quantifiers, its meaning is generally easier to represent through the use of Σ*Counts* than *FCounts*. However, there may be cases in which a Σ*Count* may be a less appropriate representation of cardinality than an *FGCount* or an *FECount*. This is particularly true of cases in which the cardinality of a set is low, i.e., is a small fuzzy number like *several*, *few*, etc. Furthermore, what should be borne in mind is that a $\Sigma Count$ is a summary of an *FGCount* and hence is intrinsically less informative.

In some of the following examples, we employ alternative counts for purposes of comparison. In others, only one type of count, usually the $\Sigma Count$, is used.

EXAMPLE 1.

$$SE \triangleq \textit{several balls most of which are large.}$$

For this semantic entity, we shall assume that *EDF* comprises the following relations:

$$\begin{aligned} EDF \triangleq \quad & BALL\,[Identifier;\, Size] + \\ & LARGE\,[Size;\, \mu\,] + \\ & SEVERAL\,[Number;\, \mu\,] + \\ & MOST\,[Proportion;\, \mu\,]\,. \end{aligned}$$

In this *EDF*, the first relation has n rows and is a list of the identifiers of balls and their respective sizes; in *LARGE*, μ is the degree to which a ball of size *Size* is large; in *SEVERAL*, μ is the degree to which *Number* fits the description *several;* and in *MOST*, μ is the degree to which *Proportion* fits the description *most*.

The test which yields the compatibility of *SE* with *ED* and thus defines the meaning of *SE* depends on the definition of fuzzy set cardinality. In particular, using the sigma-count, the test procedure may be stated as follows:

1. Test the constraint induced by *SEVERAL:*

$$\tau_1 = {}_{\mu}SEVERAL[Number = n],$$

which means that the value of *Number* is set to n and the value of μ is read, yielding the test score τ_1 for the constraint in question.

2. Find the size of each ball in *BALL:*

$$Size_i = {}_{Size}BALL[Identifier = Identifier_i],$$

$$i = 1, \ldots, n.$$

3. Test the constraint induced by *LARGE* for each ball in *BALL:*

$$\mu_{LB}(i) = {}_{\mu}LARGE[Size = Size_i].$$

4. Find the sigma-count of large balls in *BALL:*

$$\Sigma Count(LB) = \Sigma_i \mu_{LB}(i).$$

5. Find the proportion of large balls in *BALL:*

$$PLB = \frac{1}{n}\Sigma_i \mu_{LB}(i).$$

6. Test the constraint induced by *MOST:*

$$\tau_2 = {}_{\mu}MOST[Proportion = PLB].$$

7. Aggregate the partial test scores:

$$\tau = \tau_1 \wedge \tau_2,$$

where τ is the overall test score. The use of the min operator to aggregate τ_1 and τ_2 implies that we interpret the implicit conjunction in *SE* as the cartesian product of the conjuncts.

The use of fuzzy cardinality affects the way in which τ_2 is computed. Specifically, the employment of *FGCount* leads to:

$$\tau_2 = sup_i(FGCount\,(LB) \cap nMOST),$$

which expressed in terms of the membership functions of *FGCount (LB)* and *MOST* may be written as

$$\tau_2 = sup_i(\mu_{FGCount\,(LB)}(i) \wedge \mu_{MOST}(\frac{i}{n})) \quad .$$

The rest of the test procedure is unchanged.

EXAMPLE 2.

$$SE \triangleq several\ large\ balls \quad .$$

In this case, we assume that the *EDF* is the same as in Example 1, with *MOST* deleted.

As is pointed out in Zadeh (1981a), the semantic entity in question may be interpreted in different ways. In particular, using the so-called compartmentalized interpretation in which the constraints induced by *SMALL* and *SEVERAL* are tested separately, the test procedure employing the *FGCount* may be stated as follows:

1. Test the constraint induced by *SEVERAL:*

$$\tau_1 \triangleq {}_{\mu}SEVERAL[Number = n] \quad .$$

2. Find the size of the smallest ball:

$$SSB \triangleq {}_{Size}\min{}_{Size}(BALL) \quad ,$$

in which the right-hand member signifies that the smallest entry in the column *Size* of the relation *BALL* is read and assigned to the variable *SSB* (Smallest Size Ball).

3. Test the constraint induced by *LARGE* by finding the degree to which the smallest ball is large:

$$\tau_2 \triangleq {}_{\mu}LARGE[Size = SSB].$$

4. Aggregate the test scores:

$$\tau = \tau_1 \wedge \tau_2.$$

EXAMPLE 3.

$p \triangleq$ ***Hans has many acquaintances*** **and** ***a few close friends most of whom are highly intelligent.***

Assume that the *EDF* comprises the following relations:

ACQUAINTANCE [*Name 1; Name 2;* μ] +

FRIEND [*Name 1; Name2;* μ]+

INTELLIGENT [*Name;* μ]+

MANY [*Number;* μ]+

FEW [*Number;* μ]+

MOST [*Proportion;* μ].

In *ACQUAINTANCE*, μ is the degree to which *Name1* is an acquaintance of *Name2;* in *FRIEND*, μ is the degree to which *Name1* is a friend of *Name2;* in *INTELLIGENT*, μ is the degree to which *Name* is intelligent; *MANY* and *FEW* are fuzzy quantifiers of the first kind, and *MOST* is a fuzzy quantifier of the second kind.

The test procedure may be stated as follows:

1. Find the fuzzy set of Hans' acquaintances:

$$HA \triangleq {}_{Name1 \times \mu} ACQUAINTANCE[Name2 = Hans],$$

which means that in each row in which *Name2* is Hans, we read *Name1* and μ and form the fuzzy set *HA*.

