MEASURES OF DEVIANCE IN NON-CLASSICAL LOGICS

by

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INTRODUCTION

This is a summary of the talks given in a Panel Discussion Session of the 11th International Symposium on Multiple Valued Logic that took place in Oklahoma City, Oklahoma, U.S.A, May 27-29, 1981.

The title of the panel was: Measures of Deviance in Non Classical Logics, and the speakers were:

1st Prof. Ennc Trillas, from the Universitat Politècnica of Barcelona, Spain. The title of his talk was: "Measures of Fuzziness: Introductory Words."

2nd Dr. Teresa Riera, from the Universitat Politècnico of Barcelona, Spain. The title of her talk was: "Deviance Associated to a Special Kind of Variables."

3rd Prof. Ronald Yager, from Ione College of New York, U.S.A. The title of his talk was: "Measurement of Properties on Fuzzy Sets and Possibility Distributions."

4th Prof. Settimo Termini, from the Laboratorium di Gibernetica of Naples, Italy. The title of his talk was: "The Formalization of Vagueness Some (Epistemo) Logical Problems."

The panel was mainly prepared from the Computer Science Department of the University of California at Berkeley where the organizer Dr. Teresa Riera was a visitor the time before the symposium. Authors thank warmly Professor L.A. Zadeh (University of California, Berkeley) who has made possible that this summary come out.

The organizer thanks ISHVL-81 for including the session in the symposium. She is in debt with the speakers first for taking part in the panel and second for kindly writing their talks. She also thanks all people who attended the session, they were who asked for the talks

to be published as a first step to carry out a further debate which due to time shortage could not take place after the talks. They all together made it possible.

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MEASURES OF FUZZINESS: INTRODUCTORY WORDS

by

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1. I am not sure if a good way to start this Panel is to take a quick look through the part of the history of the subject in which I have been working; but the chairmen ask me for this and I will try to do it.

2. ONCE UPON A TIME there were two research fellows in Naples who were interested in both epistemological problems in Physics and new methodologies related to Cybernetics.

That is perhaps the reason why, after knowing Zadeh's 1968 paper "Probability Measures of Fuzzy Events" (Jour. Math. An. and App. 23, 421-427), they have got the idea to regard fuzzy sets from a thermodynamic-like point of view by comparing the internal disorder in statistical dynamics with the fuzziness.

These people were Aldo DeLuca and Settimo Termini and they were those who gave the definition of "entropy" (in the celebrated sense of DeLuca and Termini, of coursel) as a Measure of the Degree of Fuzziness for a fuzzy subset of a ground set X (the universe of discourse) defined by a mapping

$$D:\underset{\sim}{P}(X) \to R^{+}$$

satisfying the following three axioms:

(1) $D(A)=0$ if and only if A is a crisp subset of X, that is A P(X).

(2) $D(A)<D(1/2)$, for any $A \in \underset{\sim}{P}(X)$, being 1/2 the fuzzy set constantly equal to 1/2, that is $1/2(x)=1/2$ for any $x \in X$.

(3) D is increasing with respect to the SHARPENED PARTIAL ORDER $\leq_s$, given by

$$A \leq_s B \text{ iff } \begin{cases} A(x) \leq B(x), \text{ when } B(x) \leq 1/2 \\ A(x) \geq B(x), \text{ when } B(x) > 1/2 \end{cases}$$

that is $A \leq_s B$ implies $D(A) \leq D(B)$.

(4) $D(A) = D(\overline{A})$, for an $A \in \underset{\sim}{P}(X)$, being $\overline{A}$ the "complement" of A given by $\overline{A}(x) = 1 - A(x)$,

D is called a symmetrical entropy. Many people prefer to consider always entropies as symmetrical entropies.

In a certain way, entropy try to measure chaos due to the non- of the fuzzy set associated with an ill-defined class. In fact, this is an important problem still opened in the theory of Fuzzy Sets and that, in my opinion, needs to be carefully studied from several points of view.

It is to be remarked that if $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function such that $f(0) = 0$, and D any entropy then $f \circ D$ is also an entropy. For instance, being all $a_i > 0$, there are entropies: $a_0 + a_1 D + a_2 D^2 + \dots + a_n D^n$. In the same way $D/1+D$ is an entropy.

Analogously if $F : [0,1]^n \to [0,1]^n$ is a function strictly increasing with respect to the order $\leq_s$ (defined as above but in $[0,1]^n$) and such that verifies

$$F(\overset{n}{1/2,1/2,\dots,1/2}) = (\overset{n}{1/2,1/2,\dots,1/2})$$

$$F(\{0,1\}^n) = \{0,1\}^n$$

$$F([0,1]^n - \{0,1\}^n) = [0,1]^n - \{0,1\}^n,$$

then Dof is also an entropy. For example, functions defined by

$$f(t_1,t_2,\ldots,t_n) = (1-t_1,1-t_2,\ldots,1-t_n)$$

$$f_\sigma(t_1,t_2,\ldots,t_n) = (t_{\sigma(1)},t_{\sigma(2)},\ldots,t_{\sigma(n)}),$$ being $\sigma \in G_n$, are functions of this kind.

In this point it is convenient to remark that the four axioms of DeLuca and Tarmini do not characterize universally the entropy. In fact, there are a lot of different types of entropies to be choosen by the user (if they need to!). Actually in their early work (1972), entitled, "A definition of non-probabilistic entropy in the setting of fuzzy sets theory," Information and Control, 20(4), 301-312, DeLuca and Termini only use a particular D given by

$$D(A) = \sum_{i=1}^{n} [\mu(A(x_i))+\mu(1-A(x_i))],\ \mu(x) = -\,x \log_2 x,$$

being $X = \{x_1,x_2,\ldots,x_n\}$, which is very closed to the probabilistic entropy of Shannon.

3. It is necessary to remark that although the "logarithmic entropy" does not seem inadequate to convey a first idea of "equally fuzzy" it is clesely related to the interdependence properties of classical sets. For instance if the entropy of the product fuzzy set is required to be the sum of its factor entropies, we have a condition such that $\mu(x.y) = x\mu(y) + y\mu(x)$ on μ, which, under some suitable hypothesis of regularity, has a solution $\mu(x) = (\,cx\log x\,)$. On the other hand, entropies like this are related with the property of being "valuation" of the lattice $\underset{\sim}{P}(x)$ (in which union is pointwise defined through the function max and intersection through min), and one can

easily prove that:

"The only entropies D that are valuations of the pointwise lattice $\underset{\sim}{P}(x)_1$ are those that have the form

$$D(A) = \sum_{i=1}^{n} d_i(A(x_i)),$$

being d_i functions from [0,1] into R^+ such that:

$d_i(t) = 0$ iff $t = 0$

d_i is strictly increasing in [0,1/2]

d_i is strictly decreasing in (1/2,1]

d_i attains its maximum in $t = 1/2$."

Such entropies are invariant under the group G_n if and only if $d_1 = d_2 = \ldots = d_i$ and the only symmetrical are those that satisfy $d_i(t) = d_i(1-t)$ for all i and any $t \in [0,1]$. In the 1979 paper by C. Alsina and myself, entitled "Sur les mesures du degré de flou," Stochastica, III-1, 81-84, we also study entropies that are valuations in "sharpened" lattices $Sh(A) = \{B \in \underset{\sim}{P}(x);\ B \leq_S A\}$.

If we call $\overline{Sh(A)} = \{B \in \underset{\sim}{P}(x);\ \bar{B} \in Sh(A)\}$, then, as it is proved in the 1978 paper by T. Riera and myself, entitled "Entropies in Finite Fuzzy Sets," Information Sciences, 15, 159-168, for any $A \in \underset{\sim}{P}(x) - Sh(1/2) \cup \overline{Sh(1/2)}$ there exists a unique $\sigma_A \in G_r$, $\sigma_A(1,2,\ldots,n) = (i_1,\ldots,i_p,i_{p+1},\ldots,i_n)$, being $p = \#A^{-1}([0,1/2]) < n$, such that

$$A(x_{i_1}) \leq \ldots \leq A(x_{i_p}) \leq 1/2 < A(x_{i_{p+1}}) \leq \ldots \leq A(x_{i_n}).$$

With such machinery we have the following theorem, which is longer to write than to prove, and that it is possible to be considered as a central result in the subject.

