ANALYSIS OF LINEARIZED DECOUPLED POWER FLOW APPROXIMATIONS FOR STEADY STATE SECURITY ASSESSMENT

by

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ANALYSIS OF LINEARIZED DECOUPLED POWER FLOW APPROXIMATIONS FOR STEADY STATE SECURITY ASSESSMENT

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ABSTRACT

An analytic study of various approximations of the power flow equations for electric power systems is presented. The approximate models examined are the decoupled power flow, the linearized decoupled power flow (including the DC load flow) and the adjoint network sensitivity model, all of which are commonly used in steady-state security assessment. Error bounds on the difference between the solution of each of the approximate models and the solution of the full power flow are derived. The results are applied to the steady-state contingency analysis problem, resulting in a proposed new approach to the problem.

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I. INTRODUCTION

The steady-state behavior of a power transmission network is modelled by a set of nonlinear algebraic equations referred to as the power flow [1,2]. Their role in power systems is the basic mathematical model for many aspects of power systems planning and operation. Included in these categories are the problem of state estimation, security assessment, optimal economic operation, optimal transmission and generation expansion and power system control and stability assessment. Most of the work on power flow has consentrated on the numerical aspects of obtaining a solution [see 1 and its references]. Both Newton-Raphson techniques [3] and the fast decoupled load flow [4] have been successfully employed and their convergence properties are well known from both numerical studies [1] and a few theoretical investigations [5,6].

The power flow equations are nonlinear and often large scale (as many as three thousand unknowns). Approximate models are often substituted for the power flow in certain applications [7,8,9,10,11]. In this paper, we present an analytic investigation of the effects of the error introduced by various approximations. Two such commonly used approximate models are considered in Section 2. They are the <u>decoupled power flow</u> and the <u>linearized decoupled power flow</u>. The decoupled power flow model is based on observations on many typical power systems concluding that whereas the interactions between real power flows and phase angles and between reactive power flows and voltage magnitudes are strong, the interactions between real power and voltage magnitude and between reactive power and phase angles are weak. In Section 3, we derive sufficient conditions on the solution of the decoupled power flow for the full power flow to have a solution and a bound between the two solutions. Similar results are presented in Section 4 relating the solution of the linearized decoupled power flow to the existence of a solution to the decoupled power flow as well as an error bound between the two solutions. Direct application of these results to the problem of finding steady-state security regions [12] is also indicated in Section 3.

One problem of current interest in the operations context where approximate models are employed is contingency analysis. Here, the concern is with the steady-state performance of the system with respect to a large list of conjectured events, called contingencies. Typical contingencies are loss of a generator or removal of a transmission line from the base case. Of particular interest is the ability of the contingent network to supply the load demand while operating within the equipment limitations. This is termed the (steady-state) security of the contingent system. For large systems and typically long lists of contingencies, it is computationally impractical to solve the power flow for each contingency. Instead, it has been proposed to use either decoupled power flow [7,11] or the adjoint network sensitivity approach [8,9,13]. The latter model is examined in Sections 5 and 6. Here sufficient conditions on the solution of the sensitivity model for the decoupled power flow to have a solution and a bound between the two solutions are derived.

In Section 7, an example of how the analytic results of this paper can be applied is discussed. Specifically, a scheme for contingency analysis is proposed. In this scheme, using one of the approximate power flow models, contingencies can be classified as one of secure, insecure or "unknown" using both the approximate solution and the error bounds. This scheme is completely reliable: no insecure contingency can be classified as secure and vice-versa. Moreover, no power flow solutions are rquired for those contingencies classified as either secure or insecure cases.

The analytic tools used in this paper are degree theory and fixed point theorems [14, Ch. 6]. These have been successfully used in the study of steady-state [12] and dynamic [15] security regions of power systems. The techniques used here are similar to those in [2].

The notation used in this paper is standard. If $G \subset \mathbb{R}^n$, then $\dot{G}$, G^o and $\overline{G}$ denote the boundary, interior and closure of G. For $x,y \in \mathbb{R}^n$, $x \leqq y$ implies $x_i \leq y_i$ for each $i = 1,\ldots,n$ where x_i denotes the i-th component of x. If $x \in \mathbb{R}^n$ then $[x]$ is the nxn diagonal matrix with i, i-th entry x_i. For the matrix $A \in \mathbb{R}^{n\times m}$, $[A]_{ij}$ means the i,j-th entry. For $x \in \mathbb{R}^n$, $\|x\|_1$ and $\|x\|_\infty$ are the ℓ_1 and ℓ_∞ norms, $\sum_{i=1}^{n} |x_i|$ and $\max \{|x_i| \,|\, i=1,\ldots,n\}$ respectively. For $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, $\overline{B}_\infty(x,\lambda)$ is the set $\{y \in \mathbb{R}^n | \|x-y\|_\infty \leq \lambda\}$ while $B_\infty(x,\lambda)$ is its interior. The statement "$x := E$" means x is defined by the expression E.

2. POWER FLOW MODELS AND APPROXIMATIONS

2.1 Power Flow Equations

The steady-state behavior of a power transmission network is represented by the power flow equations. These are derived by modeling the transmission equipment as branches of a linear time invariant RLC circuit in sinusoidal steady-state. The nodes are called buses and represent generator stations and load-center substations.

Suppose the transmission network has N + 1 buses and let Y be the

(N+1) X (N+1) node (bus) admittance matrix, with $Y_{ki} = G_{ki} + jB_{ki}$ as the k,i-th element. Throughout this work, the following assumptions are made.

(A1) The network is connected.

(A2) The matrix Y is symmetric and in particular $B_{ki} = B_{ik}$. □

Remarks. 1. In assumption A1, the term connected has the following meaning. For each pair of buses, there is a path of lines, not incident on the ground node, between the two buses and each line in the path has nonzero B_{ki}.

2. Assumption A2 holds if there are no phase shifting transformers in the system. □

Using the standard model of transmission lines and transformers [2, p. 189 and p. 122], we have

Fact 1. $G_{kk} > 0$, $B_{kk} < 0$; $G_{ki} \leq 0$, $B_{ki} \geq 0$ for $i \neq k$

$$|B_{kk}| \geq \sum_{\substack{i=0 \\ i \neq k}}^{N} B_{ki} \quad \text{and} \quad G_{kk} \geq \sum_{\substack{i=0 \\ i \neq k}}^{N} |G_{ki}|$$ □

We now derive the power flow equations. Let E_k be the voltage phasor at bus k and $\hat{S}_k = \hat{P}_k + j\hat{Q}_k$ be the injected couplex power at bus k. E and $\hat{S}$ are vectors of E_k and $\hat{S}_k$ respectively. Then we have

$$\hat{S}^* = [E^*]YE \tag{1}$$

where $[E^*]$ is a diagonal matrix with k, k-th entry E_k^*, the complex conjugate of E_k. There are three types of buses.

(i) Slack bus: At this bus, the voltage magnitude and phase angle are specified. The latter is set to zero. This bus is numbered 0, and it usually corresponds to a generator bus.

(ii) PQ bus: At this bus, the injected real and reactive powers ($\hat{P}_k$ and $\hat{Q}_k$) are specified. The subscripts $\{1,\ldots,N_Q\}$ correspond to all PQ buses in the network. They are normally load buses.

(iii) PV bus: At this bus, the real injected power ($\hat{P}_k$) and the voltage mangitude ($|E_k|$) are specified. PV buses are usually generator buses and are numbered $N_Q + 1,\ldots, N$.

Let $E_k = V_k e^{j\theta_k}$ and $\theta_{ki} := \theta_k - \theta_i$. It is easy to show that equation (1) can be expressed as

$$\sum_{i=0}^{N} V_k V_i (G_{ki} \sin \theta_{ki} - B_{ki} \cos \theta_{ki}) = \hat{Q}_k \quad k = 1,\ldots, N_Q \tag{2}$$

$$\sum_{i=0}^{N} V_k V_i (G_{ki} \cos \theta_{ki} + B_{ki} \sin \theta_{ki}) = \hat{P}_k \quad k = 1,\ldots, N \tag{3}$$

Here, V_0, $V_{N_Q+1},\ldots,V_N$ and θ_0 are assumed fixed. The unknowns are $V := (V_1,\ldots,V_{N_Q})^T \in \mathbb{R}^{N_Q}$ and $\theta := (\theta_1,\ldots,\theta_N)^T \in \mathbb{R}^N$ and they are the state variables. The variables $\hat{P} := (\hat{P}_1,\ldots,\hat{P}_N)^T \in \mathbb{R}^n$ and $\hat{Q} := (\hat{Q}_1, \ldots,\hat{Q}_{N_Q})^T \in \mathbb{R}^{N_Q}$ are the inputs.

Equations (2) and (3) are known as the power flow equations. We shall define the power flow functions $Q : \mathbb{R}^n \times \mathbb{R}^{N_Q} \to \mathbb{R}^{N_Q}$ and $P : \mathbb{R}^N \times \mathbb{R}^{N_Q} \to \mathbb{R}^N$ by the expressions on the left hand sides of (2) and (3), respectively. That is, the k-th component of Q is Q_k where

$$Q_k(\theta,V) := \sum_{i=0}^{N} V_k V_i (G_{ki} \sin \theta_{ki} - B_{ki} \cos \theta_{ki}) \tag{4}$$

for $k = 1,\dots,N_Q$ and the k^{th} component of P is P_k where, for $k = 1,\dots,N$,

$$P_k(\theta,V) := \sum_{i=0}^{N} V_k V_i (G_{ki} \cos \theta_{ki} + B_{ki} \sin \theta_{ki}) \tag{5}$$

2.2 Decoupled Power Flow Equations

The following simplifying assumptions are used to produce the decoupled power flow expressions.

(SA1): The line conductances are negligible i.e. $G_{ki} = 0$.

(SA2): The phase angles across branches, θ_{ki}, are small so that $\cos \theta_{ki} \simeq 1$ and $\sin \theta_{ki} \simeq \theta_{ki}$.

(SA3): Voltage magnitudes, V_k, are close to unity and do not thus affect real power flows.

Under these simplifying assumptions, Q is approximated by $\tilde{Q} : \mathbb{R}^{N_Q} \to \mathbb{R}^{N_Q}$ and P by $\tilde{P} : \mathbb{R}^N \to \mathbb{R}^N$ where the k-th components of $\tilde{Q}$ and $\tilde{P}$ are given by

$$\tilde{Q}_k(V) := -\sum_{k=0}^{N} V_k V_i B_{ki} \qquad k = 1, \dots, N_Q \tag{6}$$

$$\tilde{P}_k(\theta) := \sum_{k=0}^{N} B_{ki}(\theta_k - \theta_i) \qquad k = 1, \dots, N \tag{7}$$

In this paper,

$$\tilde{Q}(\tilde{V}) = \hat{Q} \tag{8}$$

$$\tilde{P}(\tilde{\theta}) = \hat{P} \tag{9}$$

will be referred to as the <u>decoupled power flow</u> equations. Solutions to them, $\tilde{V} \in \mathbb{R}^{N_Q}$, $\tilde{\theta} \in \mathbb{R}^N$, are often [16] used as approximations to V and θ, the solutions of power flow equation (2) and (3). In Section 3, it is shown that, under certain conditions on $\tilde{V}$ and $\tilde{\theta}$, the power flow has a solution and a bound on the error between it and $\tilde{V}$ and $\tilde{\theta}$ is derived.

Sometimes the equations

$$\sum_{i=0}^{N} V_k V_i B_{ki}(\theta_k - \theta_i) = \hat{P}_k \qquad k = 1, \ldots, N \tag{10}$$

are used in place of $\tilde{P}(\theta) = \hat{P}$ in the decoupled power flow. In this paper, attention will be restricted to $\tilde{P}(\theta) = \hat{P}$, although an analysis similar to the one presented here can be performed for the more complicated model. The P-θ equations used here are commonly referred to as the DC load flow equations.

