DESIGN OF MULTIVARIABLE FEEDBACK SYSTEMS
WITH SIMPLE UNSTABLE PLANT

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ABSTRACT

This paper proposes a design methodology for distributed linear multivariable feedback systems with simple unstable plants (a simple unstable plant has either first or second order unstable poles). The methodology developed provides a global parametrization of all realizable compensators which stabilize a given simple unstable plant. A design example is given to show that this methodology can be used to generate, in an appropriate computer aided design environment, controllers which are optimal with respect to designer-specified criteria. Additionally, it is shown that the nature of the design methodology gives geometric insight into the dynamics of the process whereby an unstable plant is stabilized.

I. Introduction

In design of linear multivariable feedback systems, it is often desirable to be able to characterize the class of all proper controllers which stabilize a given plant. Characterizations of this class have been studied throughout the literature: first by Youla, et al. [You. 1], and subsequently by many others [Per. 1], [Des. 1], [Sae. 1]. For the case in which the plant is stable, a particularly simple and convenient parametrization of this class has been developed [Des. 2], [Zam. 1]. This parametrization has been shown to be very useful in computer-aided-design: many practical design constraints can be easily imposed using a design methodology based on this parametrization [Gus. 1]. This paper will show how this parametrization and thus the design methodology may be extended to distributed linear multivariable feedback systems with unstable plants. Although the theory will be developed only for plants whose unstable poles are first or second order (in the Laurent expansion around that pole), it can be extended to arbitrary unstable distributed plants.

We will also show that the results obtained have a geometric flavor; they can be used to give insight into the mechanism by which an unstable plant is stabilized.

Finally, the design methodology is applied to an example plant in such a way as to yield a closed-loop system with a decoupled I/O map which satisfies several inequality constraints, and minimizes a cost function. The inequality constraints and cost functions represent certain practical design goals such as avoiding plant saturation, and desensitizing the closed-loop response to output disturbances and/or plant perturbations. This example demonstrates that the design methodology can indeed be extended to unstable plants.

The paper is organized as follows:

Section II describes the feedback system and some of its basic properties.

Section III developes the theory for the case in which the plant has only first order unstable poles.

Section IV develops the theory for the case in which the plant has first and second order unstable poles.

Section V presents the design example.

Section VI contains the conclusions.

Special notations and definitions

For $\sigma \in \mathbb{R}$ (typically $\sigma < 0$), $\mathbb{C}_{\sigma+}$ denotes the closed right half plane $\mathrm{Re}(s) \geq \sigma$. $f \in A(\sigma)$ iff $f(t) = f_a(t) + \sum_{i=0}^{\infty} f_i \delta(t-t_i)$ where $f_a : \mathbb{R} \to \mathbb{R}$ with $f_a(t) = 0$ for $t < 0$, $t \mapsto f_a(t) \exp(-\sigma t) \in L_1$; $t_0 = 0$, $t_i > 0$, $\forall i > 0$; $\forall i$, $f_i \in \mathbb{R}$ and $i \mapsto f_i \exp(-\sigma t_i) \in \ell_1$. $f \in A_-(\sigma)$ iff, for some $\sigma_1 < \sigma$, $f \in A(\sigma_1)$. $\hat{f}$ denotes the Laplace transform of f. $\hat{A}_-(\sigma) := \{\hat{f} : f \in A_-(\sigma)\}$. $\hat{A}_-^{\infty}(\sigma)$, ($\hat{A}_{-,o}(\sigma)$, resp.), denotes the subset of $\hat{A}_-(\sigma)$ consisting of those f that are bounded away from zero at infinity in $\mathbb{C}_{\sigma+}$, ($\hat{f}$ that go to zero at infinity in $\mathbb{C}_{\sigma+}$, resp.).
$\hat{B}(\sigma) := [\hat{A}_-(\sigma)][\hat{A}_-^{\infty}(\sigma)]^{-1}$, the commutative algebra of fractions $\hat{g} = \hat{n}/\hat{d}$ where $\hat{n} \in \hat{A}_-(\sigma)$ and $\hat{d} \in \hat{A}_-^{\infty}(\sigma)$ [Cal. 1], [Cal.3]; for the general treatment see [Jac. 1, Sec. 7.2], [Bou. 1, Chap. II, Sec. 2].
$\hat{B}_o(\sigma) := [\hat{A}_{-,o}(\sigma)][\hat{A}_-^{\infty}(\sigma)]^{-1}$. $A_- := A_-(0)$, $\hat{B} := \hat{B}(0)$. Let $A \in \mathbb{C}^{m \times n}$, then $\bar{\sigma}[A]$:= the largest singular value of A [Ste. 1]. If S is a set, then $\mathcal{E}(S)$ denotes the set of matrices whose elements are in S.

For $H \in \hat{B}(\sigma)$, $H : s \mapsto H(s)$, H' denotes the derivative of H with respect to $s : H'(s) := \frac{d}{ds} H(s)$, $\forall s \in \mathbb{C}$.

II. System Description

Throughout, we consider the closed-loop system S(P,C), shown in Figure 1. We define [Zam. 1]

$$Q := C(I+PC)^{-1} \tag{2.1}$$

Equivalently,

$$C = Q(I-PQ)^{-1} \tag{2.2}$$

Then, we have $H_{yu} : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ given by:

$$H_{yu} = \begin{bmatrix} C(I+PC)^{-1} & -CP(I+CP)^{-1} \\ PC(I+PC)^{-1} & P(I+CP)^{-1} \end{bmatrix} = \begin{bmatrix} Q & -QP \\ PQ & P(I-QP) \end{bmatrix} \tag{2.3}$$

Also, $H_{eu} : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is given by:

$$H_{eu} = \begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix} = \begin{bmatrix} I-PQ & -P(I-QP) \\ Q & I-QP \end{bmatrix} \tag{2.4}$$

Note that $H_{y_2 d_0} : d_0 \mapsto y_2$ is given by:

$$H_{y_2 d_0} = (I+PC)^{-1} = I - PQ \tag{2.5}$$

Suppose that $P \in \mathcal{E}(\hat{\mathcal{B}}_o(\sigma_o))$. In this case, (2.1) and (2.3) imply that [Des. 2, Thm. 1, p. 410]

$$C \in \mathcal{E}(\hat{\mathcal{B}}(\sigma_o)) \Leftrightarrow Q \in \mathcal{E}(\hat{\mathcal{B}}(\sigma_o)) \tag{2.6}$$

Definition 2.1. The closed-loop system S(P,C) is said to be $\hat{A}_-(\sigma_0)$-stable if and only if $H_{yu} \in E(\hat{A}_-(\sigma_0))$

III. Stabilization of Simple Unstable Plants: First Order Pole(s)

We consider unstable multivariable plants which satisfy the following assumptions:

(A1) $P(s) \in \hat{B}_0(\sigma_0)^{n_o x n_i}$ (i.e., P is str. proper and $P \in \hat{B}(\sigma_0)^{n_o x n_i}$)

(A2) $P(s) = \frac{1}{s-\lambda_1} U_1 V_1^T + P_\alpha(s)$, where $\lambda_1 \in \mathbb{C}_{\sigma_0 +}$; $U_1 \in \mathbb{C}^{n_o x r}$ and $V_1 \in \mathbb{C}^{n_i x r}$ have rank r; and $P_\alpha(s) \in \hat{A}_{-,o}(\sigma_0)^{n_o x n_i}$.

