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ABSTRACT

Assuming that * is any operation defined on a product set X x Y and taking values on a set Z, it can be extended to fuzzy sets by means of Zadeh's extension principle. Given a fuzzy subset C of Z, it is here shown how to solve equation A * B = C (or A * B $\subseteq$ C) when a fuzzy subset A of X (or a fuzzy subset B of Y) is given. The methodology we provide includes, as a special case, the resolution of fuzzy arithmetical operations, i.e. when * stands for +, -, x or ÷, extended to fuzzy numbers (fuzzy subsets of the real line). The paper is all along illustrated with examples in fuzzy arithmetic.

Key-words. Fuzzy equations, extension principle, fuzzy arithmetic.

To Professor Lotfi A. Zadeh.

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Research sponsored in part by the National Science Foundation Grants MCS79-06543 and IST-8018196.

1. INTRODUCTION

Fuzzy numbers and fuzzy arithmetic were introduced by Zadeh (1975) to analyze and manipulate approximate numerical values. It is natural that adding a number '*approximately* 4' to a number '*approximately* 9' yields a number '*approximately* 13,' but how to combine these fuzzy numbers taking into account the fuzziness attached to the various '*approximately's*? One way of doing so is by using Zadeh's extension principle[1] which allows the domain of the definition of a mapping or of an operation to be extended from points to fuzzy sets.

Fuzzy arithmetic[2] has been extensively studied by some authors like Dubois and Prade (1978, 1980, 1981), Jain (1976), Baas and Kwakernaak (1977), Nahmias (1978), Mizumoto and K. Tanaka (1976, 1979), Yager (1977, 1980), Dijkman, van Haeringen and de Lange (1980), H. Tanaka (1981).

In Dubois and Prade (1980) one may find a useful algorithm for a practical computation of extended operations, illustrated by the addition of fuzzy numbers. So that one can compute direct as well as inverse operations on fuzzy numbers. But a problem, discussed by Mizumoto and Tanaka (1979) and Yager (1980), consists in the non existence of inverse fuzzy numbers for the usual arithmetical operations. That is, in the case of fuzzy numbers, equation $A + X = C$ can not be solved by $X = C - A$. This is the problem we here propose to solve in some sense. We say in 'some sense' for, after a previous work, we noticed that our results on

[1] The extension principle which is defined in (Zadeh, 1975, Part I) was implicit in his original 1965 paper on Fuzzy Sets, without being named so. Despite its simple formulation, it has proved to be a very seminal concept.

[2] Note that fuzzy arithmetic broadens interval arithmetic and can be used in problems related to error calculus and many others in numerical analysis.

fuzzy arithmetic could be enlarged to any $*$-operation extended to fuzzy sets, acting on any sets not necessarily equal to $\mathbb{R}$. So that we propose here a general resolution of fuzzy equations of extended operations. Moreover, we will define a special type of solution (when they exist of course) which is the greatest one. It is easy to show that the problem has a solution and to exhibit the greatest one, just like in problems of compositions of fuzzy relations (Sanchez, 1976, 1977, 1979). Note that, on the contrary to interval arithmetic, there is not uniqueness for the solutions. In fact, in interval arithmetic, solution to $A + X = C$ is trivial, for $[a_1,a_2] + [b_1,b_2] = [a_1+b_1,a_2+b_2]$, moreover, the calcellation law (Moore, 1979) holds for interval addition i.e. $A + B_1 = A + B_2$ implies $B_1 = B_2$.

2. THE EXTENDED $*$ OPERATION

Let $*$ be an operation defined on a product set $X \times Y$ and taking values on a set Z. To all (x,y) in $X \times Y$, $*$ associates an element z, in Z, which is denoted $x * y$.

The $*$ operation can be extended to fuzzy sets by means of the following extension principle (Zadeh, 1975).