2.3 P-θ Equations

Suppose that there are ℓ lines in the network, excluding shunts. Let $A \in \mathbb{R}^{N \times \ell}$ be the reduced node incidents matrix of the network obtained by deleting all shunts and the ground node. The slack bus is taken as reference. Let $[y] \in \mathbb{R}^{\ell \times \ell}$ be the diagonal matrix with elements $\{B_{ki} | k \neq i; k,i=0,\ldots,N\}$ and define $J_p \in \mathbb{R}^{N \times N}$ by

$$J_p := A[y]A^T \tag{11}$$

It is simple to show that

$$\tilde{P}(\theta) = J_p \theta \tag{12}$$

and that, under assumptions A1 and A2, J_p is nonsingular. Thus, we can define $z_p \in \mathbb{R}_+$ by

$$z_p := \max\{e_k^T J_p^{-1} e_k | k=1,\ldots,N\} \tag{13}$$

where $e_k := (0,\dots,0,1,0,\dots,0)^T$, with the 1 in the k-th position.

Finally, we define $\mathcal{R}_\theta \subset \mathbb{R}^N$ by

$$\mathcal{R}_\theta := \{\theta \in \mathbb{R}^N | -\delta \leqq A^T\theta \leqq \delta\} \tag{14}$$

for some fixed[1] $\delta \in \mathbb{R}_+^\ell$. $\mathcal{R}_\theta$ is a polytope such that the angle difference across the m-th line is less than δ_m. $\mathcal{R}_\theta$ might be interpreted as the security constraint set on θ imposed by thermal limitations on line current flow [12]. However, in general, we do not assume this to be the case. $\mathcal{R}_\theta$ will be used as a region of validity for the approximations. Thus it will usually be larger than the security constraint set.

2.4 Q-V Equations

Consider the network $\mathcal{N}_Q$ obtained by deleting from the power transmission network the slack bus, all PV buses (i.e. $\{0,N_Q+1,\dots,N\}$) all lines incident with them and all shunts. In general, $\mathcal{N}_Q$ will not be connected. Suppose it has s separate parts, $\mathcal{N}_{Q1}, \dots, \mathcal{N}_{Qs}$, and that the PQ buses are numbered so that

the buses of $\mathcal{N}_{Q1}$ are $\{1,\dots,N_{Q1}\}$

the buses of $\mathcal{N}_{Q2}$ are $\{N_{Q1}+1,\dots,N_{Q2}\}$

$\vdots$

the buses of $\mathcal{N}_{Qs}$ are $\{N_{Q(s-1)}+1,\dots,N_Q\}$

Using the above bus numbering system, it can be seen that the Q-V relationship of the decoupled power flow, $\tilde{Q}$, further decouples into s separate functions. Consider bus k in $\mathcal{N}_{Qj}$. The decoupled approximation

[1] We assume that $0 < \delta_i < \frac{\pi}{2}$ $\forall$ i = 1, ..., ℓ.

to the reactive power injection is

$$\tilde{Q}_k(V) = - \sum_{i \in N_{Qj}} V_k V_i B_{ki} - \sum_{i=0,N_Q+1}^{N} V_k V_i B_{ki} \tag{15}$$

which depends only on V_i for buses i in N_{Qj} (V_i for PV and slack buses are assumed constant). As an immediate consequence, the Jacobian of $\tilde{Q}$ will be block diagonal, each block corresponding to a separate part of N_Q.

We thus partition all the relevant variables and functions according to which separate part they belong.

$$(V^1,\ldots,V^s) := V \tag{16}$$

$$(\tilde{Q}^1(V^1),\ldots,\tilde{Q}^s(V^s)) := \tilde{Q}(V) \tag{17}$$

$$(\overline{Q}^1(V^1),\ldots,\overline{Q}^s(V^s)) := \overline{Q}(V) \tag{18}$$

$$(Q^1(\theta,V),\ldots,Q^s(\theta,v)) := Q(\theta,V) \tag{19}$$

where V^j, $\tilde{Q}^j(V^j)$, $\overline{Q}^j(V^j)$ and $Q^j(\ ,V) \in \mathbb{R}^{n_{Qj}}$, and n_{Qj} is the number of buses in N_{Qj}.

We further define, for each separate part N_{Qj}, j = 1, ..., s, the associated network N^e_{Qj} by the following procedure (see Figure 1).

(1) Let $M_j \subset \{0,N_Q+1,\ldots,N\}$ be a set of PV or slack buses such that for each $k \in M_j$ there is a bus i in N_{Qj} such that

$$B_{ki} > 0$$

(2) Append to N_{Qj} a fictitious bus f_j. For each $k \in M_j$, find a bus

i in N_{Qj} such that $B_{ki} > 0$ and append a line of value B_{ki} to N_{Qj} between bus i and bus f_j. □

For each $j = 1,\ldots,s$, let A_{Qj} be the reduced node incidence matrix of N^e_{Qj}, taking f_j as the reference node. Let $[y_{Qj}]$ be the diagonal matrix of the B_{ki}'s such that ki is a line in N^e_{Qj}, ordered so as to be consistent with A_{Qj}. Define $B_{Qj} \in \mathbb{R}^{n_{Qj} \times n_{Qj}}$ to be the node admittance of N^e_{Qj}, i.e.,

$$B_{Qj} := A_{Qj}[y_{Qj}]A^T_{Qj} \tag{20}$$

Since N^e_{Qj} is connected and $[y_{Qj}]$ is positive definite diagonal, B_{Qj} is positive definite symmetric [17, p. 768] and thus nonsingular. Hence, we can define z_{qj} to be the smallest diagonal entry in B^{-1}_{Qj}, i.e.,

$$z_{qj} := \max\{e^T_k B^{-1}_{Qj} e_k \mid k=1,\ldots,n_{Qj}\} \tag{21}$$

where e_k is as before except $e_k \in \mathbb{R}^{n_{Qj}}$.

Let $V^m, V^M \in \mathbb{R}^{N_Q}$ such that $V^m_i < V^M_i \;\; \forall\, i = 1, \ldots, N_Q$ and define $R_V \subset \mathbb{R}^{N_Q}$

$$R_V := \{V \in \mathbb{R}^{N_Q} \mid V^m \leqq V \leqq V^M\} \tag{22}$$

The set R_V has the same form as the constraints on voltage magnitude imposed by security of operation [12] and, in the results below, it can be interpreted as such. However, it is not necessary that this be the case as R_V will be the region of validity of the results. The following assumption will be made throughout this work.

(A3): $\forall\, V \in R_V$, $j \in \{1,\ldots,s\}$ and for each bus k of N_{Qj}

$$- \sum_{i=0}^{N} V_k V_i B_{ki} \geq V_k^2 \sum_{i \in N_{Qj}^e} B_{ki} \tag{23}$$

$\square$

Remark. Suppose $V_i = 1 \;\; \forall\; i = 0, 1, \ldots, N$. Then, by fact 1,

$$- V_k^2 B_{kk} \geq \sum_{\substack{i=0 \\ i \neq k}}^{N} V_k V_i B_{ki} \tag{24}$$

from which Eq. (23) easily follows. Now suppose that for each bus k in N, either

$$- B_{kk} > \sum_{\substack{i=0 \\ i \neq k}}^{N} B_{ki} \tag{25}$$

or there exists $i \in \{0, N_Q+1, \ldots N\}$ such that $B_{ki} > 0$. Then there exists a V^m and V^M with $V_i^m < V_i^M \;\; \forall\; i \in \{1, \ldots, N_Q\}$ such that Assumption A3 holds. $\square$

For $V \in \mathbb{R}^{N_Q}$, let $J_q(V) \in \mathbb{R}^{N_Q \times N_Q}$ be the Jacobian of $\tilde{Q}$ i.e.

$$J_q(V) := \frac{\partial}{\partial V} \tilde{Q}(V) \tag{26}$$

Its elements are, for $k, i = 1, \ldots, N_Q$,

$$[J_q(V)]_{ki} = - V_k B_{ki} \qquad k \neq i \tag{27}$$

$$[J_q(V)]_{kk} = - V_k B_{kk} - \sum_{i=0}^{N} V_i B_{ki} \tag{28}$$

We can now establish the following useful lemma.

Lemma 1

Under assumptions A1, A2 and A3, $\forall\; V \in R_V$, $J_q(V)$ is nonsingular.

Proof

Suppose $V \in \mathcal{R}_V$, $j \in \{1,\dots,s\}$, k is a bus in $\mathcal{N}_{Qj}$ and $u \in \mathbb{R}^{N_Q}$. Then the k-th component of $J_q(V)u$ is

$$(J_q(V)u)_k = (-2V_kB_{kk} - \sum_{\substack{i=0\\ i\neq k}}^{N} V_iB_{ki})u_k - \sum_{\substack{i=1\\ i\neq k}}^{N_Q} V_kB_{ki}u_i$$

$$= V_k[(\sum_{i\in\mathcal{N}_{Qj}^e} B_{ki})u_k - \sum_{\substack{i\in\mathcal{N}_{Qj}\\ i\neq k}} B_{ki}u_i]$$

$$+ u_k[-2V_kB_{kk} - \sum_{\substack{i=0\\ i\neq k}}^{N} V_iB_{ki} - \sum_{\substack{i\in\mathcal{N}_{Qj}^e\\ i\neq k}} V_kB_{ki}] \tag{29}$$

Partitioning V, u and $J_q(V)$ into

$$(V^1,\dots,V^s) := V \tag{30}$$

$$(u^1,\dots,u^s) := u \tag{31}$$

$$J_q^j := \frac{\partial}{\partial V^j}\tilde{Q}^j(V^j) \tag{32}$$

we get

$$u^{jT}[V^j]^{-1}J_q^ju^j = (u^j)^TB_{Qj}u^j$$

$$+ \sum_{k\in\mathcal{N}_{Qj}} \frac{u_k^2}{V_k^2}[-2V_k^2B_{kk} - \sum_{\substack{i=0\\ i\neq k}}^{N} V_kV_iB_{ki} - \sum_{\substack{i\in\mathcal{N}_{Qj}^e\\ i\neq k}} V_k^2B_{ki}]$$

$$\geq (u^j)^T B_{Qj} u^j$$

$$> 0 \quad \forall\, u^j \neq 0 \tag{33}$$

$\Rightarrow J_q^j$ is nonsingular.

But $J_q(V) = \text{diag}\{J_q^1, \ldots, J_q^s\}$

Thus $J_q(V)$ is nonsingular. $\square$

<u>Corollary 1</u>. Under Assumptions A1, A2 and A3, either

(1) $\forall\, V \in \mathcal{R}_V \quad \det\{J_q(V)\} > 0$

or

(2) $\forall\, V \in \mathcal{R}_V \quad \det\{J_q(V)\} < 0$

<u>Proof</u>

$V \to \det\{J_q(V)\}$ is a continuous function of $\mathcal{R}_V \to \mathbb{R}$ which is never zero and $\mathcal{R}_V$ is connected and compact. $\square$

2.5 <u>Linearized Decoupled Power Flow</u>

The decoupled load flow dependency of reactive power on voltage magnitude, $\tilde{Q}$, is nonlinear and it is often convenient to linearize it. To this end, V^0 is taken as a fixed vector in $\mathbb{R}^{N_Q}$ and the Jacobian $J_q(V^0) \in \mathbb{R}^{N_Q \times N_Q}$ is used to linearize it. The linearized decoupled power flow expression is then $\overline{Q} : \mathbb{R}^{N_Q} \to \mathbb{R}^{N_Q}$ defined by

$$\overline{Q}(V) = \tilde{Q}(V^0) + J_q(V^0)(V - V^0) \tag{34}$$

and $\tilde{P}$ (which is already linear). The linearized decoupled power flow equations are

$$\overline{Q}(\overline{V}) = \hat{Q} \tag{35}$$

$$\tilde{P}(\tilde{\theta}) = \hat{P} \tag{36}$$

where $\overline{V}$ and $\tilde{\theta}$ are approximations to the solution of the power flow equations, (2) and (3). In Section 4, it is shown that, under certain conditions, if $\overline{V}$ solves Eq. (35) then the decoupled power flow will have a solution, $\tilde{V}$ and a bound on the difference between $\overline{V}$ and $\tilde{V}$ is derived. This can be coupled with the results of Section 3, to derive sufficient conditions for the existence of a solution of the power flow equations and a bound on the difference between it and the solution of the linearized decoupled power flow.

