ALGEBRAIC THEORY OF LINEAR MULTIVARIABLE FEEDBACK SYSTEMS

by

C. A. Desoer and C. L. Gustafson

Memorandum No. UCB/ERL M82/90

17 December 1982

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

ALGEBRAIC THEORY OF LINEAR MULTIVARIABLE FEEDBACK SYSTEMS

C. A. Desoer and C. L. Gustafson

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Abstract

This paper presents an algebraic theory for analysis and design of linear multivariable feedback systems. The theory is developed in an algebraic setting sufficiently general to include, as special cases, continuous and discrete time systems, both lumped and distributed. Designs are implemented by construction of a controller with two vector inputs and one vector output. Use of controllers of this type is shown to generate convenient stability results, and convenient global parametrizations of all I/O maps and all disturbance-to-output maps achievable, for a given plant, by a stabilizing compensator. These parametrizations are then used to show that any such I/O map and any such disturbance-to-output map may be simultaneously realized by choice of an appropriate controller.

In the special case of lumped systems, it is shown that the design theory can be reduced to manipulations involving polynomial matrices only. The resulting design procedure is thus shown to be more efficient computationally.

Finally, the problem of asymptotically tracking a class of input signals is considered in the general algebraic setting. It is shown that the classical results on asymptotic tracking can be generalized to this setting. Additionally, sufficient conditions for robustness of asymptotic tracking, and robustness of stability are developed.

Research sponsored by a grant from Aerospace Corporation and National Science Foundation Grant ECS-8119763.

I. Introduction

A subject of great interest in the design of linear multi-input multi-output systems has been the characterization of all designs which can be achieved by a stabilizing controller for a given plant. Such results have been developed for the lumped continuous and discrete time cases; first by Youla, et al. [You. 1] and later by Pernebo [Per. 1] and others [Sae. 1] [Che. 2] [Vid. 2]. By using an algebraic formulation, Desoer, et al. [Des. 1] generalized such results greatly - to include the distributed continuous and discrete time cases, among others. And in a similar algebraic structure, a particularly flexible and convenient method for stable plants, was suggested by Zames [Zam. 1], developed by Desoer et al. [Des. 2], and used in computer-aided design by Gustafson et al. [Gus. 1]. All of these methods give their results in a parametrized form; by appropriate selection of a particular matrix, any design achievable by a stabilizing controller may be realized.

This paper presents an algebraic design procedure which generalizes the above results in several ways:

(i) The algebraic structure is more general than that of [Des. 1], because it enables one to design with non-square plants and controllers. In addition, the algebraic structure characterizes the class of plants for which an algebraic realizability condition on the controller can be included in the parametrization of stabilizing controllers. This is accomplished through use of the Jacobson radical [Zam. 1].

(ii) The parametrizations of [Per. 1] and [You. 1] for the lumped continuous and discrete time cases are extended, by the use of our algebraic methods, to a great number of additional cases (see Table I).

In addition, the algebraic formulation allows great simplification of the stability argument. Finally, using a transformation of the type proposed by [Per. 1], [Vid. 1], it is shown how design in the lumped continuous or discrete time cases can be reduced to manipulating only polynomial matrices.

The method by which these results are achieved involves construction of a controller with two vector-inputs and one vector-output [Per. 1], [Åst. 1]. This resulting closed-loop system is thus so constructed as to give a multivariable interpretation of Horowitz's two-degrees of freedom design [Hor. 1].

Also, a set of sufficient conditions for the robust stability of this feedback configuration is presented, much as in [Chen 1].

Additionally, the asymptotic tracking problem [Cal. 3], [Cal. 4], is considered; we show that known results, including the internal model principle [Won. 1], can be generalized to the abstract algebraic structure used in the design parametrizations. A unification of the theory of asymptotic tracking is thus achieved, for many interesting cases (see Table 1).

Thus, this paper achieves a unification of design parametrization theories for the canonical design settings of linear multivariable system theory.

The paper is organized as follows:

Section II defines the algebraic design structure and the closed-loop system under considerations.

Section III presents the main results: the stability theorem and the design parametrization theorems.

Section IV specializes the results of Section III to the lumped case, and shows how the design theory then need only consider polynomial matrices.

Section V discusses the robustness of stability and the asymptotic tracking problem.

Section VI contains the conclusions.

Special notations and definitions:

a := b means a denotes b. θ_{mxn} denotes the mxn zero matrix.

For definitions of standard algebraic terms, see [Jac. 1], particularly chapters 1-3, [Sig. 1] or [Mac. 1].

If H is a ring, then $E(H)$ denotes the set of matrices having all entries in H.

$\mathbb{R}(s)$ denotes the set of real rational functions in s. $\mathbb{R}_p(s)$ denotes the set of proper rational functions: those that remain bounded as $|s| \to \infty$. $\mathbb{R}_{p,o}(s)$ denotes the set of strictly proper rational functions: i.e., the proper rational functions tending to zero as $|s| \to \infty$. $\mathbb{R}_U(s)$ denotes the rational functions analytic in the region $U \subset \mathbb{C}$.

$\mathbb{R}_{\{o\}}(\lambda)$ denotes the set of real rational functions analytic at $\lambda = 0$. $\mathbb{R}_{\{o\},o}(\lambda)$ denotes the set of real rational functions having the value zero at $\lambda = 0$.

II. Preliminaries

2.1. Algebraic Theory

Roughly speaking, the algebraic structure developed here consists of a) H, a ring of scalar transfer functions; b) I, a multiplicative subset of H; c) $G := [H][I]^{-1}$, the ring of fractions over H and I; and d) J, the set of units in H, i.e., $m \in J \Rightarrow m^{-1} \in H$.

It is helpful to keep in mind a simple example, while studying the detailed definitions below: H is the ring of scalar, exponentially stable, proper rational functions in s; I is the subset of H whose elements tend to a non-zero constant as $|s| \to \infty$; G is then the ring of scalar, proper rational functions; and J is the ring of proper, exp. stable rational functions with no zeros in $\mathbb{C}_+$ nor at infinity.

In the general formulation, these terms are defined as follows:

H: An entire ring (integral domain): i.e., a commutative ring with no zero divisors. Let 0 and 1 denote the additive and multiplicative identities, respectively.

$\tilde{G}$: The field of fractions over H [Jac. 1, Sec. 2.9]: i.e., a field whose elements are the pairs $(n,d) =: n/d$, where $n,d \in H$, and $d \neq 0$, and are subject to the equivalence relation $n_1/d_1 = n_2/d_2 \Leftrightarrow n_1d_2 = n_2d_1$. (In the example above, $\tilde{G}$ is the field of rational functions).

I: A multiplicative subset of H: i.e., $I \subset H$, $0 \notin I$, and $x,y \in I \Rightarrow xy \in I$. Without loss of generality, let $1 \in I$.

$G := \{n/d \in \tilde{G} : n \in H,\ d \in I\}$, a subring of $\tilde{G}$.

$J := \{m \in H : m^{-1} \in H\}$, the ring of units in H.

Additionally, we consider the following structure, known as the Jacobson radical of G [Nai. 1] [Bou. 1].

$G_s := \{x \in G : (1+xy)^{-1} \in G,\ \forall\, y \in G\}$. It can be shown that G_s is an ideal[1] of G; thus $x \in G_s \Rightarrow xy \in G_s$, $\forall\, y \in G$ (note that in the example above, G_s is the set of strictly proper transfer functions).

In addition, let $\mathbb{F}$ be a field; typically $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. We assume

[1]Thus, G_s is also an additive subgroup of G. Some authors [Sig. 1] refer to G_s as a ring; others as a "rng" - a ring without unit [Jac. 1].

that $(H, \mathbb{F})$ and $(G, \mathbb{F})$ form vector spaces over $\mathbb{F}$ (i.e., multiplication by scalars is defined on $\mathbb{F} \times H$ and on $\mathbb{F} \times G$, and the axioms of vector spaces are satisfied). Additional examples of the algebraic structure above are given in Table I.

Comments: (a) Since by assumption, $1 \in I$, we can identify $n \in H$, and $n/1 \in G$; hence we view H as a subring of G and we write nd^{-1} for n/d. (b) By construction of G, every element of I has an inverse in G. (c) Since both H and G are commutative rings, both $(H, \mathbb{F})$ and $(G, \mathbb{F})$ are commutative algebras over $\mathbb{F}$ [Nai. 1].

2.2. Coprime Factorizations

Definition 2.1

Let $H \in G^{m \times n}$. We say that $N_{hr}D_{hr}^{-1}$ $(D_{h\ell}^{-1}N_{h\ell})$ is a right-coprime factorization (r.c.f.) (left-coprime factorization (ℓ.c.f.), resp.) of H, if and only if

(i) $H = N_{hr}D_{hr}^{-1}$ $(D_{h\ell}^{-1}N_{h\ell}$, resp.);

(ii) $N_{hr} \in H^{m \times n}$, $D_{hr} \in H^{n \times n}$ $(D_{h\ell} \in H^{m \times m}$, $N_{h\ell} \in H^{m \times n}$, resp.), and $\det D_{hr} \in I$ ($\det D_{h\ell} \in I$, resp.);

(iii) (N_{hr}, D_{hr}) are right-coprime (r.c.), i.e., $\exists\, U_r \in H^{n \times m}$ and $V_r \in H^{n \times n}$, such that $U_rN_{hr} + V_rD_{hr} = I_n$;

((iii)' $(D_{h\ell}, N_{h\ell})$ are left-coprime (l.c.), i.e., $\exists\, U_\ell \in H^{n \times m}$ and $V_\ell \in H^{m \times m}$, such that $N_{h\ell}U_\ell + D_{h\ell}V_\ell = I_m$, resp.).

Comment: Recently, Vidyasagar et al., gave a set of sufficient conditions for the existence of coprime factorizations [Vid. 2, Thm. 2.1]; it is easily seen that all of the examples in Table I satisfy these conditions.

In this paper, we assume the existence of coprime factorizations throughout.

III. Design Theory

3.1. Problem Description

We consider the system S shown in Figure 1. Given a plant P, we wish to design a controller C. We will require the following assumptions at various points in this paper.

Assumptions on System S:

(P1) $P \in G_s^{n_o \times n_i}$ (3.1)

(P2) $N_{pr}D_{pr}^{-1}$ is a r.c.f. of P, with $U_{pr}, V_{pr} \in E(H)$ satisfying

$$U_{pr}N_{pr} + V_{pr}D_{pr} = I_{n_i} \tag{3.2}$$

(C1) $C \in \tilde{G}^{n_i \times (n_v + n_o)}$ is given by $D_{c\ell}^{-1}[N_{\pi\ell} \vdots N_{f\ell}]$, with

$$D_{c\ell} \in H^{n_i \times n_i},\ N_{\pi\ell} \in H^{n_i \times n_v},\ N_{f\ell} \in H^{n_i \times n_o} \tag{3.3}$$

(C2) $C \in G^{n_i \times (n_v + n_o)}$ has a ℓ.c.f. $D_{c\ell}^{-1}[N_{\pi\ell} \vdots N_{f\ell}]$, with

$$D_{c\ell} \in H^{n_i \times n_i},\ N_{\pi\ell} \in H^{n_i \times n_v},\ N_{f\ell} \in H^{n_i \times n_o} \tag{3.4}$$

Comment: (C2) $\Rightarrow$ (C1).