Assumption (A2) means that P has one first-order unstable pole with MacMillan degree r.

The following theorem gives a global parametrization of all $\hat{A}_-(\sigma_0)$-stable closed-loop systems S(P,C) with $C \in \hat{B}(\sigma_0)^{n_i x n_o}$. ⇔

Theorem 3.1. Let P satisfy (A1) and (A2). Let $C := Q(I-PQ)^{-1}$. Then S(P,C) is $\hat{A}_-(\sigma_0)$-stable, with $C \in \hat{B}(\sigma_0)^{n_i x n_o}$

(i) Q is $\hat{A}_-(\sigma_0)$-stable (i.e., $Q \in E(\hat{A}_-(\sigma_0))$) (3.3)

(ii) $Q(\lambda_1)U_1 = \theta_{n_i x r}$ and $V_1^T Q(\lambda_1) = \theta_{r x n_o}$ (3.4)

(iii) Either $H_{y_2 u_1}(\lambda_1)U_1 = U_1$ (3.5)

or $V_1^T H_{e_2 u_2}(\lambda_1) = \theta_{r x n_i}$ (3.6)

Comments: (a) Note that for some Q's satisfying conditions (i)-(iii), the corresponding compensator $C = Q(I-PQ)^{-1}$ is also unstable: so, in such cases, the theorem guarantees stability of S(P,C) with P and C unstable.

(b) Condition (ii) amounts to guaranteeing that the unstable pole at $s = \lambda_1$ in $P(s)$ is cancelled by Q in the expressions PQ and QP. Similarly, condition (iii) guarantees that the same pole will not appear in P(I-QP).

(c) The resulting feedback system is $\hat{A}_-(\sigma_0)$-stable, and is robustly stable: indeed, suppose that $C = Q(I-PQ)^{-1}$ is either minimally realized, or realized with only $\mathring{\mathbb{C}}_{\sigma_0-}$ hidden modes, then, for the nominal system, the closed-loop characteristic function is bounded away from zero in $\mathbb{C}_{\sigma_0+}$. Now, if small perturbations are allowed on P and C, then† the closed-loop characteristic function undergoes only small perturbations, and consequently, by the usual argument based on Rouché's theorem, the perturbed characteristic function is bounded away from 0 in $\mathbb{C}_{\sigma_0+}$; hence the perturbed closed-loop system remains $\hat{A}_-(\sigma_0)$-stable.

(d) Let $\xi \in \mathbb{R}^r$, $u := U_1\xi \in R(U_1)$, then (3.5) implies that $H_{y_2u_1}(\lambda_1)u = u$, for all such u; i.e., at the frequency λ_1, the I/O map, $H_{y_2u_1}$, has unity gain in those directions in which P(s) has a pole at $s=\lambda_1$.

(e) An input $u_2(s) \in \hat{A}_-(\sigma_0)^{n_i}$ produces an output $y_2(s) = H_{y_2u_2}(s)u_2(s) = P(s)H_{e_2u_2}(s)u_2(s)$; thus $y_2(s)$ has a pole at $s = \lambda_1 \in \mathbb{C}_{\sigma_0+}$ with residue $U_1V_1^TH_{e_2u_2}(\lambda_1)u_2(\lambda_1)$. Hence, condition (3.6) implies that for all such inputs $H_{y_2u_2}(s)u_2(s) = P(I-QP)(s)u_2(s)$ has no pole at $s = \lambda_1$: i.e., none of these inputs can excite λ_1, the unstable pole of the plant. Condition (3.6) can thus be interpreted as: at the frequency λ_1, the closed-loop

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†It is crucial to note that it is C -- and not Q -- that is built into the feedback system. Consequently, the characteristic function of S(P,C) may be written -- using obvious notation for the coprime factorizations -- as $\chi = \det(D_{cl}D_{pr}+N_{cl}N_{pr})$ [Cal. 1].

map $H_{e_2u_2} : u_2 \mapsto e_2$ (from the input u_2 to e_2, the input of the plant) has a transmission zero in the subspace from which it is possible to excite the unstable pole of the plant.

(f) Straightforward calculation shows that if conditions (i) and (ii) hold, then (3.5), (3.6) and

$$V_1^T Q'(\lambda_1) U_1 = I_r$$

are equivalent.

proof:

Since Q is $\hat{A}_-(\sigma_0)$-stable, it is analytic at $s = \lambda_1$. Thus, we can write the Taylor's series for Q(s) at $s = \lambda_1$:

$$Q(s) = Q(\lambda_1) + (s-\lambda_1)\, Q'(\lambda_1) + O(s-\lambda_1)^2 \tag{3.7}$$

So,

$$(PQ)(s) = [(\tfrac{1}{s-\lambda_1})U_1V_1^T + P_\alpha(s)][Q(\lambda_1) + (s-\lambda_1)Q(\lambda_1) + O(s-\lambda_1)^2] \tag{3.8}$$

$$= (\tfrac{1}{s-\lambda_1})U_1V_1^T Q(\lambda_1) + \tilde{H}_{y_2u_1}(s)$$

where $\tilde{H}_{y_2u_1}(s)$ is the sum of the remaining terms. Note that $\tilde{H}_{y_2u_1}$ is $\hat{A}_-(\sigma_0)$-stable, since $P_\alpha(s)$ and $Q(s)$ are $\hat{A}_-(\sigma_0)$-stable. Now, by (3.4),

$$(PQ)(s) = \tilde{H}_{y_2u_1}(s)$$

Thus, PQ is $\hat{A}_-(\sigma_0)$-stable.