3. ANALYSIS OF DECOUPLING APPROXIMATION

In this section, the relationship between the solutions of the power flow equations and the decoupled power flow equations is analyzed. It is shown that if the decoupled power flow equations have a solution in $\mathcal{R}_\theta \times \mathcal{R}_V$ then, under certain conditions, the power flow equations also have a solution in $\mathcal{R}_\theta \times \mathcal{R}_V$. In fact, it is shown that a power flow solution falls within a hyperbox, centered on the decoupled power flow solution. The dimensions of this hyperbox form a bound on the error introduced into the power flow solution by decoupling.

Before giving the main result, the following definitions are required. For each k, i = 0, 1, ..., N, let

$$\phi_{ki} := \max\{|1-V_kV_i| \,\big|\, V_k \in \{V_k^m, V_k^M\}, V_i \in \{V_i^m, V_i^M\}\} \tag{37}$$

where $V_k^m := V_k^M := V_k$ for k = 0, N_Q+1, ..., N. Also

$$\delta_{ki} := \begin{cases} \delta_r & \text{if line } r \text{ is between buses } k \text{ and } i \\ 0 & \text{if } k = i \quad \text{or} \quad B_{ki} = 0 \end{cases} \tag{38}$$

Let

$$\varepsilon_p := \sum_{k=1}^{N} \sum_{i=0}^{N} [\delta_{ki}\phi_{ki}+(\delta_{ki}-\sin\,\delta_{ki})V_k^M V_i^M]B_{ki} + V_k^M V_i^M |G_{ki}| \tag{39}$$

and, for each j = 1, ..., s, let

$$\varepsilon_{qj} := \sum_{k\in N_{Qj}} \frac{V_k^M}{V_k^m} \sum_{\substack{i=0\\ i\neq k}}^{N} V_i^m[(1-\cos\,\delta_{ki})B_{ki}+\sin\,\delta_{ki}|G_{ki}|] \tag{40}$$

<u>Remark</u>. It is easy to show that

$$\varepsilon_p \geq \max\{\|\tilde{P}(\theta)-P(\theta,V)\|_1 | \theta \in \mathcal{R}_\theta,\ V \in \mathcal{R}_V\} \tag{41}$$

$$\varepsilon_{qj} \geq \max\{\|[U^j]^{-1}(\tilde{Q}^j(V^j)-Q^j(\theta,V))\|_1 | \theta \in \mathcal{R}_\theta,\ U,\ V \in \mathcal{R}_V\} \tag{42}$$

where U is partitioned in the same way as V and $[U^j]$ is a diagonal $n_{Qj} \times n_{Qj}$ matrix with U^j as its diagonal entries. □

Let $\mathcal{R}_V^o$ and $\mathcal{R}_\theta^o$ denote the interiors of $\mathcal{R}_V$ and $\mathcal{R}_\theta$, respectively.

<u>Theorem 1</u>. (Decoupling Analysis)

Suppose that $\hat{P} \in \mathbb{R}^N$ and $\hat{Q} \in \mathbb{R}^{N_Q}$ and that the decoupled power flow equations

$$\begin{aligned} \tilde{P}(\tilde{\theta}) &= \hat{P} \\ \tilde{Q}(\tilde{V}) &= \hat{Q} \end{aligned} \tag{43}$$

have a solution $(\tilde{\theta},\tilde{V}) \in \mathcal{R}_\theta \times \mathcal{R}_V$. Further, suppose that Assumptions A1, A2 and A3 hold and that

$$\begin{aligned} &\bar{B}_\infty(\tilde{V}^1, z_{q1}\,\varepsilon_{q1}) \times \ldots \times \bar{B}_\infty(\tilde{V}^s, z_{qs}\,\varepsilon_{qs}) \subset \mathcal{R}_V^o \\ &\bar{B}_\infty(\tilde{\theta}, z_p\varepsilon_p) \subset \mathcal{R}_\theta^o \end{aligned} \tag{44}$$

Consider the power flow equations

$$P(\alpha,U) = \hat{P}$$
$$Q(\alpha,U) = \hat{Q} \qquad (45)$$

Under the above conditions

(1) Existence. The power flow equations have at least one solution (α,U) in $R_\theta \times R_V$.

(2) Bound. This solution satisfies

$$\|\alpha-\tilde{\theta}\|_\infty \leq z_p \varepsilon_p$$
$$\|U^j-\tilde{V}^j\|_\infty \leq z_{qj}\varepsilon_{qj} \qquad j = 1, \ldots, s \qquad (46)$$

The parameters z_p, z_{qj}, ε_p and ε_{qj} are defined in Eqs. (13), (21), (39) and (40), respectively. □

Remark. U^j and $\tilde{V}^j$ are the partitionings of U and $\tilde{V}$ respectively. That is

$$(U^1,\ldots,U^s) := U \qquad (47)$$

$$(\tilde{V}^1,\ldots,\tilde{V}^s) := V \qquad (48)$$

where U^j, $\tilde{V}^j \in \mathbb{R}^{n_{Qj}}$ $\forall$ j = 1, ..., s

Similarly,

$$\overline{B}_\infty(\tilde{V}^1,z_{q1}\varepsilon_{q1}) \times \ldots \times \overline{B}_\infty(\tilde{V}^s,z_{qs}\varepsilon_{qs})$$
$$= \{V \in \mathbb{R}^{N_Q} \mid \|V^j-\tilde{V}^j\|_\infty \leq z_{qj}\varepsilon_{qj},\ j = 1,\ldots,s\} \qquad (49)$$

where $\|V^j-\tilde{V}^j\|_\infty$ is the infinity norm of $V^j - \tilde{V}^j$ in $\mathbb{R}^{n_{Qj}}$. □

Proof

Theorem 1 is established by proving a series of claims. First, the problem is posed as an existence of solution within an open set. In the first claim, s + 1 sufficient conditions for the existence problem are derived using the theory of degree of mappings [14, Ch. 6]. In the second and third claims, it is shown that these conditions are satisfied.

For any $\varepsilon > 0$, let

$$\begin{aligned} \beta_p^\varepsilon &:= z_p \varepsilon_p + \varepsilon \\ \beta_{qj} &:= z_{qj}\varepsilon_{qj} + \varepsilon \qquad j = 1, \ldots, s \end{aligned} \tag{50}$$

For any $(\theta,V) \in \mathbb{R}^N \times \mathbb{R}^{N_Q}$, let

$$S_\varepsilon(\theta,V) := B_\infty(\theta,\beta_p^\varepsilon) \times B_\infty(V^1,\beta_{q1}^\varepsilon) \times \ldots \times B_\infty(V^s,\beta_{qs}^\varepsilon) \tag{51}$$

Since $\bar{S}_0(\tilde{\theta},\tilde{V}) \subset (R_\theta \times R_V)^o$, $\exists\ \bar{\varepsilon} > 0 \quad \forall\ \varepsilon \in (0,\bar{\varepsilon})$

$$\bar{S}_\varepsilon(\tilde{\theta},\tilde{V}) \subset R_\theta \times R_V$$

Our approach is to show that, $\forall\ \varepsilon \in (0,\bar{\varepsilon})$, the power flow equations have a solution in $S_\varepsilon(\tilde{\theta},\tilde{V})$ and thus, they have a solution in $\bar{S}_0(\tilde{\theta},\tilde{V})$. We thus fix an ε in $(0,\bar{\varepsilon})$.

Claim 1

Consider conditions (CP) and (CQj) for j = 1, ..., s defined below.

CP: $\forall\ t \in [0,1]$, $(\theta,V) \in \bar{S}_\varepsilon(\tilde{\theta},\tilde{V})$ such that $\|\theta-\tilde{\theta}\|_\infty = \beta_p^\varepsilon$

$$\tilde{P}(\theta) - \tilde{P}(\tilde{\theta}) \neq t\{P(\theta)-P(\theta,V)\}$$

CQj: $\forall\ t \in [0,1]$, $(\theta,V) \in \bar{S}_\varepsilon(\tilde{\theta},\tilde{V})$ such that $\| V^j - \tilde{V}^j \|_\infty = \beta^\varepsilon_{qj}$

$$\tilde{Q}^j(V^j) - \tilde{Q}^j(\tilde{V}^j) \neq t\{\tilde{Q}^j(V^j) - Q^j(\theta,V)\}$$

If (CP) and (CQj) are satisfied $\forall j = 1, \ldots, s$, then the power flow equations (45) have a solution in $S_\varepsilon(\tilde{\theta},\tilde{V})$. □

Proof of Claim 1

Define the homotopy

$$H : \mathbb{R}^N \times \mathbb{R}^{N_Q} \times [0,1] \rightarrow \mathbb{R}^N \times \mathbb{R}^{N_Q} \text{ by}$$

$$H(\theta,V,t) := (1-t)(\tilde{P}(\theta),\tilde{Q}(V)) + t(P(\theta,V),Q(\theta,V)) \tag{52}$$

Thus $H(\cdot,0)$ is the decoupled power flow expression and $H(\cdot,1)$ is the full power flow expression.

By the conditions of this theorem, the equation

$$H(\theta,V,0) = (\hat{P},\hat{Q}) \tag{53}$$

has at lease one solution is $S_\varepsilon(\tilde{\theta},\tilde{V})$. Further, by Corollary 1, the Jacobians of $H(\theta,V,0)$ at all solutions in $S_\varepsilon(\tilde{\theta},\tilde{V})$ will be nonsingular and have determinants of one sign. Thus[2] [14, p. 152]

$$\deg[H(\cdot,0), S_\varepsilon(\tilde{\theta},\tilde{V}), (\hat{P},\hat{Q})] \neq 0 \tag{54}$$

Thus, by the homotopy invariance theorem [14, p. 156], if

[2] Strictly speaking, in order that the degree in (54) be defined, we need to show that (53) has no solution on the boundary of $S_\varepsilon(\tilde{\theta},\tilde{V})$. However, this is established as a by-product of the subsequent analysis.

$$H(\theta,V,t) \neq (\hat{P},\hat{Q}) \qquad \forall\ t \in [0,1], \quad (\theta,V) \in \partial S_\varepsilon(\hat{\theta},\hat{V}) \tag{55}$$

then

$$\deg[H(\cdot,1), S_\varepsilon(\tilde{\theta},\tilde{V}), (\hat{P},\hat{Q})] \neq 0 \tag{56}$$

and from the Kronecker existence theorem [14, p. 161], the equation

$$H(\theta,V,1) = (\hat{P},\hat{Q}) \tag{57}$$

has a solution in $S_\varepsilon(\tilde{\theta},\tilde{V})$. Thus, Eq. (55) is sufficient and it is simple to show that CP and CQj imply Eq. (55). □

Claim 2

CP is satisfied.

Proof of Claim 2

Let $(\theta,V) \in \overline{S}_\varepsilon(\tilde{\theta},\tilde{V})$ such that $\|\theta-\tilde{\theta}\|_\infty = \beta_p^\varepsilon$. Let $t \in [0,1]$ and $\gamma := \theta - \tilde{\theta}$. Then,

$$\tilde{P}(\theta) - \tilde{P}(\tilde{\theta}) \neq -\ t(P(\theta,V) - \tilde{P}(\theta)) \tag{58}$$

$$\Leftarrow \quad \gamma^T(\tilde{P}(\tilde{\theta}+\gamma)-\tilde{P}(\tilde{\theta})) > |\gamma^T(P(\theta,V)-\tilde{P}(\theta))| \tag{59}$$

$$\Leftarrow \quad \gamma^T J_p \gamma > \|\gamma\|_\infty \ \|P(\theta,V)-\tilde{P}(\theta)\|_1 \tag{60}$$

But, from lemma A (see Appendix).

$$\gamma^T J_p \gamma \geq \frac{1}{z_p} \|\gamma\|_\infty^2 = \frac{(\beta_p^\varepsilon)^2}{z_p} \tag{61}$$

and since $(\theta,V) \in R_\theta \times R_V$, from Eq. (41)

$$\|\gamma\|_\infty \; \|P(\theta,V)-\tilde{P}(\theta)\|_1 \leq \beta_p^\varepsilon \varepsilon_p \tag{62}$$

Since $\dfrac{(\beta_p^\varepsilon)^2}{Z_p} > \beta_p^\varepsilon \varepsilon_p \quad \forall\ \varepsilon > 0$, Eqs. (61) and (62) establish (60) and thus CP. □

Claim 3

CQj is satisfied $\forall\ j = 1, \ldots, s$.