THEOREM

If we consider the famuly of functions:

a) $\theta : \mathbb{R}^+ \times \mathbb{R}^+$, strictly increasing and null in (0,0).

b) $\theta_1^p : [0,1/2]^p \quad \mathbb{R}^+$, strictly increasings and null in (0,...,0)

c) $\theta_2^{n-p} : (1/2,1]^{n-p} \quad \mathbb{R}^+$, strictly decreasings and null in (1,...,1)

satisfying

$$\theta(\theta_1^p(1/2,\ldots,1/2),\theta_2^{n-p}(1/2,\ldots,1/2)) \leq \theta_1^n(\underset{\sim}{1/2}),\ 1 \leq p \leq n-1$$

$$\theta_2^n(\underset{\sim}{1/2}) \leq \theta_1^n(\underset{\sim}{1/2}),$$

the function $D : \underset{\sim}{P}(X) \to \mathbb{R}^+$, defined by

$$D(A) = \begin{cases} \theta_1^n(A), \text{ if } A \in Sh(\underset{\sim}{1/2}) \\ \theta_2^n(A), \text{ if } A \in \overline{Sh(\underset{\sim}{1/2})} \\ \theta(\theta_1^p(A(x_{i_1}),\ldots,A(x_{i_p})),\theta_2^{n-p}(A(x_{i_{p+1}}),\ldots,A(x_{i_p})), \\ \qquad \text{if } A \in P(X) - Sh(\underset{\sim}{1/2}) \cup Sh(\underset{\sim}{1/2}). \end{cases}$$

is an entropy in the sense of DeLuca and Termini.

This theorem opens the way to find a great number of possible entropies which will be chosen depending on the required properties. But not any entropy is of this kind, for instance in the case of X having two elements, the function

$$f(x,y) = 1 - \sqrt{2}\sqrt{(x-1/2)^2+(y-1/2)^2}$$

generates

$$D_f(A) = f(A(x_1),A(x_2))$$

and D_f does not satisfy the conditions of the theorem and it is an entropy. The same happesn with some fuzzy-indices introduced by A. Kaufmann, and it was just this kind of functions which made me study entropies that come from norms. I did it in the 1979 paper by C. Sanchis and myself, entitled "Sobre entropies de conjuntos borrosos deducidas de metricas," (in Spanish), Estadistica Española, 82, 17-25. The results of this paper generalize the before-mentioned "indices of fuzziness," of Professor Kaufmann.

Among the entropies satisfying the previous theorem we have called algebraic-entropies (not a very good name, sure!) those such that

$$D(A) = (D(A^0),D(A^1)),$$

being

$$A^0(x) = \begin{cases} 0, \text{ if } A(x) \in [0,1/2] \\ A(x), \text{ otherwise} \end{cases}, \quad A^1(x) = \begin{cases} A(x), \text{ if } A(x) \in [0,1/2] \\ 1, \text{ otherwise}, \end{cases}$$

and among such algebraic-entropies the most "popular" are the so-called θ-* entropies, given by

$$D(A) = \underset{i=1}{\overset{n}{\theta}} [a_i * \phi(A(x_i))],$$

being

1) All $a_i > 0$

2) ($\mathbb{R}^+$,θ,$\leq$) a commutative ordered semigroup such that

θ(x,s) = 0 iff r = s = 0.

3) * an isotone operation for the pointwise order such that

x * s = 0 iff r = 0 or s = 0.

4) φ a function from [0,1] into $\mathbb{R}^+$ such that

φ(t) = 0 iff t ∈ {0,1}.

θ is non-decreasing in [0,1/2]

φ is non-increasing in (1/2,1]

φ attains a maximum at t = 1/2.

Particular cases are obtained by considering

θ	*	kind	name
+	·	$D(A) = \sum[a_i \cdot \phi(A(x_i))]$	sum-prod
+	∧		
∨	·		
∨	∧	$D(A) = \vee[a_i \wedge (A(x_i)]$	max-min

4. The case in which the ground set X is not finite was first studied by J. Knopfmacher in "on Measures of Fuzziness," Jour. Math Anal. Appl., 49 (1975), 529-534, generalizing sum-prod entropies by using Lebesque integral.

The case in which X is denumerable was studied by A. DeLuca and S. Termini in "On the convergence of entropy measures of a fuzzy set," Kybemeter, 6 (1977), 219-227, and also by E. Trillas and T. Riera in "Sobre entropies para los mbeonjnitos difusos de $\mathbb{N}$," (Spanish), to be published in the Proceedings of the "VI Jornadas Matematicas Hispano-Lusas," held in Aveiro (Portugal) in 1978.

In the 1979 paper by N. Batle and myself entitled, "Entropy and fuzzy integral," Jour. Math. Anal. Appl., 69, 469-474, it is intended to generalize the max-min entropies by using the so-called Sugeno's fuzzy integral.

A nice survey, with new ideas, about the subject can be found in "Entropy and Energy Measures of a Fuzzy Set," by A. DeLuca and S. Termini, published in Advances in Fuzzy Set Theory and Applications, North Holland, 1979, edited by M. M. Gupta, R. K. Ragade and R. R. Yager.

This is, more or less, the situation till now. I can tell you in advance that in brief it will appear in some new papers on the subject containing interesting contributions.

5. Perhaps some of your are wondering what is the connection between entropy measures for fuzzy sets and the title of this Panel. Well, there is a tentative answer.

In fact, entropy measures for fuzzy sets can be regarded as a first attempt to introduce something like a way to CONTROL BOOLEANITY, or something like a measure of derivance from booleanity. Probably the papers by R. R. Yager on fuzziness in lattices (to be published shortly) will show a first extension of the primitive idea.

On the other hand, if it is a class of objects p, q,... (propositions, for instance) having a DeMORGAN ALGEBRA structure, and it can be embedded in a certain $\underset{\sim}{P}(X)$, for some suitable X, then it is possible to define the "booleanity of $p \in A$" as a real number $D(P)$, being $\underline{P}$ the corresponding image of $\underset{\sim}{P}$ in $P(X)$ and D a convenient entropy in $\underset{\sim}{P}(X)$. This is what we are doing in the case in which it is exactly $\underset{\sim}{P}(X)$ and the objects are fuzzy sets.

Really it is not very easy to prove a representation theorem of DeMorgan Algebras by means of fuzzy sets, but a particular case can be found in the Ph.D. thesis by Luiz Monteiro, "Algebras de Łukasiewicz trivalentes monâdices," (Spanish), published in 1974, in Notas De Logica Matematica, No. 32, by the Instituto de Matematica, Universidad Nacional Del Sur, Bahia Blanca, Argentina. Also some initial results for the general case are expected to be found in a paper by N. Batle and J. Grane, entitled, "Ideals in the Algebra $\underset{\sim}{P}(X)$," to be published shortly in Stochastica.

6. Let me finish with some additional ideas related to the last point. Let $\xi_1, \xi_2, \ldots, \xi_n$ be proportional variables and $\wedge$, $\vee$, $>$,... logical convectives. As you know, well formed formulae (wff) are obtained from the ξ_i, $i = 1,\ldots,m$ together with the logical convectives following formation rules and it is designed by $Y_n = \phi(\xi_1,\ldots,\xi_m)$, where it is the number of proportional variables that actually take part on $\phi(\xi_1,\ldots,\xi_m)$. Clearly each ξ_i is a wff and so are $\xi_i \vee \xi_j$, $\xi_i \wedge \xi_j$, $> \xi_i$,... etc.

If it is possible to take [0,1] to be the set of truth values of the wff, and if

$$v^{(j)}(\xi_i) = x_{ij}$$

is a truth value for ξ_i, in order to know the truth values of a wff Y_n for every assignment $v^{(j)}(\xi_i) \in [0,1]$ $(1 \leq i \leq \ldots, 1 \leq j \leq \ldots)$ of the variables that takes part in Y_n, we postulate the existence of a unique function $f_{Y_n} : [0,1]^u \to [0,1]$ such that, if

$$v^{(j_1)}(\xi_1),\ldots,v^{(j_n)}(\xi_n)$$

are truth values of the $\xi_1,\ldots,\xi_n$ appearing in Y_n, then

$$f_{Y_n}(v^{(j_1)}(\xi_1),\ldots,v^{(j_n)}(\xi_n))$$

is the truth value of Y_n. Indeed, f_{Y_n} depends on a finite number of variables (the variables that actually appear in Y_n) and it should be consistent with the two valued proportional calculus, that is, f_{Y_n} should contain the binary truth table of Y_n, as for example it happens when

$$f_{\xi_1 \vee \xi_2}(x_1,x_2) = \max\{x_1,x_2\}$$

$$f_{\xi_1 \wedge \xi_2}(x_1,x_2) = \min\{x_1,x_2\}$$

$$f_{\neg\xi}(x) = 1-x,$$

and with the equivalence $Y_n \equiv Z_p$ iff $f_{Y_n} = f_{Z_p}$ (which, as it is well known, implies $n = p$). In the quotient set we can consider the soft structure $\underset{\sim}{P}([0,1]^N)$ as we have announced in the 1-st paragraph.