2. Count the number of Hans' acquaintances:

$$CHA \triangleq \Sigma Count(HA).$$

3. Find the test score for the constraint induced by *MANY*:

$$\tau_1 = {}_{\mu}MANY[Name1 = CHA].$$

4. Find the fuzzy set of friends of Hans:

$$FH \triangleq {}_{Name1 \times \mu} FRIEND[Name2 = Hans].$$

5. Intensify *FH* to account for *close* (Zadeh (1978b)):

$$CFH \triangleq {}^2FH.$$

6. Determine the count of close friends of Hans:

$$CCFH \triangleq \Sigma Count({}^2FH).$$

7. Find the test score for the constraint induced by *FEW*:

$$\tau_2 \triangleq {}_{\mu}FEW[Number = CCFH].$$

8. Intensify *INTELLIGENT* to account for *highly*. (We assume that this is accomplished by raising *INTELLIGENT* to the third power.)

$$HIGHLY.INTELLIGENT = {}^{3}INTELLIGENT.$$

9. Find the fuzzy set of close friends of Hans who are highly intelligent:

$$CFH.HI \triangleq CFH \cap {}^{3}INTELLIGENT.$$

10. Determine the count of close friends of Hans who are highly intelligent:

$$CCFH.HI \triangleq \Sigma Count(CFH \cap {}^{3}INTELLIGENT).$$

11. Find the proportion of those who are highly intelligent among the close friends of Hans:

$$\gamma \triangleq \frac{\Sigma Count(CFH \cap {}^{3}INTELLIGENT)}{\Sigma Count(CFH)}.$$

12. Find the test score for the constraint induced by *MOST:*

$$\tau_3 \triangleq {}_{\mu}MOST[Proportion = \gamma].$$

13. Aggregate the partial test scores:

$$\tau = \tau_1 \wedge \tau_2 \wedge \tau_3.$$

The test described above may be expressed more concisely as a logical form which is semantically equivalent to p. The logical form may be expressed as follows:

$$p \leftrightarrow Count(\,_{Name1 \times \mu}ACQUAINTANCE[Name2 = Hans]) \text{ is } MANY \wedge$$

$$Count(\,_{Name1 \times \mu}{}^{2}FRIEND[Name2 = Hans]) \text{ is } FEW \wedge$$

$$Prop(\,^{3}INTELLIGENT /\ _{Name1 \times \mu}\,^{2}FRIEND[Name2 = Hans])\ is\ MOST$$

where $\wedge$ denotes the conjunction.

EXAMPLE 4.

Consider the proposition

$p \triangleq$ *Over the past few years Nick earned far more than most of his close friends.*

In this case, we shall assume that *EDF* consists of the following relations:

$$\begin{aligned} EDF \triangleq\ & INCOME\,[Name;\ Amount;\ Year]+ \\ & FRIEND\,[Name;\ \mu]+ \\ & FEW\,[Number;\ \mu]+ \\ & FAR.MORE\,[Income1;\ Income2;\ \mu]+ \\ & MOST\,[Proportion;\ \mu]. \end{aligned}$$

Using the sigma-count, the test procedure may be described as follows:

1. Find Nick's income in $Year_i$, i = 1,2,..., counting backward from present:

$$IN_i \triangleq\ _{Amount}INCOME[Name = Nick;\ Year = Year_i].$$

2. Test the constraint induced by *FEW:*

$$\mu_i \triangleq\ _{\mu}FEW[Year = Year_i].$$

3. Compute Nick's total income during the past few years:

$$TIN = \Sigma_i \mu_i IN_i ,$$

in which the μ_i play the role of weighting coefficients.

4. Compute the total income of each $Name_j$ (other than Nick) during the past several years:

$$TIName_j = \Sigma_i \mu_i IName_{ji},$$

where $IName_{ji}$ is the income of $Name_j$ in $Year_i$.

5. Find the fuzzy set of individuals in relation to whom Nick earned far more. The grade of membership of $Name_j$ in this set is given by:

$$\mu_{FM}(Name_j) = {}_{\mu}FAR.MORE[Income\,1 = TIN;\ Income\,2 = TIName_j].$$

6. Find the fuzzy set of close friends of Nick by intensifying the relation *FRIEND:*

$$CF = {}^2FRIEND,$$

which implies that

$$\mu_{CF}(Name_{j)} = (\ {}_{\mu}FRIEND[Name\ = Name_j])^2.$$

7. Using the sigma-count, count the number of close friends of Nick:

$$\Sigma Count\,(CF) = \Sigma_j \mu^2{}_{FRIEND}(Name_j).$$

8. Find the intersection of *FM* with *CF*. The grade of membership of $Name_j$ in the intersection is given by

$$\mu_{FM \cap CF}(Name_j) = \mu_{FM}(Name_j) \wedge \mu_{CF}(Name_j).$$

9. Compute the sigma-count of $FM \cap CF$:

$$\Sigma Count\,(FM \cap CF) = \Sigma_j \mu_{FM}(Name_j) \wedge \mu_{CF}(Name_j).$$

10. Compute the proportion of individuals in FM who are in CF:

$$\rho \triangleq \frac{\Sigma Count\,(FM \cap CF)}{\Sigma Count\,(CF)}.$$

11. Test the constraint induced by $MOST$:

$$\tau = {}_{\mu}MOST[Proportion = \rho],$$

which expresses the overall test score and thus represents the desired compatibility of ρ with the explanatory database.

For the proposition under consideration, the logical form has a more complex structure than in Example 3. Specifically, we have

$$Prop((\Sigma_j \mu_j / Name_j) \;/\; {}^{2}FRIEND[Name2 = Nick]) \text{ is } MOST$$

where

$$\mu_j = {}_{\mu}FAR.MORE[Income1 = TIN; Income2 = TIName_j]$$

where $Name_j \neq Nick$ and

$$TIN = \Sigma_i \mu_{FEW}(i)\,{}_{Amount}INCOME[Name = Nick; Year = Year_i]$$

and

$$TIName_j = \Sigma_i \mu_{FEW}(i)\,{}_{Amount}INCOME[Name = Name_j; Year = Year_i]$$

EXAMPLE 5.

$$p \triangleq \textit{They like each other.}$$

In this case there is an implicit fuzzy quantifier in p which reflects the understanding that not all members of the group referred to as *they* must necessarily like each other.

Since the fuzzy quantifier in p is implicit, it may be interpreted in many different ways. The test described below represents one such interpretation and involves, in effect, the use of an *FCount*.

Specifically, we associate with p the *EDF*

$$\begin{aligned} EDF \triangleq \quad & THEY[Name]+ \\ & LIKE\,[Name1;\, Name2;\, \mu]+ \\ & ALMOST.ALL\,[Proportion;\, \mu], \end{aligned}$$

in which *THEY* is the list of names of members of the group to which p refers; *LIKE* is a fuzzy relation in which μ is the degree to which *Name1* likes *Name2;* and *ALMOST.ALL* is a fuzzy quantifier in which μ is the degree to which a numerical value of *Proportion* fits a subjective perception of the meaning of *almost all.*

Let μ_{ij} be the degree to which $Name_i$ likes $Name_j$, $i \neq j$. If there are n names in *THEY*, then there are $(n^2 - n)$ μ_{ij}'s in *LIKE* with $i \neq j$. Denote the relation *LIKE* without its diagonal elements by $LIKE^*$.