Definition 1. Let A be a fuzzy subset of X and B be a fuzzy subset of Y, then the extension principle allows to define a fuzzy subset $C = A * B$ of Z as follows, in the case of noninteractive variables.[3]

[3] The concept of interaction (Zadeh, 1975) between variables is analogous to the dependence of random variables. Here it is assumed that $\mu_{(A,B)}(x,y) = \mu_A(x) \wedge \mu_B(y)$ which implies that x and y are noninteractive.

$$\forall z \in Z,\ \mu_{A*B}(z) = \sup_{\substack{x \in X, y \in Y \\ x*y = z}} \mu_A(x) \wedge \mu_B(y) \tag{1}$$

where, as usual, μ_A is the membership function of a fuzzy set A and '$\wedge$' denotes the min operation.

Note that, in (1), it is assumed that given z in Z, there exists x in X and y in Y such that $x * y = z$. But if that would not be the case $\mu_{A*B}(z)$ would be defined as being equal to 0, following the usual mathematical convention that $\sup_{(x,y) \in \emptyset} f(x,y) = 0$, in the present case.

Remarks. i) When A and B are crisp (i.e. non fuzzy) sets, A being a subset of X and B a subset of Y, $A * B$ reduces to a subset C of Z defined as

$$A * B = \{z \in Z | z = x * y,\ x \in A,\ y \in B\}.$$

ii) When $X = Y = Z = \mathbb{R}$ and $*$ stands for addition on numbers, $A * B$ stands then for the addition $A + B$ of two sets as defined by H. Minkowski in 1911.

iii) When $X = Y = Z = I(\mathbb{R})$ i.e., the set of interval numbers, or more simply intervals, in the real line $\mathbb{R}$, and when $*$ stands for example for addition, subtraction, multiplication or division, the study of $A * B$ corresponds to Interval Analysis (Moore, 1966).

Returning now to (1), given the fuzzy sets A and B, $C = A * B$ is then uniquely determined by definition. The problem we are going to investigate is the inverse one, i.e., from the knowledge of the fuzzy sets C and A (or C and B) find, when it exists, a fuzzy set B (or A) such that $C = A * B$.

Note that $*$ is not necessarily a commutative operation. Moreover, even if for a couple (x,y), the result $x * y$ is well defined, that may not be necessarily the case for $y * x$. The sequel of this paper will be mostly oriented towards the case where C and A are given (equation $A * X = C$). The case where C and B are given (equation $X * B = C$) can be easily derived by adapting the formulae.

We recall that $*$ is any operation here. A large and practical application of our study is of course fuzzy arithmetic when A, B and C are fuzzy numbers, i.e., fuzzy subsets of $\mathbb{R}$, and when $*$ is one of the following usual operations +, -, x, ÷. Difficulties arise in fuzzy arithmetic as pointed out by Mizumoto and Tanaka (1979), Dubois and Prade (1980) and Yager (1980). For example, $A + B = C$ does not imply $B = C - A$ as illustrated in Figure 1, where

$$\forall z \in R,\ \mu_{A+B}(z) = \sup_{\substack{x \in \mathbb{R}, y \in \mathbb{R} \\ x+y=z}} \mu_A(x) \wedge \mu_B(y); \qquad (2)$$

$$\forall z \in \mathbb{R},\ \mu_{A-B}(z) = \sup_{\substack{x \in \mathbb{R}, y \in \mathbb{R} \\ x-y=z}} \mu_A(x) \wedge \mu_B(y). \qquad (3)$$

The following properties are easy to check.

The extension of $*$ to fuzzy sets is <u>inclusion monotonic</u>. In other terms, for B_1 and B_2 fuzzy subsets of Y,

$$B_1 \subseteq B_2 \Rightarrow A * B_1 \subseteq A * B_2; \qquad (4)$$

for A_1 and A_2 fuzzy subsets of X,

$$A_1 \subseteq A_2 \Rightarrow A_1 * B \subseteq A_2 * B. \qquad (5)$$

In fact, the extension of $*$ is distributive over the union of fuzzy sets, i.e.,

$$A * (B_1 \cup B_2) = (A*B_1) \cup (A*B_2); \tag{6}$$

$$(A_1 \cup A_2) * B = (A_1*B) \cup (A_2*B). \tag{7}$$

Moreover,

$$A * (B_1 \cap B_2) \subseteq (A*B_1) \cap (A*B_2); \tag{8}$$

$$(A_1 \cap A_2) * B \subseteq (A_1*B) \cap (A_2*B). \tag{9}$$

A direct consequence of (6) and (7) is that the set of solutions of $A * X = C$ (or of $X * B = C$), if non void, is an upper semi-lattice. But, in view of formulae (6) to (9), the set of solutions of $A * X \subseteq C$ (or of $X * B \subseteq C$) is a lattice.