Proof of Claim 3

Let $j \in \{1, \ldots, s\}$, $t \in [0,1]$, $(\theta,V) \in \overline{S}_\varepsilon(\tilde{\theta},\tilde{V})$ such that $\|V^j-\tilde{V}^j\|_\infty = \beta_{qj}^\varepsilon$. Let $u := V - \tilde{V} = (u^1,\ldots,u^j,\ldots,u^s)$. Thus $\|u^j\|_\infty = \beta_{qj}^\varepsilon$. It is thus required to show that

$$\tilde{Q}^j(\tilde{V}^j+u^j) - \tilde{Q}(\tilde{V}^j) \neq t\{\tilde{Q}^j(V^j)-Q^j(\theta,V)\} \tag{63}$$

This is achieved by establishing that

$$(u^j)^T[\tilde{V}^j + \tfrac{1}{2}\, u^j]^{-1}\{\tilde{Q}^j(\tilde{V}^j+u^j)-\tilde{Q}^j(\tilde{V}^j)\}$$

$$> |(u^j)^T[\tilde{V}^j + \tfrac{1}{2}\, u^j]^{-1}\{\tilde{Q}^j(V^j)-Q^j(\theta,V)\}| \tag{64}$$

which is sufficient for Eq. (63).

To examine the left hand side of (64), let $k \in N_j$ and by Taylor's theorem [18, p. 190], we note that

$$\tilde{Q}_k(\tilde{V}+u)-\tilde{Q}_k(\tilde{V}) = \int_0^1 \tilde{Q}_k'(\tilde{V}+\lambda u)d\lambda\ u \tag{65}$$

Define $\tilde{U}(\lambda) := \tilde{V} + \lambda u \in \mathbb{R}^{N_Q}$ and $\tilde{U}_i(\lambda) := V_i$, $i = 0, N_Q + 1, \ldots, N$. Thus

$$u_k[\tilde{Q}_k(\tilde{V}+u)-\tilde{Q}_k(\tilde{V})] = u_k \int_0^1 \tilde{U}_k(\lambda)d\lambda[v_k(\sum_{\substack{i\in N^e_{Qj}\\ i\neq k}} B_{ki})- \sum_{\substack{i\in N_{Qj}\\ i\neq k}} u_i B_{ki}]$$

$$+ u_k^2 \int_k^1 [-2\tilde{U}_k(\lambda)B_{kk}- \sum_{\substack{i\in N^e_{Qj}\\ i\neq k}} \tilde{U}_k(\lambda)B_{ki}$$

$$- \sum_{\substack{i=0\\ i\neq k}}^{N} \tilde{U}_i(\lambda)B_{ki}]d\lambda \tag{66}$$

Noting that $\tilde{U}(\lambda) \in R_V$ for all $\lambda \in [0,1]$, by Assumption A3, the second term is positive and thus

$$u_k[\tilde{Q}_k(\tilde{V}+u)-\tilde{Q}_k(\tilde{V})] \geq u_k(\tilde{V}_k+ \frac{1}{2} u_k)[u_k(\sum_{\substack{i\in N^e_{Qj}\\ i\neq k}} B_{ki})- \sum_{\substack{i\in N_{Qj}\\ i\neq k}} u_i B_{ki}]$$

$$\Rightarrow \quad (u^j)^T[\tilde{V}^j+\frac{1}{2}u^j]^{-1}(\tilde{Q}^j(V^j)-\tilde{Q}^j(\tilde{V}^j)) \geq (u^j)^T B_{Qj} u^j$$

$$\geq \frac{1}{z_{qj}} \|u^j\|_\infty^2$$

$$= \frac{(\beta^\varepsilon_{qj})^2}{z_{qj}} \tag{67}$$

where the second inequality follows from Lemma A (see Appendix).

Now consider the right hand side of Eq. (62)

$$|(u^j)^T[\tilde{V}^j+ \frac{1}{2} u^j]^{-1}\{\tilde{Q}^j(V^j)-Q^j(\theta,V)\}|$$

$$\leq \|u^j\|_\infty \|[\tilde{V}^j+\frac{1}{2}u^j]^{-1}\{\tilde{Q}^j(V^j)-Q^j(\theta,V)\}\|_1$$

$$\leq \beta^\varepsilon_{qj}\varepsilon_{qj} \tag{68}$$

which follows from Eq. (42) and the fact that $\tilde{V}^j + \frac{1}{2}u^j \in R_V$ and $(\theta,V) \in R_\theta \times R_V$.

Since $\frac{(\beta^\varepsilon_{qj})^2}{z_{qj}} > \beta^\varepsilon_{qj}\varepsilon_{qj}$ $\forall\ \varepsilon > 0$, Eqs. (67) and (68) establish (64) and thus CQj. □

Thus CP and CQj, j = 1, ..., s hold for $\varepsilon \in (0,\bar{\varepsilon})$ and Theorem 1 is established. □

In many situations it would be useful to calculate the errors in the real power flows in the transmission lines introduced by the decoupling approximation. Line flows are particularly significant since they are limited by equipment constraints and area interchange agreements. Bounds on the decoupling errors in line flows can be obtained directly from Theorem 1. However, the following Corollary will lead to a tighter bound. The proof is omitted as it is a minor modification of the proof of Theorem 1. First, however, a few definitions are required. Let $\bar{y} \in \mathbb{R}_+$ be defined by

$$\bar{y} := \min\{y_m | m = 1,\dots,\ell\} \tag{69}$$

and let $\tau_p \in \mathbb{R}^\ell_+$ have m-th component

$$\tau_{pm} := \frac{\varepsilon_p}{\sqrt{y_m \bar{y}}} \tag{70}$$

Here $y \in \mathbb{R}^\ell$ is the vector of B_{ki} values $(k \neq i)$ used in the definition of J_p. Let $C(\tilde{\theta},\tau_p) \subset \mathbb{R}^N$ be defined by

$$C(\tilde{\theta},\tau_p) := \{\theta \in \mathbb{R}^N | -\tau_p \leqq A^T(\theta-\tilde{\theta}) \leqq \tau_p\} \tag{71}$$

The corollary is stated for the lossless case (i.e., $G_{ki} = 0$) for simplicity. A similar but more complicated result can be derived in the same fashion in the more general case.

Corollary 2. Suppose that the consitions of Theorem 1 hold except that, instead of $\overline{B}_{\infty}(\theta, Z_p\varepsilon_p) \subset R_\theta^0$, we have that

$$C(\tilde{\theta}, \tau_p) \subset R_\theta^0 \tag{72}$$

Also, suppose that $G_{ki} = 0$ for k, i = 0, 1, ..., N.

1. Existence. The power flow equations have at least one solution (α, U) in $R_\theta \times R_V$.
2. Bounds. This solution satisfies

$$-\tau_p \leqq A^T(\alpha - \tilde{\theta}) \leqq \tau_p \tag{73}$$

$$\|U^j - \tilde{V}^j\|_\infty \leq z_{qj}\varepsilon_{qj} \qquad j = 1, \ldots, s \tag{74}$$

The parameters z_{qj}, ε_{qj} and τ_p are defined in Eqs. (21), (40) and (70) respectively. The set $C(\tilde{\theta}, \tau_p)$ is defined in Eq. (71). □

Remarks. 1. Equation (73) implies that the error in the voltage angle difference across transmission line m ($m \in \{1, \ldots, \ell\}$) is smaller than τ_{pm}.

2. The steady state security region is defined to be a set of power injections (P,Q) such that the power flow equations (45) have a solution (α, u) which lies in the security constraint set $R_\theta \times R_V$ [12]. Theorem 1 above, combined with the results of [12], can be used to find a steady state security region. We describe below the basic ideas of the

approach. Let the security constraint set be defined by

$$\mathcal{R}_\theta := \{\theta : -\delta \leq A^T\theta \leq \delta\} \tag{75}$$

$$\mathcal{R}_V := \{V : V^m \leq V \leq V^M\} \tag{76}$$

Let

$$\mathcal{R}^o_{\tilde{\theta}} := \{\tilde{\theta} : -\tilde{\phi}1 \leq \tilde{\theta} \leq \tilde{\phi}1\} \tag{77}$$

where

$$\begin{aligned} \tilde{\phi} &:= \phi - z_p \varepsilon_p \\ \phi &:= \min_j \frac{1}{2}\delta_j \\ 1 &:= (1,\ldots,1)^T \end{aligned} \tag{78}$$

and

$$\mathcal{R}_{\tilde{V}} := \{\tilde{V} : \tilde{V}^m \leq \tilde{V} \leq \tilde{V}^M\} \tag{79}$$

where

$$\begin{aligned} \tilde{V}^M_i &:= V^M_i - z_{qj}\varepsilon_{qj} \\ \tilde{V}^m_i &:= V^m_i + z_{qj}\varepsilon_{qj} \end{aligned} \tag{80}$$

with bus i in the separate part j. For simplicity, let us consider the systems without PV buses.

<u>Corollary</u> Consider a power system with only PQ buses and the slack bus, and assume that there are no constraints on slack bus injections. If the power injections (P,Q) satisfy

$$\tilde{Q}_k(\tilde{V}^m) \le Q_k \le \tilde{Q}_k(\tilde{V}^M) \tag{81}$$

$$|P_k| \le \frac{\tilde{\phi}}{N} \frac{1}{X_{kk}} \tag{82}$$

then the power flow equations have a solution on the security constraint set $R_\theta \times R_V$.

Proof. If conditions 81 and 82 are satisfied, applying Theorems 1 and 2 of [12], we know that there exists a solution of the decoupled power flow equations in $R^o_{\tilde{\theta}} \times R_{\tilde{V}}$. Theorem 1 above then implies that there exists a solution of the power flow equations in $R_\theta \times R_V$.

4. ANALYSIS OF LINEARIZING APPROXIMATION

In Section 2.5, the linearized decoupled power flow expression $\overline{Q} : \mathbb{R}^{N_Q} \to \mathbb{R}^{N_Q}$ was defined by

$$\overline{Q}(V) = \tilde{Q}(V^o) + J_q(V^o)(V-V^o) \tag{83}$$

where V^o is fixed in $\mathbb{R}^{N_Q}$ and $J_q(V^o) = \frac{\partial}{\partial V} \tilde{Q}(V^o)$. In the decoupled power flow model, the real power-voltage angle relationship (i.e. the DC load flow, $\tilde{P}$) is linear and is thus not considered here. In this section, we examine the relationship between the solution of the linearized decoupled power flow equations

$$\overline{Q}(\overline{V}) = \hat{Q} \tag{84}$$

and the solution of the decoupled power flow equations

$$\tilde{Q}(\tilde{V}) = \hat{Q} \tag{85}$$

Suppose $V^o \in R_V$. Then by Lemma 1, Eq. (84) has a unique solution, $\overline{V}$. We derive conditions on $\overline{V}$ under which Eq. (85) has a solution, $\tilde{V}$, and a bound on the difference between $\overline{V}$ and $\tilde{V}$. It is then possible to combine the results of this section with those of Section 3 to obtain bounds on the difference between the solution of the full power flow and the linearized decoupled power flow.