The test procedure which yields the overall test score τ may be described as follows:

1. Count the number of members in *THEY:*

$$n \triangleq Count(THEY) \quad .$$

2. Compute the *FGCount* of $LIKE^*$:

$$C \triangleq FGCount(LIKE^*).$$

Note that in view of (2.6), C may be obtained by sorting the μ elements of $LIKE^*$ in descending order, which yields $LIKE^*\downarrow$. Thus,

$$FGCount(LIKE^*) = NLIKE^*\downarrow.$$

3. Compute the height (i.e., the maximum value) of the intersection of C and the fuzzy number $(n^2 - n)\, ALMOST.ALL$:

$$\tau = sup(FGCount(LIKE^*) \cap (n^2 - n) ALMOST.ALL)$$

The result, as shown in Figure 6, is the overall test score.

The last two examples in this Section illustrate the application of test-score semantics to question-answering. The basic idea behind this application is the following.

Suppose that the answer to a question, q, is to be deduced from a knowledge base which consists of a collection of propositions:

$$KB = \{p_1, \ldots, p_n\} \quad . \tag{4.1}$$

Furthermore, assume that the p_i are noninteractive and that each p_i induces a possibility distribution, Π^i, which is characterized by its possibility distribution function, π^i, over a collection of base variables $X = \{X_1, \ldots, X_m\}$. This implies (a) that p_i , $i = 1, \ldots, n$, translates into the possbility assignment equation

$$p_i \rightarrow \Pi^i_{(X_1, \ldots, X_m)} = F_i \quad ,$$

where F_i is a fuzzy subset of U, the cartesian product of the domains of $X_1, \ldots, X_m$, i.e.,

$$U = U_1 \times \cdots \times U_m \quad ,$$

in which U_i is the domain of X_i ; and (b) that the collection KB induces a combined possibility distribution Π whose possibility distribution function is given by

$$\pi_{(X_1,\ldots,X_m)} = \pi^1_{(X_1,\ldots,X_m)} \wedge \cdots \wedge \pi^n_{(X_1,\ldots,X_m)} \quad . \tag{4.2}$$

In test-score semantics, the translation of a question is a procedure which expresses the answer to the question as a function of the explanatory database. In terms of the framework described above, this means that the answer is expressed as a function of $(X_1, \ldots, X_m)$, i.e.,

$$ans\,(q) = f\,(X_1, \ldots, X_m) \quad .$$

Thus, given the possibility distribution Π over U and the function f, we can obtain the possibility distribution of $ans(q)$ by using the extension principle. In more specific terms, this reduces to the solution of the nonlinear program:

$$\mu_{ans\,(q)}(v) = \max_{(u_1,\ldots,u_m)} \pi_{(X_1,\ldots,X_m)}\ (u_1, \ldots, u_m) \tag{4.3}$$

subject to

$$v = f\,(u_1, \ldots, u_m) \quad ,$$

where u_i denotes the generic value of X_i and $u_i \varepsilon U_i$, $i = 1, \ldots, m$. An example of such a program which we have encountered earlier is provided by (3.64).

In many cases, the nonlinear program (4.3) has special features which reduce it to a simpler problem which can be solved by elementary means. This is what happens in the following examples.

EXAMPLE 6.

$p_1 \triangleq$ *There are about twenty graduate students in his class.*

$p_2 \triangleq$ *There are a few more undergraduate students than graduate students in his class.*

$q \triangleq$ *How many undergraduate students are there in his class?*

Let C_g, C_u and D denote, respectively, the number of graduate students, the number of undergraduate students, and the difference between the two counts, so that

$$C_u = C_g + D.$$

Applying the quantifier extension principle to this relation, we obtain

$$ans\,(q) = about\ 20 \oplus few \quad ,$$

where *ans (q)*, *about 20* and *few* are fuzzy numbers which represent the possibility distributions of C_u, C_g and D, respectively. Using the addition rule for fuzzy numbers (see Appendix), the membership function of *ans (q)* may be expressed more explicitly as

$$\mu_{ans\,(q)}(v) = sup_u\,(\mu_{ABOUT20}(u) \wedge \mu_{FEW}(v-u)) \quad .$$

EXAMPLE 7.

$p \triangleq$ *Brian is much taller than most of his close friends*

$q \triangleq$ *How tall is Brian?*

Following the approach described earlier, we shall (a) determine the possibility distribution induced by p through the use of test-score semantics; (b) express the answer to q as a function defined on the domain of the possibility distribution; and (c) compute the possibility distribution of the answer.

To represent the meaning of p, we assume, as in Examples 4 and 5, that the *EDF* comprises the following relations:

$$EDF \triangleq \quad POPULATION\,[Name;\ Height] +$$
$$MUCH.TALLER\,[Height\ 1;\ Height2;\ \mu] +$$
$$MOST\,[Proportion;\ \mu]\ .$$

For this *EDF*, the test procedure may be described as follows:

1. Determine the height of each $Name_i$, $i = 1, \ldots, n$,, in *POPULATION:*

$$HN_i = {}_{Height}POPULATION[Name = Name_i] \quad ,$$

and, in particular,

$$HB \triangleq {}_{Height}POPULATION[Name = Brian] \quad .$$

2. Determine the degree to which Brian is much taller than $Name_i$:

$$\mu_{BMT}(Name_i) \triangleq {}_{\mu}MUCH.TALLER[Height\,1 = HB; Height\,2 = HN_i] \quad .$$

3. Form the fuzzy set of members of *POPULATION* in relation to whom Brian is much taller:

$$BMT \triangleq \Sigma_i \mu_{BMT}(Name_i)\ /\ Name_i \quad , \quad Name_i \neq Brian \quad .$$

4. Determine the fuzzy set of close friend of Brian by intensifying *FRIEND :*

$$CFB = {}^2(\,{}_{\mu \times Name\,1}FRIEND[Name\,2 = Brian])$$

which implies that

$$\mu_{CFB}(Name_i) = (\,{}_{\mu}FRIEND[Name\,1 = Name_i:\ Name\,2 = Brian])^2 \quad , \quad Name_i \neq Brian \quad .$$

5. Find the proportion of BMT 's in CFB 's:

$$\Sigma Count(BMT / CFB) = \frac{\Sigma_i \mu_{BMT}(Name_i) \wedge \mu_{CFB}(Name_i)}{\Sigma_i \mu_{CFB}(Name_i)} \; .$$

6. Find the test score for the constraint induced by the fuzzy quantifier *most:*

$$\tau = {}_{\mu}MOST[Proportion = \Sigma Count(BMT / CFB)] \; .$$

This test score represents the overall test score for the test which represents the meaning of p. Expressed as a logical form, the test may be represented more compactly as:

$$Prop(BMT / \; {}^2({}_{\mu \times Name\,1}FRIEND[Name\,2 = Brian]) \text{ is } MOST \; ,$$

where the fuzzy set BMT is defined in Steps 2 and 3.