So, each equivalence class (Y_n) of wff can be "represented" by a fuzzy set $f_{Y_n} : [0,1]^n \to [0,1]$, and the booleanity of the wff can be controlled by using suitable entropy measures in $[0,1]^n$. We can take advantage of the powerful structure of $[0,1]^n$ by using entropies derived from a distance or from a norm in $\mathbb{R}^n$.

7. With all this, here you have some ideas that I think it could be interesting to discuss.

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DEVIANCE ASSOCIATED TO A SPECIAL KIND OF VARIABLES

by

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1. I am going to talk about how we can use entropy measures as a first step to describe a certain kind of variables defined in a universe of discourse S in which we have previously distinguished a special family of fuzzy sets that we consider modelizes the fuzzy environment in which our problem takes place.

2. What I am going to explain can be regarded as continuation of the paper by E. Trillas and myself, entitled "On a special kind of variables in fuzzy environment," presented last year in Evanston and published in the ISMVL-80 proceedings 149-152.

In it we suggested to modelize the fuzzy environment through the concept of fuzzy algebra, that is through DeMorgan algebras closed by Watanabe transformations, and we showed how special real functions relative to such fuzzy algebras could be defined in S.

3. First of all let me recall briefly some of these concepts as they are basic to get our present aims.

3.1. Let $\underset{\sim}{P}(S)$ be the srt of all fuzzy sets on S. We call <u>Watanabe transformation</u> to any mapping $W : \underset{\sim}{P}(S) \to \underset{\sim}{P}(S)$ such that, for each $A \in \underset{\sim}{P}(S)$, satisfies:

(i) If $A(x) > \lambda$, then $WA(x) > \lambda$

(ii) If $A(x) < \lambda$, then $WA(x) < \lambda$

(iii) If $A(x) = \lambda$, then $WA(x) = \lambda$

being $x \in S$ and WA the image of A by W.

So, by one of such transformations a fuzzy set A is changed into another WA whose values are kept either above, below or in the symmetry level (given by the negation function* which defines de complement in $\tilde{P}(S)$) as they were in the fuzzy set A.

Actually, what is behind this definition is the idea also supported by Watanabe in his work (see: "Fuzzification and Invariance," Proc. Int. Conf. Cyber. Soc. 2, 947-951, Tokyo, 1978), that any algebra with fuzzy sets should be invariant by slight changes in their membership functions and in this sense, they have a special importance, quantities and properties invariant by any transformation by which, using Watanabe's words the "rather yes" is changed into the "rather yes," the "rather no" is changed into the "rather no" and by which the indeterminance is preserved.

In our before mentioned paper we established this concept in terms of equivalence classes by defining when two fuzzy sets, in this case A and WA, were "more or less fuzzy at the level λ" which is exactly the same idea.

3.2. Now, we will go quickly through the definition of fuzzy algebra.

As I said at the begining fuzzy algebras are DeMorgan algebras closed by Watanabe transformations, that is a <u>fuzzy algebra</u> A is a family of fuzzy sets on S satisfying the following conditions:

*See, "Sobre funciones de negaciónen en la teoria de Conjuntos Di fusos," by E. Trillas in <u>Stochastica</u>, III-1, 47-60 (1979).

(FA 1) If $A \in \mathcal{A}$, then $\bar{A} \in \mathcal{A}$ (being $\bar{A}$ the complement of A).

(FA 2) If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$ (and as a consequence $A \cap B \in \mathcal{A}$).

(FA 3) If $A \in \mathcal{A}$, then any $B \in W(A)$ is $B \in \mathcal{A}$ (being $W^{\lambda}(A)$ the set of all Watanabe-transformed from A with respect to λ).

To get more operativity we add:

(FA 4) If $A \in \mathcal{A}$, then $\underline{A} \in \mathcal{A}$ (being $\underline{A}$ the nearest crisp to A, that is the classical set with characteristic function defined for any $x \in S$ as $\underline{A}(x) = 1$ when $A(x) > \lambda$ and $A(x) = 0$ when $A(x) \leq \lambda$.

(FA 5) $\underset{\sim}{\lambda} \in \mathcal{A}$ (being $\underset{\sim}{\lambda}(x) =$ for any $x \in S$)

A fuzzy algebra modelizes the fuzzy environment in which our problem takes place in the sense that it contains all the fuzzy events that we can observe as far as our problem is concerned.

They have special importance the <u>fuzzy algebras generated by fuzzy partitions</u>. We say that a fuzzy partition $\mathcal{P}$ generates a fuzzy algebra $\mathcal{A}(\mathcal{P})$ if $\mathcal{A}(\mathcal{P})$ is the intersection of all fuzzy algebras that contain $\mathcal{P}$. $\mathcal{A}(\mathcal{P})$ is the smallest fuzzy algebra of $\underset{\sim}{\mathcal{P}}(S)$ containing $\mathcal{P}$.

3.3. We define a <u>fuzzy partition</u> $\mathcal{P}$ with respect to the level λ (λ-fuzzy partition) as a collection of fuzzy sets P_i satisfying the following axioms:

(FP 1) $P_i \neq \emptyset$, for all $P_i \in \mathcal{P}$.

(FP 2) For each $x \in S$ either there exist a unique P_i such that $P_i(x) > \lambda$ or $0 < \max\{P_i(x); P_i \in \mathcal{P}\} \leq \lambda$.

When $\mathcal{P}$ is finite and satisfies (FP 1) and (FP 2) we say that $\mathcal{P}$ is a finite fuzzy partition.

Let $\mathcal{P} = \{P_1,\ldots,P_n\}$ be a finite fuzzy partition. We consider $\underline{P}_i = \{x \in S; P_i(x) > \lambda\}$ and $P_i^{\lambda} = \{x \in S; P_i(x) = \lambda\}$ for each $P_i \in \mathcal{P}$, and $\hat{P} = \{x \in S;\ \max_{1 \leq i \leq n} P_i(x) < \lambda\}$. Let $I = \{1,\ldots n\}$ be a set of integers. For each subset $K \subset I$ such that $k \neq \emptyset$ we define a subset of S in the following way $T_K = \bigcap_{i \in k} P_i \cap \bigcap_{j \in K^c} \bar{P}_j$ being K^c the complementary set of K with respect to I.

It is immediately verified that:

$$Q\ \{\underline{P}_i\}_{i \in I} \cup \{T_K\}_{\substack{K \subset I \\ K \neq \emptyset}} \cup \{\hat{P}\}$$

is a classical partition of S. We say that Q is the classical partition induced by the fuzzy partition P in S.

3.4. We modelize also a fuzzy environment in the real line by defining the concept of Borel's fuzzy algebra as the fuzzy algebra $\mathcal{B}$ generated by the fuzzy intervals together with the fuzzy set $\underset{\sim}{\lambda}$. We consider a fuzzy interval as the generalization of the notion of classical interval obtained by admitting for the characteristic function all possible values between 0 and 1 except what corresponds to the symmetry level of the negation.

3.5. So, let $\langle a,b \rangle$ be an interval on the real line ($\langle a,b \rangle$ stands for either $[a,b]$, $(a,b]$,... or (a,∞)....). A fuzzy interval $I_{\langle a,b \rangle}$ is a fuzzy set on the real line such that: (i) If $\langle a,b \rangle = (a,b)$, then $I_{(a,b)}(x) < \lambda$ when $x \leq a$ or $x \geq b$ and $I_{(a,b)}(x) > \lambda$ when $a < x < b$. (ii) Similar definitions are given in other cases.