Under (P2) and (C1), the system S is completely described by

$$\begin{bmatrix} I_{n_i} & -D_{pr} \\ D_{c\ell} & N_{f\ell}N_{pr} \end{bmatrix} \begin{bmatrix} y_1 \\ \xi_p \end{bmatrix} = \begin{bmatrix} 0 & 0 & -I_{n_i} & 0 \\ N_{\pi\ell} & N_{f\ell} & 0 & -N_{f\ell} \end{bmatrix} \begin{bmatrix} v_1 \\ u_1 \\ u_2 \\ d_0 \end{bmatrix} \tag{3.7}$$

$$
\begin{bmatrix} y_1 \\ y_2 \\ e_1 \\ e_2 \end{bmatrix} = \left[\begin{array}{c|c} I_{n_i} & 0 \\ \hline 0 & N_{pr} \\ \hline 0 & -N_{pr} \\ \hline I_{n_i} & 0 \end{array}\right] \begin{bmatrix} y_1 \\ \xi_p \end{bmatrix} + \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n_o} \\ \hline 0 & I_{n_o} & 0 & -I_{n_o} \\ \hline 0 & 0 & I_{n_i} & 0 \end{array}\right] \begin{bmatrix} v_1 \\ u_1 \\ u_2 \\ d_o \end{bmatrix} \tag{3.8}
$$

<u>Comment</u>: In the five cases of Table I, Eqs. (3.7) and (3.8) can be interpreted as matrix products (in the s,z,or λ domain), or as convolution equations in time domain ($\mathbb{R}_+$ or $\mathbb{N}$).

Let $u := (v_1^T, u_1^T, u_2^T, d_0^T)^T$, $\xi := (y_1^T, \xi_p^T)^T$, and $y := (y_1^T, y_2^T, e_1^T, e_2^T)^T$. Then, we can rewrite (3.7) and (3.8) as:

$$
D\xi = N_\ell \, u \tag{3.9}
$$

$$
y = N_r\xi + Ku \tag{3.10}
$$

where the definitions of D, N_ℓ, N_r and K are obvious from (3.7) and (3.8).

<u>Definition 3.1</u>

For any $D_{c\ell} \in H^{n_i \times n_i}$, and any $N_{f\ell} \in H^{n_i \times n_o}$, define

$$
D_h := D_{c\ell}D_{pr} + N_{f\ell}N_{pr} \in H^{n_i \times n_i} \tag{3.11}
$$

<u>Definition 3.2</u>

S is called <u>H-stable</u> iff $H_{yu} : u \mapsto y$ defined by (3.9) and (3.10) satisfies $H_{yu} \in E(H)$.

If we assume that (P2) and (C1) hold, and that det $D \in I$, then, from (3.9) and (3.10), we have

$$
H_{yu} = N_r D^{-1} N_\ell + K \in E(G) \tag{3.12}
$$

It is easy to show that if (P1), (P2) and (C1) hold, $D^{-1} \in \varepsilon(G)$.

Comment: Definition 3.2 makes sense because a) each subsystem input may be manipulated independently by some exogeneous input (i.e. component of u), and b) each subsystem output is part of the output vector y. Consequently, in the rational function case, for example, if neither C nor P have unstable hidden modes then all the zeros of the characteristic polynomial of the system S are in the stable region of the complex plane if and only if S is H-stable.

Definition 3.3

A controller C is said to be admissible for the plant P iff C satisfies (C2), and the resulting system S is H-stable.

Theorem 3.1 (Admissibility of C)

Consider the system S with P satisfying (P2), and C to be specified later. Under this assumption,

(i) If P satisfies (P1), and if, for some $D_{c\ell} \in H^{n_i x n_i}$ and $N_{f\ell} \in H^{n_i x n_o}$, det $D_h \in J$, then det $D_{c\ell} \in I$, and hence, for any $N_{\pi\ell} \in H^{n_i x n_v}$, the controller $C := D_{c\ell}^{-1}[N_{\pi\ell} \vdots N_{f\ell}]$ is admissible for P.

(ii) If C is an admissible compensator for P, then det $D_h \in J$.

Comments: (a) In statement (i), $C \in E(G)$ is part of the conclusion. $P \in E(G_s)$ and det $D_h \in J$ guarantee the H-stability of S and $C \in E(G)$. (b) In statement (ii), P is not assumed to have its elements in G_s. (c) The corollary below is a slightly weaker form of Theorem 3.1.

Corollary 3.1

Let S satisfy (P2) and (C2). Then, S is H-stable $\Longleftrightarrow$ det $D_h \in J$.

Proof of Theorem 3.1

(i) First, we will show that det $D_{c\ell} \in I$ and consequently that $C := D_{c\ell}^{-1}[N_{\pi\ell} \vdots N_{f\ell}]$ is well-defined and satisfies (C2), for any $N_{\pi\ell} \in H^{n_i x n_v}$.

By (P1) and (P2), we have $P \in G_s^{n_o x n_i}$, and $PD_{pr} = N_{pr}$ with $D_{pr} \in H^{n_i x n_i} \subset G^{n_i x n_i}$. Now, since $G_s \subset G$ is an ideal and hence is closed under addition, it follows that $N_{pr} \in G_s^{n_o x n_i}$. From (3.11), we obtain

$$D_h^{-1} D_{c\ell} D_{pr} = I - D_h^{-1} N_{f\ell} N_{pr}$$

Taking determinants of both sides, we can easily obtain

$$\det D_{c\ell} = \det D_h (\det D_{pr})^{-1} \det(I - D_h^{-1} N_{f\ell} N_{pr}) \tag{3.14}$$

By assumption, $(\det D_h)^{-1} \in H \subset G$, and $\det D_{pr} \in H \subset G$. We will now show that $\det(I-D_h^{-1}N_{f\ell}N_{pr})$ is invertible in G, thus showing that $(\det D_{c\ell})^{-1} \in G$.

We know that $N_{pr} \in G_s^{n_o x n_i} \cap H^{n_o x n_i}$, $N_{f\ell} \in H^{n_i x n_o}$, and $D_h^{-1} \in H^{n_i x n_i}$, hence, $D_h^{-1} N_{f\ell} N_{pr} \in G_s^{n_i x n_i}$. Now, by definition of determinant (for $A \in G^{mxm}$, with the ij$\underline{th}$ element of A denoted a_{ij}, the determinant of A is defined by $\det A := \sum_\sigma (\text{sgn } \sigma)\, a_{1\sigma(1)}, a_{2\sigma(2)}, \ldots, \sigma_{m\sigma(m)}$ where σ is a permutation function on the integers 1, 2, ..., m), and by the fact that G_s is a subring of G, there exists $g \in G_s$, such that

$$\det(I - D_h^{-1} N_{f\ell} N_{pr}) = 1 + g$$

By the definition of G_s, $(1+g)^{-1} \in G$, and (3.14) shows that

$$(\det D_{c\ell})^{-1} \in G \tag{3.15}$$

and hence that $\det D_{c\ell} \in I$. We can now show that C satisfies (C2) for any $N_{\pi\ell} \in H^{n_i x n_v}$, since, by (3.15), $D_{c\ell}^{-1} \in E(G)$ [Jac. 1; Thm. 2.1, p. 94] and thus $C \in E(G)$, and since $(D_h^{-1}D_{c\ell})^{-1}$ $(D_h^{-1}[N_{\pi\ell}\vdots N_{f\ell}])$ is a ℓ.c.f. of C. The second conclusion follows from (3.11), and the fact that $D_h^{-1} \in E(H)$.

Second, we will show that the system S, now well-defined by P and C, is H-stable. As a result, C will be admissible for P.

We have shown that C satisfies (C2). Thus, since P satisfies (P2), the system S is described by (3.7)-(3.10). By performing block elementary row operations (in the ring H) on the matrix $\dot{D}$ in (3.9), we obtain:

$$\det D = \eta \cdot \det(D_{c\ell}D_{pr}+N_{f\ell}N_{pr}) = \eta \cdot \det D_h \in H \quad (\text{where } \eta = \pm 1)$$

Thus, from our assumption, $\eta \cdot \det D_h = \det D \in J$, implying that $(\det D)^{-1} \in H$. Now, since H is a commutative ring, $D^{-1} \in E(H)$ [Jac. 1; Thm. 2.1, p. 94]. Consequently, since $N_r \in E(H)$, $N_\ell \in E(H)$, it is clear that $(N_r D^{-1} N+K) \in E(H)$. So, by (3.12),

$$H_{yu} \in E(H) \text{ and S is } H\text{-stable}$$

Thus, by definition, $C := D_{c\ell}^{-1}[N_{\pi\ell}\vdots N_{f\ell}]$ is admissible for P, for any $N_{\pi\ell} \in H^{n_i x n_v}$.

(ii) We prove that $\det D_h \in J$ in two steps:

First, we prove (D,N_ℓ) are ℓ.c. and (N_r,D) are r.c. By (P2), (N_{pr},D_{pr}) are r.c. and by (C2), $(D_{c\ell},[N_{\pi\ell}\vdots N_{f\ell}])$ are ℓ.c., hence there exist U_{pr}, V_{pr}, $V_{c\ell}$, $U_{\pi\ell}$, $U_{f\ell} \in E(H)$ such that:

$$U_{pr}N_{pr} + V_{pr}D_{pr} = I_{n_i} \tag{3.21}$$

$$D_{c\ell}V_{c\ell} + [N_{\pi\ell}\vdots N_{f\ell}] \begin{bmatrix} U_{\pi\ell} \\ \hdashline U_{f\ell} \end{bmatrix} = I_{n_o} \tag{3.22}$$

From (3.21) and (3.22), we can check that (3.23) holds:

$$\left[\begin{array}{c|c} I_{n_i} & -D_{pr} \\ \hline D_{c\ell} & N_{f\ell}N_{pr} \end{array}\right]\left[\begin{array}{c|c} V_{c\ell}D_h-D_{pr} & V_{c\ell} \\ \hline -I_{n_i} & 0 \end{array}\right]+\left[\begin{array}{c|c|c|c} 0 & 0 & -I_{n_i} & 0 \\ \hline N_{\pi\ell} & N_{f\ell} & 0 & -N_{f\ell} \end{array}\right]\left[\begin{array}{c|c} U_{\pi\ell}D_h & U_{\pi\ell} \\ \hline U_{f\ell}D_h & U_{f\ell} \\ \hline V_{c\ell}D_h-I_{n_i} & V_{c\ell} \\ \hline 0 & 0 \end{array}\right]=\left[\begin{array}{c|c} I_{n_i} & 0 \\ \hline 0 & I_{n_o} \end{array}\right] \tag{3.23}$$

Rewrite this as:

$$D\tilde{V}_\ell + N_\ell\tilde{U}_\ell = I_{n_o+n_i} \tag{3.24}$$

Since by (3.23), $\tilde{V}_\ell$, $\tilde{U}_\ell \in E(H)$, (D,N_ℓ) are ℓ.c. Also, from (3.21), (3.22), we can check that

$$\left[\begin{array}{c|c} D_{c\ell}-D_hV_{pr} & -I_{n_i} \\ \hline -V_{pr} & 0 \end{array}\right]\left[\begin{array}{c|c} I_{n_i} & -D_{pr} \\ \hline D_{c\ell} & N_{f\ell}N_{pr} \end{array}\right]+\left[\begin{array}{c|c|c|c} D_hV_{pr} & D_hU_{pr} & 0 & I_{n_i} \\ \hline V_{pr} & U_{pr} & 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_{n_i} & 0 \\ \hline 0 & N_{pr} \\ \hline 0 & -N_{pr} \\ \hline I_{n_i} & 0 \end{array}\right]=\left[\begin{array}{c|c} I_{n_i} & 0 \\ \hline 0 & I_{n_o} \end{array}\right] \tag{3.26}$$

Rewrite this as:

$$\tilde{V}_rD + \tilde{U}_rN_r = I_{n_o+n_i} \tag{3.27}$$

Since by (3.26), $\tilde{V}_r$, $\tilde{U}_r \in E(H)$, (N_r,D) are r.c.

Second, we prove that $\det D_h \in J$ by contradiction.