To place in evidence the variables which are constrained by p, it is expedient to rewrite the expression for τ as follows:

$$\tau = \mu_{MOST}\left[\frac{\Sigma_i \mu_{MT}(h_B, h_i) \wedge \mu_{FB}{}^2(Name_i)}{\Sigma_i \mu_{FB}{}^2(Name_i)}\right] \; ,$$

in which h_B is the height of Brian; h_i is the height of $Name_i$; $\mu_{FB}(Name_i)$ is the degree to which $Name_i$ is Brian's friend; $\mu_{MT}(h_B, h_i)$ is the degree to which Brian is much taller than $Name_i$; and μ_{MOST} is the membership function of the quantifier *most*.

Now the variables $X_1, \ldots, X_m$ are those entries in the relations in the explanatory database which are the arguments of τ, with the value of τ representing their joint possibility $\pi_{(X_1, \ldots, X_m)}$. In the example under consideration, these variables are the values of $h_i \triangleq Height(Name_i)$, $i = 1, \ldots, n$; the values of $\mu_{MT}(h_B, h_i)$; the values of $\mu_{FB}(Name_i)$; and the values of

$\mu_{MOST}(Proportion)$, where *Proportion* is the value of the argument of μ_{MOST} in the expression for τ .

Since we are interested only in the height of Brian, it is convenient to let $X_1 \triangleq h_B = Height(Brian)$. With this understanding, the possibility distribution function of *Height* (*Brian*) given the values of $X_2, \ldots, X_m$ may be expressed as

$$Poss\{Height(Brian) = u \mid X_2, \ldots, X_m\} =$$

$$\mu_{MOST}\left[\frac{\Sigma_i \mu_{MT}(u, h_i) \wedge \mu_{FB}{}^2(Name_i)}{\Sigma_i \mu_{FB}{}^2(Name_i)}\right] .$$

where the range of the index i in Σ_i excludes $Name_i = Brian$. Correspondingly, the unconditional possibility distribution function of *Height (Brian)* is given by the projection of the possibility distribution $\Pi_{(X_1, \ldots, X_m)}$ on the domain of X_1. The expression for the projection is given by the supremum of the possibility distribution function of $(X_1, \ldots, X_m)$ over all variables other than X_1 (Zadeh (1978ab)). Thus

$$Poss\{Height(Brian) = u\} =$$

$$sup_{(X_2, \ldots, X_m)} \mu_{MOST}\left[\frac{\Sigma_i \mu_{MOST}(u, h_i) \wedge \mu_{FB}{}^2(Name_i)}{\Sigma_i \mu_{FB}{}^2(Name_i)}\right] .$$

EXAMPLE 8.

$p_1 \triangleq$ *Most Frenchmen are not tall*

$p_2 \triangleq$ *Most Frenchmen are not short*

$q \triangleq$ *What is the average height of a Frenchman?*

Because of the simplicity of p_1 and p_2 , the constraints induced by the premises may be found directly. Specifically, using (3.46), p_1 and p_2 may be replaced by the semantically equivalent premises

$p'_i \triangleq$ *ant most Frenchmen are tall*

$p'_i \triangleq$ *ant most Frenchmen are short.*

To formulate the constraints induced by these premises, let $h_1, \ldots, h_n$ denote the heights of $Frenchman_1, \ldots, Frenchman_n$, respectively. Then, the test scores associated with the constraints in question may be expressed as

$$\tau_1 = \mu_{ANT\ MOST}(\frac{1}{n}\Sigma_i \mu_{TALL}(h_i))$$

and

$$\tau_2 = \mu_{ANT\ MOST}(\frac{1}{n}\Sigma_i \mu_{SHORT}(h_i)) \quad ,$$

where

$$\mu_{ANT\ MOST}(u) = \mu_{MOST}(1-u) \quad , \quad u \varepsilon [0,1] \quad ,$$

and μ_{TALL} and μ_{SHORT} are the membership functions of *TALL* and *SHORT*, respectively. Correspondingly, the overall test score may be expressed as

$$\tau = \tau_1 \wedge \tau_2 \quad .$$

Now, the average height of a Frenchman and hence the answer to the question is given by

$$ans(q) = \frac{1}{n}\Sigma_i h_i \quad .$$

Consequently, the possibility distribution of *ans(q)* is given by the solution of the nonlinear program

$$\mu_{ans(q)}(h) = \max_{h_1, \ldots, h_n}(\tau)$$

subject to

$$h = \frac{1}{n}\Sigma_i h_i \quad .$$

Alternatively, a simpler but less informative answer may be formulated by forming the intersection of the possibility distributions of *ans(q)* which are induced separately by p'_1 and p'_2. More specifically, let $\Pi_{ans(q)|p'_1}$, $\Pi_{ans(q)|p'_2}$, $\Pi_{ans(q)|p'_1 \wedge p'_2}$ be the possibility distributions of *ans(q)* which are induced by p'_1 , p'_2 , and the conjunction of p'_1 and p'_2, respectively. Then, by using the minimax inequality (Zadeh (1971)), it can readily be shown that

$$\Pi_{ans(q)|p'_1} \cap \Pi_{ans(q)|p'_2} \supset \Pi_{ans(q)|p'_1 \wedge p'_2}$$

and hence we can invoke the entailment principle to validate the intersection in question as the possibility distribution of *ans(q)*. For the example under consideration, the possibility distribution is readily found to be given by

$$Poss\,\{ans\,(q\,) = h\,\} = \mu_{ANT\ MOST}(\mu_{TALL}(h)) \wedge \mu_{ANT\ MOST}(\mu_{SHORT}(h)) \quad .$$

Concluding Remark

As was stated in the Introduction, the basic idea underlying our approach to fuzzy quantifiers is that such quantifiers may be interpreted as fuzzy numbers -- a viewpoint which makes it possible to manipulate them through the use of fuzzy arithmetic and, more generally, fuzzy logic.