3.6. At this stage we can define such special real functions I told you at the beginning and that we call variables on the universe S relative to a fuzzy algebra A (its fuzzy environment) as functions $X : S \rightarrow \mathbb{R}$ such that $\tilde{X}^{-1} : \underset{\sim}{P}(\mathbb{R}) \rightarrow \underset{\sim}{P}(S)$ satisfies $\tilde{X}^{-1}(I) = I_0 X \in A$ for all fuzzy intervals $I \in \mathcal{B}$. That is we define them by the ordinary calculus of inverse images analogous to what is usually done by measurable functions but according to the theory of Zadeh. With adequate resort to the nearest crisp we proved that the ordinary arithmetical operations with such variables are still variables of the same kind. Obviously, when everything is limited to $P(S)$ (the set of crisp subsets of S) we obtain that our variables are ordinary random variables relative to boolean algebra of subsets of S.

The main point of our ISMVL-80 paper was that we managed to characterize such functions when the fuzzy algebras were generated by a finite fuzzy partition. The characterization theorem is the following one:

3.7. A function $X : S \to \mathbb{R}$ is a variable relative to $A(\mathcal{P})$ if and only if

$$X = \sum_{Q_i \in Q} X_i \cdot \phi_{Q_i}$$

where X_i are real numbers and $Q = \{Q_i\}$ is the classical partition induced by $\mathcal{P}$ in S.

That is X is a variable relative to $A(\mathcal{P})$ if and only if X is constant in each set of Q, in other words <u>the unique variables</u> with respect to fuzzy algebras generated by finite fuzzy partitions <u>are</u> the <u>random variables</u> with respect to a boolean algebra generated by the classical partition induced by the fuzzy partition $\mathcal{P}$ previously considered.

Now our interest has been driven to manage to describe such functions (such variables) by means of certain parameters. In this sense goes the results I am going to present now.

4. We need, first of all, a measure to evaluate the fuzzy events

of our fuzzy environment. We give it by the following definition:

An evaluation of possibility m defined in a fuzzy algebra A is a function $M : A \to \mathbb{R}^+$ such that:

(EP 1) If $A, B \in A$, $A \subseteq B$ then $m(A) \leq m(B)$.

(EP 2) If $A_n \underset{\circ}{\uparrow} A$, $A_n \in A$ for all n and $A \in A$ then $\lim_{n\to\infty} m(A_n) = m(A)$.

(EP 3) $B \in W^{\lambda}(A)$, then $m(B) = m(A)$

An evaluation of possibility is called θ-additive if

(EP 4) There exists an operation $\theta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that, for any pair $A, B \in A$ of incompatible elements ($A \cap B \subset \underset{\sim}{\lambda}$), satisfies $m(A \cup B) = \theta(m(A), m(B))$.

If θ-additivity is satisfied, it is not difficult to prove that θ should be associative, commulative, non-decreasing, with null element $m(\phi)$ and $\theta \geq \max$.

I would like to say that:

- Probabilities are evaluations of possibility θ-additive with respect to $\theta = +$.

- Fuzzy measures of M. Sugeno introduced in: "Theory of fuzzy integrals and its applications," Ph.D. Thesis, Tokyo Institute of Technology, Tokyo, 1974, or evaluations of possibility and fuzzy additivity is particular case of θ-additivity ($\theta = \vee$).

-Possibility measures of L. A. Zadeh introduced in, "Fuzzy sets as a basis for a theory of possibility," Fuzzy Sets and Systems, 1, 3-28 (1977) and used later by H. T. Nguyen in "On conditional possibility distributions," Fuzzy Sets and Systems, 1, 299-309 (1978) are evaluations of possibility Max-additive.

- Scales of Nahmias defined in "Fuzzy variables," Fuzzy Sets and

Systems, 1, 97-110 (1978) are evaluations of possibility Max-additive.

5. Now, let m be an evaluation of possibility defined in a fuzzy algebra $A(P)$ generated by a fuzzy partition P. Then, to each variable X relative to the fuzzy environment $A(P)$ a real number can be assigned by the following definition:

We call Confidence C(X) of a variable X to the real number given by

$$C(X) = \underset{i=1}{\overset{n}{\theta}}\ \alpha_i * X_i$$

being $\alpha_i = m(Q_i)$, $Q_i \in Q$ where Q is the classical partition induced by P in S, X_i the value of X in each Q_i and $*$ any isotone operation in $\mathbb{R}^+$ such that $x * y = 0$ if and only if $x = 0$ or $y = 0$.

When X is a random variable, m a probability, $\theta = \Sigma$ and $* = \cdot$, c(x) is the mathematical expectation of X.

With the only restriction of considering variables with only positive values, such parameter c(x) comes to be a θ-$*$ entropy in the sense of DeLuca-Termini (of those that Prof. Trillas has just mentioned) as you can see in the following theorem:

Let m be an evaluation of possibility defined in a fuzzy algebra $A(P)$ generated by a finite fuzzy partition P and X a variable relative to $A(P)$ such that $X(x) \in \mathbb{R}^+$ for any $x \in S$. Then, the confidence c(x) of X is a θ-$*$-entropy in the sense of DeLuca-Termini depending on X.

Proof. Let X_i be the value that X takes in $Q_i \in Q$ (Q is the classical partition induced by P in S), and $T : Q \to [0,1]$ one of the fuzzy sets that can be defined on the set Q satisfying the following conditions:

(T1) $T(Q_i) = 0$ if and only if $X_i = 0$.

(T2) $T(Q_i) = 1/2$ if and only if X_i maximum.

(T3) $0 < T(Q_i) \leq T(Q_j) < 1/2$ or $1/2 < T(Q_j) \leq T(Q_i) < 1$ if and only if $X_i \leq X_j$.

(T4) $T(Q_i) = T(Q_j)$ if and only if $X_i = X_j$.

Let RanT be the image set of T. RanT is a subset of [0,1] such that $\lambda \in$ RanT and it contains either 0 or 1 if and only if there exist $Q_i \in Q$ such that $X_i = 0$. Obviously T is well defined because of (T4).

In RanT we define a function ϕ : RanT $\to \mathbb{R}^+$ by $\phi(r_i) = X_i$, being $r_i = T(Q_i)$ and X_i the value of X in Q_i. Such ϕ satisfies:

(i) $\phi(r) = 0$ if and only if $r \in \{0,1\}$.

(ii) $\phi(r)$ is maximum if and only if $r = 1/2$.

(iii) If $r \leq r' < 1/2$ then $\phi(r) \leq \phi(r')$. If $1/2 < r \leq r'$ then $\phi(r) \leq \phi(r')$.

Let $\hat{\phi} : [0,1] \to \mathbb{R}^+$ be an extension of ϕ to the whole interval [0,1] according to the following rules:

1. If $0 \notin$ RanT, then $\hat{\phi}(0) = 0$.

   If $1 \notin$ RanT, then $\hat{\phi}(1) = 1$

2. If $r \notin$ RanT, $0 < r < 1/2$, let us consider $r', r'' \in$ RanT with $r' < r < r''$ such that between r' and r'' there are no other points of RanT . Then we define $\hat{\phi}(r') < \hat{\phi}(r) < \hat{\phi}(r'')$. In the same way if $1/2 < r < 1$ we define $\hat{\phi}(r') > \hat{\phi}(r) > \hat{\phi}(r'')$

3. If $r \in$ RanT, we define $\hat{\phi}(r) = \phi(r)$.

These are the proper rules for properties (i), (ii) and (iii) to be preserved, so $\hat{\phi}$ is an N-function and $c(x) = \underset{i=1}{\overset{n}{\theta}} \alpha_i * X_i$ $= \underset{i=1}{\overset{n}{\theta}} \alpha_i * \phi(T(Q_i)) = \underset{i=1}{\overset{n}{\theta}} \alpha_i * \hat{\phi}(T(Q_i)) = D(T)$ which means (according to what Professor Trillas has just explained) that c(x) is a θ-$*$ entropy of the fuzzy set T.

Of course there are many functions satisfying the conditions required to T. Even if we fix Tthere are many $\hat{\phi}$, but in that case there is only one ϕ.

In the random case the previous theorem says that if $X : S \quad \mathbb{R}^+$ is a random variable (with only positive values) then the mathematical expectation of X is a θ-$*$-entropy in the sense of DeLuca-Termini.

