# ON LOCATING UNSTABLE EQUILIBRIUM POINTS FOR POWER SYSTEM TRANSIENT STABILITY ANALYSIS

by

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## SUMMARY

In this work we have proposed two computationally efficient methods of locating approximate unstable equilibrium points (u.e.p.'s) of the structure preserving model [1] for transient stability analysis. The approximations employ the single machine infinite bus (SMIB) analogy and linearization of the power flow relation. In addition to these approximations, an exact three term Taylor expansion is used to obtain a quadratic bound on the error in topological Lyapunov energy in terms of errors in the approximate u.e.p.'s.

State variables for the structure preserving model are the vector $\underline{\omega}_1$ of generator angular velocities and the vector $\underline{\alpha}$ of complex bus voltage angles. Equilibrium points occur at $\underline{\omega}_1 = \underline{0}$, $\underline{\alpha} = \underline{\alpha}^e$, where $\underline{\alpha}^e$ satisfies the load flow relation $\underline{f}(\underline{\alpha}^e) = \underline{P}$. Hence u.e.p.'s are found by solving the load flow relation for a given vector of real power injections $\underline{P}$.

For the SMIB case, the vector relation $\underline{f}(\underline{\alpha}^e) = \underline{P}$ reduces to a scalar relation $b \sin \alpha^e = P$, with a stable equilibrium at $\alpha^s$ and an unstable equilibrium at $\alpha^u = \pi - \alpha^s$. Hence the u.e.p. may be viewed as a shift by $(\pi - 2\alpha^s)$ of the machine angle from its stable value. To extend this analogy to a multimachine network dynamically separating into subnetworks I and II, we define a vector $\underline{\mu}$ with components

$$\mu_k = \begin{cases} 1 & \text{if bus } k \in \text{subnetwork I} \\ 0 & \text{if bus } k \in \text{subnetwork II} \end{cases}$$

An approximate u.e.p. $\hat{\underline{\alpha}}^u$ is then found by shifting all bus angles in subnetwork I by a uniform quantity, i.e., $\hat{\underline{\alpha}}^u = \underline{\alpha}^s + (\pi - \beta)\underline{\mu}$. The scalar quantity $\beta$ is chosen to minimize $\|\underline{P} - \underline{f}(\hat{\underline{\alpha}}^u)\|^2$.

The second approximation proposed involves linearizing the load flow relation about an "initial guess" u.e.p. and inverting the resulting Jacobian matrix $J^u$. The computational burden of this inversion is reduced by showing that $J^u$ is equal to the Jacobian evaluated at the stable equilibrium, $J^s$, plus a perturbation matrix. The Inverse Matrix Modification Lemma [2] is then applied to express $[J^u]^{-1}$ as $[J^s]^{-1}$ plus a perturbation matrix. This is advantageous when calculating many u.e.p.'s, as $[J^s]^{-1}$ remains unchanged.

These approximations are used to find u.e.p.'s and evaluate the topological Lyapunov function for a simple four machine network. For the various u.e.p.'s examined, the approximations yield energy figures within 2% of exact values, which are consistent with the bounds obtained from our quadratic sensitivity analysis.

The methods proposed provide efficient calculation of approximate u.e.p.'s and their associated Lyapunov energies. We have also derived quadratic error bounds on the accuracy of these energy estimates in terms of the error in u.e.p. approximations. Together, these methods should allow use of the structure preserving model for rapid evaluation of transient stability regions for planning or security assessment.

## ACKNOWLEDGEMENT

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Abstract - This work analyzes several approximate methods for efficiently locating unstable equilibrium points (u.e.p.'s) of the structure preserving model for transient stability analysis as developed by Bergen and Hill [1]. The role of these u.e.p.'s in predicting the region of attraction for the post fault stable equilibrium points via Lyapunov techniques is discussed. The sensitivity of Lyapunov energy to error in the approximate u.e.p.'s is analyzed. The approximations are applied to a simple six bus, four generator system, with numerical results of the various approaches contrasted.

## I. INTRODUCTION

Among the problems to be faced in applying Lyapunov methods to power system transient stability analysis is that of estimating the domain of attraction for the post fault stable equilibrium point. For most direct analysis techniques proposed to date, this problem translates into one of efficiently calculating unstable equilibrium points (u.e.p.'s) of the post fault system state equations. Early efforts in this area sought to identify all system u.e.p.'s [2,3,4,5] and evaluate a scalar Lyapunov function V at each of these points to locate $V_{min}$, the minimum value of Lyapunov energy over this set. Under the assumptions used in constructing the Lyapunov function, the region of attraction for the stable post fault equilibrium point is guaranteed to contain the subregion defined by $\{\underline{x} : V(\underline{x}) < V_{min}\}$ [3,4]. The criterion that the initial state of the post fault system lies within this region thus offers a sufficient condition for the post fault system to return to a stable operating point.

Depending on the fault examined, this sufficient condition for stable operation may be very conservative [6]. More recent efforts have sought to eliminate the conservativeness of this criterion by using fault dependent methods of determining the region of attraction of the stable post fault equilibrium point [7,8]. However, most of these techniques (except the PEBS approach in [7]) still rely on evaluating V at one or more post fault u.e.p.'s.

Other researchers have proposed methods for efficient computation of V through approximate u.e.p.'s [9-12]. The goal of this paper is to apply u.e.p. approximation techniques specifically to the structure preserving model for transient stability analysis. The method proposed in [1] for finding Lyapunov energy is examined in terms of approximate u.e.p.'s, and two alternative methods are developed. All of these methods are shown to have a common basis: the single machine infinite bus (SMIB) analogy and linearization of the power flow relation. While the SMIB analogy has been widely used in previous research [9,10], the features of the structure preserving model facilitate its application in new ways.