Remembering that * is any operation, note the apparent strangeness of equation (6), for example, when * stands for division or subtraction of fuzzy numbers.

For practical computations, let us show now how one can transform the expression of $A * B$ given in (1).

Assume first that B is a (non fuzzy) singleton identified with its unique element, say b in Y, so that $\mu_B(y) = 1$ if $y = b$ and $\mu_B(y) = 0$ if $y \neq b$. Thus, equation (1) yields

$$\forall\ z \in Z,\ \mu_{A*b}(z) = \sup_{\substack{x \in X \\ x*b=z}} \mu_A(x). \tag{10}$$

Equation (10) is illustrated in Figure 2 in the case of fuzzy numbers, with * standing for +. As + admits an inverse operation in $\mathbb{R}$, (10) takes here the following simpler form.

$$\forall\ z \in Z,\ \mu_{A+b}(z) = \mu_A(z-b).$$

Let us now return to (1).

$$\forall z \in Z, \ \mu_{A*B}(z) = \sup_{\substack{x\in X, y\in Y \\ x*y=z}} \mu_A(x) \wedge \mu_B(y)$$

$$= \sup_{y\in Y}(\sup_{\substack{x\in X \\ x*y=z}} \mu_A(x)\wedge\mu_B(y))$$

$$= \sup_{y\in Y}(\mu_B(y)\wedge \sup_{\substack{x\in X \\ x*y=z}} \mu_A(x))$$

$$= \sup_{y\in Y} \mu_B(y) \wedge \mu_{A*y}(z),$$

from (10) with y playing the role of b. Hence, (1) is equivalent to (11).

$$\forall z \in Z, \ \mu_{A*B}(z) = \sup_{y\in Y} \mu_{A*y}(z) \wedge \mu_B(y). \qquad (11)$$

Analogously, exchanging the roles of A and B, we have

$$\forall z \in Z, \ \mu_{A*B}(z) = \sup_{x\in X} \mu_A(x) \wedge \mu_{B*x}(z). \qquad (12)$$

Note that with addition of fuzzy numbers, (11) takes the form $\sup_{y\in Y} \mu_A(z-y) \wedge \mu_B(y)$, referred to as "sup-min convolution" by A. Kaufmann.

3. THE α OPERATOR

In order to solve the *-equation problem on fuzzy sets we need to recall the definition of the α operator which is characteristic of Brouwerian lattices. That α operator has proved to be useful in the resolution of composite fuzzy relation equations (Sanchez, 1976; Pappis and Sugeno, 1976; Tsukamoto, 1979; Wang Pei-Zhuang and Yuan Meng, 1980; Luo Cheng-Zhong, 1981; Czogala, Drewniak and Pedrycz, 1982; and others) and we study here a particular composition, with a constraint expressed by the $*$ operation.

<u>Definition 2</u>. Given a and b in [0,1], $a \alpha b$ is defined as the greatest element x in [0,1] such that $a \wedge x \leq b$, i.e.,

$$\begin{aligned} a \,\alpha\, b &= 1 \quad \text{if} \quad a \leq b \\ &= b \quad \text{if} \quad a > b. \end{aligned} \tag{13}$$

Here are some properties of the α operator that will be used in the sequel. We recall that, as usual, 'v' denotes the <u>max</u> operation.

For all a,b in [0,1] and for all family $(b_i)_{i \in I}$ of elements of [0,1], we have

$$a \wedge (a \alpha b) \leq b; \tag{14}$$

$$a \,\alpha\, (\sup_{i \in I} b_i) \geq a \,\alpha\, b_j, \quad \forall\, b_j \in I; \tag{15}$$

$$a \,\alpha\, (a \wedge b) \geq b. \tag{16}$$

According to (13), properties (14) and (16) are directly verified. To check (15), it suffices to denote $c = \sup_{i \in I, i \neq j} b_i$, and to show that $a \,\alpha\, (c \vee b_j) \geq a \,\alpha\, b_j$.