The following definitions simplify the statement and proof of the theorem. Let $J_Q \in \mathbb{R}^{N_Q x N_Q}$ be

$$J_Q := \frac{\partial \tilde{Q}(V^o)}{\partial V} = J_q(V^o) \tag{86}$$

and $\tilde{B} \in \mathbb{R}^{N_Q x N_Q}$ be defined by $[\tilde{B}]_{ki} := B_{ki}$, $k,i = 1, \ldots, N_Q$. Thus $\tilde{B}$ is the first N_Q rows and columns of B. Using the bus numbering convention introduced in Section 2, J_Q and $\tilde{B}$ are block diagonal i.e.

$$\text{block diag}(J_Q^1,\ldots,J_Q^s) := J_Q \tag{87}$$

$$\text{block diag}(\tilde{B}^1,\ldots,\tilde{B}^s) := \tilde{B} \tag{88}$$

where J_Q^j and $B^j \in \mathbb{R}^{n_{Qj} x n_{Qj}}$, $j = 1,\ldots,s$.

For each $j = 1, \ldots, s$, define $e_j : \mathbb{R}^{n_{Qj}} \to \mathbb{R}_+$ by

$$e_j(u) = \frac{z_{qj}}{\min\{V_k^m | k \in N_j\}} \| [u]\tilde{B}^j u \|_1 \tag{89}$$

which is quadratic in u.

Theorem 2. (Linearization Analysis)

Suppose $V^o \in R_V$ and Assumptions A1, A2 and A3 hold. The linearized decoupled power flow equation

$$\bar{Q}(\bar{V}) = \hat{Q} \tag{90}$$

has a unique solution $\bar{V} = V^o + J_Q^{-1}q$ where $q := \hat{Q} - \tilde{Q}(V^o)$. Let $v := J_Q^{-1}q$ and let v and $\bar{V}$ be partitioned according to their s separate parts, i.e.,

$$(v^1,\ldots,v^s) := v \quad , \quad (\bar{V}^1,\ldots,\bar{V}^2) := \bar{V} \tag{91}$$

where $v^j, \bar{V}^j \in \mathbb{R}^{n_{Qj}}$, $j = 1, \ldots, s$.

Suppose that

$$\bar{B}_\infty(\bar{V}^1, e_1(v^1)) \times \ldots \times \bar{B}_\infty(\bar{V}^s, e_s(v^s)) \subset R_V^o \tag{92}$$

Then

1. Existence. The decoupled power flow equations

$$\tilde{Q}(\tilde{V}) = \hat{Q} \tag{93}$$

has a solution $\tilde{V} = (\tilde{V}^1,\ldots,\tilde{V}^s) \in R_V$.

2. Error Bound. For each $j = 1, \ldots, s$

$$\|\tilde{V}^j - \bar{V}^j\|_\infty \le e_j(v^j) \tag{94}$$

The parameter $e_j(v^j)$ is defined in Eq. (89). □

Proof

Define the map $\Delta : \mathbb{R}^{N_Q} \to \mathbb{R}^{N_Q}$ by

$$\Delta(\mu) := \tilde{Q}(\bar{V}+\mu) - \hat{Q} \tag{95}$$

and the region $S_V \subset \mathbb{R}^{N_Q}$ by

$$S_V := \overline{B}_\infty(0,e_1(v^1)) \times \ldots \times \overline{B}_\infty(0, e_s(v^s)) \tag{96}$$

where

$$B_\infty(0,e_j(v^j)) \subset \mathbb{R}^{n_{Qj}} \text{ for } j = 1, \ldots, s.$$

Now suppose Δ has a zero in S_V i.e. $\exists\ \mu^* \in S_V$ such that $\Delta(\mu^*) = 0$. Then the theorem is established by setting $\tilde{V} = \overline{V} + \mu^*$. Thus it is sufficient to establish that Δ has a zero in S_V. This is achieved in the following two claims. First, however, Δ and μ are partitioned, i.e.,

$$(\mu^1,\ldots,\mu^s) := \mu \tag{97}$$

$$(\Delta^1(\mu^1),\ldots,\Delta^s(\mu^s)) := \Delta(\mu) \tag{98}$$

<u>Claim 1</u>

Suppose that for each $j = 1, \ldots, s$ for all $\mu^j \in \mathbb{R}^{n_{Qj}}$ such that $\|\mu^j\|_\infty = e_j(v^j)$

$$(\mu^j)^T[\overline{V}^j + \tfrac{1}{2}\,\mu^j]^{-1}\Delta^j(\mu^j) \geq 0 \tag{99}$$

Then Δ has a zero in S_V.

<u>Proof of Claim 1</u>

It follows from the hypothesis that $\forall\ j = 1, \ldots, s$ and

$\forall\ \|\mu^j\|_\infty = e_j(v^j)$

$$\Delta^j(\mu^j) \neq -\,\omega\mu^j \qquad \forall\ \omega > 0 \tag{100}$$

Thus

$$\Delta(\mu) \neq -\,\omega\mu \qquad \forall\ \omega > 0,\ \mu \in \partial S_V \tag{101}$$

Define $\Phi : \mathbb{R}^{N_Q} \to \mathbb{R}^{N_Q}$ by

$$\Phi(\mu) := \mu - \Delta(\mu)$$

From Eq. (101)

$$\Phi(\mu) \neq \lambda\mu \quad \forall\ \lambda > 1,\ \mu \in \partial S_V$$

This is precisely the condition of the Leray-Schauder fixed point theorem [14, p. 162]. Thus Φ has a fixed point in S_V, i.e., $\exists\ \mu^* \in S_V$ such that $\Phi(\mu^*) = \mu^*$. This is equivalent to $\Delta(\mu^*) = 0$. □

Claim 2

The condition of Claim 1 is satisfied.

Proof of Claim 2

Fix $j \in \{1,\ldots,s\}$ and $\mu^j \in \mathbb{R}^{n_{Qj}}$ such that $\|\mu^j\|_\infty = e_j(v^j)$.

Now

$$\Delta^j(\mu^j) = \tilde{Q}^j(\overline{V}^j+\mu^j) - \tilde{Q}^j(\overline{V}^j)$$

$$+ \tilde{Q}^j(V^{oj}+v^j) - \tilde{Q}^j(V^{oj}) - J_V^j\ v^j \tag{102}$$

Using an argument similar to the one employed in the proof of Theorem 1

$$(\mu^j)^T[\overline{V}^j + \tfrac{1}{2}\mu^j]^{-1}(\tilde{Q}^j(\overline{V}^j+\mu^j)-\tilde{Q}^j(\overline{V}^j))$$

$$\geq \|\mu^j\|_\infty^2 \frac{1}{z_{qj}} = (e_j(v^j))^2 \frac{1}{z_{qj}} \tag{103}$$

Let $k \in N_j$ and $v_i = 0$ for $i = 0, N_Q + 1, \ldots, N$. Then

$$\tilde{Q}_k(V^o+v) - \tilde{Q}_k(V^o) - \tilde{Q}_k'(V^o)v$$

$$= - \sum_{i=0}^{N} \{(V_k^o+v_k)(V_i^o+v_i)-V_k^oV_i^o-V_k^ov_i-V_i^ov_k\}B_{ki}$$

$$= - v_k \sum_{i\in N_{Qj}} v_iB_{ki} \tag{104}$$

Thus

$$|(\mu^j)^T[\overline{V}^j+ \tfrac{1}{2}\mu^j]^{-1}(\tilde{Q}^j(V^{oj}+v^j)-\tilde{Q}^j(V^{oj})-J_V^jv^j)|$$

$$= |(\mu^j)^T[\overline{V}^j+ \tfrac{1}{2}\mu^j]^{-1}[v^j]\tilde{B}^jv^j|$$

$$\leq \|\mu^j\|_\infty \, \|[\overline{V}^j+ \tfrac{1}{2}\mu^j]^{-1}[v^j]\tilde{B}^jv^j\|_1$$

$$\leq [e_j(v^j)]^2 \frac{1}{z_{qj}}$$

$$\leq (\mu^j)^T[\overline{V}^j+ \tfrac{1}{2}\mu^j]^{-1}(\tilde{Q}^j(\overline{V}^j+\mu^j)-\tilde{Q}^j(\overline{V}^j)) \qquad \text{(by (103))}$$

Thus, by Eq. (102),

$$(\mu^j)^T[\overline{V}^j+ \tfrac{1}{2}\mu^j]^{-1}\Delta(\mu^j) \geq 0 \qquad \square$$

5. REAL POWER CONTINGENCY ANALYSIS

In this section, the decoupled power flow model is used to examine the dependency of the angles of the complex bus voltages on transmission line susceptances. We have the following application in mind. Suppose that the system is being operated securely with real power injections $\hat{P} \in \mathbb{R}^N$ and with transmission line susceptances $y \in \mathbb{R}^\ell$. We refer to this as the base case situation. In Section 2, the real power part of the decoupled power flow expression was written as

$$\tilde{P}(\theta,y) = A[y]A^T\theta \tag{105}$$

where the dependence on y is now shown explicitly. Then, according to this model, the base bus angles, $\tilde{\theta}^b$ are given by the solution of

$$\tilde{P}(\tilde{\theta}^b,y) = \hat{P} \tag{106}$$

Consider the contingency of y changing to $y + \xi$ for $\xi \in \mathbb{R}^{\ell}$. This can represent the outage or addition of any number of lines. The problem of interest is then whether the contingent network has a secure solution. That is, using the decoupled power flow model, does

$$\tilde{P}(\tilde{\theta}^c,y+\xi) = \hat{P} \tag{107}$$

have a secure solution $\tilde{\theta}^c \in \mathbb{R}^N$? Here we assume that the real power injections, $\hat{P}$, remain at their base level, $\hat{P}$.

One approach to contingency analysis - the adjoint sensitivity method - is to leave the contingent decoupled power flow equations (107) unsolved. Instead, Eq. (106) is linearized in $(\tilde{\theta}^b,y)$ and used to approximate Eq. (107). The resulting linear equation is solved to yield an approximation, $\overline{\theta}_c$, to the contingent angles, $\tilde{\theta}^c$, defined by

$$\overline{\theta}^c := \tilde{\theta}^b - [\frac{\partial}{\partial\theta}\tilde{P}(\tilde{\theta}^b,y)]^{-1}\frac{\partial}{\partial y}\tilde{P}(\tilde{\theta}^b,y)\xi \tag{108}$$

Note that [19] if $\sigma^b := A^T\tilde{\theta}^b$, then

$$\begin{aligned} \frac{\partial\tilde{P}}{\partial\theta}(\tilde{\theta}^b,y) &= A[y]A^T \in \mathbb{R}^{N\times N} \\ \frac{\partial\tilde{P}}{\partial y}(\tilde{\theta}^b,y) &= A[\sigma^b] \in \mathbb{R}^{N\times\ell} \end{aligned} \tag{109}$$

The main result of this section, Theorem 3, gives a bound on $\|\tilde{\theta}^c-\bar{\theta}^c\|_\infty$, the error between the linearized approximation and the exact solution to the contingent DC load flow.

We make the following assumption.

(A4). The contingent network is connected and all the components of $y + \xi$ are non-negative.

<u>Remark</u>. In Assumption A4, the word "connected" has the same meaning as in Assumption A1, i.e., we ignore shunts. □

Define $J_p' \in \mathbb{R}^{N\times N}$ by

$$J_p' := A[y+\xi]A^T \tag{110}$$

so that $\forall\ \theta \in \mathbb{R}^N$

$$\tilde{P}(\theta,y+\xi) = J_p'\theta \tag{111}$$

Under Assumption A4, J_p' is nonsingular. Define $z_p' \in \mathbb{R}_+$ by

$$z_p' := \max\ \{e_k^T(J_p')^{-1}e_k \mid k=1,\ldots,N\} \tag{112}$$

<u>Remark</u>. It is obviously undesirable to calculate $(J_p')^{-1}$ since this amounts to solving the contingent DC load flow. However, it is a simple matter to use the Shemon-Morrison-Woodbury formula [14, p. 50] (also known as the Matrix line Inversion lemma or Housholder's formula) to calculate the diagonal entries of $(J_p')^{-1}$ based on a few elements of J_p^{-1} without further matrix inversion [20]. The required entries of J_p^{-1} are those which can be found using the approach of [21]. Also there is no need to calculate z_p' exactly as any upper bound will suffice and simple

upper bounds, not involving matrix inversions, can be found. □

Define $K \in \mathbb{R}^{N \times \ell}$ by

$$K := - J_p^{-1} A[\sigma^b] \tag{113}$$

so that $\bar{\theta}^c = \tilde{\theta}^b + K\xi$. Note that J_p is nonsingular under Assumptions A1 and A2.