The paper is organized in the following fashion. The mathematical background necessary for the structure preserving model and topological Lyapunov function is presented in Section II. Section III explains the concepts behind the SMIB and linearization approaches to finding approximate u.e.p.'s. Sections IV and V provide the mathematical development for the new approaches proposed in this paper. Sensitivity of the topological Lyapunov function to errors in approximate u.e.p.'s is analyzed in Section VI. Numerical results of the proposed approximations applied to a simple four machine example appear in Section VII, with conclusions in Section VIII.

## II. BACKGROUND: THE STRUCTURE PRESERVING MODEL AND TOPOLOGICAL LYAPUNOV FUNCTION

The structure preserving model has as its basic assumption a new representation for system loads. Real power loads are treated as affine functions of frequency, allowing load buses to be retained in the network model. Bus voltage magnitudes are assumed constant and transfer conductances of the transmission lines are ignored. The transmission network model is augmented to include generator transient reactances as lines, with internal generator buses explicitly displayed in the network. The reader is referred to the example of Section VII for an illustration of this augmented network model.

Consider an augmented network having m generators, $\ell$ lines, and n buses, with the slack bus numbered as n. Define an n-dimensional vector $\underline{\delta}$, where the components $\delta_i$ represent complex voltage angles at buses $i = 1,2,\ldots,n$. State variables for the structure preserving model are the (n-1)-dimensional vector $\underline{\alpha}$ of bus voltage angles referenced to the slack bus, with components

$$\alpha_i = \delta_i - \delta_n \quad i = 1,2,\ldots,(n-1) \tag{1}$$

and the m-dimensional vector $\underline{\omega}_1$ with components

$$\omega_j = \dot{\delta}_j \quad j = 1,2,\ldots,m \tag{2}$$

where j indexes generator buses. Following the development in [1], the equilibrium points of the structure preserving model occur at $\underline{\omega}_1 = \underline{0}$, $\underline{\alpha} = \underline{\alpha}^e$, $\underline{\alpha}^e$ satisfying

$$\underline{f}(\underline{\alpha}^e) = \underline{P} \tag{3}$$

where

$\underline{f}(\underline{\alpha}) = A\underline{g}(\underline{\sigma})$ = (n-1) dimensional vector load flow function

A = reduced incidence matrix for augmented transmission network treated as a directed graph.

$\underline{\sigma}$ $= A^T\underline{\alpha}$ = $\ell$-dimensional vector of transmission line voltage angles

$\underline{g}(\underline{\sigma}) = [b_1\sin\sigma_1, b_2\sin\sigma_2,\ldots,b_\ell\sin\sigma_\ell]^T$ = $\ell$-dimensional vector line flow function

$b_k = \frac{E_iE_j}{X_k}$, where line k connects bus i to bus j

A Lyapunov function for the structure preserving model is given by

$$V(\underline{\alpha},\underline{\omega}_1) = 1/2\sum_{k=1}^{m} M_{kk}\omega_k^2 + \sum_{k=1}^{\ell} b_k\int_{\sigma_k^s}^{\sigma_k}(\sin u - \sin\sigma_k^s)du \qquad (4)$$

where $\sigma_k^s$ and $\sigma_k$ are components of

$$\underline{\sigma}^s = A^T\underline{\alpha}^s,\ \underline{\sigma} = A^T\underline{\alpha}$$

$\underline{\alpha}^s$ = post fault stable equilibrium angle vector

At a u.e.p., the first summation term in (4) goes to zero. It is therefore useful to define

$$W(\underline{\sigma}) = \sum_{k=1}^{\ell} b_k\int_{\sigma_k^s}^{\sigma_k}(\sin u - \sin\sigma_k^s)du \qquad (5)$$

which is simply the second summation term of (4). Note that W is a function of the state vector $(\underline{\alpha},\underline{\omega}_1)$ only through $\underline{\sigma} = A^T\underline{\alpha}$.

In general, exact solution for $\underline{\alpha}^e$ satisfying $\underline{P} = \underline{f}(\underline{\alpha}^e)$ will require considerable computation; suitable approximations will be addressed in the later portions of this paper. However, for the special case of $\underline{P} = \underline{0}$ such solutions have a very simple form. By analogy with this $\underline{P} = \underline{0}$ case, the authors in [1] imply a relationship between u.e.p.'s and critical cutsets which will be further developed here.

Consider a cutset i separating the augmented network into subnetwork I and subnetwork II as pictured in Figure 1. For convenience, we will assume that all lines in the cutset have reference orientation directed from subnetwork I to subnetwork II. The set $C_i$ will consist of the indices of all lines in the cutset. Define the (n-1)-dimensional vector $\underline{\mu}$ with components.

$$\mu_k = \begin{cases} 1 & \text{if bus } k \in \text{subnetwork I} \\ 0 & \text{if bus } k \in \text{subnetwork II} \end{cases} \quad k = 1,\ldots,(n-1) \qquad (6)$$

In the network model, the bus voltage angle vectors $\underline{\alpha}_1^0 \triangleq \underline{\mu}\pi$ and $\underline{\alpha}_2^0 \triangleq -\underline{\mu}\pi$ then correspond to all buses in subnetwork I having angles of $\pi$ ($-\pi$ for $\underline{\alpha}_2^0$) and all buses in subnetwork II having angles of 0. It follows then that the corresponding line angle vectors $\underline{\sigma}_1^0$ and $\underline{\sigma}_2^0$ have components

$$\sigma_{k,1}^0 = [A^T\underline{\mu}\pi]_k = \begin{cases} \pi & \text{if } k \in C_i \\ 0 & \text{if } k \notin C_i \end{cases} \quad k = 1,\ldots,\ell \qquad (7)$$

$$\sigma_{k,2}^0 = [-A^T\underline{\mu}\pi]_k = \begin{cases} -\pi & \text{if } k \in C_i \\ 0 & \text{if } k \notin C_i \end{cases} \quad k = 1,\ldots,\ell \qquad (8)$$