Assume $\det D_h \notin J$. Then, since $\eta\cdot\det D_h = \det D$, $D^{-1} \notin H^{n\times n}$. Rewriting (3.24) as:

$$\tilde{V}_\ell + D^{-1}N_\ell\tilde{U}_\ell = D^{-1}$$

Since $\tilde{V}_\ell$, $\tilde{U}_\ell \in E(H)$, and $D^{-1} \notin E(H)$, we conclude that $D^{-1}N_\ell \notin E(H)$. Post-multiplying (3.27) by $D^{-1}N_\ell$, we obtain

$$\tilde{V}_r N_\ell + \tilde{U}_r N_r D^{-1} N_\ell = D^{-1} N_\ell$$

Since $\tilde{V}_r$, N_ℓ, $\tilde{U}_r \in E(H)$ and $D^{-1}N_\ell \notin E(H)$,

$$N_r D^{-1} N_\ell \notin E(H)$$

Thus, $H_{yu} = N_r D^{-1} N_\ell + K \notin E(H)$, and so S is <u>not</u> H-stable. But this is a contradiction, hence det $D_h \in J$. □

In Theorem 3.1, we have developed two relationships between the admissibility of C and det D_h. We will now use these relationships to give global parametrizations of a) the family of all I/O maps possible for a given plant with some admissible controller; and b) the family of all disturbance-to-output maps possible for a given plant with some admissible controller.

For a given system S satisfying (P2) and (C1), and det $D_h \neq 0$, the I/O map $H_{y_2 v_1} : v_1 \mapsto y_2$, and the disturbance-to-output map $H_{y_2 d_o} : d_o \mapsto y_2$ are given by:

$$H_{y_2 v_1} = N_{pr} D_h^{-1} N_{\pi\ell} \tag{3.35}$$

$$H_{y_2 d_o} = I - N_{pr} D_h^{-1} N_{f\ell} \tag{3.36}$$

The corresponding families are defined as follows:

Definition 3.4

Let $P \in \tilde{G}^{n_o x n_i}$ be a given plant; hence the specification of the controller C determines the system S. Then,

$$\mathcal{H}_{y_2 v_1}(P) := \{H_{y_2 v_1} : C \text{ is admissible for } P\} \tag{3.37}$$

$$\mathcal{H}_{y_2 d_o}(P) := \{H_{y_2 d_o} : C \text{ is admissible for } P\} \tag{3.38}$$

Theorem 3.2 (Achievable I/O Maps)

Consider a plant P satisfying (P1) and (P2). For this plant,

$$\mathcal{H}_{y_2 v_1}(P) = \{N_{pr} M : M \in \mathcal{H}^{n_i x n_v}\} \tag{3.39}$$

Comments: (a) (3.39) is a global parametrization of all I/O maps achievable by an $\mathcal{H}$-stable S, with a two-input-one-output compensator $C \in \mathcal{E}(G)$. If we can factor N_{pr} as $N_{pr}^{(u)} N_{pr}^{(s)}$ where $N_{pr}^{(s)}$ has a right inverse in $\mathcal{E}(\mathcal{H})$, and $N_{pr}^{(u)}$ is a left factor of all possible $N_{pr}^{(u)}$'s, then (3.39) can be rewritten minimally, as

$$\mathcal{H}_{y_2 v_1}(P) = \{N_{pr}^{(u)} M : M \text{ is } \mathcal{H}\text{-stable}\}$$

Pernebo [Per. 1] has discussed this for the cases where $\mathcal{H} = \mathbb{R}_U(s)$ or $\mathcal{H} = \mathbb{R}_D(z)$.

(b) Suppose that $N_{pr}^{(u)} = I$, that is, N_{pr} has a right-inverse in $\mathcal{H}$. Then Theorem 3.2 asserts that any $M \in \mathcal{H}^{n_o x n_v}$ can be achieved as the I/O map $H_{y_2 v_1}$ of the system S with the given plant P and some admissible C. In particular, with a single-input single-output plant with a 3 dB bandwidth of a few hertz, one can achieve an I/O map with a 3 dB bandwidth in the megahertz (MHz) range! This is absurd because in reality, it would require huge gains with the compensator that would cause thermal noise to saturate the plant. (For a more realistic approach to design, see [Gus. 1]). With due regard to this limitation, Thm. 3.2 is very useful for it shows precisely what are the fundamental limitations on $H_{y_2v_1}$.

Proof

I) Select any $H_v \in \mathcal{H}_{y_2v_1}(P)$. Then, there exists an admissible controller C admissible for P such that the resulting system S has $H_{y_2v_1} = H_v$. So, since (P2) holds, by Theorem 3.1 (ii), det $D_h \in \mathcal{J}$. Thus $D_h^{-1} \in \mathcal{H}^{n_i x n_i}$ [Jac. 1, p. 94] and, since C is admissible, $N_{\pi\ell} \in \mathcal{H}^{n_i x n_v}$. Let $M := D_h^{-1} N_{\pi\ell}$. Then $M \in \mathcal{H}^{n_i x n_v}$, and by (3.35), $H_{y_2v_1} = N_{pr}M$. Hence,

$$\mathcal{H}_{y_2v_1}(P) \subset \{N_{pr}M : M \in \mathcal{H}^{n_i x n_v}\} \tag{3.40}$$

II) Now, select any $L \in \mathcal{H}^{n_i x n_v}$. Let $N_{\pi\ell} := L$. From (3.2) of (P2), there are $U_{pr} \in \mathcal{H}^{n_i x n_o}$, $V_{pr} \in \mathcal{H}^{n_i x n_i}$, such that

$$U_{pr}N_{pr} + V_{pr}D_{pr} = I_{n_i}$$

Define a controller $C := D_{c\ell}^{-1}[N_{\pi\ell} \vdots N_{f\ell}]$ with $D_{c\ell} := V_{pr}$ and $N_{f\ell} := U_{pr}$. Then, $D_h = I_{n_i}$, and det $D_h = 1 \in \mathcal{J}$. Thus, since $D_{c\ell} \in \mathcal{H}^{n_i x n_i}$, $N_{f\ell} \in \mathcal{H}^{n_i x n_o}$ and (P1) and (P2) hold, by Theorem 3.1 (i), C is admissible for P. And, by (3.35),

$$H_{y_2v_1} = N_{pr}D_h^{-1}N_{\pi\ell} = N_{pr}L \tag{3.41}$$

Hence,

$$\mathcal{H}_{y_2v_1}(P) \supset \{N_{pr}M : M \in \mathcal{H}^{n_i x n_v}\}$$

The conclusion follows from (3.40) and (3.41). □

Theorem 3.3 (Achievable Disturbance-to-Output Maps)

Let P satisfy (P1) and (P2), and let P have a ℓ.c.f. $D_{p\ell}^{-1}N_{p\ell}$. For this plant,

$$H_{y_2 d_o}(P) = \{I - N_{pr}(U_{pr} + YD_{p\ell}) : Y \in H^{n_i \times n_o}\}$$

Proof

I) Select any $Y \in H^{n_i \times n_o}$. Define a system S by choosing $C := D_{c\ell}^{-1}[\theta_{n_i \times n_v} \vdots N_{f\ell}]$ where $D_{c\ell} := V_{pr} - YN_{p\ell}$ and $N_{f\ell} := U_{pr} + YD_{p\ell}$. By (P2) and our assumption, $D_{c\ell}, N_{f\ell} \in E(H)$. Then

$$D_h = (V_{pr} - YN_{p\ell})D_{pr} + (U_{pr} + YD_{p\ell})N_{pr}$$

$$= V_{pr}D_{pr} + U_{pr}N_{pr} + Y(D_{p\ell}N_{pr} - N_{p\ell}D_{pr}) \tag{3.42}$$

Since $D_{p\ell}^{-1}N_{p\ell}$ is a ℓ.c.f. of P, we have

$$D_{p\ell}^{-1}N_{p\ell} = N_{pr}D_{pr}^{-1}$$

or

$$N_{p\ell}D_{pr} = D_{p\ell}N_{pr}$$

Thus, (3.42) becomes $D_h = V_{pr}D_{pr} + U_{pr}N_{pr} = I$, and $\det D_h = 1$. Consequently, since (P1) and (P2) hold, by Theorem 3.1 (i), C is admissible for P. Thus,

$$H_{y_2 d_o} = I - N_{pr}D_h^{-1}N_{f\ell} = I - N_{pr}(U_{pr} + YD_{p\ell}) \in H_{y_2 d_o}(P)$$

Hence,

$$\mathcal{H}_{y_2d_o}(P) \supset \{I-N_{pr}(U_{pr}+YD_{p\ell}) : Y \in \mathcal{H}^{n_i x n_o}\} \tag{3.44}$$

II) Now, select any $H_d \in \mathcal{H}_{y_2d_o}(P)$. Then, for the given P, there exists an admissible C which realizes $H_{y_2d_o} = H_d$. By (C2), this C has a ℓ.c.f., say $D_{c\ell}^{-1}[N_{\pi\ell}\vdots N_{f\ell}]$. Now, by Thm. 3.1 (ii), det $D_h \in \mathcal{J}$, since C is admissible for P. Thus, $D_h^{-1} \in \mathcal{H}^{n_i x n_i}$. So, $D_h^{-1}D_{c\ell} \in \mathcal{H}^{n_i x n_i}$, and $D_h^{-1}N_{f\ell} \in \mathcal{H}^{n_i x n_o}$. Also, from (3.11),

$$D_h^{-1}D_{c\ell}\cdot D_{pr} + D_h^{-1}N_{f\ell}\cdot N_{pr} = I_{n_i}$$

Thus, subtracting (3.2),

$$(D_h^{-1}D_{c\ell}-V_{pr})D_{pr} + (D_h^{-1}N_{f\ell}-U_{pr})N_{pr} = \theta_{n_i x n_i} \tag{3.45}$$

Choose

$$Y := (D_h^{-1}N_{f\ell}- U_{pr})D_{p\ell}^{-1} \tag{3.46}$$

or, equivalently,

$$YD_{p\ell} = D_h^{-1}N_{f\ell} - U_{pr} \tag{3.47}$$

(a) We prove that $Y \in \mathcal{E}(\mathcal{H})$

By (3.45),

$$D_h^{-1}D_{c\ell} - V_{pr} = - (D_h^{-1}N_{f\ell}-U_{pr})N_{pr}D_{pr}^{-1}$$

$$= - (D_h^{-1}N_{f\ell}-U_{pr})D_{p\ell}^{-1}N_{p\ell}$$

$$= - YN_{p\ell}$$

Now, since $(D_{p\ell},N_{p\ell})$ are ℓ.c. there exists $V_{p\ell}$, $U_{p\ell} \in \mathcal{E}(\mathcal{H})$ such that

$$D_{p\ell}V_{p\ell} + N_{p\ell}U_{p\ell} = I_{n_o}$$

Thus,

$$Y = Y(D_{p\ell}V_{p\ell}+N_{p\ell}U_{p\ell})$$

$$= (D_h^{-1}N_{f\ell}-U_{pr})V_{p\ell} - (D_h^{-1}D_{c\ell}-V_{pr})U_{p\ell}$$

So, since $D_h^{-1}N_{f\ell}$, $D_h^{-1}D_{c\ell}$, U_{pr}, V_{pr}, $U_{p\ell}$, $V_{p\ell} \in E(H)$, it follows that $Y \in E(H)$.

(b) We prove that the given H_d is of the required form $I - N_{pr}(U_{pr}+YD_{p\ell})$.