Theorem 3

Let $\tilde{\theta}^b$ be the solution of the base case DC load flow equations

$$\tilde{P}(\tilde{\theta}^b, y) = \hat{P} \tag{114}$$

and suppose that assumptions A1, A2 and A4 hold.

1. Existence. The DC load flow equations for the contingent case

$$\tilde{P}(\tilde{\theta}^c, y+\xi) = \hat{P} \tag{115}$$

have a unique solution $\tilde{\theta}^c \in \mathbb{R}^N$.

2. Bound. Let $\bar{\theta}^c$ be obtained by the sensitivity method (Eq. (108)), then

$$\|\tilde{\theta}^c - \bar{\theta}^c\|_\infty \leq z_p' \, \|A[\xi]A^T K\xi\|_1 \tag{116}$$

where z_p' and K are defined in Eqs. (112) and (113) respectively. □

Proof

Define $h : \mathbb{R}^N \to \mathbb{R}^N$ by

$$h(\mu) := \tilde{P}(\bar{\theta}^c + \mu, y+\xi) - \tilde{P}(\tilde{\theta}^b, y) \tag{117}$$

It is sufficient to show that h has a zero in $\overline{B}_\infty(0,\beta)$ where

$$\beta := z_p' \| A[\xi] A^T K \xi \|_1 \tag{118}$$

This in turn is implied by [14, 6.3.4, p. 163]

$$\mu^T h(\mu) \geq 0 \qquad \forall \; \|\mu\|_\infty = \beta \tag{119}$$

Observe that

$$\mu^T h(\mu) = \mu^T \{\tilde{P}(\overline{\theta}^c + \mu, y+\xi) - \tilde{P}(\overline{\theta}^c, y+\xi)\}$$

$$+ \mu^T \{\tilde{P}(\tilde{\theta}^b + K\xi, y+\xi) - \tilde{P}(\tilde{\theta}^b, y)\} \tag{120}$$

Examining the first term in Eq. (120),

$$\mu^T \{\tilde{P}(\overline{\theta}^c + \mu, y+\xi) - \tilde{P}(\overline{\theta}^c, y+\xi)\}$$

$$= \mu^T J_p' \mu$$

$$\geq \frac{1}{z_p'} \|\mu\|_\infty^2 = \frac{\beta^2}{z_p'} \tag{121}$$

where the inequality follows from Lemma A in the Appendix. Similarly for the second term

$$|\mu^T \{\tilde{P}(\tilde{\theta}^b + K\xi, y+\xi) - \tilde{P}(\tilde{\theta}^b, y)\}|$$

$$\leq \|\mu\|_\infty \; \|A[y+\xi] A^T (\tilde{\theta}^b + K\xi) - A[y] A^T \tilde{\theta}^b\|_1$$

$$= \beta \|A[\xi] A^T K \xi\|_1 \tag{122}$$

$$= \frac{\beta^2}{z_p'} \tag{123}$$

where (122) follows from (113). Equations (121) and (123) establish (119) through (120). □

6. REACTIVE POWER CONTINGENCY

In this section, the dependency of the magnitude of bus voltages on transmission line susceptances is examined using the decoupled power flow model. Existence results and bounds similar to those in the previous section are denied, the application again being contingency analysis.

Suppose the transmission system has $\tilde{\ell}$ lines, including shunts and let $y \in \mathbb{R}^{\tilde{\ell}}$ be the vector of transmission line susceptances including shunts. Note that for reactive power considerations, shunts are significant in the decoupled model.

Let $\tilde{A}$ be the reduced bus incidence matrix of the entire network with the ground node taken as reference. Thus $\tilde{A} \in \mathbb{R}^{(N+1) x \ell}$. The columns of $\tilde{A}$ are assumed to be ordered compatibly with the ordering of $\tilde{y}$, and the rows are ordered so that rows 1 through N refer to buses 1 through N and row N+1 refers to bus 0 (the slack bus). Let

$$\tilde{A}_V := \text{first } N_Q \text{ rows of } \tilde{A}$$

$$\tilde{A}_R := \text{rows } N_Q+1 \text{ through } N+1 \text{ of } \tilde{A}$$

So

$$A = \begin{bmatrix} \tilde{A}_V \\ \hline \tilde{A}_R \end{bmatrix}$$

and

$$\tilde{A}_V \in \mathbb{R}^{N_Q x \tilde{\ell}}, \ \tilde{A}_V \in \mathbb{R}^{(N-N_Q+1) x \tilde{\ell}}$$

Let

$$V^R := (V_{N_Q+1}, \ldots, V_N, V_0)^T \in \mathbb{R}^{N-N_Q+1}$$

Using these definitions, the decoupled power flow expression for reactive power can be written as

$$\tilde{Q}(V,\tilde{y}) = [V]\tilde{A}[\tilde{y}](\tilde{A}_V^T V+\tilde{A}_R^T V^R) \tag{124}$$

where $\tilde{Q}(V,\tilde{y})$ is written for $\tilde{Q}(V)$. Here, the explicit dependence on $\tilde{y}$ is shown. Compare this to the original expression,

$$\tilde{Q}_k(V,y) = - \sum_{k=0}^{N} V_k V_i B_{ki} \tag{125}$$

where each B_{ki}, $k \neq i$ occurs in four places (i.e., B_{ki}, B_{ik} and as a component of B_{ii} and B_{kk}).

The reactive power contingency analysis problem is as follows. The base case is the system operating securely with reactive injections $\hat{Q} \in \mathbb{R}^{N_Q}$ and with transmission line susceptances $\tilde{y} \in \mathbb{R}^{\tilde{\ell}}$. Then, $\tilde{V}^b \in \mathbb{R}^{N_Q}$, the base case bus voltage magnitudes, is, according to the decoupled power flow model, the solution of

$$\tilde{Q}(\tilde{V}^b,\tilde{y}) = \hat{Q}. \tag{126}$$

Consider, as in Section 5, the contingency of $\tilde{y}$ changing to $\tilde{y} + \tilde{\xi}$, for some $\tilde{\xi} \in \mathbb{R}^{\tilde{\ell}}$ representing the loss or addition of any number of lines. One is interested in the solution of the contingent decoupled power flow model

$$\tilde{Q}(\tilde{V}^c,\tilde{y}+\tilde{\xi}) = \hat{Q} \tag{127}$$

Rather than solving Eq. (127), Eq. (126) is linearized in $\tilde{y}$ and the resulting linear equation can be solved to yield

$$\overline{V}^c = \tilde{V}^b - [\tfrac{\partial}{\partial V} \tilde{Q}(\tilde{V}^b,\tilde{y})]^{-1} \tfrac{\partial}{\partial \tilde{y}} \tilde{Q}(V^b,\tilde{y})\tilde{\xi} \tag{128}$$

$\overline{V}^c$ is then an approximation to the contingent bus voltages $\tilde{V}^c$. The main result of this section is to show that Eq. (127) has a solution and derive a bound on the difference between $\tilde{V}^c$ and $\overline{V}^c$. First, however, Eq. (124) is used to calculate the derivatives in Eq. (128).

Let $\sigma(V) := \tilde{A}_V^T V + \tilde{A}_R^T V^R \in \mathbb{R}^{\tilde{\ell}}$, so that

$$\tilde{Q}(V,\tilde{y}) = [V]\tilde{A}_V[\tilde{y}]\sigma(V)$$

$$= [V]\tilde{A}_V[\sigma(V)]\tilde{y} \tag{129}$$

Thus

$$\frac{\partial}{\partial \tilde{y}} \tilde{Q}(V,\tilde{y}) = [V]\tilde{A}_V[\sigma(V)]$$

$$\frac{\partial}{\partial V} \tilde{Q}(V,\tilde{y}) = [V]\Gamma(V,\tilde{y}) \tag{130}$$

where $\Gamma(V,\tilde{y}) \in \mathbb{R}^{N_Q \times N_Q}$ is defined by

$$[\Gamma(V,\tilde{y})]_{ki} = -B_{ki} \qquad k \neq i \quad k, i = 1, \ldots, N_Q$$

$$[\Gamma(V,\tilde{y})]_{kk} = -2B_{kk} - \sum_{\substack{i=0 \\ i \neq k}}^{N} \frac{V_i}{V_k} B_{ki} \qquad k = 1, \ldots, N_Q \tag{131}$$

Thus, from Eq. (128),

$$\overline{V}^c = \tilde{V}^b - [\Gamma(\tilde{V}^b,\tilde{y})]^{-1}\tilde{A}_V[\sigma(\tilde{V}^b)]\tilde{\xi} \tag{132}$$

The contingent network $\tilde{Q}$ relationship is now examined in the same fashion as the base case was treated in Section 2.4. Thus B is replaced by B', the bus susceptance matrix of the contingent case. Throughout the rest of this work, Assumption A4 is presumed to be in force.

Consider the network, N_Q', obtained by deleting all the PV buses, the slack bus and all the shunts from the contingent system. Note that N_Q' may have a different line to bus incidence structure than N_Q. N_Q' in general will not be connected and will have s' separate parts N_{Q1}', ..., $N_{Qs'}'$. We will assume throughout that buses have been numbered so that

the n_{Q1}' buses of N_{Q1}' are $\{1,\ldots,N_{Q1}'\}$

the n_{Q2}' buses of N'_{Q2} are $\{N_{Q1}'+1,\ldots,N_{Q2}'\}$

$\vdots$

the $n_{Qs'}'$ buses of $N_{Qs'}'$ are $\{N_{Q(s'-1)}'+1,\ldots,N_Q\}$

Remark. This may require that the buses be re-numbered for the contingent case. However, we require the re-numbering only for the computation of the error bound. □

Using the same reasoning as in Section 2.4, we see that $\tilde{Q}(\cdot,y+\xi)$, the contingent Q-V decoupled power flow relationship, further decouples into s' separate function and that its Jacobian is block diagonal.

We thus partition all relevant functions and variables according to which separate part they pertain. Note that this may be a different partitioning to the one used in Sections 2, 3 and 4, where the partition was performed on the base case separate parts. We refer to the partitioning used in this section as the contingent partitioning. It is as follows.

$$V := (V^1,\ldots,V^{s'})$$
$$\tilde{Q}(V,\tilde{y}+\tilde{\xi}) := (\tilde{Q}^1(V^1,\tilde{y}+\tilde{\xi}),\ldots,\tilde{Q}^{s'}(V^{s'},\tilde{y}+\tilde{\xi})) \tag{133}$$

where

$$V^j,\tilde{Q}^j(V^j,\tilde{y}+\tilde{\xi}) \in \mathbb{R}^{n_{Qj}'} \qquad \forall\ j = 1,\ldots,s'$$

For each contingent separate part N'_{Qj}, $j = 1,\ldots,s'$, the contingent associated network, N'^{e}_{Qj}, is defined using an identical procedure as is used to define the base associated entwork N^{e}_{Qj} from the base separate part N_{Qj} for $j = 1,\ldots,s$ in Section 2.4. That is, for each N'_{Qj}, a group of PV or slack buses M'_j is selected so that for each bus in M'_j there is a line (of non-zero susceptance) in the contingent network connecting it to some bus in N'_{Qj}. N'^{e}_{Qj} is then constructed by appending to N'_{Qj} a fictitious bus, f'_j, and, for each bus i in M'_j, a line from f'_j to bus k in N'_{Qj} of susceptance B'_{ki} where $B'_{ki} > 0$.

Define A'_{Qj} to be the reduced bus incidence of N'^{e}_{Qj} taking f'_j as reference and let $[y'_{Qj}]$ be the diagonal matrix of B'_{ki} of lines in N'^{e}_{Qj}, numbered so as to be consistent with A'_{Qj}. Define

$$B'_{Qj} := A'_{Qj}[y'_{Qj}](A'_{Qj})^T \in \mathbb{R}^{n'_{Qj} x n'_{Qj}}$$

and

$$z'_{qj} := \min\{e_k^T(B'_{Qj})^{-1}e_k \,|\, k=1,\ldots,n'_{Qj}\} \tag{134}$$

where $e_k = (0,\ldots,0,1,0,\ldots,0)^T \in \mathbb{R}^{n'_{Qj}}$ with the one in the k^{th} position.