By applying test-score semantics to the translation of fuzzily-quantified possibilities, a method is provided for inference from knowledge bases which contain such propositions -- as most real-world knowledge bases do. The examples presented in this Section are intended to illustrate the translation and inference techniques which form the central part of our approach. There are many computational issues, however, which are not addressed by these examples. One such issue is the solution of nonlinear programs to which the problem of inference is reduced by the application of the extension principle. What is needed for this purpose are computationally efficient techniques which are capable of taking advantage of the tolerance for imprecision which is intrinsic in inference from natural language knowledge bases.

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APPENDIX

The Extension Principle

Let f be a function from U to V. The extension principle -- as its name implies -- serves to extend the domain of definition of f from U to the set of fuzzy subsets of U. In particular, if F is a finite fuzzy subset of U expressed as

$$F = \mu_1/u_1 + \ldots + \mu_n/u_n$$

then $f(F)$ is a finite fuzzy subset of V defined as

$$f(F) = f(\mu_1/u_1 + \ldots + \mu_n/u_n) \tag{A1}$$

$$= \mu_1/f(u_1) + \ldots + \mu_n/f(u_n).$$

Furthermore, if U is the cartesian product of $U_1, \ldots, U_N$, so that $u = (u^1, \ldots, u^N)$, $u^i \varepsilon U_i$, and we know only the projections of F on $U_1, \ldots, U_N$, whose membership functions are, respectively, $\mu_{FI}, \ldots, \mu_{FN}$, then

$$f(F) = \Sigma_u \mu_{FI}(u^1) \wedge \cdots \wedge \mu_{FN}(u^N) / f(u^1, \ldots, u^N), \tag{A2}$$

with the understanding that, in replacing $\mu_F(u^1, \ldots, u^N)$ with $\mu_F(u^1) \wedge \cdots \wedge \mu_{FN}(u^N)$, we are tacitly invoking the *principle of maximal possibility* (Zadeh (1975b)). This principle asserts that in the absence of complete information about a possibility distribution Π, we should equate Π to the maximal (i.e., least restrictive) possibility distribution which is consistent with the partial information about Π.

As a simple illustration of the extension principle, assume that $U = \{1, 2, \ldots, 10\}$; f is the operation of squaring; and *SMALL* is a fuzzy subset of U defined by

$$SMALL = 1/1 + 1/2 + 0.8/3 + 0.6/4 + 0.4/5 \ .$$

Then, it follows from (A2) that the *right square* of *SMALL* is given by

$$SMALL^2 \triangleq 1/1 + 1/4 + 0.8/9 + 0.6/16 + 0.4/25 \quad .$$

On the other hand, the *left square* of *SMALL* is defined by

$$^2SMALL \triangleq 1/1 + 1/2 + 0.64/3 + 0.36/4 + 0.16/5$$

and, more generally, for a subset F of U and any real m, we have

$$\mu_{m_F}(u) \triangleq (\mu_F(u))^m, \quad u \varepsilon U. \tag{A3}$$

Fuzzy Numbers *

By a fuzzy number, we mean a number which is characterized by a possibility distribution or is a fuzzy subset of real numbers. Simple examples of fuzzy numbers are fuzzy subsets of the real line labeled *small*, *approximately 8*, *very close to 5*, *more or less large*, *much larger than 6*, *several*, etc. In general, a fuzzy number is either a convex or a concave fuzzy subset of the real line. A special case of a fuzzy number is an interval. Viewed in this perspective, fuzzy arithmetic may be viewed as a generalization of interval arithmetic (Moore (1966)).

Fuzzy arithmetic is not intended to be used in situations in which a high degree of precision is required. To take advantage of this assumption, it is expedient to represent the possibility distribution associated with a fuzzy number in a standardized form which involves a small number of parameters -- usually two -- which can be adjusted to fit the given distribution. A system of

* A more detailed exposition of the properties of fuzzy numbers may be found in Dubois and Prade (1980).

standardized possibility distributions which suits this purpose in most cases of practical interest is the following (Zadeh (1975b)).

1. π -numbers. The possibility distribution of such numbers is bell-shaped and piecewise-quadratic. The distribution is characterized by two parameters: (a) the peak-point, i.e., the point at which $\pi = 1$, and (b) the bandwidth, β, which is defined as the distance between the cross-over points, i.e., the points at which $\pi = 0.5$. Thus, a fuzzy π -number, x, is expressed as (p, β), where p is the peak-point and β is the bandwidth; or, alternatively, as (p, β'), where β' is the normalized bandwidth, i.e., $\beta' = \beta/p$. As a function of u, $u \varepsilon (-\infty, \infty)$, the values of $\pi_x(u)$ are defined by the equations

$$\pi_x(u) = 0 \text{ for } u \le p-\beta \text{ and } u \ge p+\beta \tag{A4}$$

$$= \frac{2}{\beta^2}(u-p+\beta)^2 \text{ for } p-\beta \le u \le p-\frac{\beta}{2}$$

$$= 1 - \frac{2}{\beta^2}(u-p)^2 \text{ for } p-\frac{\beta}{2} \le u \le p+\frac{\beta}{2}$$

$$= \frac{2}{\beta^2}(u-p-\beta)^2 \text{ for } p+\frac{\beta}{2} \le u \le p+\beta \quad .$$

2. s-numbers. As its name implies, the possibility distribution of an s-number has the shape of an s. Thus, the equations defining an s-number, expressed as (p/β), are:

$$\pi_x(u) = 0 \text{ for } u \le p-\beta \tag{A5}$$

$$= \frac{2}{\beta^2}(u-p+\beta)^2 \text{ for } p-\beta \le u \le p-\frac{\beta}{2}$$

$$=1-\frac{2}{\beta^2}(u-p)^2 \text{ for } p-\frac{\beta}{2}\le u\le p$$

$$=1 \text{ for } u\ge p \quad ,$$

where β (the bandwidth) is the length of the transition interval from $\pi_x = 0$ to $\pi_x = 1$ and p is the left peak-point, i.e., the right end-point of the transition interval.