From (3.47),

$$D_h^{-1}N_{f\ell} = U_{pr} + YD_{p\ell}$$

Thus,

$$H_d = I - N_{pr}D_h^{-1}N_{f\ell} = I - N_{pr}(U_{pr}+YD_{p\ell})$$

So, the given H_d is in the set $\{I-N_{pr}(U_{pr}+YD_{p\ell}) : Y \in H^{n_i x n_o}\}$. Hence

$$H_{y_2d_o}(P) \subset \{I-N_{pr}(U_{pr}+YD_{p\ell}) : Y \in H^{n_i x n_o}\} \tag{3.50}$$

The conclusion follows from (3.44) and (3.50). □

Now, the parametrizations given in Theorems 3.2 and 3.3, suggest a general design scheme, which allows simultaneous realization of $H_v \in H_{y_2v_1}$ and $H_d \in H_{y_2d_o}$, for any such H_v and H_d.

Conceptual Design Algorithm 3.1

Data: $P = N_{pr}D_{pr}^{-1} = D_{p\ell}^{-1}N_{p\ell} \in G_s^{n_o x n_i}$; (N_{pr},D_{pr}) r.c., with $U_{pr}N_{pr} + V_{pr}D_{pr} = I_{n_i}$; $(D_{p\ell},N_{p\ell})$ ℓ.c.;

$H_v \in \mathcal{H}_{y_2 y_1}(P)$, $H_d \in \mathcal{H}_{y_2 d_o}(P)$ (both H_v, $H_d \in \mathcal{E}(\mathcal{H})$).

<u>Step 1</u>: Find $Y \in \mathcal{H}^{n_i \times n_o}$, such that $H_d = I - N_{pr}(U_{pr}+YD_{p\ell})$. Let

$$D_{c\ell} := V_{pr} - YN_{p\ell}$$

$$N_{f\ell} := U_{pr} + YD_{p\ell}$$

<u>Step 2</u>: Find $M \in \mathcal{H}^{n_i \times n_v}$, such that $H_v = N_{pr}M$. Let $N_{\pi\ell} := M$

<u>Step 3</u>: Choose a controller C (and thus specify a system S) by:

$$C := D_{c\ell}^{-1}[N_{\pi\ell} \vdots N_{f\ell}] \tag{3.53}$$

<u>Claim 3.1</u>: The system S, as specified by the plant P and controller C of Algorithm 3.1, satisfies the following:

(i) C is admissible for P,

(ii) $H_{y_2 v_1} = H_v$,

(iii) $H_{y_2 d_o} = H_d$.

<u>Justification of Claim</u>:

(i)
$$\begin{aligned} D_h &= D_{c\ell}D_{pr} + N_{f\ell}N_{pr} \\ &= (V_{pr}-YN_{p\ell})D_{pr} + (U_{pr}+YD_{p\ell})N_{pr} \\ &= V_{pr}D_{pr} + U_{pr}N_{pr} + Y(D_{p\ell}N_{pr}-N_{p\ell}D_{pr}) \\ &= I_{n_i} \end{aligned}$$

Thus, det $D_h = 1 \in \mathcal{J}$. So, since (P1) and (P2) are satisfied by assumption, and $D_{c\ell}$, $N_{f\ell} \in \mathcal{E}(\mathcal{H})$, by Theorem 3.1 (i), C is <u>admissible for P</u>.

(ii) For the system S, defined by P and C, we have

$$H_{y_2 d_o} = I - N_{pr} D_h^{-1} N_{f\ell}$$

$$= I - N_{pr}(U_{pr} + Y D_{p\ell}) = H_d$$

Thus, $H_{y_2 d_o} = H_d$ as required.

(iii) Similarly,

$$H_{y_2 v_1} = N_{pr} D_h^{-1} N_{\pi\ell}$$

$$= N_{pr} M = H_v$$

Thus, $H_{y_2 v_1} = H_v$ as required.

IV. Lumped Case Design Using Polynomial Subrings

4.1. Motivation

The results developed in Section III are valid for many classes of systems, some of which are listed in Table I. However, perhaps the most important classes are the first two given in Table I: the lumped continuous time case ($H = \mathbb{R}_U(s)$), and the lumped discrete time case ($H = \mathbb{R}_D(z)$). In both of these cases, H is a ring whose elements are only rational functions (in s or z, as appropriate). Ideally, however, we would like H to contain, as a subring, the ring of polynomials in either s or z. This is desirable for ease of computation, specifically: in solution of a Bezout identity (i.e., finding U_{pr}, V_{pr}, N_{pr} and D_{pr} in (3.3), given $P \in E(G)$), and in addition.

In this section, we give a computationally efficient method, of transforming design problems with $H = \mathbb{R}_U(s)$ or $H = \mathbb{R}_D(z)$ into design problems with $H = \mathbb{R}_\Lambda(\lambda)$ (with $\infty \notin \Lambda$, $\mathbb{R}_\Lambda(\lambda)$ contains all non-proper transfer functions in λ, including, of course, $\mathbb{R}[\lambda]$, the ring of polynomials in λ). Conceptually, the method is this: A transformation f, mapping s (z, respectively) into λ is defined. Then, using this change of variables, the transformation $P \in E(\mathbb{R}_{\{o\},o}(\lambda))$ of a given plant $\tilde{P} \in E(\mathbb{R}_{p,o}(s))$ ($\tilde{P} \in E(\mathbb{R}_{p,o}(z))$, resp.) is found. Next, the design methods of section III are used to generate a controller $C \in E(\mathbb{R}_{\{o\}}(\lambda))$. Finally, $\tilde{C} \in E(\mathbb{R}_p(s))$ ($\tilde{C} \in E(\mathbb{R}_p(z))$, resp.), the inverse image of the controller C is found. Details of the transformations are given in Sections 4.2 and 4.3.

Consider again the conceptual design algorithm of Section III. If $H = \mathbb{R}_\Lambda(\lambda)$, we can modify the algorithm to take advantage of the fact that $\mathbb{R}[\lambda] \subset \mathbb{R}_\Lambda(\lambda)$. This modified algorithm is presented below. Note that this algorithm is valid for either the discrete time or lumped continuous time case, once the transformation $P \in E(\mathbb{R}_{\{o\},o}(\lambda))$ has been calculated.

Conceptual Design Algorithm 4.1

Data: $P = N_{pr}D_{pr}^{-1} = D_{p\ell}^{-1}N_{p\ell} \in \mathbb{R}_{\{o\},o}(\lambda)^{n_o \times n_i}$; $(N_{pr}, D_{pr}) \in E(\mathbb{R}[\lambda])$, r.c., with $U_{pr}N_{pr} + V_{pr}D_{pr} = I_{n_i}$, and $U_{pr}, V_{pr} \in E(\mathbb{R}[\lambda])$; $(D_{p\ell}, N_{p\ell}) \in E(\mathbb{R}[\lambda])$, ℓ.c.; $H_v \in H_{y_2v_1}(P)$ and $H_d \in H_{y_2d_o}(P)$.

Step 1: Find $Y \in \mathbb{R}_\Lambda(\lambda)^{n_i \times n_o}$, such that $H_d = I_{n_o} - N_{pr}(U_{pr} + YD_{p\ell})$.

Step 2: Find a ℓ.c.f. $D_{y\ell}^{-1}N_{y\ell}$ of Y, with $N_{y\ell}, D_{y\ell} \in E(\mathbb{R}[\lambda])$. Let

$$\overline{D}_{c\ell} := D_{y\ell}V_{pr} - N_{y\ell}N_{p\ell}$$

$$\overline{N}_{f\ell} := D_{y\ell}U_{pr} + N_{y\ell}D_{p\ell}$$

Step 3: Find $M \in \mathbb{R}_\lambda(\lambda)^{n_i \times n_v}$, such that $H_v = N_{pr}M$

<u>Step 4</u>: Find a ℓ.c.f. $\bar{D}_{\pi\ell}^{-1}\bar{N}_{\pi\ell}$ of $\bar{\Pi} := D_{y\ell} M \in E(\mathbb{R}_\Lambda(\lambda))$, with $\bar{D}_{\pi\ell}, \bar{N}_{\pi\ell} \in E(\mathbb{R}[\lambda])$. Let

$$D_{c\ell} := \bar{D}_{\pi\ell}\bar{D}_{c\ell}$$

$$N_{f\ell} := \bar{D}_{\pi\ell}\bar{N}_{f\ell}$$

$$N_{\pi\ell} := \bar{N}_{\pi\ell}$$

<u>Step 5</u>: The required controller, and hence the system S, is specified by:

$$C := D_{c\ell}^{-1}[N_{\pi\ell} \vdots N_{f\ell}]$$

<u>Claim 4.1</u>: The system S, as specified by the plant P and controller C of Algorithm 4.1, satisfies the following:

(i) C is admissible for P,

(ii) $H_{y_2 v_1} = H_v$,

(iii) $H_{y_2 d_o} = H_d$.

<u>Justification of Claim</u>:

(i) $D_h = D_{c\ell}D_{pr} + N_{f\ell}N_{pr}$

$$= \bar{D}_{\pi\ell}(D_{y\ell}V_{pr}-N_{y\ell}N_{p\ell})D_{pr} + \bar{D}_{\pi\ell}(D_{y\ell}U_{pr}+N_{y\ell}D_{p\ell})N_{pr}$$

$$= \bar{D}_{\pi\ell}D_{y\ell}(V_{pr}D_{pr}+U_{pr}N_{pr}) - \bar{D}_{\pi\ell}N_{y\ell}(N_{p\ell}D_{pr}-D_{p\ell}N_{pr})$$

$$= \bar{D}_{\pi\ell}D_{y\ell} \tag{4.10}$$

Thus, $\det D_h = \det \bar{D}_{\pi\ell} \cdot \det D_{y\ell}$. By Step 4, $\bar{\Pi} \in E(\mathbb{R}_\Lambda(\lambda))$, and by Step 1, $Y \in E(\mathbb{R}_\Lambda(\lambda))$. Hence, $\bar{D}_{\pi\ell}^{-1} \in E(\mathbb{R}_\Lambda(\lambda))$ and $D_{y\ell}^{-1} \in E(\mathbb{R}_\Lambda(\lambda))$ (because

$(\bar{D}_{\pi\ell},\bar{N}_{\pi\ell})$ and $(D_{y\ell},N_{y\ell})$ are ℓ.c. pairs) and so $(\det \bar{D}_{\pi\ell})^{-1}$, $(\det D_{y\ell})^{-1}$ $\in \mathbb{R}_\Lambda(\lambda)$. So,

$$(\det D_h)^{-1} = (\det \bar{D}_{\pi\ell})^{-1}\cdot(\det D_{y\ell})^{-1} \in \mathbb{R}_\Lambda(\lambda)$$

and hence, since (P1), (P2) are satisifed by assumption, and $D_{c\ell},N_{f\ell}$ $\in E(\mathbb{R}_\Lambda(\lambda))$, C is admissible for P, by Theorem 3.1 (i).

(ii) For the system S, defined by P and C, we have

$$H_{y_2v_1} = N_{pr}D_h^{-1}N_{\pi\ell}$$

$$= N_{pr}(D_{y\ell}^{-1}\bar{D}_{\pi\ell}^{-1})\bar{N}_{\pi\ell}$$

$$= N_{pr}D_{y\ell}^{-1}\cdot D_{y\ell}M = N_{pr}M = H_v$$

Thus, $H_{y_2v_1} = H_v$ as required.

(iii) Similarly,

$$H_{y_2d_o} = I_{n_o} - N_{pr}D_h^{-1}N_{f\ell}$$

$$= I_{n_o} - N_{pr}(D_{y\ell}^{-1}\bar{D}_{\pi\ell}^{-1})\bar{D}_{\pi\ell}(D_{y\ell}U_{pr}+N_{y\ell}D_{p\ell})$$

$$= I_{n_o} - N_{pr}(U_{pr}+YD_{p\ell}) = H_d$$

Thus $H_{y_2d_o} = H_d$ as required. □

<u>Comments</u>: (a) From Eq. (4.10), it is clear that the zeros of det D_h are fixed by specification of M and Y (remember that $\bar{D}_{\pi\ell}^{-1}\bar{N}_{\pi\ell} = D_{y\ell}M$ - thus the dynamics specified by $\bar{D}_{\pi\ell}$ are those of $D_{m\ell}$ which are not included in

$D_{y\ell}$, if $D_{m\ell}^{-1}N_{m\ell}$ is a ℓ.c.f. of M). Thus the dynamics of the closed-loop system are completely specified by the choice of M and Y.