We define the region $R'_V \subset \mathbb{R}^{N_Q}$, analogously to R_V by

$$R'_V := \{V \in \mathbb{R}^{N_Q} \,|\, V^{m'} \leqq V \leqq V^{M'}\} \tag{135}$$

and we make the following assumption.

(A5): $\forall\, V \in R'_V$, for each $j \in \{1,\ldots,s'\}$ and for each bus k in N'_{Qj}

$$-\sum_{i=0}^{N} V_k V_i B'_{ki} \geq V_k^2 \sum_{i \in N^{e'}_{Qj}} B'_{ki} \tag{136}$$

□

The following lemma is the contingent analog of Lemma 1.

Lemma 2

Under assumptions A4 and A5, $\frac{\partial}{\partial V}\tilde{Q}(V,\tilde{y}+\tilde{\xi})$ is nonsingular $\forall \quad V \in \mathcal{R}'_V$.

$\square$

Under Assumptions A1, A2 and A3, if $\tilde{V}^b \in \mathcal{R}_V$ then $\Gamma(\tilde{V}^b,\tilde{y})$ is nonsingular and we can define for $\xi \in \mathbb{R}^{\ell}$ $u(\tilde{\xi})$, $g(\tilde{\xi}) \in \mathbb{R}^{N_Q}$ by

$$u(\tilde{\xi}) := -\,[\Gamma(\tilde{V}^b,\tilde{y})]^{-1}\tilde{A}_V[\sigma(\tilde{V}^b)]\tilde{\xi} \tag{137}$$

$$\begin{aligned} g(\tilde{\xi}) := {} & [V^o+u(\tilde{\xi})]\tilde{A}_V[\tilde{\xi}]\tilde{A}_V^T u(\xi) \\ & + [u(\tilde{\xi})]\tilde{A}_V[\tilde{y}]\tilde{A}_V^T u(\tilde{\xi}) \\ & + [u(\tilde{\xi})]\tilde{A}_V[\tilde{\xi}]\sigma(\tilde{V}^b) \end{aligned} \tag{138}$$

Note that since $u(\tilde{\xi})$ is linear in $\tilde{\xi}$, $g(\tilde{\xi})$ is quadratic and cubic in $\tilde{\xi}$. Using the <u>contingent</u> partitioning

$$(g^1(\tilde{\xi}),\ldots,g^{s'}(\tilde{\xi})) := g(\tilde{\xi}) \tag{139}$$

where

$$g^j(\tilde{\xi}) \in \mathbb{R}^{n'_{Qj}} \qquad j = 1, \ldots, s'.$$

We then define $d_j(\tilde{\xi}) \in \mathbb{R}_+$ by

$$d_j(\tilde{\xi}) := \frac{z'_{qj}}{\min\{V_k^{m'} \mid k \in \mathcal{N}_{Qj}^{e'}\}} \, \|g^j(\tilde{\xi})\|_1 \tag{140}$$

Theorem 4

Let $\tilde{V}^b$ be the solution of the base case decoupled reactive power flow

$$\tilde{Q}(\tilde{V}^b,\tilde{y}) = \hat{Q} \tag{141}$$

Suppose that $\tilde{V}^b \in \mathcal{R}_V$ and Assumptions A1, A2, A3, A4 and A5 hold. Let

$\overline{V}^c$ be the sensitivity approximation to the contingent voltage magnitudes defined in Eq. (128) and let

$$(\overline{V}^{c1},\ldots,\overline{V}^{cs'}) := \overline{V}^c$$

where $\overline{V}^{cj} \in \mathbb{R}^{n'_{Qj}}$ be the contingent partitioning of $\overline{V}^c$. Also, suppose

$$\overline{B}_\infty(\overline{V}^{c1},d_1(\tilde{\xi}))\text{x}\ldots\text{x}\overline{B}_\infty(\overline{V}^{cs'},d_{s'}(\tilde{\xi})) \subset (R'_V)^o \tag{142}$$

Then

1. Existence. The decoupled reactive power flow equations

$$\tilde{Q}(\tilde{V}^c,\tilde{y}+\tilde{\xi}) = \hat{Q} \tag{143}$$

have a solution $\tilde{V}^c = (\tilde{V}^{c1},\ldots,\tilde{V}^{cs'})$ (contingent partitioning) in R'_V and

2. Bounds. For each $j \in \{1,\ldots s'\}$

$$\|\overline{V}^{cj}-\tilde{V}^{cj}\|_\infty \leq d_j(\tilde{\xi}) \tag{144}$$

where $d_j(\tilde{\xi})$ is defined in Eq. (140). □

Proof

Define the map $\Delta' : \mathbb{R}^{N_Q} \to \mathbb{R}^{N_Q}$ by

$$\Delta'(\mu) := \tilde{Q}(\overline{V}^c+\mu,y+\xi)-\tilde{Q}(\tilde{V}^b,y) \tag{145}$$

and the region $S'_V \subset \mathbb{R}^{N_Q}$ by

$$S'_V := \overline{B}_\infty(0,d_1(\xi)) \text{ x}\ldots\text{x } \overline{B}_\infty(0,d_{s'}(\xi)) \tag{146}$$

The proof then proceeds along identical lines to that of Theorem 2 (linearization analysis) with Δ replaced by Δ' and S_V replaced by S'_V.

Also, contingent partitioning is used instead of the base partitioning used in Theorem 2. □

7. CONTINGENCY CLASSIFICATION

The analytic results of this paper are now applied to the contingency analysis problem, which is central to steady-state security assessment [19,22] and to transmission realiability evaluation [16]. It is presented as an example of how approximate power system models can be used with these results to yield completely reliable qualitative information. First, however, current practices in contingency analysis are briefly reviewed.

The contingency analysis problem is to test whether the system in steady state can operate within security constraints for each case in a given list of contingencies (generator and line outages). To avoid costly computation of solving a power flow for each case, the automatic contingency selection method [7,8,9,10,11,13] has been proposed. In this approach, contingencies are first ranked according to a performance index, which is defined in such a way to reflect the deviations from the desired operating conditions. Power flows are then solved to test system security of each case starting from the top of the ranking and stopped when the case does not give problems. To evaluate the performance index for each contingency, either direct substitition from an approximate power flow solution (e.g., DC load flow) is employed [11] or linearization is used to evaluate the change in performance index from the base case [8]. A novel information theoretic approach to optimal selection of the performance index and threshold is given in [23].

In [11], it is shown that contingency selection using the adjoint network sensitivity method [8] can fail to capture all the insecure contingencies. The proposed remedy was to increase the accuracy of the model (in this case, DC load flow was advocated) and to use direct substitution for performance index evaluation. Several other modifications to this basic method aimed at improving reliability have been suggested [16]. However, there remains one possible cause of failure which is intrinsic to the contingency selection method. It is the effect of the error introduced by the case of approximations. The form of the performance index cannot overcome this problem: complete reliability is thus impossible with the contingency selection approach.

To overcome this, the following contingency classification scheme is proposed. For each contingency, one of the approximate models is solved. Using the results of this paper, it is then possible to classify the contingency into one of three categories: secure, insecure or "uncertain." Suppose the sufficient conditions for power flow solution existence are satisfied. Then secure classification occurs when the approximate solution and all solutions falling within the error bound (and thus the power flow solution) are secure. Similarly, a contingency is classified as insecure when the approximate solution and all solutions within the error bound are insecure. The "uncertain" classification is applied when either the sufficient conditions on the approximate model are not satisfied or when one possible solution falling within the error bound is secure and another is insecure. For "uncertain" contingencies, a more accurate model can be used to achieve classification as either secure or insecure. In the final instance, this may include the use of a full power flow.

To illustrate this idea, consider a contingency described by changing the bus addmittance matrix elements from $\{G_{ij}+jB_{ij}\}$ to $\{G'_{ij}+jB'_{ij}\}$. Let this be equivalent to changing y to $y + \xi$ (Section 5) and $\tilde{y}$ to $\tilde{y} + \tilde{\xi}$ (Section 6). Suppose that the contingent network N'_Q has been partitioned, according to the procedure described in Section 6 and that R'_V and R'_θ have been chosen as

$$R'_V := \{V \in \mathbb{R}^{N_Q} | V^{m'} \leqq V \leqq V^{M'}\} \tag{147}$$

$$R'_\theta := \{\theta \in \mathbb{R}^N | -\delta' \leqq (A')^T \theta \leqq \delta'\} \tag{148}$$

where A' is the reduced node incidence matrix of the contingent network, ignoring shunts and taking the slack bus as reference. Suppose also that Assumptions A1, A2, A3, A4 and A5 hold and that the base case decoupled power flow has a solution $\tilde{\theta}^b$, $\tilde{V}^b$. Then $\overline{\theta}^c$ and $\overline{V}^c$, the sensitivity approximations to voltage angles and magnitudes, can be calculated from equations (108) and (128) respectively. Let λ'_p, $\lambda'_{qj} \in \mathbb{R}_+$ $j = 1, \ldots, s'$ be defined by

$$\lambda'_p := z'_p(\|A[\xi]A^T K\xi\|_1+\varepsilon'_p) \tag{149}$$

$$\lambda'_{qj} := d_j(\tilde{\xi}) + z'_{qj}\varepsilon'_{qj} \tag{150}$$

where z'_p, z'_{qj} and $d_j(\tilde{\xi})$ are defined in equations (112), (134) and (140). The parameters ε'_p and ε'_{qj} can be found from equations (39) and (40) using contingent parameters (i.e. change δ_{ki}, V^m_k, V^M_k, B_{ki}, G_{ki}, N_{Qj} to δ'_{ki}, $V^{m'}_k$, $V^{M'}_k$, B'_{ki}, G'_{ki}, N'_{Qj} respectively). Let $(\overline{V}^{c1},\ldots,\overline{V}^{cs'}) = \overline{V}^c$ be the contingent partitioning of $\overline{V}^c$.

Then, by Theorems 3 and 4 and Theorem 1 applied to the contingent network, if

$$\overline{B}_\infty(\overline{\theta}^c,\lambda_p') \subset (R_\theta')^o \tag{151}$$

and

$$\overline{B}_\infty(\overline{V}^{c1},\lambda_{q1}') \times \ldots \times \overline{B}_\infty(\overline{V}^{cs'},\lambda_{qs'}') \subset (R_V')^o \tag{152}$$

then the <u>contingent</u> power flow equations have a solution $(\alpha^c, U^c) \in R_\theta' \times R_V'$ and

$$\|\alpha^c - \overline{\theta}^c\|_\infty \leq \lambda_p' \tag{153}$$

$$\|U^c - \overline{V}^c\|_\infty \leq \lambda_{qj}' \qquad j = 1, \ldots, s' \tag{154}$$

The contingency classification procedure is then applied as follows. Suppose the security constraints on the power flow solution are that $\theta \in R^s$, $V \in R_V^s$ [12].

1) If Eqs. (151) and (152) hold and

$$\overline{B}_\infty(\overline{\theta}^c,\lambda_p') \subset R_\theta^s \tag{155}$$

$$\overline{B}_\infty(\overline{V}^{c1},\lambda_{q1}') \times \ldots \times \overline{B}_\infty(\overline{V}^{cs'},\lambda_{qs'}') \subset R_V^s \tag{156}$$

then the contingency is classified <u>secure</u>.

2) If Eqs. (151) and (152) hold and

$$\overline{B}_\infty(\overline{\theta}^c,\lambda_p') \cap R_\theta^s = \phi \tag{157}$$

$$[\overline{B}_\infty(\overline{V}^{c1},\ {}_q') \times \ldots \times \overline{B}_\infty(\overline{V}^{cs'},\lambda_{qs'}')] \cap R_V^s = \phi \tag{158}$$

then the contingency is classified <u>insecure</u>.