Such angles clearly satisfy the sufficient condition for equilibrium; namely, that $\sin\sigma_{k,1}^0 = \sin\sigma_{k,2}^0 = 0$ for $k = 1,2,\ldots,\ell$, and as shown in [1] are in fact unstable. While this approach is not guaranteed to locate all u.e.p.'s, physical reasoning suggests that the u.e.p.'s associated with network cutsets correspond to dynamic separation of the network, and therefore are most relevant to transient stability analysis. In light of their role in system separation, we shall refer to these points as separating u.e.p.'s

This simple form of solution in the $\underline{P} = \underline{0}$ case will lend insight into the approximations to be examined in subsequent sections of this paper. In the general case of $\underline{P} \neq \underline{0}$, we still assume that two separating u.e.p.'s are associated with each network cutset. One such u.e.p. is assumed to be located at $\underline{\mu}\pi + \underline{\Delta}^+$, the other at $-\underline{\mu}\pi + \underline{\Delta}^-$, where $\underline{\Delta}^+$ and $\underline{\Delta}^-$ are correction terms determined by the particular approximation employed. Suitable assumptions[1] on the power flow equation $\underline{P} = A\underline{g}(A^T\underline{\alpha})$ guarantee the existence of these solutions for a set of injections $\underline{P}$ of sufficiently small magnitude.

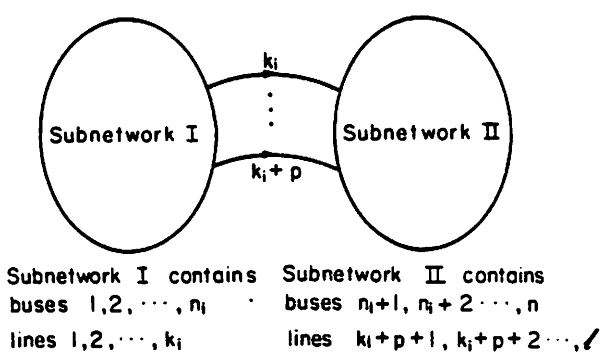


Figure 1. Network Partitioned by Cutset.

## III. U.E.P. APPROXIMATION CONCEPTS: SINGLE MACHINE INFINITE BUS ANALOGY AND LINEARIZATION

In this section, the basic concepts behind the approximation techniques developed in this paper will be analyzed. In addition, three basic forms of approximation will be proposed. These will be termed the line stretch, the single parameter stretch, and linearization approaches. The first two relate to the single machine infinite bus analogy. The third, as its name implies, relates to linearization of the power flow relation.

In the well known example of a single machine linked through a lossless transmission line to an infinite bus, the task of locating u.e.p.'s for arbitrary injections is straightforward. Suppose the stable line angle for a given injection P is $\sigma^s$. The unstable line angles closest to $\sigma^s$ are then $\sigma_1^u = \pi - \sigma^s$ and $\sigma_2^u = -\pi - \sigma^s$. These both correspond to dynamic separation of the machine from the infinite bus, which is the only possible mode of instability in this simple system. The use of the equal area criterion to investigate such separation is standard.

In the general case of a multimachine system, we wish to extend the concept of a single machine separating from the infinite bus to the case of a network dynamically separating into two approximately coherent subnetworks. Here the cutset of lines which divides the network will play the role of the one line in the single machine infinite bus (SMIB) case.

[1] As given by Inverse Function Theorem. See, for instance, pp. 125-126 [16].

The Lyapunov energy approximation developed in reference [1] applies this analogy directly to locate a vector of unstable equilibrium line angles. Consider a network separated by a directed cutset i as shown in Figure 1. As in Section II, all lines in the cutset are assumed to have reference directions oriented from subnetwork I to subnetwork II. For lines in the cutset, we follow the structure of the SMIB case line by line to yield approximations $\underline{\sigma}_1^u$ and $\underline{\sigma}_2^u$ with components $\sigma_{k,1}^u = \pi - \sigma_k^s$ and $\sigma_{k,2}^u = -\pi - \sigma_k^s$. For a cutset of lines with arbitrary orientation, these approximations may be expressed as

$$\underline{\sigma}_1^u \doteq A^T\underline{\mu}\pi + \underline{\Delta} \tag{9}$$

$$\underline{\sigma}_2^u = -A^T\underline{\mu}\pi + \underline{\Delta} \tag{10}$$

where

$$\Delta_k = \begin{cases} -\sigma_k^s & \text{if } k \in C_i \\ +\sigma_k^s & \text{if } k \notin C_i \end{cases} \quad k = 1,\dots,\ell$$

We shall refer to (9) and (10) as the line stretch approximations.

Substituting (9) and (10) into $W(\underline{\sigma})$ given in (5) yields the approximations for Lyapunov energy in [1], where the relevant value of energy for a particular cutset is taken to be the minimum of $W(\underline{\sigma}_1^u)$ and $W(\underline{\sigma}_2^u)$.

Several observations should be made about this approach. In (9) and (10) we have assumed that a free choice of $\underline{\sigma}_1^u$ and $\underline{\sigma}_2^u$ was possible. In fact any exact solution for an unstable equilibrium vector of line angles $\underline{\sigma}^u$ is constrained to satisfy $\underline{\sigma}^u = A^T\underline{\alpha}^u$. If one considers the nonlinear resistive circuit analogy for the structure preserving model described in [1], where $\sigma_k^u$ are analogous to branch voltages and $\alpha_k^u$ are analogous to node to datum voltages, this is equivalent to observing that an exact $\underline{\sigma}^u$ must satisfy Kirchoff's Voltage Law (KVL). It is clear that $\underline{\sigma}_1^u$ and $\underline{\sigma}_2^u$ may not meet this constraint. However, the fact that $\underline{P} = A\underline{g}(\underline{\sigma}_1^u) = A\underline{g}(\underline{\sigma}_2^u)$ implies that power flows (which are analogous to circuit current flows) resulting from these angles do satisfy Kirchoff's Current Law (KCL).