(b) An actual engineering design problem would probably not be formulated as a synthesis problem of the type in Algorithm 4.1, but rather as a problem of finding the best design that satisfies certain design criteria. In this case, the designer would not have a prespecified M and Y, but rather, would choose M and Y as part of the design process, and use the approach of Algorithm 4.1 to find the resulting compensator C. Such a design process can be automated by formulating the design problem as an optimization problem [Gus. 1], [May. 1].

4.2. Application to Lumped Continuous Time Case

We will utilize the results of Section 4.1 for $\tilde{P} \in \mathbb{R}_{p,o}(s)^{n_o \times n_i}$ by introducing the following transformation.

Definition 4.1

$$f : \mathbb{C}\backslash\{-\alpha\} \to \mathbb{C} \text{ is defined by } f : s \mapsto \lambda = \frac{1}{s+\alpha}$$

$$f^{-1} : \mathbb{C}\backslash\{0\} \to \mathbb{C} \text{ is defined by } f^{-1} : \lambda \mapsto s = \frac{1-\lambda\alpha}{\lambda}$$

with $\alpha \in U^c$, where $U \subset \mathbb{C}$ is the region of instability. We assume that $\infty \in U$, so that $\mathbb{R}_U(s) \subset \mathbb{R}_p(s)$.

Definition 4.2

For a given $\tilde{P} \in \mathbb{R}(s)^{n_o \times n_i}$, we define $P \in \mathbb{R}(\lambda)^{n_o \times n_i}$ by

$$P(\lambda) := \tilde{P}(f^{-1}(\lambda)) = \tilde{P}(\frac{1-\lambda\alpha}{\alpha}), \ \forall \lambda \in C \qquad (4.11)$$

It is crucial to note that the calculation of P given $\tilde{P}$ in pole-zero form, is trivial. To wit, let

$$\tilde{p}(s) = \frac{k(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)(s+p_3)}$$

Then

$$p(\lambda) = \tilde{p}\left(\frac{1-\lambda\alpha}{\lambda}\right) = \frac{k\lambda[1+\lambda(z_1-\alpha)][1+\lambda(z_2-\alpha)]}{[1+\lambda(p_1-\alpha)][1+\lambda(p_2-\alpha)][1+\lambda(p_3-\alpha)]}$$

Clearly, the inverse transformation is just as simple. It is because of the simplicity of this sort of calculation, that this design method using $H = \mathbb{R}_\Lambda(\lambda)$ is <u>computationally efficient</u> and in fact <u>less expensive</u> computationally than direct calculations with $H = \mathbb{R}_U(s)$ or $H = \mathbb{R}_D(z)$. Note that if $\tilde{P}$ is <u>not</u> given in pole-zero form, it is usually quite simple to put it in that form.

<u>Fact 4.1</u>

(i) $\tilde{P} \in \mathbb{R}(s)^{n_o \times n_i} \Longleftrightarrow P \in \mathbb{R}(\lambda)^{n_o \times n_i}$

(ii) $\tilde{P} \in \mathbb{R}_p(s)^{n_o \times n_i}$ (proper) $\Longleftrightarrow P \in \mathbb{R}_{\{o\}}(\lambda)^{n_o \times n_i}$

(iii) $\tilde{P} \in \mathbb{R}_{p,o}(s)^{n_o \times n_i}$(str. proper) $\Longleftrightarrow P \in \mathbb{R}_{\{o\},o}(\lambda)^{n_o \times n_i}$

(iv) $\tilde{P} \in \mathbb{R}_U(s)^{n_o \times n_i}$ (U-stable) $\Longleftrightarrow \begin{cases} P \in \mathbb{R}_\Lambda(\lambda)^{n_o \times n_i}, \text{ where} \\ \Lambda = f(U) \quad (\Lambda\text{-stable}) \end{cases}$

So, given $\tilde{P} \in \mathbb{R}_{p,o}(s)^{n_o \times n_i}$, we can obtain P from (4.10). Then, by the methods of Section 4.1, we can design a controller $C \in E(\mathbb{R}_{\{o\}}(\lambda))$, with a ℓ.c.f. $D_{c\ell}^{-1}[N_{\pi\ell} \vdots N_{f\ell}]$ over $\mathbb{R}[\lambda]$. Then the following procedure can be used to obtain a ℓ.c.f., over $\mathbb{R}[s]$, of $\tilde{C}(s) := C(f(s))$. Note that $\tilde{C} \in E(\mathbb{R}_p(s))$, by Fact 4.1 (ii).

(i) Find $L_1(\lambda) \in E(\mathbb{R}[\lambda])$, such that $L_1^{-1}(\lambda) \in \mathbb{R}[\lambda]$, and

$C_{hr} := L_1(\lambda)[D_{c\ell}(\lambda) \vdots N_{\pi\ell}(\lambda) \vdots N_{f\ell}(\lambda)]_{hr}$ has full row rank.

(ii) Let $\bar{D}_{c\ell} := L_1 D_{c\ell}$ (4.12)

$$\bar{N}_{f\ell} := L_1 N_{f\ell} \tag{4.13}$$

$$\bar{N}_{\pi\ell} := L_1 N_{\pi\ell} \tag{4.14}$$

(iii) Let $\tilde{L}_2(s) := \text{diag}[(s+\alpha)^{r_i}]_{i=1}^{n_i}$, where[2] (4.15)

$$r_i := \partial\rho_i[D_{c\ell}(\lambda) \vdots N_{\pi\ell}(\lambda) \vdots N_{f\ell}(\lambda)] \tag{4.16}$$

(iv) Let $\tilde{D}_{c\ell}(s) := \tilde{L}_2(s)\bar{D}_{c\ell}(f(s)) = \tilde{L}_2(s)\bar{D}_{c\ell}(\frac{1}{s+\alpha})$ (4.17)

$$\tilde{N}_{f\ell}(s) := \tilde{L}_2(s)\bar{N}_{f\ell}(f(s)) = \tilde{L}_2(s)\bar{N}_{f\ell}(\frac{1}{s+\alpha}) \tag{4.18}$$

$$\tilde{N}_{\pi\ell}(s) := \tilde{L}_2(s)\bar{N}_{\pi\ell}(f(s)) = \tilde{L}_2(s)\bar{N}_{\pi\ell}(\frac{1}{s+\alpha}) \tag{4.19}$$

<u>Remark</u>: Step (i) can always be accomplished by reduction of $[D_{c\ell}(\lambda) \vdots N_{\pi\ell}(\lambda) \vdots N_{f\ell}(\lambda)]$ to row-Hermite form [Kai. 1], [Cal. 1].

<u>Fact 4.2</u>: $\tilde{D}_{c\ell}^{-1}[\tilde{N}_{\pi\ell}(s) \vdots \tilde{N}_{f\ell}(s)]$, as constructed in (4.12)-(4.19) is a ℓ.c.f. of $\tilde{C}(s) := C(f(s))$ over $\mathbb{R}[\lambda]$.

<u>Proof</u>

Since $f^{-1}(\lambda^{r_i}) = \dfrac{1}{(s+\alpha)^{r_i}}$, it is clear that

$(s+\alpha)^{r_i} \cdot \rho_i[\bar{D}_{c\ell}(f(s)) \vdots \bar{N}_{\pi\ell}(f(s)) \vdots \bar{N}_{f\ell}(f(s))] \in E(\mathbb{R}[s])$, for $i = 1, 2, \ldots, n_o$.

Thus, $\tilde{L}_2(s)[\bar{D}_{c\ell}(f(s)) \vdots \bar{N}_{\pi\ell}(f(s)) \vdots \bar{N}_{f\ell}(f(s))] \in E(\mathbb{R}[s])$.

Now, since $(D_{c\ell}(\lambda), [N_{\pi\ell}(\lambda) \vdots N_{f\ell}(\lambda)])$ are ℓ.c., $\text{rk}[\bar{D}_{c\ell}(f(s)) \vdots \bar{N}_{\pi\ell}(f(s)) \vdots \bar{N}_{f\ell}(f(s))] = n_o$, for all $s \in \mathbb{C}\backslash\{-\alpha\}$. Thus,

---

[2] For $A \in \mathbb{R}[\lambda]^{m\times n}$, $\partial\rho_i[A]$ denotes the highest degree of any polynomial in the i^{th} row of A.

$\mathrm{rk}[\tilde{D}_{c\ell}(s) \vdots \tilde{N}_{\pi\ell}(s) \vdots \tilde{N}_{f\ell}(s)] = n_o$, for all $s \in \mathbb{C}\setminus\{-\alpha\}$, since $L_2(s)$ is non-singular for $s \in \mathbb{C}\setminus\{-\alpha\}$. And, for $s = -\alpha$, $[\tilde{D}_{c\ell}(-\alpha) \vdots \tilde{N}_{\pi\ell}(-\alpha) \vdots \tilde{N}_{f\ell}(-\alpha)]$ $= C_{hr}$, which has full row rank.

Thus, $\mathrm{rk}[\tilde{D}_{c\ell}(s) \vdots \tilde{N}_{\pi\ell}(s) \vdots \tilde{N}_{f\ell}(s)] = n_o$, for all $s \in \mathbb{C}$. Hence $\tilde{D}_{c\ell}^{-1}(s)[\tilde{N}_{\pi\ell}(s) \vdots \tilde{N}_{f\ell}(s)]$ is a ℓ.c.f. of $\tilde{C}(s) := C(f(s))$ over $\mathbb{R}[s]$.

4.3. Application to Discrete Time Case

We will utilize the results of Section 4.1 for $\tilde{P} \in \mathbb{R}(z)^{n_o \times n_i}$ by introducing the following transformation.

Definition 4.3

$g : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ is defined by $g : z \to \lambda = \frac{1}{z}$

$g^{-1} : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ is defined by $g^{-1} : \lambda \to z = \frac{1}{\lambda}$

Definition 4.4

For a given $\tilde{P} \in \mathbb{R}(z)^{n_o \times n_i}$, we define $P \in \mathbb{R}(\lambda)^{n_o \times n_i}$

$$P(\lambda) := \tilde{P}(g^{-1}(\lambda)) = \tilde{P}(\tfrac{1}{\lambda}) \qquad (4.25)$$

Fact 4.3:

(i) $\tilde{P} \in \mathbb{R}(z)^{n_o \times n_i}$ $\Longleftrightarrow$ $P \in \mathbb{R}(\lambda)^{n_o \times n_i}$

(ii) $\tilde{P} \in \mathbb{R}_p(z)^{n_o \times n_i}$ (causal) $\Longleftrightarrow$ $P \in \mathbb{R}_{\{o\}}(\lambda)^{n_o \times n_i}$

(iii) $\tilde{P} \in \mathbb{R}_{p,o}(z)^{n_o \times n_i}$ (strictly causal) $\Longleftrightarrow$ $P \in \mathbb{R}_{\{o\},o}(\lambda)^{n_o \times n_i}$

(iv) $\tilde{P} \in \mathbb{R}_D(z)$ (D-stable) $\Longleftrightarrow$ $\begin{cases} P \in \mathbb{R}_\Lambda(\lambda)^{n_o \times n_i}, \text{ where} \\ \Lambda = g(D) \quad (\Lambda\text{-stable}) \end{cases}$

Comment: As in the continuous time case, we assume that $\infty \in D$, so that $\mathbb{R}_D(z) \subset \mathbb{R}_p(z)$.