Also,

$$(QP)(s) = [Q(\lambda_1) + (s-\lambda_1)Q'(\lambda_1) + O(s-\lambda_1)^2][(\tfrac{1}{s-\lambda_1})U_1V_1^T + P_\alpha(s)]$$

$$= (\tfrac{1}{s-\lambda_1})\, Q(\lambda_1)U_1V_1^T + \tilde{H}_{y_1u_2}(s) \tag{3.10}$$

where $\tilde{H}_{y_1u_2}(s)$ is the $\hat{A}_-(\sigma_o)$-stable sum of the remaining terms. Now, by (3.4),

$$(QP)(s) = \tilde{H}_{y_1u_2}(s)$$

Thus, QP is $\hat{A}_-(\sigma_o)$-stable. Also, $H_{e_2u_2} = I - QP$ is $\hat{A}_-(\sigma_o)$-stable. Finally, since $H_{y_2u_1}(s)$ is analytic at $s = \lambda_1$,

$$\begin{aligned}[(I-PQ)P](s) &= [(I-H_{y_2u_1})P](s)\\ &= [I - H_{y_2u_1}(\lambda_1) + O(s-\lambda_1)][(\frac{1}{s-\lambda_1})U_1V_1^T + P_\alpha(s)]\\ &= (\frac{1}{s-\lambda_1})\ [I - H_{y_2u_1}(\lambda_1)]U_1V_1^T + \tilde{H}_{y_2u_2}(s)\end{aligned}$$

where $\tilde{H}_{y_2u_2}(s)$ is the $\hat{A}_-(\sigma_o)$-stable sum of the remaining terms. Now, by (3.5),

$$[(I-PQ)P](s) = \tilde{H}_{y_2u_2}(s)$$

Thus, (3.3) - (3.5) imply S(P,C) is $\hat{A}_-(\sigma_o)$-stable. Also, since $H_{e_2u_2}(s)$ is analytic at $s = \lambda_1$,

$$\begin{aligned}[P(I-QP)](s) &= (PH_{e_2u_2})(s)\\ &= [(\frac{1}{s-\lambda_1})U_1V_1^T + P_\alpha(s)][H_{e_2u_2}(\lambda_1) + O(s-\lambda_1)]\\ &= (\frac{1}{s-\lambda_1})U_1V_1^TH_{e_2u_2}(\lambda_1) + \bar{H}_{y_2u_2}(s)\end{aligned}$$

where $\bar{H}_{y_2u_2}(s)$ is the $\hat{A}_-(\sigma_o)$-stable sum of the remaining terms. Now, by (3.6),

$$[P(I-QP)](s) = \bar{H}_{y_2u_2}(s)$$

Thus, (3.3), (3.4), and (3.6) imply S(P,C) is $\hat{A}_-(\sigma_o)$-stable and it follows from (2.6) and (3.3) that $C \in \hat{B}(\sigma_o)^{n_i x n_o}$. So, for either of the conditions (3.5) or (3.6), S(P,C) is $\hat{A}_-(\sigma_0)$-stable with $C \in \hat{B}(\sigma_0)^{n_i x n_o}$.

$\Rightarrow$ By (2.3), $S(P,C) \in E(\hat{A}_-(\sigma_0))$ implies that Q, PQ, QP and P(I-QP) are $\hat{A}_-(\sigma_0)$-stable. So, condition (i) is satisfied immediately.

Proof of Condition (ii): By assumption $PQ = H_{y_2u_1}$ is $\hat{A}_-(\sigma_0)$-stable.

By calculation,

$$(PQ)(s) = [(\frac{1}{s-\lambda_1})U_1V_1^T + P_\alpha(s)]Q(s)$$

Thus, since P_α and Q are $\hat{A}_-(\sigma_0)$-stable, $P_\alpha Q$ is $\hat{A}_-(\sigma_0)$-stable, and hence $(\frac{1}{s-\lambda_1})U_1V_1^TQ(s)$ is $\hat{A}_-(\sigma_0)$-stable; thus $U_1V_1^TQ(\lambda_1) = \theta$ (if not, then $(\frac{1}{s-\lambda_1})U_1V_1^TQ(s)$ has a pole at $s = \lambda_1$, and hence is not $\hat{A}_-(\sigma_0)$-stable). And, since $rk(U_1) = r$; $N(U_1) = \{\theta_r\}$, and this condition reduces to

$$V_1^TQ(\lambda_1) = \theta_{rxn_o r} \tag{3.20}$$

Similarly, since QP is $\hat{A}_-(\sigma_0)$-stable, $(\frac{1}{s-\lambda_1})Q(s)U_1V_1^T$ is $\hat{A}_-(\sigma_0)$-stable, and thus, since $N(V_1) = \{\theta_r\}$,

$$Q(\lambda_1)U_1 = \theta_{n_i xr} \tag{3.21}$$

Proof of Condition (iii): By assumption $H_{e_2u_2}$, $H_{y_2u_1}$ and $H_{y_2u_2} = P(I_{n_i}-QP)$ are $\hat{A}_-(\sigma_0)$-stable. By calculation

$$[P(I_{n_i}-QP)](s) = (PH_{e_2u_2})(s) = [(\frac{1}{s-\lambda_1})U_1V_1^T + P_\alpha(s)]H_{e_2u_2}(s)$$

Since $P(I_{n_i}-QP)$, $H_{e_2u_2} = I_{n_i}-QP$, and P_α are $A_-(\sigma_0)$-stable, $(\frac{1}{s-\lambda_1})U_1V_1^TH_{e_2u_2}(s)$ must be $\hat{A}_-(\sigma_0)$-stable. Thus,

$$V_1^TH_{e_2u_2}(\lambda_1) = \theta_{rxn_i} \tag{3.22}$$

where we note that $N(U_1) = \{\theta_r\}$

Also,

$$[(I_{n_o}-PQ)P](s) = [(I_{n_o}-H_{y_2u_1})P](s) = [I_{n_o}-H_{y_2u_1}(s)][(\frac{1}{s-\lambda_1})U_1V_1^T+P_\alpha(s)]$$

Since $(I_{n_o}-PQ)P$, PQ, and P_α are $\hat{A}_-(\sigma_o)$-stable, $(\frac{1}{s-\lambda_1})[I_{n_o}-H_{y_2u_1}(s)]U_1V_1^T$ must be $\hat{A}_-(\sigma_o)$-stable. Thus,

$$[I - H_{y_2u_1}(\lambda_1)]U_1V_1^T = \theta$$

and,

$$H_{y_2u_1}(\lambda_1)U_1 = U_1 \tag{3.23}$$

since $N(V_1) = \{\theta_r\}$. ¤

<u>Theorem 3.2</u>. Let P satisfy (A1) and (A2). Let $C := Q(I-PQ)^{-1}$. Suppose that, in addition, P(s) has <u>full normal rank</u>, and has <u>no transmission zeros</u> at $s = \lambda_1$. U.t.c. S(P,C) is $\hat{A}_-(\sigma_o)$-stable with $C \in \hat{B}(\sigma_o)^{n_i x n_o}$ ⇔

(i) Q is $\hat{A}_-(\sigma_o)$-stable (3.24)

(ii)

for $n_o \le n_i$	for $n_o \ge n_i$	
$Q(\lambda_1)U_1 = \theta_{n_i x r}$	$V_1^TQ(\lambda_1) = \theta_{rxn_o}$	(3.25)

(iii) Either $H_{y_2u_1}(\lambda_1)U_1 = U_1$ (3.26)

or $V_1^TH_{e_2u_2}(\lambda_1) = \theta_{rxn_i}$ (3.27)

<u>Proof</u>:

⇒ This direction follows from Theorem 3.1.