6. Of course, variables such as X are not completely described by c(x). It is an opened problem to find new parameters which bring more information about variables themselves. We are thinking that energy measures described by A. DeLuca and S. Termini in their paper entitled, "Entropy and energy measures of a fuzzy set" (already quoted by Professor Trillas) can be a possible tool but this work remains still uncompleted. I ask you for suggestions, comments and criticisms on the whole work and particularly on this point. Thank you very much.

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MEASUREMENT OF PROPERTIES ON FUZZY SETS AND POSSIBILITY DISTRIBUTIONS

by

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INTRODUCTION

In [1] Zadeh introduced the concept of a fuzzy subset as a generalization of the idea of a subset. The main distinction being that an element can belong to a fuzzy subset with a partial degree of membership, any number in the unit interval as opposed to the situation for sets where the membership grades are selected from the set {0,1}.

Subsequent to the introduction of this new idea many different applications and corresponding interpretations of the idea of membership have been introduced.

One can use fuzzy sets to represent concepts, i.e., linguistic values. In this application we represent a concept by a fuzzy subset over some set of elements in which the membership grade can be interpreted as the degree to which a particular element satisfies the concept. More specifically the membership grade can be seen to be the truth of the assertion that the element has the property of the fuzzy set we are defining. In this application a matter of concern relates to the question of how well we distinguish between elements having the property and those not having the property.

Alternatively this can be seen as a question of the degree to which the concept is a binary or crisp concept. The converse of this idea is the degree of fuzziness of the concept. It is related to the satisfaction of the law of the excluded middle. A property which is fuzzy leads to complications in problems involving the determination of how many elements have the property, the cardinality of the set. The measure of fuzziness effects the degree to which you can confidently measure the cardinality of a set.

In [2] Zadeh introduced the idea of a possibility distribution derived from a fuzzy set via the possibility assignment equation.

In particular let V be a variable taking values in the set X and let A be a linguistic value represented as a fuzzy subset of X.

A proposition p is a statement of the form

P: V = A

The possibility assignment equation associated with the variable V a possibility distribution Π_V such that

$$\Pi_V : X \to [0,1]$$

in which

$$\Pi_V(x) = A(x),$$

with the understanding that $\Pi_V(x)$ is the possibility that the value of V is x under the knowledge that V = A.

The possibility distribution is a reflection of the uncertainty associated with the variable.

In discussing possibility distributions a matter of interest involves the degree to which the possibility distribution points to one element of the set X as being the manifestation of the variable V. This is an indication of the specificity of the distribution. This idea measures the information contained in the possibility distribution. It appears to be the analog in possibility theory of Shannon's concept of entropy for probability distributions.

On a Measure of Specificity

Let X be a finite set of elements and let Π be a possibility distribution associated with this set, i.e.,

$$\Pi : X \to [0,1] \ .$$

We shall here introduce a measure of specificity associated with this possibility distribution which can be taken in many regards as an analog to Shannons concept of entropy associated with probability distributions. Further details in regard to this new concept can be found in [3].

Let

$$\Pi_\alpha = \{x | \Pi(x) \geq \alpha, \ x \in \Pi\}$$

that is Π_α is the set of elements having membership of at least α, let card Π_α be the number of elements in Π_α, finally let α max be the highest value of possibility associated with the possibility distribution function Π, then the specificity associated with this distribution is defined to be

$$S(\Pi,X) = \int_0^{\alpha_{max}} \frac{1}{\text{Card } \Pi_\alpha} \, d\alpha$$

<u>Example</u>: Let X = {X1,X2,X3,X4,X5} and let Π be defined as

$\Pi(x_1)$ = .3, $\Pi(x_2$ = .7, $\Pi(x_3)$ = .8, $\Pi(x_4)$ = 1, $\Pi(x_5)$ = 1. Then

$0 \leq \alpha \leq .3$	$\Pi_\alpha = \{x_1,x_2,x_3,x_4,x_5\}$	$\text{card}\,\Pi_\alpha = 5$
$.3 \leq \alpha \leq .7$	$\Pi_\alpha = \{x_2,x_3,x_4,x_5\}$	$\text{card}\,\Pi_\alpha = 4$
$.7 \leq \alpha \leq .8$	$\Pi_\alpha = \{x_3,x_4,x_5\}$	$\text{card}\,\Pi_\alpha = 3$
$.8 \leq \alpha \leq 1$	$\Pi_\alpha = \{x_4,x_5\}$	$\text{card}\,\Pi_\alpha = 2$

$$S(\Pi,\alpha) = \int_0^1 \frac{1}{\text{Rand}}\, \Pi_\alpha\, d_\alpha$$

$$= (\tfrac{1}{5})(.3) + (\tfrac{1}{4})(.4) + (\tfrac{1}{3})(.1) + (\tfrac{1}{2})(.2) = .294$$

The following properties are satisfied by this measure

1. For all Π

   $S(\Pi,X) \in [0,1]$

2. $S(\Pi,X) = 0$ iff $\Pi(x) = 0$ for all $x \in X$

3. $S(\Pi,X) = 1$ iff $\exists\ x \in X$ such that

   $\Pi(x) = 1$ and $\Pi(y) = 0$ for all $y \neq X$.

4. If Π_1 and Π_2 are two possibility distributions defined over X such that

   1) $\exists\ x_1 \in X$ such that $\Pi_1(x_1) = 1$

   2) $\exists\ x_2 \in Z$ such that $\Pi_2(x_2) = 1$

   3) $\Pi_1(x) \leq \Pi_2(x)$ for all $x \in X$

then

$S(\Pi_1,X) \geq S(\Pi_2,X)$.

Psuedo Metrics From Co-Norms

Definition: A t-norm T is a mapping

$$T : [0,1] \times [0,1] \to [0,1]$$

s.t.

1. $T(0,0) = 0$

   $T(a,1) = a$

2. $T(a,b) = T(b,a)$

3. $T(a,b) \leq T(c,d)$ for $c \geq a$ and $d \geq b$

4. $T(a,T(b,c)) = T(T(a,b),C)$

A t-conorm S is a mapping from [0,1] x [0,1] into [0,1] which satisfies conditions 2, 3 and 4 of the above and in addition satisfies

1'. $S(1,1) = 1$

   $S(o,a) = a$

The idea of these norms was introduced into the theory of fuzzy sets by DuBois and Prade [4], Klement [5] and Alsina, Trillas and Valverde [6].

Definition: Assume F is the class of all fuzzy subsets of the set X, we define a family of mappings

$$Gs : F \to [0,1]$$

such that for each $A \in F$

$$Gs(A) - \underset{x \in X}{S} [A(x)]$$

where S is a t-conorm.

Definition: Again assuming F is the family of all fuzzy subsets of X we define the mapping DS: $F \times F \to [0,1]$

such that for each $A, B \in F$

$$Ds(A,B) = |Gs(A)-Gs(B)|.$$

Theorem: Ds is psuedo metric, i.e.

$Ds(A,A) = 0$

$Ds(A,B) \geq 0$

$Ds(A,B) = Ds(B,A)$

$Ds(A,C) \leq D(A,B) + Ds(B,C)$

Note: $Ds(A,\emptyset) = Gs(B)$

In Ref. [7] Yager has introduced a family of t norms and associated t-conorms, these are

$$Tp(x,y) = 1 - Min[1,((C1-x)^p+(1-y)^p)^{1/p}]$$

$$Sp(x,y) = Min(1,(x^p+y^p)^{1/p}).$$

Conormed Based Measures of Fuzziness

In [8] Yager has suggested that the idea of fuzziness associated with a fuzzy set can be related to the distance between the intersection of a set A and its negation $\bar{A}$, i.e. $\bar{A}(x) = 1 - A(x)$, and the null set. Since the t-norm provides a general class of intersection operations and the family Ds provides a general family of psuedo metrics this suggests a potential family of measure of fuzziness as

$$F_{\Gamma,S}(A) = Ds(A \underset{T}{\cap} \bar{A}, \Phi) = Gs(A \underset{T}{\cap} \bar{A})$$

$$= \underset{x \in Z}{S}\,[T(A(x),1-A(x))].$$

A prototypical example of this family occurs when S = max and T = min in this case

$$F_{\wedge,\vee}(A) = \underset{x \in X}{Max}\ Min[A(x),(1-A(x))].$$

Note for all T, S and A

$$F_{T,S}\ 7(A) = F_{T,S}\ (\bar{A})$$

In Ref. [9] DeLuca and Termini suggest an axiomatic definition for any measure of fuzziness. Assume F is a measure of fuzziness and A is a fuzzy subset of X then they suggest that

1. $F(A) = 0$ iff A is crisp, i.e. for all $x \in X\cdot$ $A(x) \in \{0,1\}$.