As an alternative, we may also apply the insight from the SMIB case in a slightly different fashion. Consider an approximation to a separating u.e.p. in which all bus angles in subnetwork I are shifted uniformly from their stable equilibrium values by the same quantity, and all bus angles in subnetwork II remain fixed at their stable equilibrium values. As in the previous approach, two u.e.p.'s are associated with each cutset in the network. Using the conventions outlined in Figure 1 and (6), these approximate unstable bus angles may be expressed as

$$\underline{\alpha}_1^u = \underline{\alpha}^s + \pi\underline{\mu} - 2\beta_1\underline{\mu} \tag{11}$$

$$\underline{\alpha}_2^u = \underline{\alpha}^s - \pi\underline{\mu} - 2\beta_2\underline{\mu} \tag{12}$$

Note that the precise form of the shift is chosen to facilitate computation which will be addressed in Section IV; since $\beta_1$ and $\beta_2$ are free parameters arbitrary shifts may be obtained. With this structure established, $\beta_1$ and $\beta_2$ will be chosen to approximately minimize the error in the power flow; i.e., to minimize

$$\|\underline{P} - A\underline{g}(A^T(\underline{\alpha}^s+\pi\underline{\mu}-2\beta_1\underline{\mu}))\|^2 \tag{13}$$

and

$$\|\underline{P} - A\underline{g}(A^T(\underline{\alpha}^s-\pi\underline{\mu}-2\beta_2\underline{\mu}))\|^2 \tag{14}$$

respectively. The details of this minimization are found in Section IV.

Returning to the nonlinear circuit analogy, we see that this approach yields solutions which satisfy KVL, simply because they are expressible in terms of bus angles (analogously node-to-datum voltages). The single parameter $\beta_i$ in each solution is then adjusted by the minimization process to provide a best fit to the KCL constraints imposed by the power flow relation $\underline{P} = A\underline{g}(A^T\underline{\alpha})$. This will be referred to as the single parameter stretch (SPS) approximation.

As an alternative to approaches drawn from the SMIB analogy, standard linearization techniques may be applied to the problem of locating approximate u.e.p.'s. Linearization approaches rely on a Taylor series expansion of the power flow relation $\underline{P} = A\underline{g}(A^T\underline{\alpha})$ about a suitable linearization point $\underline{\alpha}$. We first define the Jacobian of $\underline{g}(\underline{\sigma})$ as

$$J(\underline{\sigma}) = \frac{\partial}{\partial \underline{\sigma}}\,\underline{g}(\underline{\sigma}) = \mathrm{diag}[b_1\cos\sigma_1,\dots,b_\ell\cos\sigma_\ell] \tag{15}$$

The desired linearization is then given by

$$A\underline{g}(A^T\underline{\alpha}^u) = A\underline{g}(A^T\underline{\alpha}) + AJ(A^T\underline{\alpha})A^T[\underline{\alpha}^u-\underline{\alpha}] + \underline{o}(\underline{\alpha}^u-\underline{\alpha}) \tag{16}$$

where the $\underline{o}$ function represents higher order terms in the Taylor series. For $\|\underline{\alpha}^u - \underline{\alpha}\|$ small, we will assume that $\underline{o}(\underline{\alpha}^u-\underline{\alpha})$ is negligible, and define an approximation $\hat{\underline{\alpha}}^u$ satisfying

$$\underline{P} = A\underline{g}(A^T\underline{\alpha}) + AJ(A^T\underline{\alpha})A^T[\hat{\underline{\alpha}}^u-\underline{\alpha}] \tag{17}$$

Examination of equation (17) reveals that the expansion does not involve the linearization point $\underline{\alpha}$ directly, but only through the line angles $\underline{\sigma} = A^T\underline{\alpha}$. Therefore, the approximate unstable line angles located by the line stretch approximation may serve as linearization points. Denoting these angles $\underline{\sigma}_1^u$, and $\underline{\sigma}_2^u$, we recall that $A\underline{g}(\underline{\sigma}_1^u) = A\underline{g}(\underline{\sigma}_2^u) = \underline{P}$. Taking $\underline{\sigma}_1^u$ for example, (17) reduces to solving[2]

$$AJ(\underline{\sigma}_1^u)[A^T\hat{\underline{\alpha}}^u-\underline{\sigma}_1^u] = \underline{0} \tag{18}$$

Efficient computational techniques which take advantage of the network structure of this problem are examined in Section V.

## IV. COMPUTATIONAL CONSIDERATIONS: SINGLE PARAMETER STRETCH APPROACH

Consider a network with cutset i as pictured in Figure 1, with the accompanying vector $\underline{\mu}$ as defined in (6). As described in Section III, we seek parameters $\beta_1$ and $\beta_2$ which define u.e.p. approximations through (11) and (12). For initial development, we will focus on (11) and $\beta_1$. It will be demonstrated later that for a given cutset i $\beta_1 = \beta_2$, so only one parameter $\beta$ is

[2] We will assume that $AJ(\underline{\sigma}_1^u)A^T$ (which equals $AJ(\underline{\sigma}_2^u)A^T$) is nonsingular and well conditioned. Tests for these conditions are proposed in Section V.

calculated.