<u>Notation</u>. From now on, $F(U)$ will denote the class all the fuzzy subsets of a set U.

<u>Definition 3</u>. Given $A \in F(X)$, $C \in F(Z)$ and $* : X \times Y \to Z$, we define $\tilde{*} : F(Z) \times F(X) \to F(Y)$ as follows

$$\forall\, y \in Y,\ \mu_{C \tilde{*} A}(y) = \inf_{\substack{x \in X, z \in Z \\ x * y = z}} \mu_A(x)\ \alpha\ \mu_C(z). \tag{17}$$

As a guideline, let us mention that when $*$ stands for example for addition extended to fuzzy numbers, $C \tilde{*} A$ will stand for a new and non-standard subtraction ($C \tilde{+} A$ being denoted by $C \ominus A$) of the fuzzy number A from the fuzzy number C, allowing to solve equation $A + X = C$. Equation (17) is illustrated in Figure 3, where in the case of addition of fuzzy numbers, we have

$$\forall\, y \in Y,\ \mu_{C \ominus A}(y) = \inf_{x \in \mathbb{R}} \mu_A(x)\ \alpha\ \mu_{C-y}(x), \tag{18}$$

Since $\mu_C(x+y) = \mu_{C-y}(x)$ from (10).

The following property of the $\tilde{*}$ operation holds. For C_1 and C_2 fuzzy subsets of Z,

$$C_1 \subseteq C_2 \Rightarrow C_1 \tilde{*} A \subseteq C_2 \tilde{*} A. \tag{19}$$

Equation (19) is simply verified after checking that in [0,1], if $c_1 \le c_2$ then $a\ \alpha\ c_1 \le a\ \alpha\ c_2$.

4. RESOLUTION OF $*$-EQUATIONS ON FUZZY SETS

Theorem 1. For every pair of fuzzy sets $A \in F(X)$ and $C \in F(Z)$, and for $* : X \times Y \to Z$, we have

$$A * (C \tilde{*} A) \subseteq C. \tag{20}$$

In other terms, $C \tilde{*} A$ is a particular solution to $A * X \subseteq C$.

Proof. Let $U = A * (C \tilde{*} A)$ and let $z \in Z$,

$$\mu_U(z) = \sup_{\substack{x \in X, y \in Y \\ x*y=z}} \mu_A(x) \wedge \mu_{C\tilde{*}A}(y);$$

$$\mu_U(z) = \sup_{\substack{x \in X, y \in Y \\ x*y=z}} \mu_A(x) \wedge \inf_{\substack{x' \in X, z' \in Z \\ x'*y=z'}} \mu_A(x') \,\alpha\, \mu_C(z');$$

$$\mu_U(z) \leq \sup_{\substack{x \in X, y \in Y \\ x*y=z}} \mu_A(x) \wedge [\mu_A(x) \,\alpha\, \mu_C(z)];$$

$$\mu_U(z) \leq \sup_{\substack{x \in X, y \in Y \\ x*y=z}} \mu_C(z), \qquad \text{from (14)};$$

$$\mu_U(z) \leq \mu_C(z).$$ ¤

Theorem 2. For every pair of fuzzy sets $A \in F(X)$ and $B \in F(Y)$, and for $* : X \times Y \to Z$, we have

$$B \subseteq (A*B) \tilde{*} A. \tag{21}$$

Note that when $A * B = C$, we have $B \subseteq C \tilde{*} A$.

Proof. Let $V = (A*B) \tilde{*} A$ and let $y \in Y$,

$$\mu_V(y) = \inf_{\substack{x \in X, z \in Z \\ x*y=z}} \mu_A(x) \,\alpha\, \mu_{A*B}(z);$$

$$\mu_V(y) = \inf_{\substack{x \in X, z \in Z \\ x*y=z}} \mu_A(x) \,\alpha \sup_{\substack{x' \in X, y' \in Y \\ x'*y'=z}} \mu_A(x') \wedge \mu_B(y') \quad ;$$

$$\mu_V(y) \geq \inf_{\substack{x \in X, z \in Z \\ x*y=z}} \mu_A(x) \,\alpha\, [\mu_A(x) \wedge \mu_B(y)], \qquad \text{from (15);}$$

$$\mu_V(y) \geq \inf_{\substack{x \in X, z \in Z \\ x*y=z}} \mu_B(y), \qquad \text{from (16);}$$

$$\mu_V(y) \geq \mu_B(y).$$ ¤

Corollary 1. Given $A \in \mathcal{F}(X)$, $C \in \mathcal{F}(Z)$ and $* : X \times Y \to Z$, equation $A * X \subseteq C$ has always a greatest solution given by $C \tilde{*} A$. Moreover, the set of solutions of $A * X \subseteq C$ is a lattice.