So, given $\tilde{P} \in \mathbb{R}_{p,o}(z)^{n_o \times n_i}$, we can obtain P from (4.25). Then, by the methods of Section 4.1, we can design a controller $C \in E(\mathbb{R}_{\{o\}}(\lambda))$, with a ℓ.c.f. $D_c^{-1}[N_{\pi\ell} \vdots N_{f\ell}]$ over $\mathbb{R}[\lambda]$. And, since $\lambda = z^{-1}$, we can directly implement a controller $\tilde{C} \in E(\mathbb{R}_p(z))$, without taking an inverse transformation (that $\tilde{C} \in E(\mathbb{R}_p(z))$ follows directly from Fact 4.3 (ii)).

V. Robustness: Asymptotic Tracking and Stability

In this section, we consider the problem of designing, for a given plant P, a compensator C, which is admissible for P, and is robust with respect to the asymptotic tracking of a given family of inputs Ψ (See Fig. 2). This problem will be formulated and solved in the algebraic framework of section III. In developing a robustness result, we will consider the fractional perturbation approach [Chen 1], [Vid. 1] and develop sufficient conditions for the robustness of stability.

5.1. Robust Stability

The following robust stability theorem is similar to [Chen 1: Cor. 4.4], except that multiple perturbations (both plant and compensator) are considered.

Theorem 5.1 (Robust Stability)

Consider the system S, of Figure 1, with P satisfying (P2), and C admissible for P. Let D_{pr}, N_{pr}, $D_{c\ell}$, $N_{f\ell}$ and $N_{\pi\ell}$ be additively perturbed by, resp., ΔD_{pr}, ΔN_{pr}, $\Delta D_{c\ell}$, $\Delta N_{f\ell}$, $\Delta N_{\pi\ell} \in E(H)$ with $\det(D_{pr}+\Delta D_{pr})$, and $\det(D_{c\ell}+\Delta D_{c\ell}) \in I$. Let $(H, \|\cdot\|)$ be a Banach algebra and B(0;r) denote the open ball of radius r centered on 0. Now, let $\rho_{dp} > 0$, $\rho_{np} > 0$, $\rho_{dc} > 0$, $\rho_{nf} > 0$, be such that

$$\|D_h^{-1} D_{c\ell}\| \rho_{dp} + \|D_h^{-1} N_{f\ell}\| \rho_{np} + \|D_h^{-1}\| (\|D_{pr}\| \rho_{dc} + \|N_{pr}\| \rho_{nf} + \rho_{dp}\rho_{dc} + \rho_{np}\rho_{nf}) < 1 \tag{5.1}$$

U.t.c., if

$$\begin{aligned} &\Delta D_{pr} \in B(0;\rho_{dp}) \qquad\qquad \Delta D_{c\ell} \in B(0;\rho_{dc}) \\ &\qquad\qquad\qquad \text{and} \\ &\Delta N_{pr} \in B(0;\rho_{np}) \qquad\qquad \Delta N_{f\ell} \in B(0;\rho_{nf}) \end{aligned} \tag{5.2}$$

then, the perturbed system is H-stable.

<u>Proof</u>

Let $\tilde{D}_{pr} := D_{pr} + \Delta D_{pr}$, $\tilde{N}_{pr} := N_{pr} + \Delta N_{pr}$, $\tilde{D}_{c\ell} := D_{c\ell} + \Delta D_{c\ell}$, $\tilde{N}_{\pi\ell} := N_{\pi\ell} + \Delta N_{\pi\ell}$ and $\tilde{N}_{f\ell} := N_{f\ell} + \Delta N_{f\ell}$ denote the perturbed numerator and demoninator matrices of the plant and the compensator. Let the perturbed system defined by $\tilde{D}_{pr}$, $\tilde{N}_{pr}$, $\tilde{D}_{c\ell}$, $\tilde{N}_{f\ell}$ and $\tilde{N}_{\pi\ell}$ be denoted as $\tilde{S}$. In accordance with Definition 3.3, $\tilde{S}$ will be called H-stable iff $\tilde{H}_{yu} := \tilde{N}_r\tilde{D}^{-1}\tilde{N}_\ell + K$ is H-stable (where $\tilde{N}_r$, $\tilde{D}$ and $\tilde{N}_\ell$ are the perturbations of N_r, D and N_ℓ, resulting from ΔD_{pr}, ΔN_{pr}, $\Delta D_{c\ell}$, $\Delta N_{f\ell}$ and $\Delta N_{\pi\ell}$). It is thus clear that if $\tilde{D} \in E(H)$ is invertible in $E(H)$, then $\tilde{S}$ is H-stable. We prove that $\tilde{D}^{-1} \in E(H)$ as follows.

First, note that $\tilde{D}^{-1} \in E(H)$, if $[D_h^{-1}\tilde{D}_h]^{-1} \in E(H)$ where $\tilde{D}_h := \tilde{D}_{c\ell}\tilde{D}_{pr} + \tilde{N}_{f\ell}\tilde{N}_{pr}$. This follows from performing elementary row operations on $\tilde{D}$, showing that $\det \tilde{D} = \eta \cdot \det \tilde{D}_h$, where $\eta = \pm 1$, and from the fact that $D_h^{-1} \in E(H)$, by Theorem 3.1 (ii). Thus, it is sufficient to show that $[D_h^{-1}\tilde{D}_h]^{-1} \in E(H)$. Now,

$$\begin{aligned} D_h^{-1}\tilde{D}_h = I &+ D_h^{-1}D_{c\ell}\Delta D_{pr} + D_h^{-1}N_{f\ell}\Delta N_{pr} + D_h^{-1}\Delta D_{c\ell}D_{pr} \\ &+ D_h^{-1}\Delta N_{f\ell}N_{pr} + D_h^{-1}\Delta D_{c\ell}\Delta D_{pr} + D_h^{-1}\Delta N_{f\ell}\Delta N_{pr} \end{aligned} \tag{5.5}$$

And, by (5.1) and (5.2),

$$\| D_h^{-1} D_{c\ell} \Delta D_{pr} + D_h^{-1} N_{f\ell} \Delta N_{pr} + D_h^{-1} \Delta D_{c\ell} D_{pr} + D_h^{-1} \Delta N_{f\ell} N_{pr} + D_h^{-1} \Delta D_{c\ell} \Delta D_{pr}$$

$$+ D_h^{-1} \Delta N_{f\ell} \Delta N_{pr} \| < 1$$

Consequently, by (5.5) and [Die. 1, (8.3.2.1)];

$$[D_h^{-1} \tilde{D}_h]^{-1} \in E(H)$$

It thus follows that $\tilde{D}^{-1} \in E(H)$, and hence the perturbed system $\tilde{S}$ is H-stable.

Comments: (a) Clearly, this result supplies only sufficient conditions for H-stability of $S(\Delta N_{pr}, \Delta D_{pr})$. However, there are no requirements imposed on ΔN_{pr}, ΔD_{pr}, $\Delta D_{c\ell}$, $\Delta N_{\pi\ell}$, $\Delta N_{f\ell} \in E(H)$ beyond (5.2). Thus, this result allows for a more general class of perturbations than others [Cru. 1], [Pos. 1], [Zam. 1], [Doy. 1]: e.g., in the lumped case, it allows for changes in the number and the location of poles and zeros.

(b) A similar result may be obtained for the case in which a left coprime factorization of the plant and a right coprime factorization of the compensator are used. This will be utilized in the discussion of robust asymptotic tracking in Section 5.2.

5.2. Asymptotic Tracking

For the tracking problem we consider the unity-feedback configuration S_1 of Figure 2. The class of inputs Ψ, to be considered in the tracking problem, is defined as follows.

Definition 5.1

The class of Ψ of inputs to be tracked consists of vectors $\psi^{-1}u$ where $\psi \in I \backslash J$ and $u \in H^{n_i}$, with the property that for all $u \in H^{n_i}$ that are not a multiple of ψ, the vector $\psi^{-1}u \notin E(H)$.

Definition 5.3

The closed-loop system S will be said to asymptotically track the class Ψ iff $y_2 - u_1 \in H$, $\forall\, u_1 \in \Psi$.

We now present three results on the tracking problem for the configuration S_1 of Figure 2.

Theorem 5.2 (Necessary Conditions)

Let P satisfy (P2). Let C be an admissible compensator for P; thus C has a ℓ.c.f. $D_{c\ell}^{-1}N_{c\ell}$. Suppose that S_1, as specified by P and C asymptotically tracks the class Ψ. U.t.c.,

$$\text{(i) } n_i \geq n_o \tag{5.7}$$

(ii) the only common factors of $\det(N_{pr}N_{c\ell})$ and ψ are units of H.

Comment: The interpretation of (ii) for the lumped case is that PC and ψ have no zeros in common.

Proof

Let us define $D_{h\ell} \in H^{n_i \times n_i}$ by

$$D_{h\ell} := D_{c\ell}D_{pr} + N_{c\ell}N_{pr} \tag{5.9}$$

It can easily be shown (similar to Theorem 3.1 (ii)), that C admissible for P implies that $\det D_{h\ell} \in J$ (hence $D_{h\ell}^{-1} \in H^{n_i \times n_i}$). Thus, there exists a ℓ.c.f. $\tilde{D}_{c\ell}^{-1}\tilde{N}_{c\ell}$ of C such that

$$\tilde{D}_{c\ell}D_{pr} + \tilde{N}_{c\ell}N_{pr} = I_{n_i}$$

Consequently, $H_{y_2u_1} : u_1 \mapsto y_2$ in S_1, is given by

$$H_{y_2u_1} = N_{pr}\tilde{N}_{c\ell} \tag{5.10}$$

(i) Assume that $n_o > n_i$. We will show that a contradiction results. Since $n_o > n_i$,

$$\text{rk } H_{y_2u_1} \le \min(\text{rk } N_{pr}, \text{rk } \tilde{N}_{c\ell}) \le n_i < n_o$$

Thus, there exists $\gamma \in H^{n_o}$, such that [Bou. 2, Chap III, §8, Prop. 14]

$$\text{(a) } H_{y_2u_1}\gamma = \theta_{n_o} \tag{5.11}$$

$$\text{(b) } \gamma \text{ is not a multiple of } \psi \tag{5.12}$$

(If γ were a multiple of ψ, say $\gamma = \psi^k\tilde{\gamma}$, where k is the multiplicity of ψ as a factor of γ, then $H_{y_2u_1}\tilde{\gamma} = \theta$, and $\tilde{\gamma}$ would not be a multiple of ψ).

To develop the contradiction, we apply the input $u_1 = \psi^{-1}\gamma \notin E(H)$ (from (5.12)). The resulting output y_2 is given by

$$y_2 = H_{y_2u_1}\cdot u_1 = H_{y_2u_1}\cdot\psi^{-1}\gamma = \theta_n.$$

Hence, $y_2 - u_1 = \psi^{-1}\gamma \notin E(H)$, which contradicts the assumption that S_1 tracks ψ asymptotically. Thus $n_i \ge n_o$.

(ii) Consider $\tilde{N}_{c\ell}$ as defined in part (i):

$$\det(N_{pr}\tilde{N}_{c\ell}) = \det(N_{pr}N_{c\ell})\cdot\det(D_{h\ell}^{-1})$$

Since $\det(D_{h\ell}^{-1}) \in J$, we can assume, without loss of generality, that $\det(N_{pr}\tilde{N}_{c\ell}) = \det(N_{pr}N_{c\ell})$.