Ideally, $\beta$ must minimize the norm squared error in the power flow relation

$$\|\underline{P} - A\underline{g}(A^T(\underline{\alpha}^s+\underline{\mu}\pi-2\beta\underline{\mu}))\|^2 \tag{13}$$

where $\underline{\alpha}^s$ is the stable post fault bus angle vector. However, solving such a nonlinear minimization may be computationally intractable. Instead, we linearize g about $\underline{\sigma}_1^u$ to obtain

$$\|\underline{P} - A\underline{g}[A^T(\underline{\alpha}^s+\underline{\mu}\pi-2\beta\underline{\mu})]\|^2$$

$$\cong \|\underline{P} - A\underline{g}(\underline{\sigma}_1^u) - AJ(\underline{\sigma}_1^u)(A^T(\underline{\alpha}^s+\underline{\mu}\pi-2\beta\underline{\mu}) - \underline{\sigma}_1^u)\|^2 \tag{19}$$

Recalling that $A\underline{g}(\underline{\sigma}_1^u) = \underline{P}$, we may substitute into (19) to obtain a "minimization function"

$$O(\beta) = \|AJ(\underline{\sigma}_1^u)\underline{\sigma}_1^u - AJ(\underline{\sigma}_1^u)A^T(\underline{\alpha}^s+\underline{\mu}\pi-2\beta\underline{\mu})\|^2 \tag{20}$$

$$= \|AJ(\underline{\sigma}_1^u)[\underline{\sigma}_1^u - A^T(\underline{\alpha}^s+\underline{\mu}\pi-2\beta\underline{\mu})]\|^2$$

To confirm the earlier claim that $\beta_1 = \beta_2$, we observe that $J(\underline{\sigma}_1^u) = J(\underline{\sigma}_2^u)$. Thus the minimization function (20) is identical for $\beta_1$ and $\beta_2$, leading to the same value of $\beta$.

To simplify (20), we begin by defining the vector

$$\underline{v}_j = \text{jth column of the reduced incidence matrix } A \tag{21}$$

Note also that from (9) it follows

$$[\underline{\sigma}_1^u - A^T(\underline{\alpha}^s+\underline{\mu}\pi-2\beta\underline{\mu})]_k = \begin{cases} -2\sigma_k^s + 2\beta A^T\underline{\mu} & \text{for } k \in C_i \\ 0 & \text{for } k \notin C_i \end{cases} \tag{22}$$

Finally, by the definition of the diagonal Jacobian matrix $J(\underline{\sigma})$ in (15), we have

$$[J(\underline{\sigma}_1^u)]_{kk} = \begin{cases} -b_k \cos\sigma_k^s & \text{for } k \in C_i \\ +b_k \cos\sigma_k^s & \text{for } k \notin C_i \end{cases} \tag{23}$$

With these observations, (20) becomes

$$O(\beta) = \|2 \sum_{j\in C_i} b_j \cos\sigma_j^s \underline{v}_j \sigma_j^s - 2\beta \sum_{k\in C_i} b_k \cos\sigma_k^s \underline{v}_k \underline{v}_k^T \underline{\mu}\|^2 \tag{24}$$

Choosing $\beta$ to minimize $O(\beta)$ in (24) yields

$$\beta = \sum_{j\in C_i} a_j \sigma_j^s \tag{25}$$

where

$$a_j = \frac{[(b_j \cos\sigma_j^s \underline{v}_j)^T (\sum_{k\in C_i} b_k \cos\sigma_k^s \underline{v}_k \underline{v}_k^T \underline{\mu})]}{[(\sum_{j\in C_i} b_j \cos\sigma_j^s \underline{v}_j \underline{v}_j^T \underline{\mu})^T (\sum_{k\in C_i} b_k \cos\sigma_k^s \underline{v}_k \underline{v}_k^T \underline{\mu})]} \tag{26}$$

By the definitions of $\underline{\mu}$ and $\underline{v}_j$ in (6) and (21) respectively, $\underline{v}_j^T\underline{\mu}$, $j \in C_i$ satisfies

$$v_j^T \underline{\mu} = \begin{cases} 1 & \text{if line j oriented with cutset} \\ -1 & \text{if line j oriented against cutset} \end{cases} \tag{27}$$

Hence for the case of a cutset with all lines oriented with the cutset reference direction, $\sum_{j\in C_i} a_j = 1$ and (25) defines a convex combination of the stable equilibrium line angles $\sigma_j^s$, $j \in C_i$.

To illustrate this method consider the simple network and cutset illustrated in Figure 2. The $\underline{v}_j$, $j \in C_i$ vectors for this example are

$$\underline{v}_4 = [0,0,1,0,0]^T$$

$$\underline{v}_5 = [0,0,1,-1,0]^T$$

$$\underline{v}_6 = [0,1,0,-1,0]^T$$

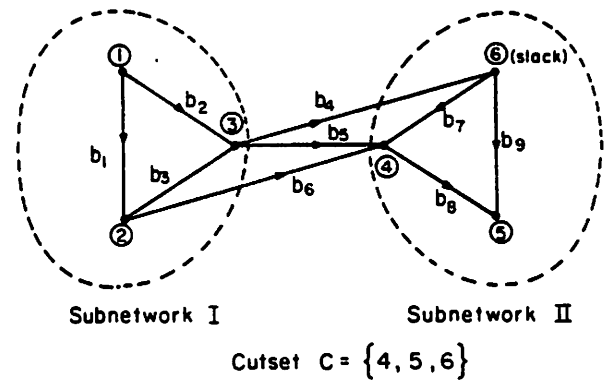


Figure 2. Network Example for Calculation of $\beta$.

Using (26) these yield

$$a_4 = (1/d)[(b_4\cos\sigma_4^s)^2 + (b_4\cos\sigma_4^2)(b_5\cos\sigma_5^s)]$$

$$a_5 = (1/d)[2(b_5\cos\sigma_5^s)^2 + (b_5\cos\sigma_5^s)(b_4\cos\sigma_4^s) + (b_5\cos\sigma_5^s)(b_6\cos\sigma_6^s)]$$

$$a_6 = (1/d)[2(b_6\cos\sigma_6^s)^2 + (b_6\cos\sigma_6^s)(b_5\cos\sigma_5^s)]$$

where

$$d = [(b_4\cos\sigma_4^s)^2 + 2(b_4\cos\sigma_4^s)(b_5\cos\sigma_5^s) + 2(b_5\cos\sigma_5^s)^2 + 2(b_5\cos\sigma_5^s)(b_6\cos\sigma_6^s) + 2(b_6\cos\sigma_6^s)^2]$$

The pattern exhibited in this example may be generalized to provide $a_j$, $j \in C_i$ for an arbitrary network.