2. If A and B are two fuzzy subsets of X such that

   when $A(x) \geq 1/2$ then $A(x) \leq B(x)$

   when $A(x) \leq 1/2$ then $A(x) \geq B(x)$

   i.e. B is crisper than A,

then

$$F(A) \geq F(B).$$

3. If B is the fuzzy subset of X defined by $B(x) = 1/2$ for all $x \in X$ then

$$F(B) > F(A).$$

It is our purpose here to investigate which of the members of our conjectured family $F_{T,S}$ satisfy the DeLuca-Termini conditions.

Before preceeding we must make some definitions and observations about T norms.

__Definition__: A T norm is said to be nilpotent if there exists some $\cap$ such that $T(a1,a2,...,an) = 0$ for $a_i \in (0,1)$.

__Note__: If T is non-nilpotent then $T(a,(1-a)) \neq 0$ for $a \in (0,1)$.

__Definition__: A t-norm T is said to be regular under complement for any $a,b \in I$

$a \wedge (1-a) \geq b \vee (1-b) \Rightarrow T(a,(1-a)) \geq T(b,(1-b))$.

Theorem

For all S and any nonnilpotent T, $T_{T,S}(A) = 0$ iff A is crips.

Proof: 1. If A is crisp then for any T,

$T(A(x),1-A(x)) = 0$ for all $x \in X$ and hence

$$\underset{x \in Z}{S}[0] = 0$$

2. If A is non-crisp there exists some $x \in X$ such that $A(x)$, $1 - A(x) \in (0,1)$. For non-nilpotent T-norms $T(A(x),(1-A(x))) = B(x) > 0$. Since for all conorms S, $\underset{x \in X}{S}(B(x)) = 0$ iff $B(x) = 0$ for all x our theorem is proved.

For the second of the DeLuca-Termini conditions we see the following theorem applies.

Theorem

Assume T is a t-norm which is regular under complement then $F_{T,S}$ satisfies Termini and DeLuca's second condition for all S.

Proof: Let A and B be two fuzzy subsets such that when $A(x) \geq 1/2$ then $B(x) \geq A(x)$ and when $A(x)$ _ 1/2 then $B(x)$ _ $A(x)$. Under this condition

$$A(x) \wedge (1-A(x)) \geq B(x)(1-B(x))$$

for all $x \in X$. If T is regular under complement then $T(A(x), (1-A(x))$ $\geq T(B(x),(1-B(x))$ for all $x \in X$. From the monotonicity property of S it follows that $F_{T,S}(A) \geq F_{T,S}(B)$.

The satisfaction of the third condition of DeLuca and Termini is more complicated. We first prove a theorem for a weaker versionof this third condition.

Theorem

If B is the fuzzy subset of X defined by $B(x) = 1/2$ for all $x \in X$ and if T is regular under complement than for all S and $A \in F$

$$F_{T,S}(B) \geq F_{T,S}(A).$$

Proof: For all $x \in X$, $(1/2,1/2) \geq A(x) \wedge (1-a(x))$, hence

$$T(B(x),(1-B(x))) \geq T(A(x),(1-A(x)))$$

for all x and the result follows from the monotinicity of S.

A thoerem showing some conditions under which the strong version of the DeLuca Termini conditions hold has been proven by Yager in [10]. We here state the theorem without proof.

Theorem

Assume X is finite set of n elements and A and B are two fuzzy subsets of X such that $A(x) = 1/2$ for all $x \in X$ and $B \neq A$. Let T be the product norm, a.b. Then $F_{T,S_p}(A) > F_{T,S_p}(B)$ for any Sp in Yager's family such that

(1) if $n \geq 4$ then $p \in [\frac{\log_2}{2} n,\infty]$

(2) if $n < 4$ then $p \in [1,\infty]$.

REFERENCES

[1] Zadeh, L. A., "Fuzzy Sets," Information and Control 8, 338-353, 1965.

[2] Zadeh, L. A., "Fuzzy Sets as a Basis for a Theory of Possibility," Fuzzy Sets and Systems 1, 3-28, 1978.

[3] Yager, R. R., "Measuring Tranquility and Anxiety in Decision Making, An Application of Fuzzy Sets," Int. J. of General Systems (to appear).

[4] Dubois, D. and Prade, H., "A Class of Fuzzy Measures Based on Triangular norms," Int. J. of General Systems 8, 1981.

[5] Klement, E. P., "Fuzzy Sigma Algebras and Fuzzy Measures with Respect to T-norms," Round Table on Fuzzy Sets, Lyon, 1980.

[6] Alsina, C., Trillas, E. and Valverde, L., "On Non-distributive Logical Connectives for Fuzzy Set Theory," Busefal 2, 18-29, 1980.

[7] Yager, R. R., "On a General Class of Fuzzy Connectives," Fuzzy Sets and Systems 4, 235-242, 1980.

[8] Yager, R. R., "On the Measure of Fuzziness and Negation, Part I: Membership in the Unit Interval," Int. J. of General Systems 5, 221-229, 1979.

[9] DeLuca, A. and Termini, S., "A Definition of Nonprobabilistic Entropy in the Setting of Fuzzy Set Theory," Information and Control 20, 301-312, 1972.

[10] Yager, R. R., "Measures of Fuzziness Based on T-Norms," Tech Report #RRY 81-02, Machine Intelligence Institute, Iona College, New Rochelle, 1981.

THE FORMALIZATION OF VAGUENESS:
SOME (EPISTEMO) LOGICAL PROBLEMS*

by

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1. Introduction

The problem of the formalization of vague concepts has attracted more and more people in recent years and has also become - rightly in my opinion - one of the standard topics of this series of Symposia on Multiple-Valued Logics. This, in fact, is related to a general trend to a strong and vigorous increase of new ideas, theories and debates in many disciplinary subjects. This trend, in subjects near the ones in which we are interested, is both cause and effect of a renewal of interests in such things as the philosophical import and the possible applications of many-valued logics, the problem of imprecision in science, the possible and provided by the techniques of modern formal logic to some problems of the empirical sciences, and so on.

For the topics which we are concerned with, the catalyst of all this has been - as it is well known to all of us - Lotfi Zadeh, and we all owe very much to him not only for his pioneering works but also for his constant activity aimed at the defence and diffusion of these ideas. His merit, however, is not limited to this. It rests also in the fact

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that as a by-product of the debate caused by the spread of these new questions, we all have been obliged to look back and rediscover a huge number of problems already present in our scientific and cultural tradition, and look at them from a new point of view and with the hope of being able to deal with them in a unified manner. From this perspective the role played by Zadeh's work is similar to that played by Kuhn's, The Structure of Scientific Revolutions, in birth of what Brown [1] calls the "new philosophy of science."

Kuhn presented the epistemological thesis on the development of scientific theories elaborated in The Structure of Scientific Revolutions as the result of his personal experience and specific work as a historian of science. The professional epistemologists observed that in order to appreciate fully the conceptions outlined there, besides comparing them with Popper's theories, one had to remember what, for instance, Hanson, Agassi, Wartofsky had already stressed; and before them Collingwood in his Essay on Metaphysics.

This new trend had been primarily brought about by some new questions in the history of science, the answers to which - though being of direct interest to this discipline - involved also an assessment of many old problems. These, in turn, needed for their clarification other tools and work already done in the epistemology, and philosophy of science, and in the history of the philosophy of science.

What Gaines has celled the exponential growth in the interest in the theory of fuzzy sets is, I think, very similar to all this. After more than five years of exponential growth in this field some epistemological reflections are needed if we are to llok carefully and critically at the paths that we can follow.

The main reason why I spoke at the beginning of "the problem of the formalization of vague concepts" and not of "the theory of fuzzy sets" <u>tout court</u> is just based on my conviction that this theory acted also as a pointer towards a lot of already existing problems, theories and trends of research. It is now that we ought to try to look at them from a general perspective aiming at focussing and individuating some unifying points.