3. z-numbers. A z-number is a mirror image of an s-number. Thus, the defining equations for a z-number, expressed as $(p\backslash\beta)$, are:

$$\pi_x(u) = 0 \text{ for } u\le p-\beta \tag{A6}$$

$$=\frac{2}{\beta^2}(u-p+\beta)^2 \text{ for } p-\beta\le u\le p-\frac{\beta}{2}$$

$$=1-\frac{2}{\beta^2}(u-p)^2 \text{ for } p-\frac{\beta}{2}\le u\le p+\beta$$

$$=0 \text{ for } u\ge p+\beta \quad ,$$

where p is the right peak-point and β is the bandwidth.

4. s/z-numbers. An s/z-number has a flat-top possibility distribution which may be regarded as the intersection of the possibility distributions of an s-number and a z-number, with the understanding that the left peak-point of the s-number lies to the left of the right peak-point of the z-number. In some cases, however, it is expedient to disregard the latter restrictions and allow an s/z-number to have a sharp peak rather than a flat top. An s/z-number is represented as an ordered pair $(p_1 / \beta_1;p_2\backslash\beta_2)$ in which the first element is an

s-number and the second element is a z-number.

5. z\s-numbers. The possibility distribution of a z-s number is the complement of that of an s/z-number. Thus, whereas an s/z-number is a convex fuzzy subset of the real line, a z\s-number is a concave fuzzy subset. Equivalently, the possibility distribution of a z\s-number may be regarded as the union of the possibility distributions of a z-number and an s-number. A z\s-number is represented as $(p_1\backslash\beta_1; p_2/\beta_2)$.

Arithmetic Operations on Fuzzy Numbers

Let $*$ denote an arithmetic operation such as addition, subtraction, multiplication or division, and let $x*y$ be the result of applying $*$ to the fuzzy numbers x and y.

By the use of the extension principle, it can readily be established that the possibility distribution function of $x*y$ may be expressed in terms of those of x and y by the relation

$$\pi_{x*y}(w) = \vee_{u,v}(\pi_x(u) \wedge \pi_y(v)), \tag{A7}$$

subject to the constraint

$$w = u*v \ , \quad u,v,w\,\varepsilon(-\infty,\infty)$$

where $\vee_{u,v}$ denotes the supremum over u, v, and $\wedge \triangleq$ min.

As a special case of a general result established by Dubois and Prade (Dubois & Prade (1980)) for so-called L-R numbers, it can readily be deduced from (A7) that if x and y are numbers of the same type (e.g., π-numbers), then so are $x+y$ and x-y. Furthermore, the characterizing parameters of $x+y$ and x-y depend in a very simple and natural way on those of x and y. More specifically, if $x = (p, \beta)$ and $y = (q, \gamma)$, then

$$(p,\beta) + (q,\gamma) = (p+q\ ,\ \beta+\gamma)$$

$$(p\,/\,\beta) + (q\,/\,\gamma) = (p+q\ /\ \beta+\gamma)$$

$$(p\setminus\beta) + (q\setminus\gamma) = (p+q\setminus\beta+\gamma)$$

$$(p_1/\,\beta_1 : p_2\setminus\beta_2)+(q_1/\,\gamma_1 : q_2\setminus\gamma_2)$$

$$=(p_1+q_1/\,\beta_1+\beta_2 : p_2+q_2\setminus\gamma_1+\gamma_2)$$

$$(p,\beta) - (q,\gamma) = (p-q\ ,\ \beta+\gamma)$$

and similarly for other types of numbers.

In the case of multiplication, it is true only as an approximation that if x and y are π -numbers then so is $x \times y$. However, the relation between the peak-points and normalized bandwidths which is stated below is exact:

$$(p,\beta')\times (q,\gamma') = (p \times q,\beta' + \gamma'). \tag{A8}$$

The operation of division, x/y, may be regarded as the composition of (a) forming the reciprocal of y, and (b) multiplying the result by x. In general, the operation $1/y$ does not preserve the type of y and hence the same applies to x/y. However, if y is a π -number whose peak point is much larger than 1 and whose normalized bandwidth is small, then $1/y$ is approximately a π -number defined by

$$1/(p,\beta) \cong (1/p,(\beta'/p')) \tag{A9}$$

and consequently

$$(p,\beta)/(q,\gamma) \cong (p/q,(\beta'/p + \gamma'/q')). \tag{A10}$$

As a simple example of operations on fuzzy numbers, suppose that x is a π -number (p, β) and y is a number which is much larger than x. The question is: What is the possibility distribution of y?

Assume that the relation $y >> x$ is characterized by a conditional possibility distribution $\Pi_{(y|x)}$ (i.e., the conditional possibility distribution of y given x) which for real values of x is expressed as an s-number

$$\Pi_{(y|x)} = (q(x)/\gamma(x)) \tag{A11}$$

whose peak-point and bandwidth depend on x.

On applying the extension principle to the composition of the binary relation >> as defined by (A11) with the unary relation x, it is readily found that y is an s-number which is approximately characterized by

$$y = (q(p)/[q(p)-q(p-\beta)]). \tag{A12}$$

In this way, then, the possibility distribution of y may be expressed in terms of the possibility distribution of x and the conditional possibility distribution of y given x.

Because of the reproducibility property of possibility distributions, the computational effort involved in the manipulation of fuzzy numbers is generally not much greater than that required in interval arithmetic. The bounds on the results, however, are usually appreciably tighter because in the case of fuzzy numbers the possibility distribution functions are allowed to take intermediate values in the interval [0,1], and not just 0 or 1, as in the case of intervals.