Define

$$a_j = \begin{cases} t_j / \sum_{k\in C_i} t_k & \text{if line j directed with cutset} \\ -t_j / \sum_{k\in C_i} t_k & \text{if line j directed against cutset} \end{cases} \tag{28}$$

$$t_j = s_j(b_j\cos\sigma_j^s)^2 + b_j\cos\sigma_j^s(\sum_{k\in I_j} b_k\cos\sigma_k^s) \tag{29}$$

$$s_j = \begin{cases} 1 & \text{if line j incident on slack bus} \\ 2 & \text{otherwise} \end{cases} \tag{30}$$

$I_j$ = set of all indices of lines in $C_i$ incident on a common bus with line j.

It is useful to note that these $a_j$ coefficients are dependent only on the cutset examined and network parameters and therefore will not change with varying

$\underline{P}$. In some sense, the $a_j$ coefficient quantifies the strength of the interconnection between the buses linked by the jth line, taking into account the effect of neighboring lines. As may be seen in (25), if $a_j$ is large, implying a strong interconnection, then $\beta$ is strongly influenced by $\hat{\sigma}_j^s$. Conversely, a weak connection across line j implies that $\hat{\sigma}_j^s$ has little influence on $\beta$. From the structure of (25), it also follows that the absolute value of $\beta$ is bounded above by the absolute value of the largest line angle in the cutset.

## V. COMPUTATIONAL CONSIDERATIONS: LINEARIZATION APPROACH

As described in Section III, the linearization approximation reduces to solving (18) for $\hat{\underline{\alpha}}^u$, which may be expressed as

$$\hat{\underline{\alpha}}^u = [AJ(\underline{\sigma}_1^u)A^T]^{-1}AJ(\underline{\sigma}_1^u)\underline{\sigma}_1^u \tag{31}$$

The compuational burden lies in inverting the matrix $[AJ(\underline{\sigma}_1^u)A^T]$. To ease this task, we observe by (9) and (15) that $J(\underline{\sigma}_1^u) - J(\underline{\sigma}^s)$ is a diagonal matrix with components

$$[J(\underline{\sigma}_1^u)-J(\underline{\sigma}^s)]_{kk} = \begin{cases} -2b_k\cos\sigma_k^s & \text{if } k \in C_i \\ 0 & \text{if } k \notin C_i \end{cases} \quad k = 1,2,\ldots,\ell \tag{32}$$

Using (21) and (32) it follows that

$$AJ(\underline{\sigma}_1^u)A^T = AJ(\underline{\sigma}^s)A^T - 2\sum_{j\in C_i} b_j\cos\sigma_j^s \underline{v}_j\underline{v}_j^T \tag{33}$$

In this form, the matrix $AJ(\underline{\sigma}_1^u)A^T$ is expressed as a perturbation of the matrix $AJ(\underline{\sigma}^s)A^T$. It is also useful to note that $AJ(\underline{\sigma}^s)A^T$ takes the form of a node admittance matrix for a network of positive resistors. Standard results of circuit theory guarantee that such a matrix is nonsingular. With these observations, the Inverse Matrix Modification Lemma may be applied to express $[AJ(\underline{\sigma}_1^u)A^T]^{-1}$ as a function of $[AJ(\underline{\sigma}^s)A^T]^{-1}$. A simple iterative application of this technique is presented here. For more detailed analysis of applications of this lemma, the reader is referred to [13].

The form of the Inverse Matrix Modification Lemma most appropriate to our problem is obtained from [14, p. 655]

$$(M+\underline{r}\underline{s}^T)^{-1} = M^{-1} - \frac{M^{-1}\underline{r}\underline{s}^T M^{-1}}{1+\underline{s}^T M^{-1}\underline{r}} \tag{34}$$

where

$M \in \mathbb{R}^{n\times n}$

$\underline{r},\underline{s} \in \mathbb{R}^n$

This formula will be applied iteratively to calculate $AJ(\underline{\sigma}_1^u)A^T$.

Assume $C_i$ consists of p line indices $\{k_1,k_2,\ldots,k_p\}$.

Define

$$M_0 = AJ(\underline{\sigma}^s)A^T \tag{35}$$

$$M_h = AJ(\underline{\sigma}^s)A^T + \sum_{j=1}^{h} b_{k_j}\cos\sigma_{k_j}^s \underline{v}_{k_j}\underline{v}_{k_j}^T$$

$$= M_{h-1} + b_{k_h}\cos\sigma_{k_h}^s \underline{v}_{k_h}\underline{v}_{k_h}^T \quad h = 1,2,\ldots p \tag{36}$$

Applying (31)

$$M_0^{-1} = [AJ(\underline{\sigma}^s)A^T]^{-1} \tag{37}$$

$$M_h^{-1} = M_{h-1}^{-1} - \frac{2b_{k_h}\cos\sigma_{k_h}^s (M_{h-1}^{-1}\underline{v}_{k_h})(\underline{v}_{k_h}^T M_{h-1}^{-1})}{1+2b_{k_h}\cos\sigma_{k_h}^s \underline{v}_{k_h}^T M_{h-1}^{-1}\underline{v}_{k_h}} \tag{38}$$

Applying this iteration p times

$$(AJ(\underline{\sigma}_1^u)A^T)^{-1} = M_p^{-1} \tag{39}$$

After the initial cost of calculating $[AJ(\underline{\sigma}^s)A^T]^{-1}$, this approach requires only order p vector multiplications for calculating $[AJ(\underline{\sigma}_1^u)A^T]^{-1}$. In addition, this formulation allows a test for $AJ(\underline{\sigma}_1^u)A^T$ singular or ill-conditioned through the magnitude of the scalar terms in the denominator of (38). It should be emphasized that $AJ(\underline{\sigma}^s)A^T$ is calculated only once for a given post fault configuration, simplifying calculation for a large number of separating u.e.p.'s.

