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ON STABILIZATION AND THE EXISTENCE  
OF COPRIME FACTORIZATIONS

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V. Anantharam

Memorandum No. UCB/ERL M82/61

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Abstract

Let  $H$  be an integral domain and  $F$  its quotient field. We show there are plants over  $F$  which have no stable coprime factorizations, but can be stabilized in a stable closed loop. This answers a question posed by Vidyasagar, Schneider and Francis.

Our purpose is to develop an example which answers a question posed by Vidyasagar-Schneider and Francis [4]. Admittedly the example is somewhat artificial from a system theoretic standpoint, but it should prove of assistance in subsequent investigations.

### I. The Problem

Let  $H$  be an integral domain and  $F$  its quotient field. If  $P \in F^{n \times m}$  and  $C \in F^{m \times n}$  are such that the closed loop system [Fig. 1] is stable, then the input to error matrix

$$H_{eu} = \begin{bmatrix} (I_n + PC)^{-1} & -P(I_m + CP)^{-1} \\ C(I_n + PC)^{-1} & (I_m + CP)^{-1} \end{bmatrix} \quad (1.1)$$

is such that  $H_{eu} \in H^{(n+m) \times (n+m)}$  (by definition  $H$  consists of stable scalar plants - see [4]). It may be readily proved that the matrix

$$\begin{bmatrix} C & 0 \\ 0 & P \end{bmatrix}$$

always has both a right coprime fractional representation and

a left coprime fractional representation when the closed loop is stable [4].

The question posed in [4] is the following: Is it always necessary that  $C$  and  $P$  individually have coprime factorizations when the closed loop is stable? An answer is of great importance, because the bane of fractional representation theory is the question of existence of coprime representations.

We construct an example which answers the question in the negative. The key idea is that our integral domain  $H$  is not a unique factorization domain.

## II. The Example

Let  $H = \mathbb{Z}[\sqrt{-5}] \approx \mathbb{Z}[x]/(x^2 + 5)$  where  $\mathbb{Z}$  is the ring of integers,  $\mathbb{Z}[x]$  the ring of polynomials in  $x$  with integral coefficients,  $(x^2 + 5)$  is the ideal generated by  $x^2 + 5 \in \mathbb{Z}[x]$  and  $/$  means we form the quotient ring. (For all undefined terms see any good book on Algebra e.g. [1], [2] or [3]). Then  $H$  is an integral domain because the ideal  $(x^2+5)$  is prime. It is also not a unique factorization domain, as can be verified from

$$6 = (1+\sqrt{-5})(1-\sqrt{-5}) = 2 \cdot 3 \quad (2.1)$$

Our example is scalar. Let  $p = \frac{1+\sqrt{-5}}{2} \in F$  and  $c = \frac{1-\sqrt{-5}}{-2} \in F$ . We verify that

$$H_{eu} = \begin{bmatrix} -2 & (1+\sqrt{-5}) \\ -(1-\sqrt{-5}) & -2 \end{bmatrix} \in H^{2 \times 2} \quad (2.2)$$

We will now show that  $p$  has no stable coprime factorization (similar arguments work for  $c$ ). First, we have to find all possible fractional representations for  $p$ . Let

$$p = \frac{1+\sqrt{-5}}{2} = \frac{a + b\sqrt{-5}}{c + d\sqrt{-5}} \quad a, b, c, d \in \mathbb{Z} \quad (2.3)$$

Multiplying we get the conditions

$$\begin{aligned} c + d &= 2b \\ c - 5d &= 2a \end{aligned} \quad (2.4)$$

which on simplification yield

$$\begin{aligned} 2a - 2b + 6d &= 0 \\ c &= 2a + 5d \end{aligned} \tag{2.5}$$

It is easy to see that all possible solutions in integers of the above equations are given by

$$(a,b,c,d) = \alpha(1,1,2,0) + \beta(3,0,1,-1) \quad \alpha, \beta \in \mathbb{Z} \tag{2.6}$$

Indeed, given any solution  $(a,b,c,d)$  let  $\alpha = b$  and  $\beta = -d$ .