⇐ $(n_o \le n_i)$ By Theorem 3.1, to establish that S(P,C) is $\hat{A}_-(\sigma_o)$-stable, we need only prove $V_1^TQ(\lambda_1) = \theta_{rxn_o}$.

Since P(s) has full normal rank, it has a right inverse $P^{-R}(s)$, which is analytic in a neighborhood of λ_1, because P(s) has no transmission zero at $s = \lambda_1$. Furthermore, since $PQ = H_{y_2u_1}$, for any such right inverse, there exists $R(s) \in \hat{A}_-(\sigma_0)^{n_i \times n_o}$ such that $PR = \theta_{n_o \times n_o}$ and,

$$Q(s) = P^{-R}(s)H_{y_2u_1}(s) + R(s) \tag{3.29}$$

Now, using (A2), we obtain

$$I_{n_o} = (PP^{-R})(s) = (\frac{1}{s-\lambda_1})U_1V_1^TP^{-R}(s) + (P_\alpha P^{-R})(s), \quad \forall s \in \mathbb{C} \tag{3.30}$$

$$\theta_{n_o \times n_o} = (PR)(s) = (\frac{1}{s-\lambda_1})U_1V_1^TR(s) + (P_\alpha R)(s), \quad \forall s \in \mathbb{C} \tag{3.31}$$

Since $P_\alpha(s)$, $P^{-R}(s)$, and $R(s)$ are analytic at $s = \lambda_1$, and since $N(U_1) = \{\theta_r\}$, (3.30) and (3.31) imply

$$V_1^TP^{-R}(\lambda_1) = \theta_{r \times n_o} \quad \text{and} \quad V_1^TR(\lambda_1) = \theta_{r \times n_o} \tag{3.32}$$

By (3.24) and (A2), $H_{y_2u_1} = PQ$ is analytic in $\mathbb{C}_{\sigma_o+}$, except possibly at $s = \lambda_1$, where it may at most have a first order pole. Suppose $H_{y_2u_1}$ has such a pole: since the columns of $P^{-R}(s)$ are linearly independent, and since $R(s)$ is analytic at λ_1, then by (3.29), Q would also have a pole at $s = \lambda_1$. This contradicts (3.24). Hence <u>$H_{y_2u_1}$ is analytic at $s = \lambda_1$</u>. Consequently, equation (3.32) in equation (3.29) gives

$$V_1^TQ(\lambda_1) = \theta_{r \times n_o} \tag{3.33}$$

So, by Theorem 3.1, $S(P,C)$ is $\hat{A}_-(\sigma_o)$-stable, and $C \in \hat{B}(\sigma_o)^{n_i \times n_o}$. The proof for $n_o \geq n_i$ is dual to the above. ¤

<u>Corollary 3.3</u>. Let the assumptions of Theorem 3:2 hold. If, in addition, P is <u>square</u> ($n_o=n_i$), then the necessary and sufficient conditions reduce to (i) and (iii) (note that (ii) is omitted).

Proof: The proof follows easily from that of Theorem 3.2. ¤

Remark: Note that the theorems of this section can be easily extended to the class of plants satisfying the two assumptions (A1) and (A3), namely,

(A1) $P(s) \in \hat{B}_o(\sigma_o)^{n_o x n_i}$

(A3) $P(s) = \sum_{k=1}^{p} \frac{1}{s-\lambda_k} U_k V_k + P_\alpha(s)$, where, for each $1 \le k \le p$,

$\lambda_k \in \mathbb{C}_{\sigma_o+}$; $U_k \in \mathbb{C}^{n_o x r_k}$, $V_k \in \mathbb{C}^{n_i x r_k}$ have rank r_k; and $P_\alpha(s) \in \hat{A}_{-,o}(\sigma_o)^{n_o x n_i}$

For P satisfying (A1) and (A3), S(P,C) will be $\hat{A}_-(\sigma_o)$-stable with $C \in \hat{B}(\sigma_o)^{n_i x n_o}$ ⇔ The conditions (i)-(iii) of Theorem 3.1 are satisfied for λ_k, U_k and V_k, for each $1 \le k \le p$.

So, in theory, the results of this section may be used to stabilize any unstable plant whose instability results from first order $\mathbb{C}_{\sigma_o+}$-poles.

IV. Stabilization of Simple Unstable Plants: Second Order Pole(s)

We consider unstable multivariable plants which satisfy the following assumptions:

(A1) $P(s) \in \hat{B}_o(\sigma_o)^{n_o x n_i}$

(A4) $P(s) = (\frac{1}{s-\lambda_1})^2 R_{12} + (\frac{1}{s-\lambda_1}) R_{11} + P_\alpha(s)$,

where $\lambda_1 \in \mathbb{C}_{\sigma_o+}$; for $j = 1,2$, $R_{1j} = U_{1j} V_{1j}^T$, and $U_{1j} \in \mathbb{C}^{n_o x r_j}$, $V_{1j} \in \mathbb{C}^{n_i x r_j}$ each have rank r_j; and $P_\alpha(s) \in \hat{A}_{-,o}(\sigma_o)^{n_o x n_i}$. Assumption (A4) means that P has one second order unstable pole with MacMillan degree equal to $\text{rk}\begin{bmatrix} R_{11} & R_{12} \\ R_{12} & 0 \end{bmatrix}$.

The following theorem gives a global parametrization of all $\hat{A}_-(\sigma_o)$-stable closed-loop systems S(P,C) with $C(s) \in \hat{B}(\sigma_o)^{n_i x n_o}$.