In order to develop a contradiction, assume that $\det(N_{pr}\tilde{N}_{c\ell})$ and ψ have a common factor $v \in H$. Let k_1 denote the multiplicity of v as a factor of $\det(N_{pr}\tilde{N}_{c\ell})$. Consequently, there exist $\tilde{\psi}$, m, $\tilde{m} \in H$, such that

$\psi = \tilde{\psi}\cdot v$

$\det(N_{pr}\tilde{N}_{c\ell}) = m\cdot v = \tilde{m}\cdot v^{k_1}$

We will construct an input $u_1 \in \psi$, such that $y_2 - u_1 \notin E(H)$ where y_2 is the output resulting from the input u_1.

Consider the matrix $N_{pr}\tilde{N}_{c\ell} \in H^{n_o x n_o}$. If $rk(N_{pr}\tilde{N}_{c\ell}) < n_o$, then, as in part (i), we can find $\gamma \in H^{n_o}$ satisfying (5.11) and (5.12). The input $u_1 := \psi^{-1}\gamma \notin E(H)$ then yields $y_2 = 0$, and thus $y_2 - u_1 \notin E(H)$.

So, suppose $rk(N_{pr}\tilde{N}_{c\ell}) = n_o$. Then, $\det(N_{pr}\tilde{N}_{c\ell}) \neq 0$, and thus the expression

$$I_{n_o}\cdot\det(N_{pr}\tilde{N}_{c\ell}) = Adj(N_{pr}\tilde{N}_{c\ell})\cdot N_{pr}\tilde{N}_{c} \tag{5.15}$$

yields

$$\det[Adj(N_{pr}\tilde{N}_{c\ell})] = [\det(N_{pr}\tilde{N}_{c\ell})]^{n_o-1} \neq 0 \tag{5.16}$$

Using (5.16), we will show that some element of $Adj(N_{pr}\tilde{N}_{c\ell})$ has v as a factor with a multiplicity which is <u>strictly less</u> than k_1, (the multiplicity of v as a factor of det $(N_{pr}\tilde{N}_{c\ell})$): if not, then every term in the summation

$$\det[Adj(N_{pr}\tilde{N}_{c\ell})] = \sum_{\sigma} sgn(\sigma)\cdot n_{1\sigma(1)}n_{2\sigma(2)} \cdots n_{n_o\sigma(n_o)}$$

(where n_{ij} denotes the $ij^{\underline{th}}$ element of $Adj(N_{pr}\tilde{N}_{c\ell})$) would have $v^{n_o k_1}$ as a factor, implying that $\det[Adj(N_{pr}\tilde{N}_{c\ell})]$ would have $v^{n_o k_1}$ as a factor, contradicting (5.16), which indicates that v has multiplicity of only $k_1(n_o-1)$ as a factor of $\det[Adj(N_{pr}\tilde{N}_{c\ell})]$.

Let $\tilde{\beta} \in H^{n_o}$ one of the columns (say the $\ell^{\underline{th}}$ column) of $Adj(N_{pr}\tilde{N}_{c\ell})$

containing the element which has multiplicity of v as factor strictly less than that of $\det(N_{pr}\tilde{N}_{c\ell})$. Let k_2 be the <u>least</u> multiplicity of v as a factor of any of the elements of $\tilde{\beta}$. Then, $k_1 > k_2$ and $k_1 - 1 \geq k_2$.

Define $\beta := v^{-k_2}\tilde{\beta}$. Then, one element of β (say the j^{th} element, denoted β_j) does not have v as a factor, and additionally, $\beta \in H^{n_o}$.

We can now define the input u_1 by,

$$u_1 := \psi^{-1}\tilde{\psi}\beta \quad \text{where} \quad \tilde{\psi} \in H,\ \beta \in H^{n_o}$$

$$= v^{-1}\beta \tag{5.18}$$

Clearly, $u_1 \notin E(H)$, and thus $u_1 \in \Psi$. The resulting output y_2 is given by

$$y_2 = N_{pr}\tilde{N}_{c\ell}u_1 \qquad \text{(from (5.10))}$$

Thus,

$$\mathrm{Adj}(N_{pr}\tilde{N}_{c\ell})\cdot y_2 = \det(N_{pr}\tilde{N}_{c\ell})\cdot v^{-1}\beta$$

by (5.15) and (5.18). Equivalently,

$$\mathrm{Adj}(N_{pr}\tilde{N}_{c\ell})\cdot y_2 = \tilde{m}v^{k_1-1}\cdot\beta, \qquad \text{by (5.14)}$$

$$= \tilde{m}v^k\cdot\tilde{\beta}, \qquad \text{by definition of } \beta$$

where $k := k_1 - 1 - k_2 \geq 0$; hence $v^k \in H$. Now, by (5.16), $\mathrm{Adj}(N_{pr}\tilde{N}_{c\ell})$ is invertible in $\tilde{G}^{n_o \times n_o}$; hence,

$$y_2 = \tilde{m}v^k\cdot[\mathrm{Adj}(N_{pr}\tilde{N}_c)]^{-1}\cdot\tilde{\beta}$$

$$= \tilde{m}v^k\cdot e_\ell$$

where $e_\ell \in H^{n_o}$ has a one in the $\ell^{\underline{th}}$ position, where ℓ is the column number of $\tilde{\beta}$ in $\text{Adj}(N_{pr}\tilde{N}_{c\ell})$, and has zeros in all other positions.

Thus, $y_2 \in H^{n_o}$; hence, $y_2 - u_1 \notin E(H)$, since $u_1 \notin E(H)$. But this contradicts the assumption that S_1 tracks the class Ψ asymptotically, consequently, $\det(N_{pr}N_{c\ell})$ and ψ have no common factors which are not units of H. □

We will now present a set of conditions which are sufficient to guarantee that S_1 asymptotically tracks the class Ψ. Additionally, we will show that the same conditions are sufficient for the robust asymptotic tracking of that class: i.e., these conditions guarantee that S_1 will still asymptotically track the class Ψ under fractional perturbations of the type considered in Section 5.1.

We will require two additional assumptions on the system S_1:

(P2') $D_{p\ell}^{-1}N_{p\ell}$ is a ℓ.c.f. of P, with $V_{p\ell}$, $U_{p\ell} \in E(H)$

satisfying:

$$D_{p\ell}V_{p\ell} + N_{p\ell}U_{p\ell} = I_{n_o} \tag{5.20}$$

(C2') $C \in G^{n_i x n_o}$ has a r.c.f. $N_{cr}D_{cr}^{-1}$

We will say that C is right admissible for P if the resulting closed-loop system is H-stable, and C satisfies (C2').

Theorem 5.3 (Sufficient Conditions)

Let P satisfy (P2'). Let C be right admissible for P; thus C has a r.c.f. $N_{cr}D_{cr}^{-1}$. If D_{cr} is such that $D_{cr} = \psi D_c$, for some $D_c \in H^{n_o x n_o}$, then the system S asymptotically tracks the class ψ.

Proof

Let us here define $D_{hr} \in H^{n_o x n_o}$ by:

$$D_{hr} := D_{p\ell}D_{cr} + N_{p\ell}N_{cr}$$

It can easily be shown (similar to Theorem 3.1 (ii)), that C right admissible for P implies that det $D_{hr} \in J$. Thus,

$$D_{hr}^{-1} \in H^{n_o \times n_o}. \tag{5.23}$$

The closed-loop map $H_{e_1u_1} : u_1 \mapsto e_1$ is given by

$$H_{e_1u_1} = D_{cr}D_{hr}^{-1}D_{p\ell}$$

$$= D_cD_{hr}^{-1}D_{p\ell}\,\psi, \quad \text{by assumption}$$

Now, consider an imput $u_1 = \psi^{-1}u \in \Psi$ (note $u \in H^{n_i}$). Under application of this input, the resulting output y_2 and the resulting error e_1 are given by

$$y_2 - u_1 = e_1 = H_{e_1u_1} \cdot u_1$$

$$= D_cD_{hr}^{-1}D_{p\ell}\psi \cdot \psi^{-1}u$$

$$= D_cD_{hr}^{-1}D_{p\ell}u$$

Thus, for any $u \in H^{n_i}$, $y_2 - u_1 \in E(H)$, by (5.23). Since u_1 is an arbitrary member of the class Ψ, it follows that S_1 asymptotically tracks the class Ψ.

Comment: This result shows that the "internal model principle" can be generalized to an algebraic setting which includes the canonical examples of Table I.

Theorem 5.4 (Robust Asymptotic Tracking)

Let the assumptions of Theorem 5.3 hold. Consider arbitrary changes in the plant, $N_{p\ell} \to \tilde{N}_{p\ell}, D_{p\ell} \to \tilde{D}_{p\ell}$ such that $(\tilde{D}_{p\ell}, \tilde{N}_{p\ell})$ are ℓ.c. and arbitrary changes in the controller C, $\tilde{N}_{cr} \to N_{cr}$, $D_c \to \tilde{D}_c$, such that $\tilde{C} := \tilde{N}_{cr}\tilde{D}_{cr}^{-1}$ is right admissible for $\tilde{P} := \tilde{D}_{p\ell}^{-1}\tilde{N}_{p\ell}$. U.t.c., the perturbed system $\tilde{S}_1$, specified by $\tilde{P}$ and $\tilde{C}$, asymptotically tracks the class Ψ.

Proof

Follows the same steps as the proof of Theorem 5.3.

Comment: If one allows only plant perturbations, then a necessary and sufficient condition for the set of all perturbed systems $\tilde{S}_1$, for which H-stability is maintained, to track the class Ψ asymptotically, is that the compensator C satisfy the internal model principle, namely $D_{cr} = \psi D_c$, with $D_c \in H^{n_o \times n_o}$.

The following corollary provides sufficient conditions for robust asymptotic tracking of the class Ψ, which are similar to the conditions for robust stability, given in Theorem 5.1.

Corollary 5.4

Let the assumptions of Theorem 5.3 hold. Let $D_{p\ell}$, $N_{p\ell}$, D_c and N_{cr} be additively perturbed by, respectively, $\Delta D_{p\ell}, \Delta N_{p\ell}, \Delta D_c, \Delta N_{cr} \in E(H)$, with $\det(D_{p\ell}+\Delta D_{p\ell}), \det[\psi(D_c+\Delta D_c)] \in I$. Let $(H, \|\cdot\|)$ be a Banach algebra. Now, let $\rho_{dp} > 0$, $\rho_{np} > 0$, $\rho_{dc} > 0$, $\rho_{nc} > 0$, be such that

$$\|\psi D_{hr}^{-1} D_{p\ell}\|\rho_{dc} + \|D_{hr}^{-1} N_{p\ell}\|\rho_{nc} + \|D_{hr}\|^{-1}(\|\psi D_c\|\rho_{dp} + \|N_{cr}\|\rho_{np} + \|\psi\|\rho_{dc}\rho_{dp} + \rho_{nc}\rho_{np}) < 1$$

U.t.c., if

$$\Delta D_{p\ell} \in B(0;\rho_{dp}) \qquad \Delta D_c \in B(0;\rho_{dc})$$

and

$$\Delta N_{p\ell} \in B(0;\rho_{np}) \qquad \Delta N_{cr} \in B(0;\rho_{nc})$$

then the perturbed system $\tilde{S}_1$ is H-stable, and asymptotically tracks the class ψ.

VI. Conclusions

This paper has presented an algebraic design theory for linear multivariable feedback systems which leads to the following results:

(i) The use of an algebraic structure achieves a unification of the canonical design settings of modern control theory, including the lumped and distributed cases, for both continuous and discrete time systems (see Table I).

(ii) The results presented generalize earlier results [Des. 1], using a similar algebraic structure, to the case of non-square plants and controllers. Additionally, this paper gives, for the algebraic case, simpler and more elegant derivations of the achievable I/O and disturbance-to-output maps.

(iii) As in [Per. 1] it is shown that in the lumped case (continuous or discrete time), the algebraic design procedures may be reduced to manipulations of polynomial matrices, which is more desirable than the alternative: manipulation of matrices of rational functions.