A general epistemological discussion on the problem of the presence of vagueness and imprecision in science and of the most fruitful ways of dealing with it involves, of course, also a re-examination of the foundations of the theory of fuzzy sets intended as a specific methematical theory, a need which has recently been stressed also by Goe Goguen [2]. This is, in my opinion, a positive fact. The raising of foundational and conceptual questions witnesses, in fact, strong vitality - the more so if the questions which are raised point to new trends that are present in contemporary philosophy of science.

What I shall do in the present talk is to argue - in a very brief manner - in favor of the thesis according to which:

<u>any theory of vagueness, in order to make fully explicit its problems and potentialities, has to develop a dialogue with classical sciences</u>.

This dialogue, however, is hampered by the claim that <u>the real world is fuzzy</u>, usually considered as one of the strongest justifications for the theory of fuzzy sets and shared also by some of the people working on the philosophical aspects of vaguesness (see, for instance, [3]).

In the following I shall briefly outline an epistemological view of vagueness that indicates how to found the theory of fuzzy sets without use of this ontological assumption (Section 2) and shall then outline the role of the measures of fuzziness in this setting (Section 3).

2. Vagueness, Fuzziness and Ontological Assumptions

A preliminary clarificationof the terms used in the title of the paper (or, at least, of the way in which those terms are used there) can perhaps help in understanding better the aims of the talk.

First of all, the term "vagueness" is used in the philosophical sense given to it by Russel and Black and not in the ordinary languages sense. It is then very similar to Zadeh's use of the general term "fuzziness," in that it is distinct from ambivalence and lack of specificity. There is, however, one reason why I think that the term "vagueness" has to be preferred at an informal level. The reason is that it is useful to have two distinct terms for indicating the explicandum and the explicatum of a given notion. Now, linguistically, the term "fuzziness" is too much related to the explicatum provided by the theory of fuzzy sets to be a neutral candidate for denoting the explicandum. The term "vagueness" presents also some disadvantages, for instance, its pejorative use in colloquial discourse. What I see in favor of its use is, however, apart from the lack of other available terms, its specific and technical use in philosophy which corresponds exactly to what we have in mind and need in this context.

Secondly, the word "(epistemo) logical" is used in order to point out and stress that, by dealing with the problem of vagueness and the ways of treating it with the rigour that is now customary in modern science, one is forced to take into consideration both logical (technical, formal, absolute, context free) and epistemological (conceptual, relative, context-dependent) aspects. This is, in my view, a very interesting and innovative circumstance especially as compared to such sterile attitudes towards science such as divisions into watertight compartments. Let me add, incidentally, that this is more positive the more one locally uses (and takes account of) the rigorous results and the methodological rigor of modern formal sciences.

As I have already said, a formal approach to vagueness can manifest all the potential richness and eepth only by means of and through a dialogue and comparison with the other sciences.

We have to acknowledge that a dialogue of this kind between classical theories and the theory of fuzzy sets has not been fully developed. In my opinion this dialogue has been hindered by the development of an ontological view that is radically different from the basic (naive) ontology of classical science. According to this view, widely shared in the fuzzy sets community, a good and solid foundation for the theory of fuzzy sets could be obtained only by admitting that vagueness is in the world - namely that it is the real world that is actually vague and fuzzy. It is obvious that this thesis provides an immediate foundation and justification for the theory of fuzzy sets.

However, once we take this thesis seriously (as we should) it can be seen to be not at all as innocent as it first seemed. It is only at a first level of approximation that it provides us with an easy foundation for the theory of fuzzy sets. If we take it seriously we are left with a lot of difficult additional problems intertwined with our theory, since the assumption that vagueness is in the world - which can be described as an essentialistic attitude - involves a deep and drastic change in perspective in our way of looking at the world forcing us to enrich it with a host of attributes (precision, imprecision, crispness, fuzziness) usually considered as pertaining to the realm of language and description.

On the other hand, if we succeed in showing that the role of the theory of fuzzy sets or of other formal theories of vagueness can be defended without the help of presuppositions different from the ones of "classical" science, then we are much more free to defend the actual results of these theories for what they are. Moreover, this perspective can also help us to appreciate more properly, and even to discover, the innovative points that have arisen and developed outside the classical approaches.

What I propose then is to interpret and develop all the theory of fuzzy sets - and, for what is of interest here, also any other theory of fuzziness and vagueness - without making any ontological committment regarding the existence of fuzzy entities.

Incidentally, I think that the notion of a real world that is essentially fuzzy and vague (as well as the one that the world is precise) come out from the (oldfashioned) idea that what science (or, in general, human knowledge) does is to represent reality as it is. (See what D. Bohm has written in this respect, for instance in [4]).

This point certainly deserves a more detailed treatment, both in itself and with respect to the following point. However it is impossible to go into this deeply here.

In my view the really interesting feature of present attempts at the formalizationof vagueness is that they - implicitly - break this representational tradition and present the challenging issue of providing rules for a correct functioning and handling of a certain language and theory in new domains (in which their validity begins to become unclear). The challenge is that to behave in that way <u>can be more rational</u> than trying to construct a model aiming primarily at <u>representing</u> correctly the "permanent" things in a considered piece of reality.

What can, in fact, happen in this last case is that giving a priority to "following the rules" for a <u>correct representation</u> one loses sight of the meaningful features of the particular piece of reality considered, so that the model comes out actually uncorrelated with these meaningful features. All that leads then to revising the notion of model; and first, therefore, to clarifying what a classical model is. This will be briefly outlined in the following section. In the remaining part of this section we shall clarify a little our thesis of "vagueness without ontological assumptions."

As a matter of fact, one can give a correct and satisfactory characterization of vagueness, independent from ontological assumptions, starting from reflections concerning language. Intuitively, one says that a predicate of a language L is vague if it is not possible to determine precisely its extension (i.e., a set, in the classical sense). Thus, "vague" is a (vague!) predicate whose extension is in turn a set of predicates, trying to capture a central feature of their relation to the domain of the intended model. This position, which stresses the epistemological aspects of vagueness, dates back essentially to the classical 1923 paper by Russell.

According to this epistemological view, the notions of vagueness (and of precision) capture a feature of the relation between a predicate and a domain of application, or of a notion and the context in which it is used. Therefore, there are no inherently inexact, fuzzy or vague notions (and equally, there are no inherently exact or precise notions). A predicate that is vague if applied to a certain domain might have a very precise application, with no doubtful cases, in another domain. This means also, of course, that predicates with a precise extension within a given domain - even those embodying numerical parameters - can be used vaguely. But this is not especially strange; it is what was done, for instance, during the initial stages of development of new theories of the physical world.[1] In these stages the normal practice of the scientists is that practice strongly characterized by J. L. Synge [5] as the "cuckoo process:" the embryo of the new theory is inserted into the body of the previous theory borrowing from the latter in an undogmatic way whichever notions can be useful. This process is likely to create discrepancies (and then vagueness), which subsequently the scientist tries to locate and then eliminate.[2] In both scientific practice and that of daily life what is really found is then both fuzzy

[1] According to the view expressed by Lakatos in his Proofs and Refutations, this is true not only of empirical sciences but also of mathematics.

[2] Let me observe that the previous point is strictly related to one of the most debated topics of contemporary philosophy of science, namely, "the commensurability of theories" and "the dynamics of science." I shall not enter into this tangle of difficult problems. However I wish to note that a discussion on the presence of vagueness in science acquires its full significance only in the setting of a dynamic view of science. Conversely the problems of philosophy of science previously mentioned cannot but benefit by having at their disposal detailed conceptual analysis of vague concepts and the elaboration of formal instruments for dealing with them.

and precise uses of the same notions, depending on the domains in which we utilize them. In this perspective then the distinction precise/ imprecise becomes a fluctuating one.

It is true that classically one tends to eliminate all eliminable vagueness; but this is a policy worth following also according to the new perspective proposed by the theory of fuzzy sets: the difference with the classical attitude being only (but it is an "only" that makes a difference) that we wish to have appropriate tools for dealing with vagueness when this is not eliminable in a certain specific situation. (And it is not important that in principle it could be, or, in the future, it actually will be eliminated).

From this perspective the distinction between the classical and fuzzy approaches is not that the first refers to a precise stuff while the second to a fuzzy stuff; indeed our ontological opinion towards the worlds simply have no relevance on the issue in question.