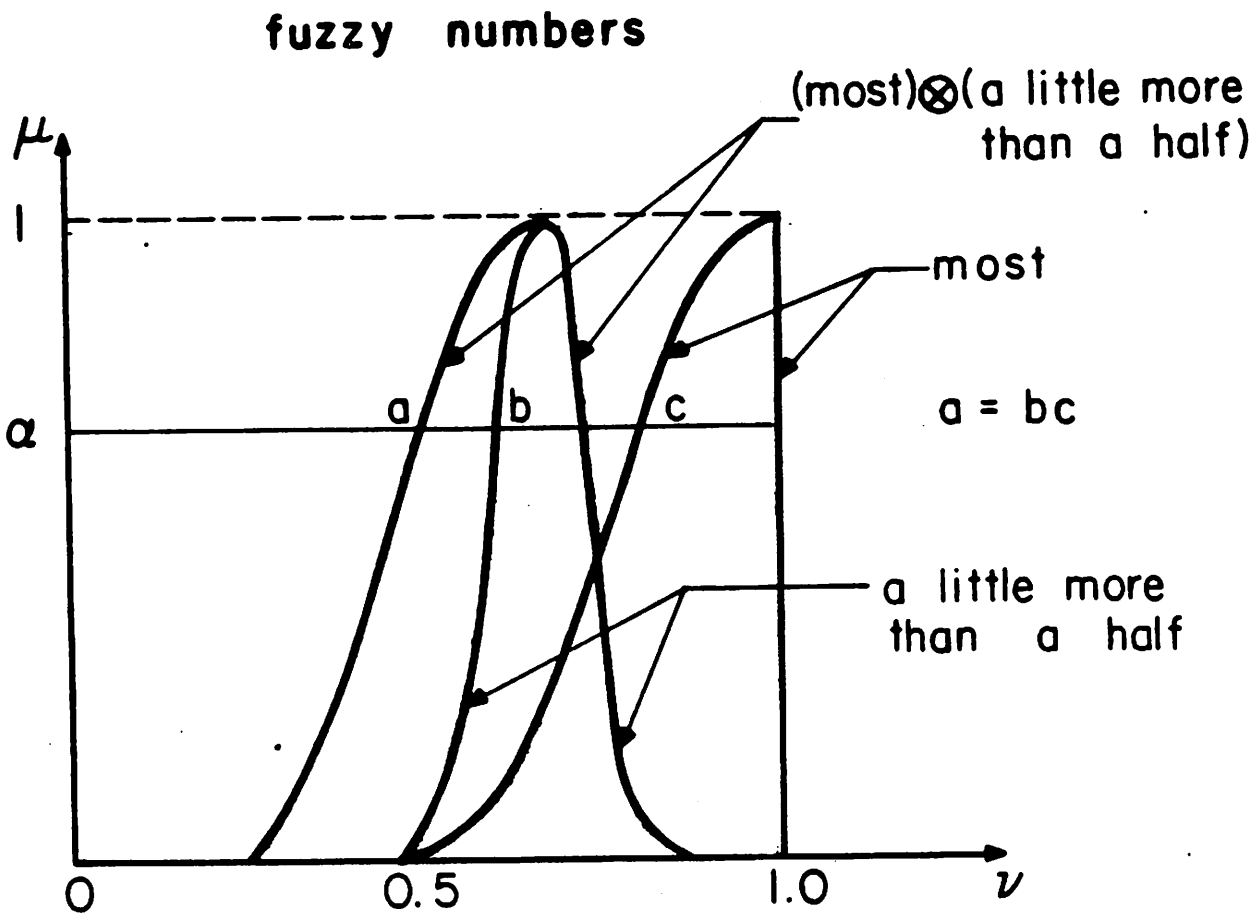

Fig. 1. The intersection/product syllogism with fuzzy quantifiers.

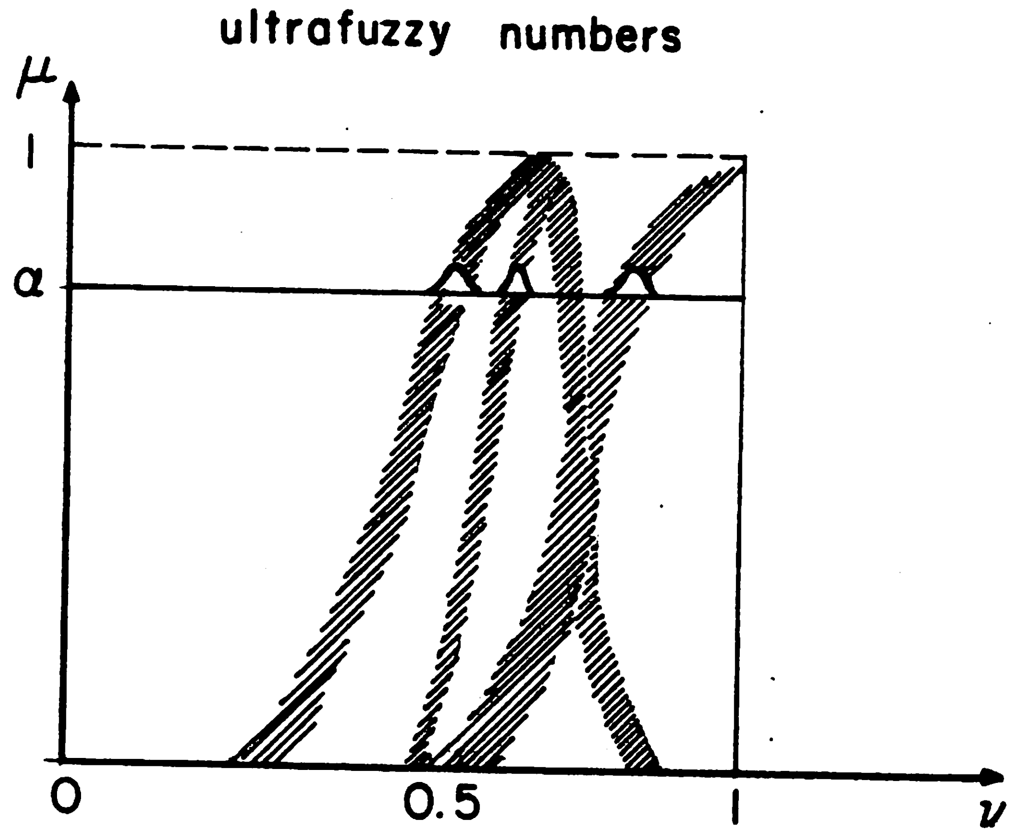

Fig. 2. The intersection/product syllogism with ultrafuzzy quantifiers.