(iv) The robustness theory shows that the achieved designs are robust with respect to plant and controller perturbations.

(v) The theory of asymptotic tracking and robust asymptotic tracking are generalized to the algebraic setting. This includes generalization of the so-called "internal model principle" [Won. 1].

VII. Acknowledgements

The authors thank Dr. M. J. Chen and Mr. C. A. Lin for many valuable and stimulating discussions, and Professor P. P. Varaiya for his questions on the tracking problem.

TABLE I

	Lumped Continuous time	Lumped Discrete time	Distributed Continuous time	Distributed Discrete time	λ-Generalized Polynomials	Multivariate Rational Functions
$\tilde{G}$	$\mathbb{R}(s)$	$\mathbb{R}(z)$			$\mathbb{R}(\lambda)$	
G	$\mathbb{R}_p(s)$	$\mathbb{R}_p(z)$	$\hat{B}(\sigma_o)$	$\tilde{b}(\rho_o)$	$\mathbb{R}_{\{o\}}(\lambda)$	
G_s	$\mathbb{R}_{p,o}(s)$	$\mathbb{R}_{p,o}(z)$	$\hat{B}_o(\sigma_o)$	$\tilde{b}_o(\rho_o)$	$\mathbb{R}_{\{o\},o}(\lambda)$	
H	$\mathbb{R}_U(s)$, $\infty \in U$	$\mathbb{R}_D(z)$, $\infty \in D$	$\hat{A}_-(\sigma_o)$	$\tilde{\ell}_{1-}(\rho_o)$	$\mathbb{R}_\Lambda(\lambda)$, $\infty \notin \Lambda$	
I	$p \in \mathbb{R}_U(s)$ s.t. $p^{-1} \in \mathbb{R}_p(s)$	$p \in \mathbb{R}_D(z)$ s.t. $p^{-1} \in \mathbb{R}_p(z)$	$\tilde{\hat{A}}^{\infty}_-(\sigma_o)$	$\tilde{\ell}^{\infty}_{1-}(\rho_o)$	$p \in \mathbb{R}_\Lambda(\lambda)$ s.t. $p^{-1} \in \mathbb{R}_{\{o\}}$	
J	$p \in \mathbb{R}_U(s)$ s.t. $\lvert p(s)\rvert > 0$ $\forall\ s \in U$	$p \in \mathbb{R}_D(z)$ s.t. $\lvert p(z)\rvert > 0$ $\forall\ z \in D$	$p \in \hat{A}_-(\sigma_o)$ s.t. $\lvert p(s)\rvert > 0$ $\forall\ s \in C_{\sigma_o+}$	$p \in \tilde{\ell}_{1-}(\rho_o)$ s.t. $\lvert p(z)\rvert > 0$ $\forall\ z \in D(\rho_o)^C$	$p \in \mathbb{R}_\Lambda(\lambda)$ s.t. $\lvert p(\lambda)\rvert > 0$ $\forall\ \lambda \in \Lambda$	
Reference	[Cal. 1-3]	[Che. 1]	[Cal. 1-2]	[Che. 1]	[Per. 1]	

References

[Ast. 1] K. J. Åstrom, "Robustness of a design method based on assignment of poles and zeros," IEEE Trans. on Atuomat. Control, vol. AC-25, pp. 588-591, June 1980.

[Bou. 1] N. Bourbaki, Commutative Algebra, Reading, Mass: Addison-Wesley, 1970.

[Bou. 2] N. Bourbaki, Algèbre, Paris: Hermann and Co., 1970.

[Cal. 1] F. M. Callier and C. A. desoer, "An algebra of transfer functions for distributed time-invariant systems," IEEE Trans. on Circuits and Systems, vol. CAS-25, pp. 651-662, Sept. 1978; corrections in vol. CAS-26, pp. 360, May 1979.

[Cal. 2] F. M. Callier and C. A. Desoer, "Simplifications and clarifications on the paper 'An algebra of transfer functions for distributed time-invariant systems,'" IEEE Trans. on Circuits and Systems, vol. CAS-27, pp. 320-323, April 1980.

[Cal. 3] F. M. Callier and C. A. Desoer, "Stabilization, tracking and disturbance rejection in multivariate convolution systems," Annales de la Société Scientifique de Bruxelles, T. 94, I, pp. 7-51, 1980.

[Cal. 4] F. M. Callier and C. A. Desoer, Multivariable Feedback Systems, Springer-Verlag: New York-Heidelberg'Berlin, 1982.

[Chen 1] M. J. Chen and C. A. Desoer, "Algebraic Theory for Robust Stability of Interconnected Systems: Necessary and Sufficient Conditions," University of California, Berkeley, Memo No. UCB/ERL M81/89, December 14, 1981.

[Che. 1] V. H. L. Cheng and C. A. Desoer, "Discrete time convolution control systems," Int. Journal of Control, vol. 36, No. 3, pp. 367-407, 1982.

[Che. 2] L. Cheng and J. B. Pearson, "Frequency domain synthesis of multivariable linear regulators," IEEE Trans. on Automat. Control, vol. AC-26, pp. 194-202, Feb. 1981.

[Cru. 1] J. B. Cruz, Jr., J. S. Frendenberg, and D. P. Looze, "A relationship between sensitivity and stability of multivariable feedback systems," IEEE Trans. on Automat. Control, vol. AC-26, pp. 66-74, Feb. 1981.

[Des. 1] C. A. Desoer, R. W. Liu, J. Murray, and R. Saeks, "Feedback system design: the fractional representation approach to analysis and synthesis," IEEE Trans. Automat. Control, vol. AC-25, pp. 399-412, 1980.

[Des. 2] C. A. Desoer and M. J. Chen, "Design of multivariable feedback systems with stable plant," IEEE Trans. on Automat. Control, vol. AC-26, pp. 408-415, April 1981.

[Des. 3] C. A. Desoer and C. L. Gustafson, "Design of Multivariable Feedback Systems with Simple Unstable Plant," University of California, Berkeley, Memo No. UCB/ERL M82/60, March 12, 1982.

[Doy. 1] J. C. Doyle and G. Stein, "Multivariable feedback design: Concepts for a classical/modern synthesis," IEEE Trans. on Automat. Control, vol. AC-26, pp. 4-16, Feb. 1981.

[Gus. 1] C. L. Gustafson and C. A. Desoer, "Controller design for linear multivariable feedback systems with stable plants, using optimization with inequality constraints," to appear in Int. Journal of Control.

[Hor. 1] I. M. Horowitz, Synthesis of Feedback Systems, Academic Press: New York and London, 1963.

[Jac. 1] N. Jacobson, Basic Algebra I, San Francisco: W. H. Freeman and Company, 1974.

[Kai. 1] T. Kailath, Linear Systems, Prentice Hall: Englewood Cliffs, N. J., 1980.

[Mac. 1] S. MacLane and G. Birkoff, Algebra, 2nd ed., Reading, Mass: Addison-Wesley, 1979.

[May. 1] D. Q. Mayne, E. L. Polak, and A. L. M. Sangiovanni-Vincentelli, "Computer-aided design via optimization: A review," Automatica, vol. 18, No. 2, pp. 147-155, March 1982.

[Nai. 1] M. A. Naimark, Normed Algebra, 2nd ed., Noordhoff Publ, Groningen, Netherlands, 1972.

[Per. 1] L. Pernebo, "An algebraid theory for the design of controllers for linear multivariable feedback systems," IEEE Trans. Automat. Control, vol. AC-26, pp. 171-194, Feb. 1981.

[Pos. 1] I. Posthelwaite, J. M. Edmunds, and A. G. MacFarlane, "Principal gains and phases in the analysis of linear multivariable feedback systems," IEEE Trans. on Automat. Control, vol. AC-26, pp. 32-46, Feb. 1981.

[Sal. 1] R. Saeks and J. Murray, "Fractional representation, algebraic geometry and the simultaneous stabilization problem," IEEE Trans. on Automat. Control, vol. AC-27, August 1982.

[Sig. 1] L. E. Sigler, Algebra, Springer-Verlag: New York-Heidelberg-Berlin, 1976.

[Vid. 1] M. Vidyasagar, "On the use of right-coprime factorizations in distributed feedback systems containing unstable subsystems," IEEE Trans. on Circuits and Systems, vol. CAS-25, pp. 916-921, Nov. 1978.

[Vid. 2] M. Vidyasagar, H. Schneider, and B. Francis, "Algebraic and topological aspects os stabilization," IEEE Trans. on Automat. Control, vol. AC-27, pp. 880-895, August 1982.

[Won. 1] W. M. Wonham, Linear Multivariable Control: A Geometric Approach, 2nd ed. Springer-Verlag: New York-Heidelberg-Berlin, 1979.

[You. 1] D. C. Youla, H. A. Jabr, and J. J. Bongiorno, Jr., "Modern Wiener-Hopf design of optinal controllers -- Part II: The multivariable case," IEEE Trans. on Automat. Control, vol. AC-21, pp. 319-338., June 1976.

[Zam. 1] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses," IEEE Trans. on Automat. Control, vol. AC-26, pp. 301-320, April 1981.

Figure Captions

Figure 1. The feedback system S.

Figure 2. The feedback system S_1.

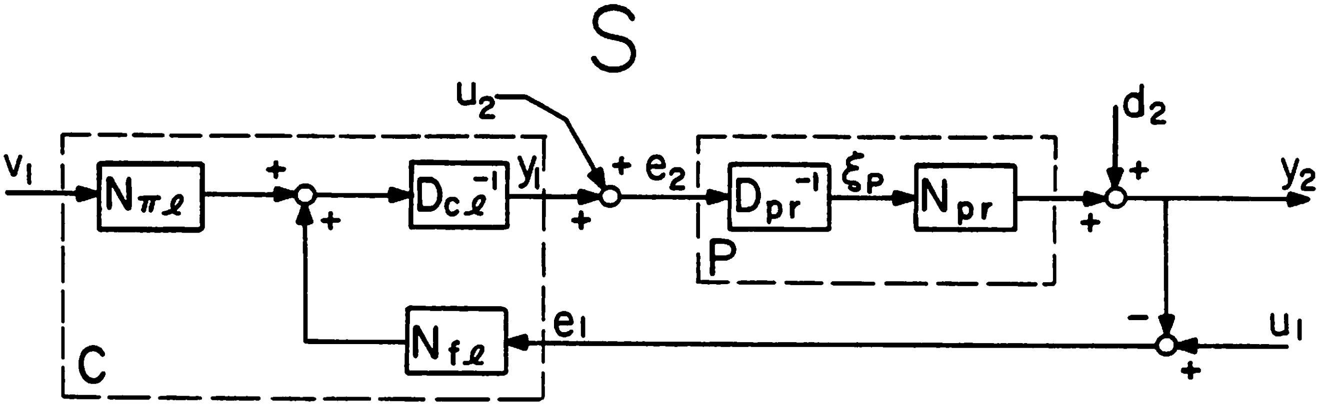

Fig. 1

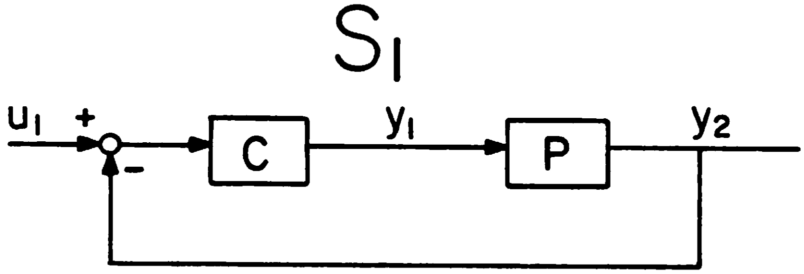

Fig. 2