## VI. SENSITIVITY OF TOPOLOGICAL LYAPUNOV FUNCTION TO ERRORS IN APPROXIMATE U.E.P.'s

Previous research in estimating critical values of V for Lyapunov techniques has observed the relative insensitivity of Lyapunov energy to small deviations about the exact u.e.p.'s [9,10]. Specifically in [9], using a linearized analysis of V about a u.e.p. $\underline{x}$, $\Delta V$ critical $\cong [\frac{\partial V}{\partial \underline{x}}]^T \Delta x$, it was observed that $\|\frac{\partial V}{\partial \underline{x}}\|^2$ will in general be "small."

For the topological Lyapunov function, this sensitivity analysis can be carried further with an exact three term Taylor expansion. Let $\underline{\alpha}^u$ denote the true u.e.p. angles in question, and $\Delta\underline{\alpha}^u$ denote the deviation of the approximation $\hat{\underline{\alpha}}^u$ from this value; $\Delta\underline{\alpha}^u = \underline{\alpha}^u - \hat{\underline{\alpha}}^u$. The stable post fault equilibrium angles will be denoted $\underline{\alpha}^s$. It is also convenient to define the corresponding line angle quantities $\underline{\sigma}^u = A^T\underline{\alpha}^u$, $\Delta\underline{\sigma} = A^T\Delta\underline{\alpha}$, and $\underline{\sigma}^s = A^T\underline{\alpha}^s$. An exact three term Taylor expansion of $W(\underline{\sigma}^u+\Delta\underline{\sigma})$ about $\underline{\sigma}^u$ as found in [15, p. 190] yields

$$W(\underline{\sigma}^u + \Delta\underline{\sigma}) = W(\underline{\sigma}^u) + [\underline{g}(\underline{\sigma}^u) - \underline{g}(\underline{\sigma}^s)]^T A^T \Delta\underline{\alpha} + \Delta\underline{\alpha}^T A \int_0^1 (1-s)J(\hat{\underline{\sigma}}^u+s\Delta\underline{\sigma})A^T\Delta\underline{\alpha}\, ds \tag{40}$$

The linear term in (40) may be re-expressed as

$$[\underline{g}(\underline{\sigma}^u) - \underline{g}(\underline{\sigma}^s)]^T A^T \Delta\underline{\alpha} = [A\underline{g}(\underline{\sigma}^u) - A\underline{g}(\underline{\sigma}^s)]^T \Delta\underline{\alpha} \tag{41}$$

By definition, both stable and unstable equilibrium angles must satisfy the power flow relation defined in (3) so we have $\underline{P} = A\underline{g}(\underline{\sigma}^u) = A\underline{g}(\underline{\sigma}^s)$. Therefore the linear term in (40) must be zero.

With this observation we may express the error in the Lyapunov energy by

$$|W(\underline{\sigma}^u+\Delta\underline{\sigma}) - W(\underline{\sigma}^u)| = |\Delta\underline{\alpha}^T A \int_0^1 (1-s)J(\hat{\underline{\sigma}}^u+s\Delta\sigma)A^T\Delta\underline{\alpha}\, ds| \quad (42)$$

To bound this integral in terms of known quantities, we need only observe from the definition (15) of $J(\underline{\sigma})$ that for any $\underline{x}, \underline{y} \in \mathbb{R}^\ell$, $|\underline{x}^T J(\underline{y})\underline{x}| \leq \underline{x}^T J(\underline{0})\underline{x}$. Equation (42) then yields

$$|W(\underline{\sigma}^u+\Delta\underline{\sigma}) - W(\underline{\sigma}^u)| \leq \frac{1}{2}\, \Delta\underline{\alpha}^T A J(\underline{0}) A^T \Delta\underline{\alpha} = \frac{1}{2}\, \Sigma b_k (\Delta\sigma_k)^2 \quad (43)$$

In applying (43) we need bounds on $\|\Delta\underline{\sigma}\|$ or $\|\Delta\underline{\alpha}\|$. Promising results for obtaining these bounds in the case of approximate stable equilibrium angles are available in [17], but their practical extension to the case of u.e.p.'s remains an open area for research.

## VII. APPLICATIONS TO A SAMPLE NETWORK

For purposes of illustration, a four generator, six bus system similar to that presented in [4] will be analyzed. To accommodate the structure preserving model, internal generator buses are introduced explicitly into the network. Furthermore, we shall assume that all bus voltage magnitudes are 1.0 p.u., so $b_k = 1/x_k$. The resulting system is shown in Figure 3.

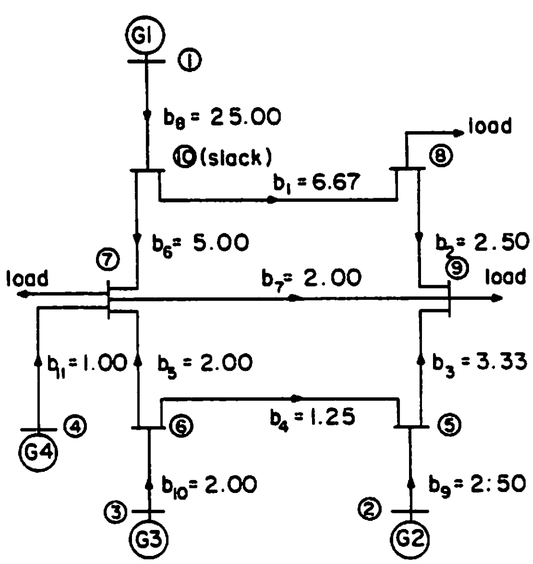


Figure 3. Structure Preserving Model for Four Machine Network.