Proof. From (20) in Theorem 1, $C \tilde{*} A$ is a solution to $A * X \subseteq C$, let us show that it is the greatest one. Let $B \in \mathcal{F}(Y)$ such that $A * B \subseteq C$. From (19), we have $(A*B) \tilde{*} A \subseteq C \tilde{*} A$. Finally, (21) in Theorem 2 yields $B \subseteq C \tilde{*} A$.

The fact that the set of solutions of $A * X \subseteq C$ is a lattice was already pointed out as a result of (6) and (8). ¤

Corollary 2. For $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ and $C \in \mathcal{F}(Z)$, we have

$$A * B \subseteq C \quad \text{iff} \quad B \subseteq C \tilde{*} A. \tag{22}$$

Proof. 'If $A * B \subseteq C$ then $B \subseteq C \tilde{*} A$' was already shown in the proof of Corollary 1. Let us now assume that $B \subseteq C \tilde{*} A$. Hence, from (4) we have $A * B \subseteq A * (C \tilde{*} A)$. Finally, (20) in Theorem 1 yields $A * B \subseteq C$. ¤

Let us now state a fundamental theorem in our study.

Theorem 3. Given the fuzzy sets $A \in \mathcal{F}(X)$, $C \in \mathcal{F}(Z)$ and an operation $* : X \times Y \to Z$, equation

$$A * X = C$$

has a solution if, and only if,

$$A * (C \tilde{*} A) = C.$$

Moreover, when $C \tilde{*} A$ is a solution, then it is the greatest one and the set of solutions is an upper semi-lattice.

An analogous theorem holds, of course, for equation $X * B = C$.

<u>Proof</u>. Let us assume that $B \in F(Y)$ is a solution to $A * X = C$, i.e., $A * B = C$. Hence, from (21) we have $B \subseteq C \tilde{*} A$. But (4) yields $A * B \subseteq A * (C \tilde{*} A)$, i.e., $C \subseteq A * (C \tilde{*} A)$, so that $A * (C \tilde{*} A) = C$ from (20)

When $A * X = C$ has a solution, then $C \tilde{*} A$ is the greatest one, as a direct application of (21).

The fact that the set of solutions of $A * X = C$, when non void, is an upper semi-lattice was already pointed out as a result of (6). ¤

<u>Remarks</u>. i) Theorem 3 is not only interesting in the search for solutions to equation $A * X = C$ but, for example, after a computation of $A * B = C$, one may be willing to know how far B can be expanded (in the fuzzy inclusion sense) so that the result, C, is unchanged. The upper bound for B is of course given by $C \tilde{*} A$.

ii) When there is a unique solution to $A * X = C$, that solution is simply retrieved from $C \tilde{*} A$. For example, in the case of a unique solution dealing with the addition of fuzzy numbers (see Figure 1 for an illustration), one has $A + B = C \Rightarrow B = C \ominus A$, where $C \ominus A$ denotes $C \tilde{+} A$.

iii) In arithmetic or, even in interval arithmetic, one has

$A + B_1 = A + B_2$ if, and only if, $B_1 = B_2$. This property may no longer be true in fuzzy arithmetic. For example, one may have $A + B_1 = A + B_2 = C$, where $B_2 = C \ominus A$, and $B_1 \neq B_2$ as in the following illustration where $B_1 \subset B_2$ (strict inclusion).