Theorem 4.1. Let P satisfy (A1) and (A4). Let $C := Q(I-PQ)^{-1}$. Then, S(P,C) is $\hat{A}_-(\sigma_o)$-stable, with $C \in \hat{B}(\sigma_o)^{n_i x n_o}$ $\Leftrightarrow$

(i) Q is $\hat{A}_-(\sigma_o)$-stable (i.e., $Q \in E(\hat{A}_-(\sigma_o))$) (4.3)

(ii) $Q(\lambda_1)U_{12} = \theta_{n_i x r_2}$ and $V_{12}^T Q(\lambda_1) = \theta_{r_2 x n_o}$ (4.4)

(iii) $Q(\lambda_1)R_{11} = -Q'(\lambda_1)R_{12}$ and $R_{11}Q(\lambda_1) = -R_{12}Q'(\lambda_1)$ (4.5)

(iv) Either $H_{y_2u_1}(\lambda_1)U_{12} = U_{12}$ (4.6)

or $V_{12}^T H_{e_2u_2}(\lambda_1) = \theta_{r_2 x n_i}$ (4.7)

(v) Either $R_{11} = H_{y_2u_2}(\lambda_1)R_{11} + H'_{y_2u_1}(\lambda_1)R_{12}$ (4.8)

or $\theta_{n_o x n_i} = R_{11}H_{e_2u_2}(\lambda_1) + R_{12}H'_{e_2u_2}(\lambda_1)$ (4.9)

Comments: (a) Conditions (ii) and (iii) are used to prove $\hat{A}_-(\sigma_o)$-stability of PQ and QP, and are implied by the same. Conditions (iv) and (v) are similarly related to the $\hat{A}_-(\sigma_o)$-stability of P(I-QP).

(b) If R_{12} is full row (column, resp.) rank, then (4.6) ((4.7), resp.) becomes

$$H_{y_2u_1}(\lambda_1) = I_{n_o} \quad (4.10)$$

$$(H_{e_2u_2}(\lambda_1) = \theta_{n_i x n_i}, \text{ resp.}) \quad (4.11)$$

Also, (4.8) ((4.9), resp.) becomes

$$H'_{y_2u_1}(\lambda_1)R_{12} = \theta_{n_o x n_i} \quad (4.12)$$

$$(H'_{e_2u_2}(\lambda_1)R_{12} = \theta_{n_i x n_i}, \text{ resp.}) \quad (4.13)$$

(c) Straightforward calculations show that if conditions (i)-(iii) hold, then (4.6), (4.7) and

$$R_{12} = R_{12}Q'(\lambda_1)R_{11} + \frac{1}{2} R_{12}Q''(\lambda_1)R_{12}$$

are equivalent.

(d) Similarly, if conditions (i)-(iii) hold, then (4.8), (4.9) and

$$R_{11} = R_{11}Q'(\lambda_1)R_{11} + \frac{1}{2} [R_{12}Q''(\lambda_1)R_{11} + R_{11}Q''(\lambda_1)R_{12}] + \frac{1}{6} R_{12}Q'''(\lambda_1)R_{12}$$

are equivalent.

Proof:

Since Q(s) is $\hat{A}_-(\sigma_0)$-stable, we can write its Taylor's series expansion around $s = \lambda_1$:

$$Q(s) = Q(\lambda_1) + (s-\lambda_1)\, Q'(\lambda_1) + O(s-\lambda_1)^2 \tag{4.14}$$

So,

$$(PQ)(s) = [(\frac{1}{s-\lambda_1})^2 R_{12} + (\frac{1}{s-\lambda_1})R_{11} + P_\alpha(s)]Q(s)$$

$$= (\frac{1}{s-\lambda_1})^2 R_{12}Q(\lambda_1) + (\frac{1}{s-\lambda_1})[R_{11}Q(\lambda_1) + R_{12}Q'(\lambda_1)] + \tilde{H}_{y_2u_1}(s)$$

where $\tilde{H}_{y_2u_1}$ is $\hat{A}_-(\sigma_0)$-stable, since P_α and Q are $\hat{A}_-(\sigma_0)$-stable. Now, by (4.4) and (4.5),

$$(PQ)(s) = \tilde{H}_{y_2u_1}(s)$$

Thus, PQ is $\hat{A}_-(\sigma_0)$-stable.

Also,

$$(QP)(s) = Q(s)[(\frac{1}{s-\lambda_1})^2 R_{12} + (\frac{1}{s-\lambda_1})R_{11} + P_\alpha(s)]$$

$$= (\frac{1}{s-\lambda_1})^2 Q(\lambda_1)R_{12} + (\frac{1}{s-\lambda_1})[Q(\lambda_1)R_{11} + Q'(\lambda_1)R_{12}] + \tilde{H}_{y_1u_2}(s)$$

where $\tilde{H}_{y_1u_2}$ is $\hat{A}_-(\sigma_0)$-stable, since P_α and Q are $\hat{A}_-(\sigma_0)$-stable. Now, by (4.4) and (4.5),

$$(QP)(s) = \tilde{H}_{y_1u_2}(s)$$

Thus, QP is $\hat{A}_-(\sigma_0)$-stable.

Next,

$$[P(I_{n_i}-QP)](s) = (PH_{e_2u_2})(s) = [(I_{n_o}-H_{y_2u_1})P](s)$$

So, since $H_{e_2u_2}(s)$ and $H_{y_2u_1}(s)$ are $\hat{A}_-(\sigma_0)$-stable, by the above proof, they are analytic at $s = \lambda_1$. Thus they each have Taylor's series expansions at $s = \lambda_1$, and hence,

$$[P(I_{n_i}-QP)](s) = \left(\frac{1}{s-\lambda_1}\right)^2 R_{12}H_{e_2u_2}(\lambda_1) + \left(\frac{1}{s-\lambda_1}\right)[R_{11}H_{e_2u_2}(\lambda_1)+R_{12}H'_{e_2u_2}(\lambda_1)] + \tilde{H}_{y_2u_2}$$

$$= \left(\frac{1}{s-\lambda_1}\right)^2 [I_{n_o}-H_{y_2u_1}(\lambda_1)]R_{12} + \left(\frac{1}{s-\lambda_1}\right)[(I_{n_o}-H_{y_2u_1}(\lambda_1))R_{11}$$

$$+ H'_{y_2u_1}(\lambda_1)R_{12}] + \bar{H}_{y_2u_2}(s)$$

where $\tilde{H}_{y_2u_2}$ and $\bar{H}_{y_2u_2}$ are $\hat{A}_-(\sigma_0)$-stable, since $H_{e_2u_2}$, $H_{y_2u_1}$ and P_α are $\hat{A}_-(\sigma_0)$-stable. Now, by the uniqueness of the Laurent expansion for $[P(I_{n_i}-QP)](s)$ about $s = \lambda_1$,

$$R_{12}\, H_{e_2u_2}(\lambda_1) = [I_{n_o}-H_{y_2u_2}(\lambda_1)]R_{12}$$

$$R_{11}H_{e_2u_2}(\lambda_1) + R_{12}H'_{e_2u_2}(\lambda_1) = [I_{n_o}-H_{y_2u_1}(\lambda_1)]R_{11} + H'_{y_2u_1}(\lambda_1)R_{12}$$

Note that this implies $\tilde{H}_{y_2u_2} = \bar{H}_{y_2u_2}$. Thus, it is clear that any of the four possible combinations of conditions in (iv) and (v) implies that

$$[P(I_{n_i}-QP)](s) = \tilde{H}_{y_2u_2}(s)$$

Thus, $P(I_{n_i}-QP)$ is $\hat{A}_-(\sigma_0)$-stable, in any case.