Thus, all possible fractional representations of  $p$  are given by

$$p = \frac{(\alpha+3\beta) + \alpha\sqrt{-5}}{(2\alpha+\beta) - \beta\sqrt{-5}} \tag{2.7}$$

where  $\alpha, \beta \in \mathbb{Z}$ , at least one nonzero.

Next we prove that no such representation can be coprime over  $H$ . Suppose the contrary. Then there exist integers  $u, v, \omega, x$  such that

$$(u+v\sqrt{-5})((\alpha+3\beta)+\alpha\sqrt{-5})+(\omega+x\sqrt{-5})((2\alpha+\beta)-\beta\sqrt{-5}) = 1 \tag{2.8}$$

Multiplying out gives the conditions

$$\begin{bmatrix} \alpha & \alpha+3\beta & -\beta & 2\alpha+\beta \\ \alpha+3\beta & -5\alpha & 2\alpha+\beta & 5\beta \end{bmatrix} \begin{bmatrix} u \\ v \\ \omega \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.9}$$

This has an immediate module theoretic interpretation. To establish contradiction, we wish to prove that the submodule of  $\mathbb{Z}^2$  generated over  $\mathbb{Z}$  by the column vectors of the  $2 \times 4$  matrix above can never contain  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , whatever the values of  $\alpha, \beta$ .

Let us multiply the above equation on the left by the  $\mathbb{Z}$ -unimodular

matrix  $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ . Since

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (2.10)$$

we have the equivalent problem about the integer matrix

$$\begin{bmatrix} \alpha & \alpha+3\beta & -\beta & 2\alpha+\beta \\ 3\beta & -6\alpha-3\beta & 2\alpha+2\beta & 4\beta-2\alpha \end{bmatrix} \quad (2.11)$$

i.e. about the submodule spanned by its columns, over  $\mathbb{Z}$ .

By forming  $\mathbb{Z}$ -linear combinations of the columns of the above matrix, one can check that the vectors  $\begin{bmatrix} \alpha \\ 3\beta \end{bmatrix}$  and  $\begin{bmatrix} \beta \\ -2(\beta+\alpha) \end{bmatrix}$  generate the submodule we are interested in. Thus an equivalent problem is to show that it is impossible to find integers  $m$  and  $n$  such that

$$m \begin{bmatrix} \alpha \\ 3\beta \end{bmatrix} + n \begin{bmatrix} \beta \\ -2(\beta+\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (2.12)$$

If  $\alpha$  and  $\beta$  have a nontrivial common divisor, this is clearly impossible. So we assume  $\gcd(\alpha, \beta) = 1$ . Now, the only choices of  $m$  and  $n$  making the first row zero are  $m = k\beta$   $n = -k\alpha$  with some  $k \in \mathbb{Z}$ . If  $k \neq \pm 1$ , it appears as a common factor of the second row, so once again it is impossible to satisfy (2.12). So assume  $k = \pm 1$ . We get

$$\pm (3\beta^2 + 2\alpha\beta + 2\alpha^2) = 1$$

If either  $\alpha = 0$  or  $\beta = 0$  this is clearly impossible. If  $|\beta| \geq |\alpha|$  then, since  $3\beta^2 + 2\alpha\beta > 0$  and  $2\alpha^2 > 0$  we get



$$| \pm (3\beta^2 + 2\alpha\beta + 2\beta^2) | > 1 \quad (2.13)$$

If  $|\alpha| > |\beta|$  then, since  $2\alpha^2 + 2\alpha\beta > 0$  and  $3\beta^2 > 0$  we once again get (2.13). Thus we have proved the impossibility of satisfying (2.12). It follows that  $p$  has no coprime fractional representations

### III. Conclusion (The Answer)

If  $H$  is permitted to be an arbitrary integral domain as in [4] it is possible to stabilize plants which have no coprime factorizations.

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