Example. Let $X = Y = Z = \mathbb{Z}$ and let $*$ stand for the addition of integers. A, B_1, C and B_2 are defined by their membership functions described in Table 1 where

$$A + B_1 = C,$$

$$B_2 = C \ominus A \quad \text{and} \quad A + B_2 = C,$$

but the membership functions of B_1 and B_2 differ on points 3 and 4 in $\mathbb{Z}$, so that $B_1 \subset B_2$.

$\mathbb{Z}$	...	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
A	...	0	0	0	0.2	0.7	1	0.2	0.7	1	0	0	0	...					
B_1	...	0	0	0.2	1	0.3	0	0.2	1	0.3	0	0	0	...					
C	...	0	0	0	0.2	0.2	0.7	1	0.3	0.7	1	1	0.3	0.7	1	0.3	0	0	...
B_2	...	0	0	0.2	1	0.3	0.3	0.3	1	0.3	0	0	0	...					

Table 1. Illustration for $A + B_1 = A + B_2 \not\Rightarrow B_1 = B_2$.

5. *-EQUATIONS AND o-EQUATIONS

When we introduced the α operator in this paper, we pointed out that we were studying a particular case of compositional equations. In order to clarify this statement, let us now relate the $*$-equation problem investigated in this paper to the more general problem of composite fuzzy relation equations.

Given $A \in F(X)$ and $* : X \times Y \to Z$, we define a fuzzy relation, $(A*)$, from Y to Z, i.e., a fuzzy subset of $Y \times Z$, by its membership function:

$$\forall (y,z) \in Y \times Z, \ \mu_{(A*)}(y,z) = \mu_{A*y}(z), \tag{23}$$

where $A * y$ is defined in (10).

From (11) and (23), $A * B$ can be given by the following expression, equivalent to (1).

$$\forall z \in Z, \ \mu_{A*B}(z) = \sup_{y \in Y} \mu_B(y) \wedge \mu_{(A*)}(y,z)$$

$$= \mu_{(A*)oB}(z),$$

where 'o' stands for the sup-min composition of fuzzy relations or, more precisely here, for the sup-min composition of a fuzzy relation with a fuzzy set.

Note that some authors write 'B o (A*)' instead of '(A*) o B,' it is only a matter of notation which depends upon the <u>classical</u> concept that 'o' is supposed to extend to fuzzy sets.

Finally, we have

$$A * B = (A*) \circ B. \tag{24}$$

Adapting now Sanchez (1976), we derive the following result.

<u>Theorem 3'</u>. Given $C \in F(Z)$, $* : X \times Y \to Z$ and $(A*) \in F(Y \times Z)$ we have

$$\mathcal{B} = \{B \in F(Y) \mid (A*) \circ B = C\} \neq \emptyset$$

if, and only if,

$(A*) \circledS C \in \mathcal{B}$; then it is the greatest element in $\mathcal{B}$, where,

$$\forall\, y \in Y,\ \mu_{(A*)ⓐC}(y) = \inf_{z \in Z} \mu_{(A*)}(y,z)\ \alpha\ \mu_C(z). \tag{25}$$

Hence, from (23) and (10) we have

$$\mu_{(A*)ⓐC}(y) = \inf_{z \in Z}[(\sup_{\substack{x \in X \\ x*y=z}} \mu_A(x))\ \alpha\ \mu_C(z)]. \tag{26}$$

Now, from a direct application of (13) in Definition 2, one may check that $\forall\, y \in Y$, $\forall\, z \in Z$,

$$(\sup_{\substack{x \in X \\ x*y=z}} \mu_A(x))\ \alpha \mu_C(z) = \inf_{\substack{x \in X \\ x*y=z}} [\mu_A(x)\ \alpha \mu_C(z)]. \tag{27}$$

Note that (27), with no $*$-constraint and in the finite case, corresponds to the following property of the α operator.

For all a, b and c in [0,1], we have

$$(a \vee b)\ \alpha\ c = (a \alpha c) \wedge (b \alpha c). \tag{28}$$

From (26) and (27) we then derive, $\forall y \in Y$,

$$\mu_{(A*)ⓐC}(y) = \inf_{\substack{x \in X, z \in Z \\ x*y=z}} \mu_A(x)\ \alpha\ \mu_C(z)$$

$$= \mu_{C \tilde{*} A}(y), \qquad \text{from (17).}$$

Hence,

$$(A*)ⓐC = C \tilde{*} A, \tag{29}$$

So that our present results are consistent with the more general ones.