So, since Q is $\hat{A}_-(\sigma_0)$-stable by assumption, and we have proved PQ, QP and $P(I_{n_i}-QP)$ $\hat{A}_-(\sigma_0)$-stable, it follows, by (2.3) and (2.6), that $S(P,C)$ is $\hat{A}_-(\sigma_0)$-stable, and $C \in \hat{B}(\sigma_0)^{n_i x n_o}$, respectively.

⇒ By (2.3), SP(P,C) $\hat{A}_-(\sigma_0)$-stable implies that Q, PQ, QP and $P(I_{n_i}-QP)$ are $\hat{A}_-(\sigma_0)$-stable. So, condition (i) is satisfied immediately.

Proof of Conditions (ii) and (iii): By assumption, $PQ = H_{y_2 u_1}$ is $\hat{A}_-(\sigma_0)$-stable. By calculation,

$$(PQ)(s) = [(\frac{1}{s-\lambda_1})^2 R_{12} + (\frac{1}{s-\lambda_1})R_{11} + P_\alpha(s)]\ Q(s)$$

Thus, since P_α is $\hat{A}_-(\sigma_0)$-stable, $P_\alpha Q$ is $\hat{A}_-(\sigma_0)$-stable, and hence $[(\frac{1}{s-\lambda_1})^2 R_{12} + (\frac{1}{s-\lambda_1})R_{11}]\ Q(s)$ is $\hat{A}_-(\sigma_0)$-stable. Using the Taylor's series expansion for Q(s) at $s = \lambda_1$, this condition reduces to

$$R_{11}Q(\lambda_1) + R_{12}Q'(\lambda_1) = \theta_{n_o x n_o} \tag{4.25}$$

$$R_{12}Q(\lambda_1) = \theta_{n_o x n_o}$$

and, since $rk(U_{12}) = r_2$, $N(U_{12}) = \{\theta_{r_2}\}$, and thus

$$V_{12}^T Q(\lambda_1) = \theta_{r_2 x n_o} \tag{4.26}$$

Similarly, QP $\hat{A}_-(\sigma_0)$-stable implies that $Q(s)[(\frac{1}{s-\lambda_1})^2 R_{12} + (\frac{1}{s-\lambda_1})R_{11}]$ is $\hat{A}_-(\sigma_0)$-stable. So, using again the Taylor's series expansion for Q(s) at $s = \lambda_1$,

$$Q(\lambda_1)R_{11} + Q'(\lambda_1)R_{12} = \theta_{n_i x n_i} \tag{4.27}$$

$$Q(\lambda_1)R_{12} = \theta_{n_i x n_i}$$

and since $N(V_{12}) = \{\theta_{r_2}\}$,

$$Q(\lambda_1)U_{12} = \theta_{n_i x r_2} \tag{4.28}$$

Proof of Conditions (iv) and (v): By assumption, $H_{e_2u_2}$, $H_{y_2u_1}$ and $P(I_{n_i}-QP) = H_{y_2u_2}$ are $\hat{A}_-(\sigma_o)$-stable. By calculation,

$[P(I_{n_i}-QP)](s) = (PH_{e_2u_2})(s) = [(\frac{1}{s-\lambda_1})^2 R_{12} + (\frac{1}{s-\lambda_1})R_{11} + P_\alpha(s)]H_{e_2u_2}(s)$.

Since P_α and $H_{e_2u_2} = I - QP$ are $\hat{A}_-(\sigma_o)$-stable, $[(\frac{1}{s-\lambda_1})^2 R_{12} + (\frac{1}{s-\lambda_1})R_{11}]H_{e_2u_2}(s)$ must be $\hat{A}_-(\sigma_o)$-stable. Thus, as with Q above, in (4.25) and (4.26),

$$R_{11}H_{e_2u_2}(\lambda_1) + R_{12}H'_{e_2u_2}(\lambda_1) = \theta_{n_o x n_i} \tag{4.29}$$

$$V_{12}^T H_{e_2u_2}(\lambda_1) = \theta_{r_2 x n_i} \tag{4.30}$$

Also,

$$[P(I_{n_i}-QP)](s) = [(I_{n_o}-H_{y_2u_1})P](s)$$
$$= [I_{n_o} - H_{y_2u_1}(s)][(\frac{1}{s-\lambda_1})^2 R_{12} + (\frac{1}{s-\lambda_1})R_{11} + P\ (s)]$$

Since P_α and $H_{y_2u_1} = PQ$ are $\hat{A}_-(\sigma_o)$-stable, $[I_{n_o} - H_{y_2u_1}(s)][(\frac{1}{s-\lambda_1})^2 R_{12} + (\frac{1}{s-\lambda_1})R_{11}]$ must be $\hat{A}_-(\sigma_o)$-stable. Thus, as with Q above, in (4.27) and (4.28),

$$[I_{n_o} - H_{y_2u_1}(\lambda_1)]R_{11} + H'_{y_2u_1}(\lambda_1)R_{12} = \theta_{n_o x n_i}$$

$$[I_{n_o} - H_{y_2u_1}(\lambda_1)]R_{12} = \theta_{n_o x n_i}$$

Rearranging, and noting $N(V_{12}) = \{\theta_{r_2}\}$,

$$R_{11} = H_{y_2u_1}(\lambda_1)R_{11} + H'_{y_2u_1}(\lambda_1)R_{12} \tag{4.31}$$

$$U_{12} = H_{y_2u_1}(\lambda_1)U_{12} \tag{4.32}$$

¤

<u>Theorem 4.2</u>. Let P satisfy (A1) and (A4). Let $C := Q(I-PQ)^{-1}$. Suppose that, in addition, P(s) has <u>full normal rank</u>, and has <u>no transmission zeros</u> at $s = \lambda_1$. S(P,C) is $\hat{A}_-(\sigma_0)$-stable with $C \in \hat{B}(\sigma_0)^{n_o x n_i}$ ⇔