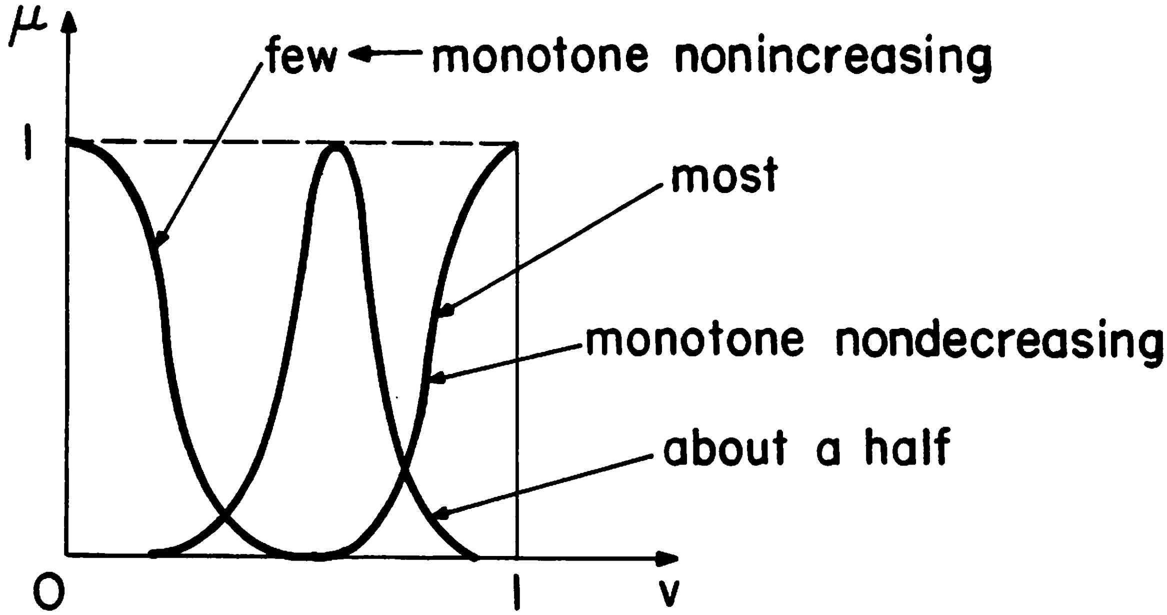

Fig. 3. The fuzzy quantifier few as an antonym of most.

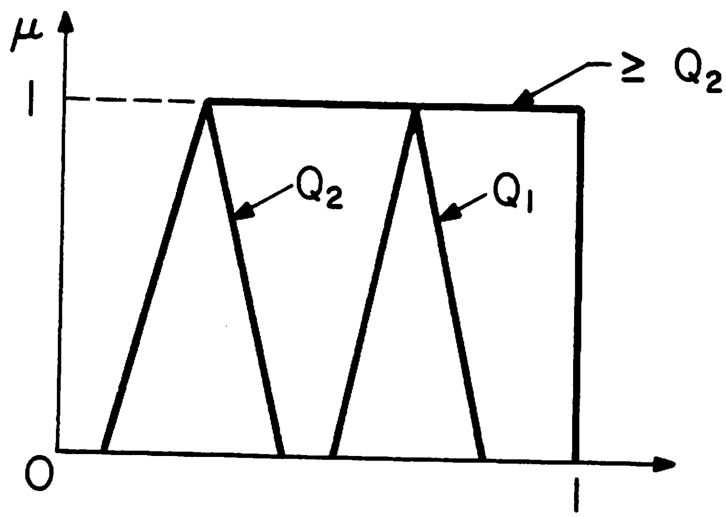

Fig. 4. The possibility distribution of Q_2 and Q_1, with $Q_2 \leqslant Q_1$.

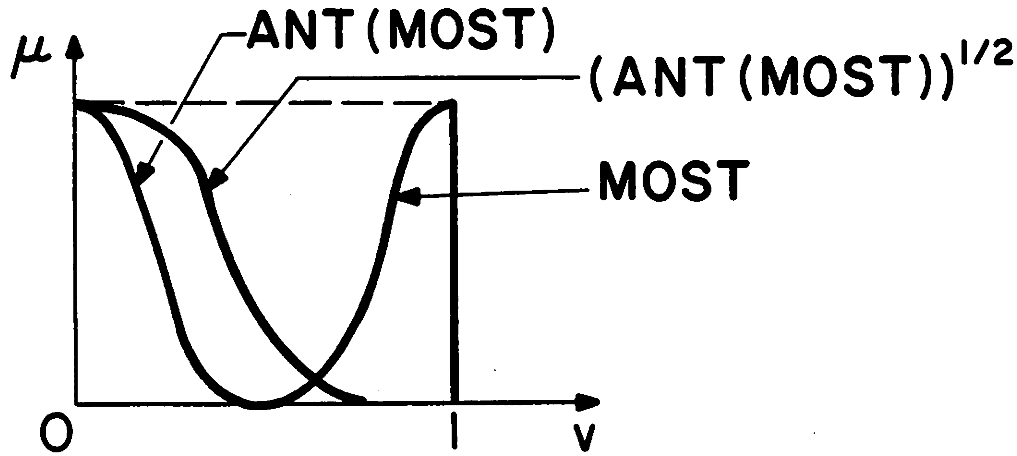

Fig. 5. The possibility distribution of the fuzzy quantifiers <u>most</u>, <u>ant</u> <u>most</u> and (<u>ant</u> <u>most</u>)$^{0.5}$.

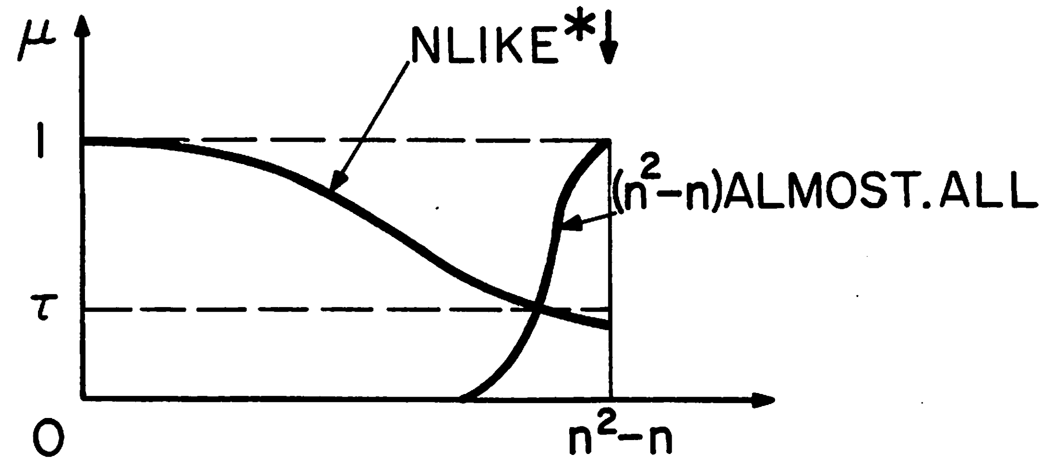

Fig. 6. Computation of the test score for "They like each other."