3) If neither 1) or 2) above holds, then the contingency is classified <u>uncertain</u>.

If the uncertain classification occurs, then the contingent decoupled power flow can be solved for $\tilde{\theta}^c$ and $\tilde{V}^c$. Let $(\tilde{V}^{c1},\ldots,\tilde{V}^{cs'}) = \tilde{V}^c$ be the contingent partitioning of $\tilde{V}^c$. Let $\tilde{\lambda}_p := z_p'\varepsilon_p'$ and $\tilde{\lambda}_{qj} := z_{qj}'\varepsilon_{qj}'$, $j = 1, \ldots, s'$. Then from Theorem 1 applied to the contingent case, if

$$\overline{B}_\infty(\tilde{\theta}^c,\tilde{\lambda}_p) \subset (\mathcal{R}_\theta')^o \tag{159}$$

$$\overline{B}_\infty(\tilde{V}^{c1},\tilde{\lambda}_{q1}) \times \ldots \times \overline{B}_\infty(\tilde{V}^{cs'},\tilde{\lambda}_{qs'}) \subset (\mathcal{R}_V')^o \tag{160}$$

then the contingent power flow has a solution $(\alpha^c,U^c) \in \mathcal{R}_\theta' \times \mathcal{R}_V'$ and

$$\|\alpha^c-\tilde{\theta}^c\|_\infty \leq \tilde{\lambda}_p \tag{161}$$

$$\|U^{cj}-\tilde{V}^{cj}\|_\infty \leq \tilde{\lambda}_{qj} \qquad j = 1, \ldots, s' \tag{162}$$

Thus the contingency classification scheme could continue by repeating steps 1), 2) and 3) with $\overline{\theta}^c$, $\overline{V}^c$, λ_p' and λ_{qj}' replaced by $\tilde{\theta}^c$, $\tilde{V}^c$, $\tilde{\lambda}_p$ and $\tilde{\lambda}_{qj}$ respectively. For the remaining uncertain cases, a full power flow analysis is required.

The advantage of this scheme is that the results are exact. If a contingency is classified as secure or insecure then it is secure or insecure respectively. The use of the bounds extracts completely reliable information from the approximate solutions. The use of these results is not restricted to contingency analysis - in fact they should find application in any area where approximate power flow models are used. Similar results for the transient stability model and an application to dynamic security assessment can be found in [15].

8. CONCLUSIONS

In this paper, an analysis of various approximate power flow models was presented. The results include the following:

- the error bound between the solution of the DC load flow equations and the solution of the full power flow equations
- the error bound between the solution of the decoupled power flow equations and the solution of the full power flow equations
- the error bound between the solution using adjoint netowrk sensitivity method and the solution of the DC load flow equations

A sensitivity method for reactive power flow is proposed in Sec. 6. Similar error bounds are obtained for reactive power flow approximations.

A sample application to contingency analysis is presented to illustrate the use of these results. These results should be useful in the analysis of other applications where approximate models of power flow are used, for example, optimal power flow [24, 25, 26] and state estimation [27].

APPENDIX

In this appendix, the problem of finding

$$\min\{v^T A[y] A^T v \mid \|v\|_\infty = \beta\}$$

where $A \in \mathbb{R}^{n \times \ell}$ is the reduced node incidence matrix of some network, N, and $[y] \in \mathbb{R}^{\ell \times \ell}$ is a diagonal positive semidefinite matrix is examined. Noting that $G := A[y]A^T$ is the node conductance matrix of an ℓ-branch, n-node electric circuit, concepts from electrical circuit theory are employed.

Lemma A

Let $A \in \mathbb{R}^{n \times \ell}$ be the reduced node incidence matrix of an n-node ℓ-branch network, N and $[y] \in \mathbb{R}^{\ell \times \ell}$ be a diagonal positive semidefinite matrix. Let $G := A[y]A^T \in \mathbb{R}^{n \times n}$. With each branch, i, of N, we associate $[y]_{ii}$ as a conductance so that G is a node conductance matrix. Suppose the resulting electrical circuit is connected (i.e. between each node of N and the reference node there is a path of branches, each with non-zero $[y]_{ii}$). Then

$$\min\{v^T G v \mid \|v\|_\infty = \beta\} = \frac{\beta^2}{z}$$

where

$$z := \max\{[G^{-1}]_{kk} \mid k = 1, \ldots, n\} \qquad \square$$

Proof

Let $\alpha \in \mathbb{R}_+$, $k \in \{1, \ldots, n\}$ and consider the problem

$$P_k := \inf\{v^T G v \mid v_k = \alpha\}$$

Since G is strictly positive definite [17, p. 768], P_k has a unique global minimizer [28, p. 226]. From a simple application of Lagrange multiples, the minimizer of P_k is

$$v^k = \frac{\alpha}{z_k} G^{-1} e_k$$

where

$$z_k := [G^{-1}]_{kk}$$

$$e_k := (0,\ldots,0,1,0,\ldots,0)^T \in \mathbb{R}^n \text{ with the 1 in the k-th position.}$$

Consider the j-th component of v^k, v_j^k

$$|v_j^k| = \frac{|\alpha|}{z_k} |[G^{-1}]_{jk}|$$

$$\leq \frac{|\alpha|}{z_k} [G^{-1}]_{kk} = |\alpha|$$

where the inequality follows directly from the fact that the voltage gain of a circuit of strictly passive linear resistive circuit is less than unity [17, p. 777]. Thus $\|v^k\|_\infty \leq |\alpha|$ and v^k solves

$$P_k' = \min\{v^T G v \mid \|v\|_\infty \leq |\alpha|, v_k = \alpha\}$$

and by substitution $P_k' = \frac{\alpha^2}{z_k}$. The lemma follows. □

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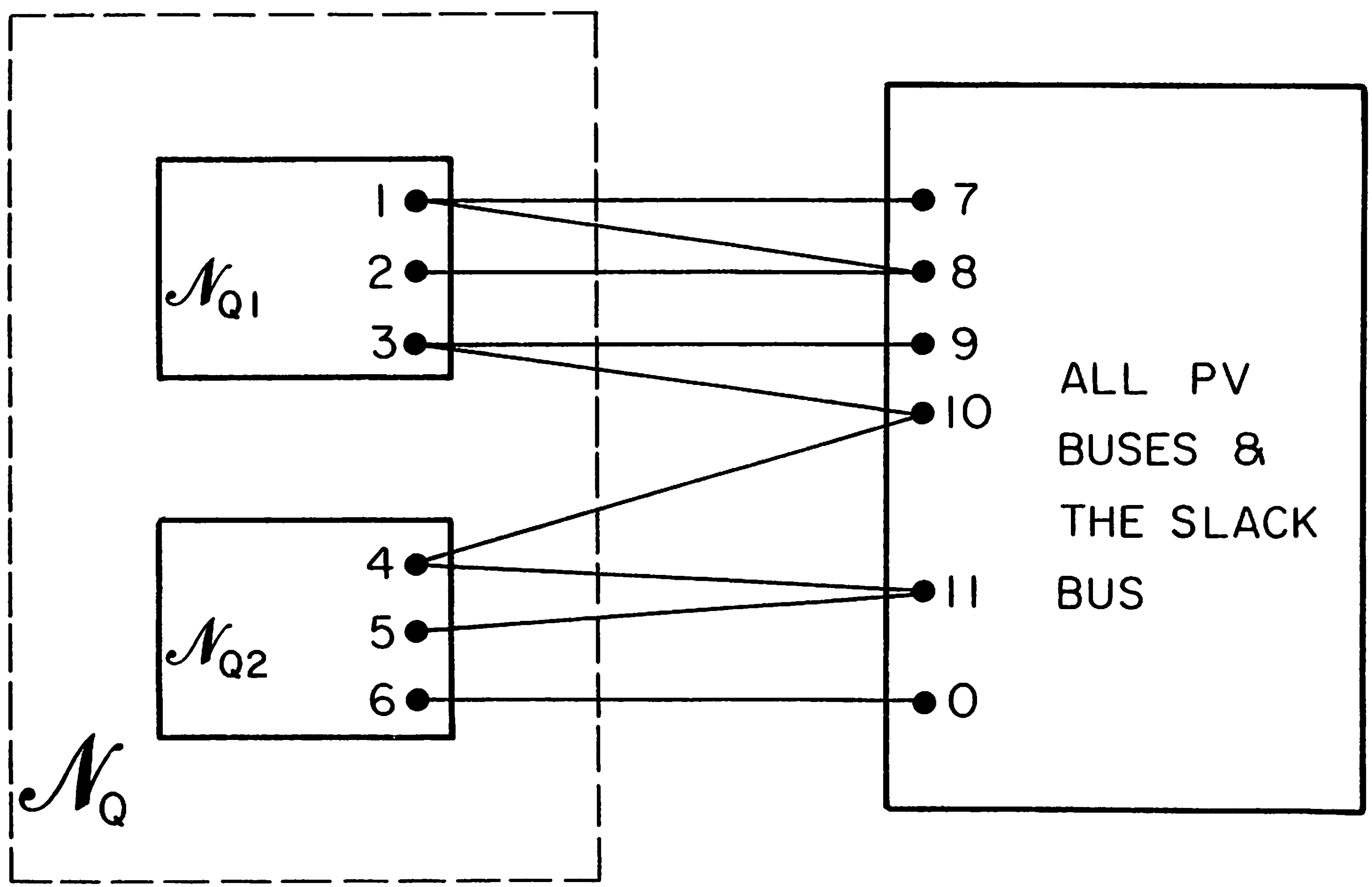
FIGURE CAPTION

Figure 1: Example of constructionof associated network.

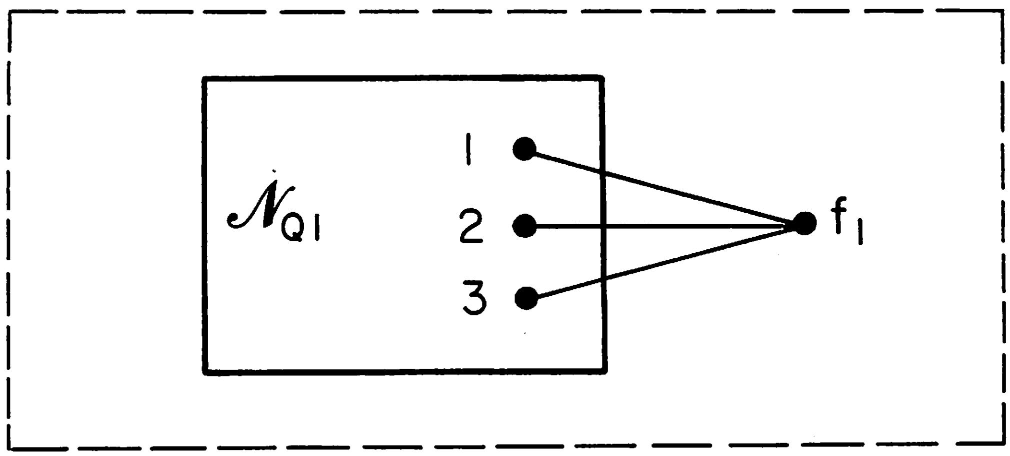
(a) Original network showing all buses but only those lines which connect a PV (or slack) bus to a PQ bus. Note that $N = 11$, $N_Q = 6$ and $n_{Q1} = 3$, $n_{Q2} = 3$, $s = 2$. M_1 can be taken as any non-empty subset of $\{7,8,9,10\}$ and M_2 as any non-empty subset of $\{10,11,0\}$.

(b) Possible construction of the associated network N^e_{Q1}. Suppose we choose $M_1 = \{7,8,9\}$ then the value of the lines $(1,f_1)$, $(2,f_1)$ and $(3,f_1)$ are $B_{1,7}$, $B_{2,8}$ and $B_{3,9}$ respectively. Other choices are possible.



$\mathcal{N}_{Q1}$
1
2
3
$\mathcal{N}_{Q2}$
4
5
6
$\mathcal{N}_Q$
7
8
9
10
11
0
ALL PV
BUSES &
THE SLACK
BUS

(a)



(b)