(i) Q is $\hat{A}_-(\sigma_0)$-stable (4.33)

(ii)

for $n_o \le n_i$	for $n_o \ge n_i$	
$Q(\lambda_1)U_{12} = \theta_{n_i x r}$	$V_{12}^T Q(\lambda_1) = \theta_{r_2 x n_o}$	(4.34)
$Q(\lambda_1)R_{11} = -Q'(\lambda_1)R_{12}$	$R_{11}Q(\lambda_1) = -R_{12}Q'(\lambda_1)$	(4.35)

(iii) Either $H_{y_2u_1}(\lambda_1)U_{12} = U_{12}$ (4.36)

or $V_{12}^T H_{e_2u_2}(\lambda_1) = \theta_{r_2 x n_i}$ (4.37)

(iv) Either $R_{11} = H_{y_2u_1}(\lambda_1)R_{11} + H'_{y_2u_1}(\lambda_1)R_{12}$ (4.38)

or $\theta_{n_o x n_i} = R_{11}H_{e_2u_2}(\lambda_1) + R_{12}H'_{e_2u_2}(\lambda_1)$ (4.39)

<u>Proof</u>:

⇒ This direction clearly follows from Theorem 4.1.

⇐ ($n_o \le n_i$) By Theorem 4.1, to establish that S(P,C) is $\hat{A}_-(\sigma_0)$-stable, we need only prove $V_{12}^T Q(\lambda_1) = \theta_{r_2 x n_o}$ and $R_{11}Q(\lambda_1) = -R_{12}Q'(\lambda_1)$.

Since P(s) has full normal rank, it has a right inverse $P^{-R}(s)$, which is analytic in a neighborhood of λ_1, because P(s) has no transmission zero at $s = \lambda_1$. Furthermore, since $PQ = H_{y_2u_1}$, for any such right inverse, there exists $R(s) \in \hat{A}_-(\sigma_0)^{n_i x n_o}$, such that $PR = \theta_{n_o x n_o}$, and

$$Q(s) = P^{-R}(s)H_{y_2u_1}(s) + R(s) \tag{4.40}$$

Now, using (A4), we obtain

$$I_{n_o} = (PP^{-R})(s) = \left(\frac{1}{s-\lambda_1}\right)^2 R_{12}P^{-R}(s) + \left(\frac{1}{s-\lambda_1}\right)R_{11}P^{-R}(s) + (P_\alpha P^{-R})(s), \quad \forall s \in \mathbb{C} \tag{4.41}$$

$$\theta_{n_o x n_o} = (PR)(s) = \left(\frac{1}{s-\lambda_1}\right)^2 R_{12}R(s) + \left(\frac{1}{s-\lambda_1}\right)R_{11}R(s) + (P_\alpha R)(s), \quad \forall s \in \mathbb{C} \tag{4.42}$$

Since $P_\alpha(s)$, $P^{-R}(s)$ and $R(s)$ are analytic at $s = \lambda_1$, and since $N(U_{12}) = \{\theta_{r_2}\}$, (4.41) and (4.42) imply

$$V_{12}^T P^{-R}(\lambda_1) = \theta_{r_2 x n_o} \quad \text{and} \quad V_{12}^T R(\lambda_1) = \theta_{r_2 x n_o} \tag{4.43}$$

$$R_{11}P^{-R}(\lambda_1) = -R_{12}(P^{-R})'(\lambda_1) \quad \text{and} \quad R_{11}R(\lambda_1) = -R_{12}R'(\lambda_1) \tag{4.44}$$

By (4.33) and (A4), $H_{y_2u_1} = PQ$ is analytic in $\mathbb{C}_{\sigma_0+}$, except possibly at $s = \lambda_1$. Suppose $H_{y_2u_1}(s)$ had a pole at $s = \lambda_1$: since the columns of $P^{-R}(s)$ are linearly independent, and since $R(s)$ is analytic at $s = \lambda_1$, then by (4.40), Q would have a pole at $s = \lambda_1$. But this contradicts (4.33). Hence $H_{y_2u_1}(s)$ is analytic at $s = \lambda_1$. Consequently, by (4.43) and (4.40),

$$V_{12}^T Q(\lambda_1) = \theta_{r_2 x n_i}$$

In addition, since $Q(s)$ is $\hat{A}_-(\sigma_0)$-stable, it is analytic at $s = \lambda_1$, and hence, by (4.40),

$$Q'(s) = [(P^{-R})'H_{y_2u_1}](s) + (P^{-R}H'_{y_2u_1})(s) + R'(s)$$

So,

$$R_{12}Q'(\lambda_1) = R_{12}[(P^{-R})'(\lambda_1)]H_{y_2u_1}(\lambda_1) + R_{12}P^{-R}(\lambda_1)H'_{y_2u_1}(\lambda_1) + R_{12}R'(\lambda_1)$$

$$= R_{12}[(P^{-R})'(\lambda_1)]H_{y_2u_1}(\lambda_1) + R_{12}R'(\lambda_1), \text{ by (4.43)}$$

$$= -R_{11}[(P^{-R})(\lambda_1)]H_{y_2u_1}(\lambda_1) - R_{11}R(\lambda_1), \text{ by (4.44)}$$

$$= -R_{11}Q(\lambda_1) \qquad , \text{ by (4.44)}$$

So, by Theorem 4.1, S(P,C) is $\hat{A}_-(\sigma_0)$-stable and $C \in \hat{B}(\sigma_0)^{n_i x n_o}$. The proof for $n_o \geq n_i$ is dual to the above. ¤

Corollary 4.3. Let the assumptions of Theorem 4.2 hold. If, in addition, P is square ($n_o=n_i$), then the necessary and sufficient conditions reduce to (i) and (iii) (note that (ii) is omitted).