6. CONCLUDING REMARKS

Our main concern was here to develop some basic tools for applications involving fuzzy equations. The present paper will be followed by a companion one (Sanchez, 1982) containing results we presented at NAFIP-1, too. Dealing with fuzzy numbers, '$\tilde{+}$' will serve to define a non-standard subtraction '$\ominus$' reducing fuzziness comparatively to the usual subtraction '-'. For example, in the case of Figure 1, we shall have $C \ominus A = B$. Analogous results will also hold for '$\tilde{x}$' redefined as '$\oplus$'.

Let us now indicate some points that could be developed in further investigations.

The min operator '$\wedge$' in (1) can be replaced by a different triangular norm[4] (briefly t-norm) and $A * X = C$ can be studied in the case of t-related variables, see Alsina and Nguyen (1978), Dubois and Prade (1981) for extended operations by means of sup-t-norms.

Fuzzy equations can be enlarged to fuzzy sets with membership functions taking values in Brouwerian lattices, see Sanchez (1976), or to ultra-fuzzy sets, i.e., fuzzy sets with fuzzy set valued membership functions, as suggested by Zadeh, and with a potential application to the computation of fuzzy quantifiers in natural languages, see Zadeh (1982).

Finally, the fuzziness of solutions to $A * X = C$ can be investigated as in the work of Di Nola and Pedrycz (1982) or Di Nola and Sessa (1982).

[4]t-norms have extensively been explored by Schweizer and Sklar (1960) following Menger's work (1942), in the field of statistical metric spaces. t-norms in connection with fuzzy sets have been studied by Alsina, Trillas and Valverde (1980), Klement (1980), Dubois and Prade (1980) among others.

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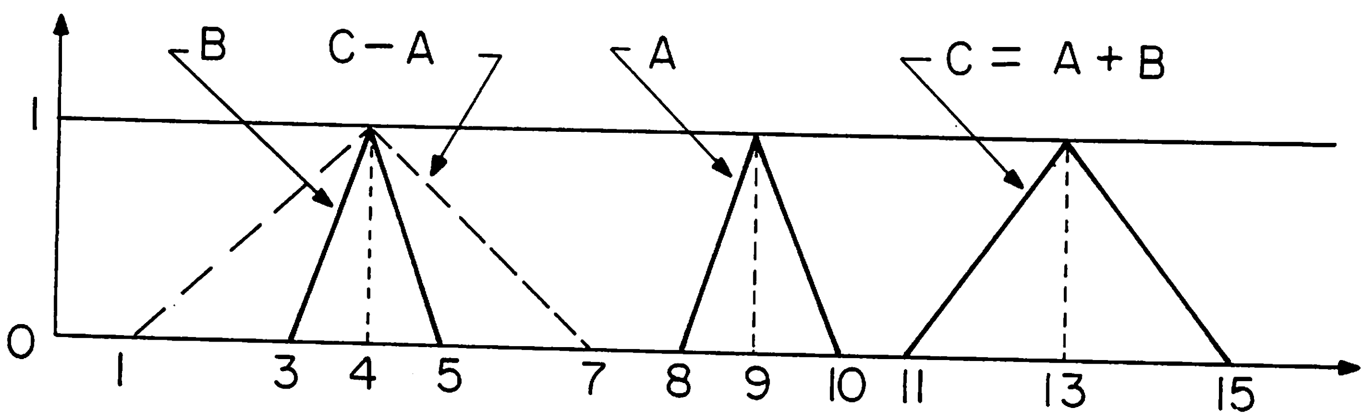

Figure 1. Illustrative example for A + B = C $\not\Rightarrow$ B = C - A.