To illustrate the various approximation methods, six test cases were considered. For two different power injection vectors, $\underline{P}_1$ and $\underline{P}_2$, three different cutsets were examined. For each case, the separating u.e.p. angles in the vicinity of $\underline{\mu}\pi$ were located by the three approximation methods outlined in this paper. Note that the procedures would be similar for locating the second set of separating u.e.p. angles in the vicinity of $-\underline{\mu}\pi$ for any of the cutsets. As a benchmark against which these approximations would be judged, the sets of u.e.p. angles were also found by a standard Newton-Raphson (N-R) iterative technique. For each approximation method, three quantities were calculated: the Lyapunov energy associated with the approximate angles $\hat{\underline{\alpha}}^u$ (or line angles $\hat{\underline{\sigma}}_1^u$ for line stretch approximation); the Euclidean norm of the difference between approximate angles $\hat{\underline{\alpha}}^u$ and "exact" (N-R) angles $\underline{\alpha}^u$, $\|\Delta\underline{\alpha}\| = \|\underline{\alpha}^u - \hat{\underline{\alpha}}^u\|$; and the error bound on V as predicted by the Section VI analysis, denoted as $|\overline{\Delta V}|$. Note that all angle quantities are expressed in radians.

The results of the calculations are summarized in Tables 1 through 3. The first section of each table shows results for u.e.p. angles found in the case of "small" injections

$\underline{P}_1$ = [0.332,0.200,0.3-0,0.100,0.,0.,-0.210,-0.312, -0.410]$^T$.

The remaining section of each table shows results for "moderate" injections

$\underline{P}_2$ = [0.600,0.250,0.300,0.120,0.,0.,-0.520,-0.350,-0.400]$^T$.

TABLE 1

Test Conditions: Cutset Lines {4,5}

Injections $\underline{P}_1$

| | SPS | Linearized | Line Stretch[3] | Exact (N-R) |
|---|---|---|---|---|
| V | 5.589 | 5.588 | 5.587 | 5.588 |
| $\|\Delta\underline{\alpha}\|$ | 0.078 | 0.012 | 0.118 | - |
| $\|\overline{\Delta V}\|$ | 0.004 | 0.001 | 0.011 | - |

Injections $\underline{P}_2$

| | SPS | Linearized | Line Stretch[3] | Exact (N-R) |
|---|---|---|---|---|
| V | 5.590 | 5.587 | 5.585 | 5.607 |
| $\|\Delta\underline{\alpha}\|$ | 0.147 | 0.074 | 0.201 | - |
| $\|\overline{\Delta V}\|$ | 0.030 | 0.020 | 0.046 | - |

TABLE 2

Test Conditions: Cutset Lines {3,5}

Injections $\underline{P}_1$

| | SPS | Linearized | Line Stretch[3] | Exact (N-R) |
|---|---|---|---|---|
| V | 9.139 | 9.142 | 9.140 | 9.142 |
| $\|\Delta\underline{\alpha}\|$ | 0.121 | 0.0196 | 0.099 | - |
| $\|\overline{\Delta V}\|$ | 0.020 | 0.001 | 0.015 | - |

Injections $\underline{P}_2$

| | SPS | Linearized | Line Stretch[3] | Exact (N-R) |
|---|---|---|---|---|
| V | 8.996 | 8.998 | 8.989 | 9.023 |
| $\|\Delta\underline{\alpha}\|$ | 0.165 | 0.148 | 0.159 | - |
| $\|\overline{\Delta V}\|$ | 0.060 | 0.065 | 0.060 | - |

[3]Bus angle quantities $\hat{\underline{\alpha}}^u$ necessary for the $\Delta\underline{\alpha}$ and $|\overline{\Delta V}|$ calculations were obtained by performing a psuedo-inverse on the line stretch approximation $\underline{\sigma}_1^u$. $\hat{\underline{\alpha}}^u = [AA^T]^{-1}A\underline{\sigma}_1^u$.

TABLE 3

Test Conditions: Cutset Lines {3,4}

Injections $\underline{P}_1$

| | SPS | Linearized | Line Stretch[3] | Exact (N-R) |
|---|---|---|---|---|
| V | 8.515 | 8.399 | 8.565 | 8.399 |
| $\|\Delta\alpha\|$ | 0.876 | 0.043 | 1.038 | - |
| $\|\overline{\Delta V}\|$ | 0.707 | 0.003 | .997 | - |

Injections $\underline{P}_2$

| | SPS | Linearized | Line Stretch[3] | Exact (N-R) |
|---|---|---|---|---|
| V | 8.366 | 8.249 | 8.412 | 8.249 |
| $\|\Delta\alpha\|$ | 0.877 | 0.036 | 1.039 | - |
| $\|\overline{\Delta V}\|$ | 0.705 | 0.003 | 0.995 | - |

## VIII. CONCLUSIONS

In this paper we have proposed several approximate methods for locating unstable equilibrium points for the structure preserving model. While further experience with these approaches is necessary before a final evaluation can be made, the results of Section VII appear very promising. The single parameter stretch method in particular appears to offer accurate prediction of Lyapunov energy with minimal computation.

The linearization method appears to offer somewhat more accurate location of the vectors of u.e.p. angles, but the improvement in the accuracy of the scalar Lyapunov function is negligible. Depending on the application, this advantage may or may not justify the additional computational requirements.

Overall, these techniques should allow use of the structure preserving model and topological Lyapunov function for rapid evaluation of transient stability regions.

## IX. ACKNOWLEDGEMENT

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