The difference lies in the attitude of the two approaches towards the possibility of elaborating meaningful instruments to treat the vagueness present in the theory, and towards the relevance of these tools for identifying and solving the central problems of the given theory - in all those cases in which as a matter of fact a certain vagueness is present.

3. A Role for the Theories of Measures of Fuzziness

One could rightly ask what is the relevanceof all this to the specific topic of this Special Session. My point of view is the following one.

The stated program of reinterpreting all the results of the theory of fuzzy sets without making any ontological assumption implicitly requires that when we tackle a certain problem through a fuzzy approach we have already tried to eliminate all eliminable vagueness. The fuzziness that we are concerned with is then really that bare unavoidable minimum which comes out either from the incompleteness of our informationand our knowledge of the subject or system under consideration or from the imperfect fit (discrepancy) of the (conceptual or mathematical) tools that are used and the domain in which they are used.

It seems natural then to ask how much vagueness remains in the model; and that is what a theory of measures of fuzziness is designed to answer.

Let us call classical a situation in which a complete information is, at least in principle, obtainable with respect to the parameters chosen asmeaningful in the theory and expressed in the language determined by the tools used to construct the model.

Asking to know how much residual fuzziness is in the fuzzy model corresponds explicitly to asking how distant our model is from a classical one. The classical model, in this context, then remains (and must remain) as an ideal at which we aim; and in all cases in which we cannot achieve it we have to measure how distant from it we are.

Theories of measures of fuzziness would then represent a bridge for passing from fuzzy theories to classical theories or at least for relating them; moreover they are an active basis for the coexistence of both kinds of thoeries.

Of course, for the previous thesis on the role assigned to the measures of fuzziness tohave more than exhortatory value it must be articulated better and preliminarily, it will need to rely on a more detailed analysis of the notion of the classical.

I shall not (and could not for reasons of brevity) provide this here. However besides stressing the importance that an analysis and a clearcut definition of the classical would have in the setting of an epistemological view of vagueness, I which to call the attention to one more point.

The theory of measures of fuzziness - as presented above in the setting of an epistemological view of vagueness - is charged with two burdens. First it has to show to measure the <u>fuzziness present in the model</u>. Secondly it has to measure <u>how much</u> the model departs from a classical one.

In a good theory of the measures of fuzziness these two quantities should turn out to be the same.

Let me observe that this demand is less innocent than it seems; I attribute to it a noteworthy importance since it does not usually hold in the current interpretations of the notions of <u>vague</u> and <u>classical</u>. For instance a stochastic model is considered classical with respect to the role of its defining parameters, but vagueness remains.

Conversely a model of quantum theory would be structurally non-classical without presenting, according to rhe usual interpretation, any vagueness.

I find particularly interesting and challenging our being naturally led to consider the "degree of non-classicity" of a model and the quantity of fuzziness or inexactness that it contains as two (interchangeable) aspects of one and the same problem. And it is noteworthy that this comes out just in an approach, like the one outlined above, that gives full citizenship to vagueness at each stage of the of the process of constructing and developing a model but asbstains from making any ontological assumption about (the nature of) the vagueness which is present.

Both traditionally and in a fuzzy approach rroted in an essentialistic attitude we would regard the two previous problems as different.

The theory of measures of fuzziness would (and could) then be just a formal tool allowing a not purely qualitative development of the epistemological thesis of vagueness: it could then strengthen the project of precipitating a confrontation and comparison with some classical sciences.

But are there really in the recent developments of some of these sciences elements that justify provoking such a confrontation? One could generally refer to all the epistemological problems raised by microphysics and quantum theory, but there exist moreover two specific examples which I want to mention. One is borrowed from the theory of automata and is due to von Neumann; the other is from the foundations of physics and is due to Wheeler. In my view they show a particularly strong relationship with the topic of measures of fuzziness.

Wheeler's program tries to work out a basic theory from which to derive the actual properties and manifestations of space-time. His main idea is that the geometry of space-time (as we can know it at a physical level) has to come out of an underlying structure (a "pregeometry"), which cannot itself be modelled by means of geometrical intuitions and extrapolations.

But if we want to go beyond the notions of space and time there is very little left that we cna use for our purpose. So it is very difficult to answer the question asked by Wheeler, "Out of what 'pregeometry' are the geometry of space and space-time are built?" [6].

The little left to which one could refer is elementary logic (in particular the calculus of propositions). And, in fact, the first proposal based on these concepts is ecpressed by Wheeler in the following terms: "make a statistical analysis of the calculus of propositions in the limit where the number of propositions is great and most of them are long. Ask if parameters force themselves on one's attention in this analysis, 1) analogous in some small measure to the temperature and entropy of statistical mechanics, but 2) so much more numerous and dynamic in character that they reproduce the continuum of everyday physics." However Wheeler disconsolately ends by saying that "a later analysis found nothing in mathematical logic supportive of this proposal."

The other example to which I want to draw your attention is von Neumann's attempt to construct a "logic of automata." Let me quote the following passages:

"Everybody who has worked in formal logic will confirm that it is one of the technically most refractory parts of mathematics. The reason for this is that it deals with rigid all-or-none concepts and has very little contact with the continuous concept of the real of of the complex number, that is with mathematical analysis." "(In the theory of automata) the operations of logic will have to be treated by procedures which allow exceptions (malfunctions) with low but non-zero probabilities. All of this will lead to theories which are much less rigidly of an all-or-none nature than past and present formal logic. They will be of a much less combinatorial and much more analytical character."

"This new system of formal logic will move closer to another discipline which has been little linked in the past with logic. This is thermodynamics, primarily in the form it was received from Boltzmann, and it is that part of theoretical physica which comes nearest in some of its aspects to manipulating and measuring information" [7].

Let me stress that what von Neumann says in the first quotation is very similar to what Wheeler regrets and the second quotation outlines what Wheeler would have been very glad to find already developed; the difference being, mainly, that von Neumann seems more optimistic about the feasibility of this project.

Let me stress finally that both the previous programs aim at something that is conceptually strictly related to the underlying core of the theory of fuzzy sets and, specifically, to what a fullfledged and fully developed theory of the measures of fuzziness would do.

It is then natural to presume that an understanding of the problems in the above fields can help in developing this last theory in the most promising directions. In the same way one can naturally presume that new formal developments in the theory of measures of fuzziness could be of the greatest use in a reanalysis of Wheeler's and von Neumann's approaches, which have been abandoned (partly) also of the lack of suitable formal instruments.

In conclusion, a program that aims at the construction of a quantitative calculus for dealing with situations of <u>incomplete description</u> and <u>information</u> is both epistemologically rooted in the best classical tradition and potentially useful for the actual problems of the hard sciences.

What both Wheeler and von Neumann are searching for respectively in the further development of physics and the toeory of automata is in my view very similar to some of the things that the theory of fuzzy sets is searching for and trying to develop, in that all are trying to construct something that takes as a starting point situations of incomplete knowledge.

The demand for the kind of tools that the fuzzy community wants to develop is then very strong also in those sciences that belong to the classical tradition.

So there are, in principle, objective, favorable conditions for establishing a fruitful dialogue between "classical" science and these new approaches. But in order that this dialogue actually begins one has to create the widest possible common background.

From this perspective, then, it is useful to have no (unnecessary) estrascientific assumptions that are not shared by both speakers.

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I do not know how much he will share the thesis outlined in the previous pages: what is certain, however, is that it came from taking very seriously his ideas while trying to reconcile them with my image of science.

REFERENCES

[1] H. I. Brown, Perception, Theory, Commitment, The University of Chicago Press, 1979.

[2] J. A. Goquen, "Fuzzy sets and the social nature of truth," in Advances in Fuzzy Sets Theory and Applications, Gupta, Ragade and Yager, Eds., North-Holland, 1979.

[3] B. Rolf, "A Theory of Vagueness," J. of Philosophical Logic, 9, pp. 315-325, 1980.

[4] D. Böhm, Fragmentation and Wholeness, The Van Leer Jerusalem Foundation, Jerusalem, 1976.

[5] J. L. Synge, Relativity: The Special Theory, North-Holland, 1965.

[6] J. A. Wheeler, "Pregeometry: Motivations and Prospects," in Quantum Theory and Gravitation, A. R. Marlow, Ed., Academic Press, 1980.

[7] J. Von Neumann, Collected Works, McMillan, New York, 1963.