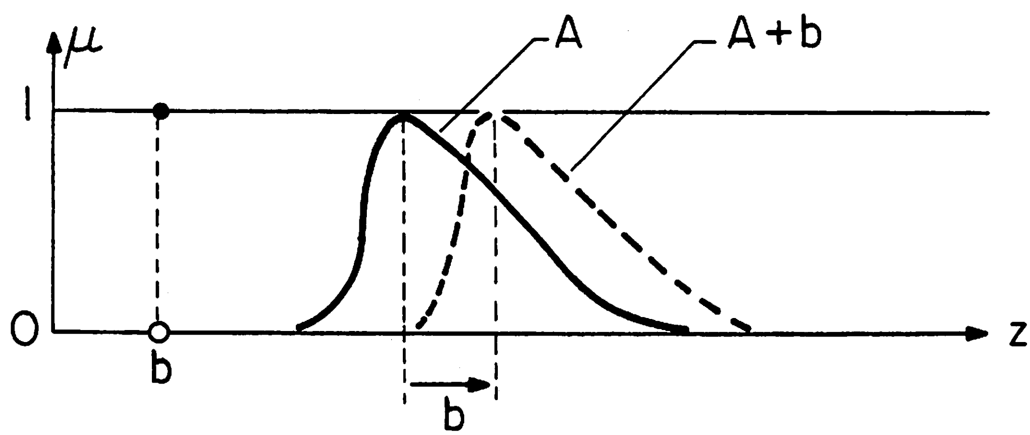

Figure 2. Illustration of A + B when B = b in $\mathbb{R}$.

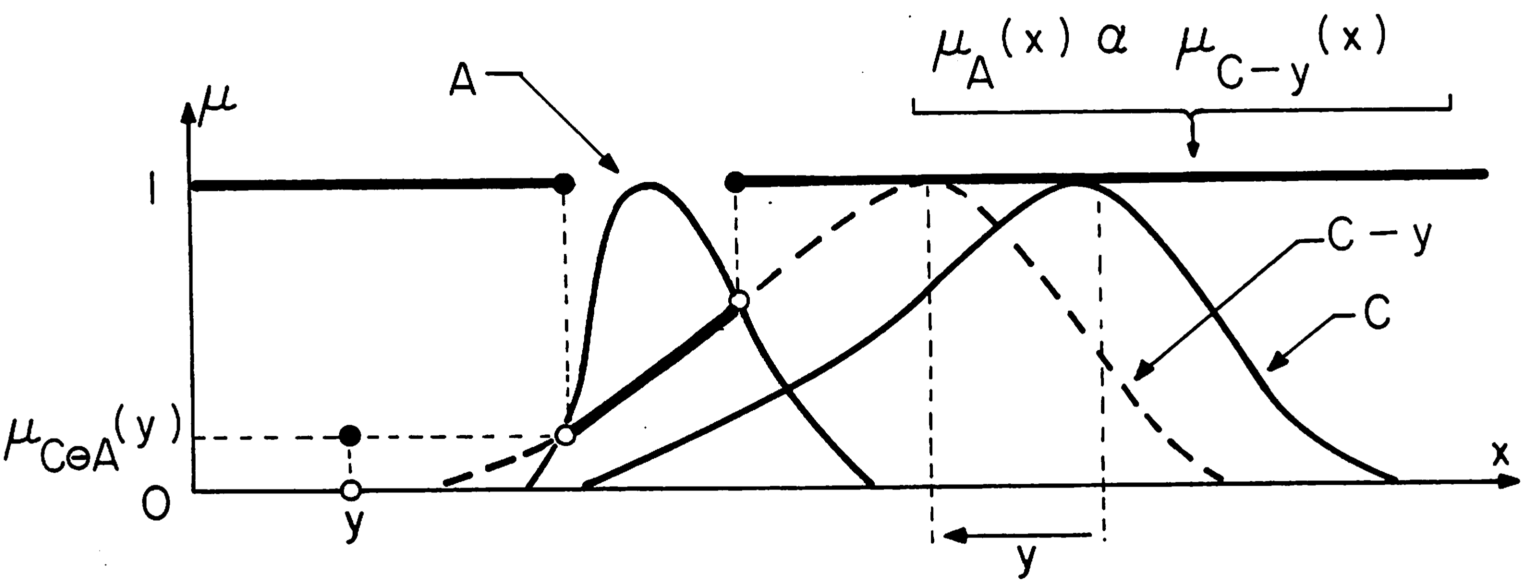

Figure 3. Illustration of C $\tilde{\div}$ A (denoted C ⊝ A) in the case of fuzzy numbers.