Proof. The proof follows easily from that of Theorem 4.2. ¤

Remark: Note that the theorems of this section can be easily extended to the class of plants satisfying the two assumptions (A1) and (A5), namely,

(A1) $P(s) \in \hat{B}_o(\sigma_o)^{n_o x n_i}$

(A5) $P(s) = \sum_{k=1}^{p} [(\frac{1}{s-\lambda_k})^2 R_{k2} + (\frac{1}{s-\lambda_k})R_{k1}] + P_\alpha(s)$

where, for each $1 \leq k \leq p$, $\lambda_k \in \mathbb{C}_{\sigma_o+}$; for j = 1,2, $R_{kj} = U_{kj}V_{kj}^T$, for $U_{kj} \in \mathbb{C}^{n_o x r_{kj}}$, $V_{kj} \in \mathbb{C}^{n_i x r_{kj}}$ each have rank r_{kj}; and $P_\alpha(s) \in \hat{A}_{-,o}(\sigma_o)^{n_o x n_i}$

For P satisfying (A1) and (A5), S(P,C) will be $\hat{A}_-(\sigma_o)$-stable with $C \in \hat{B}(\sigma_o)^{n_i x n_o}$ ⇔ The conditions (i)-(v) of Theorem 4.1 are satisfied for λ_k, R_{k1}, R_{k2}, U_{k2} and V_{k2} for each $1 \leq k \leq p$.

V. Design Example

Design Example 5.1. Lumped unstable plant

We consider the following plant:

$$P(s) = \frac{1}{s^2}\begin{bmatrix} 8453.9 - \frac{(45.6)^2}{\alpha(s)} & 22.6 - \frac{(0.9)(45.6)}{\alpha(s)} \\ 22.6 - \frac{(0.9)(45.6)}{\alpha(s)} & 7731.1 - \frac{(0.9)^2}{\alpha(s)} \end{bmatrix}^{-1} \tag{5.1}$$

$$=: \frac{1}{s^2}\left(P_o - \frac{1}{\alpha(s)} AA^T\right)^{-1} \tag{5.2}$$

$$\text{where } \alpha(s) := 1 + (2\sqrt{13.8} \times 0.0014)\,\frac{1}{s} + (13.8)\,\frac{1}{s^2} \tag{5.3}$$

It can be easily shown that P satisfies the assumptions of Corollary 4.3. Thus, since R_{12} is full rank, it is clear, by comment (b) of Theorem 4.1, that the conditions of Corollary 4.3 are satisfied by:

$$H_{y_2u_1}(s) = \text{diag}\left[\frac{(\omega_{oi}^2 - 2\zeta_i\omega_{oi}p_i)s + \omega_{oi}^2 p_i}{(s^2+2\zeta_i\omega_{oi}s+\omega_{oi}^2)(s-p_i)}\right]_{i=1,2} \tag{5.4}$$

$$=: \text{diag}\left[\frac{n_i}{d_i}\right]_{i=1,2} \tag{5.5}$$

where ω_{oi}, ζ_i, p_i are free parameters for $i = 1,2$ (subject to stability constraints). Thus the compensator defined by

$$C = Q(I-PQ)^{-1}$$

$$= P^{-1}H_{y_2u_1}(I-H_{y_2u_1})^{-1}$$

$$= s^2\left(P_o - \frac{1}{\alpha(s)} AA^T\right) \text{diag}\left[\frac{n_i}{d_i-n_i}\right]_{i=1,2} \tag{5.7}$$

yields an exponentially stable closed-loop system S and satisfies $C \in \hat{B}(\sigma_o)^{n_i \times n_o}$.

The values of ω_{oi}, ζ_i, and p_i for $i = 1,2$, were chosen by solving the following optimization problem

$$\min_{z}\ \max_{\omega}\ \bar{\sigma}[H_{y_2 d_o}(j\omega)] \text{ over } \omega \in [.01, .5] \qquad (5.8)$$

subject to:

$$\max_{\omega}\ \bar{\sigma}[Q(j\omega)] \leq 3.10^4 \text{ over } \omega \in [.1,50] \qquad (5.9)$$

$$\omega_{oi} \geq 2 \quad \text{for } i = 1,2 \qquad (5.10)$$

$$p_i \leq -1 \quad \text{for } i = 1,2 \qquad (5.11)$$

$$\zeta_i \geq 0.5 \quad \text{for } i = 1,2 \qquad (5.12)$$

The solution obtained was:

$\zeta_1 = .501 \qquad \zeta_2 = .7$

$\omega_{o1} = 2.22 \qquad \omega_{o2} = 2.2$

$p_1 = -1.58 \qquad p_2 = -1.2$

with $\max_{\omega\in[.01,.5]} \bar{\sigma}[H_{y_2 d_o}(j\omega)] = .0772$ at this point.

This solution has the interpretation that the maximum desensitization possible for the given saturation bound has been achieved, where $\max_{\omega\in[.01,.5]} \bar{\sigma}[H_{y_2 d_o}(j\omega)]$ is the measure of desensitization, and $\max_{\omega\in[.1,50]} \bar{\sigma}[Q(j\omega)]$ is the measure of saturation.

Similar optimal design problem formulations may be found in the literature [Doy. 1], [May. 1], [Hor. 1], [Gus. 1]. The solution to this problem was found using RATTLE/DELIGHT, a software package used for formulation and solution of optimization problems [Nye 1] .

VI. Conclusions

We have developed a design theory for distributed linear multivariable feedback systems with simple unstable plants which accomplishes the following:

(i) Gives global parametrizations for all compensators $C \in E(\hat{B}(\sigma_0))$ which stabilize the closed-loop system S for a given plant P with either first or second order unstable poles. We have also indicated that such parametrizations are available for plants with unstable poles of third or higher order.

(ii) Gives geometric insight into the dynamics of the process by which an unstable plant is stabilized.

(iii) Allows extension of a existing design methodology for stable plants to the case of unstable plants. The application of this methodology is especially simple when the plant is only simply unstable, because when the unstable poles are only first or second order, the number of constraints on the design is small.

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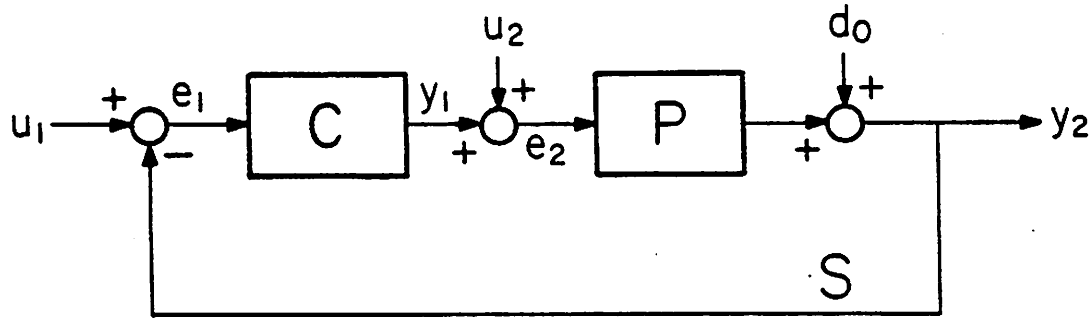

Figure 1