COMPUTER-AIDED DESIGN OF STRUCTURES SUBJECT TO EIGENVALUE INEQUALITY CONSTRAINTS

by

Y.Y. Wardi and E. Polak

Memorandum No. UCB/ERL M82/91

19 August 1982

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Computer-Aided Design of Structures Subject to Eignevalue Inequality Constraints

by

Y.Y. Wardi and E. Polak

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Abstract

This paper presents a nondifferentiable optimization algorithm for the solution of structural optimal design problems with eigenvalue inequality constraints. The algorithm is shown to be convergent both in the L_∞ and in the sequence space topologies.

Research sponsored by the National Science Foundation Grants ECS 79-13138 and CEE 81-05790.

1. Introduction

A major goal in the design of elastic structures is the reduction of resonances. Since damping increases with frequency, resonances can be kept within acceptable limits by ensuring that the lowest natural frequency of the structure being designed lies above a certain threshold. Optimization offers powerful tools for coping with this design constraint on the natural frequencies as well as with the constraints imposed by various other performance requirements.

The structural design problems discussed in this paper involve the design of the cross section of vibrating strings, beams, membranes and plates. Thus the design parameter is an element of L^∞ and hence infinite dimensional. The state equations describing the frequency response of these structures are elliptic boundary value problems, so that the state space is also infinite dimensional. These boundary value problems have an infinite number of eigenvalues which form a countable subset of $\mathbb{R}$. The eigenvalues are the squares of the natural frequencies of the structures. The literature which is relevant to the solution of eigenvalue constrained optimal design problems is not very large. It was shown in [O,T1, T2] that strings, beams and plates can have multiple eigenvalues. Under mild assumptions, distinct eigenvalues are always Frechet differentiable in the design parameter. However, multiple eigenvalues may or may not be differentiable. The sensitivity properties of multiple eigenvalues were studied in [H1, H2, H4, S1, C1, C2, C3]. A calculus for nondifferentiable functions and general optimality conditions for nondifferentiable optimization can be found in [C4, C5, C6]. Infinite dimensional optimization problems present special difficulties since either they may fail to have a solution, or the sequences constructed by an algorithm in the process of their solution may fail to have accumulation points in the space in which the problem is defined. One way out of this predicament is to recast the problems in terms of minimizing sequences. A theory dealing with nondifferentiable algorithm construction is given in [P2, P7].

Optimality conditions for optimization problems with eigenvalue constraints can be obtained in various ways. In this paper and in [P3], the optimality conditions are expressed in terms of the Clarke [C4] generalized gradients. In [M1] Mazur and Mroz show that repeated eigenvalues lead to nondifferentiable problems and present an elegant treatment of appropriate optimality criteria. In [C2, H2], we find the same optimality conditions as in the present paper, expressed in a different form because they were derived by specializing some general results of Pshenichnyi [P9]. In [J1] we find the solution of a design problem in which the multiple eigenvalues are differentiable.

Since multiple eigenvalues are nondifferentiable, optimal design problems involving eigenvalue constraints require nondifferentiable optimization algorithms for their solution. General purpose nondifferentable optimization algorithms are extremely cumbersome because they require the accumulation of bundles of generalized gradients [P2] in search direction computation. This is a process which is too complex and too ill-conditioned numerically to implement in the solution of structural optimal design problems. Forturnately, in [P7] we find a theory which enables one to exploit problem structure in designing special purpose nondifferentiable optimization algorithms and in [C7, P3] we find algorithms which are consistent with this theory. In conjunction with the theory in [P7], these finite dimensional algorithms can be used as a guide in the development of an algorithm for optimization problems with constraints on the eigenvalues of an elliptic boundary value problem.

This paper presents a generalization of the algorithm in [P3] and deals with optimal design problems involving constraints on nondifferentiable multiple eigenvalues. Although only vibrating strings, beams and plates are considered explicitely, the results are easily extended to any optimal design problem with constraints on the eigenvalues of an elliptic boundary value problem.

2. Formulation of the Design Problem

It has been shown in [H4] that a vibrating string, a vibrating beam, a vibrating membrane and a vibrating plate, all lead to an eigenvalue problem of the form

$$a_u(y,v) = \lambda b_u(y,v) \quad \forall v \in V \tag{2.1}$$

where u is the design variable, V is an appropriate Sobolev space and a_u, b_u are bilinear forms on V. A solution $(\lambda,y) \in \mathbb{R} \times V - \{0\}$ which satisfies (2.1) is an eigenvalue-eigenvector pair for the system. It was demonstrated in [H4] that the eigenvalue-eigenvector equations for the four problems mentioned above, as well as for other structural design problems have quite similar mathematical structures. Consequently, without loss of generality, we confine our discussion to the physical set up of a clamped-clamped vibrating beam of constant width. In the context of clamped-clamped beam of uniform width, the most general design variable is $u = (\rho,E,h)$ where ρ is the density of the material, E is its Young's modulus and h(x) is the height of the beam at a distance x from one end of the beam. We simplify exposition without loss of generality by assuming that ρ and E are fixed, so that $u = h$. Our formulas simplify further if we scale relevant quantities so that the length of the beam is 1 unit and $E/\rho = 12$. Once this is done, we get that

$$a_h(y,v) \triangleq \int_\Omega h^3 y_{xx} v_{xx} d\mu \tag{2.2a}$$

$$b_h(y,v) \triangleq \int_\Omega hyv d\mu , \tag{2.2b}$$

where $\Omega = [0,1]$, the height $h \in L^\infty(\Omega)$, μ is the Lesbesgue measure on Ω, and the Sobolev space V is taken to be $H_0^2(\Omega)$. Hence (2.1) becomes

$$\int_\Omega h^3 y_{xx} v_{xx} d\mu = \lambda \int_\Omega hyv d\mu . \tag{2.3}$$

A typical design problem consists of minimizing the weight of the beam subject to the constraints a) that the height of the beam exceeds a given minimum height and b) that all the natural frequencies of the beam exceed a given minimum value. Consequently, we define the cost function $f : L^\infty(\Omega) \to \mathbb{R}$ by

$$f(h) \triangleq \int_\Omega h d\mu \ . \tag{2.4}$$

$\bar{h} > 0$ given, we specify the constraint on the height of the beam through the function $\phi : L^\infty(\Omega) \to \mathbb{R}$ defined by

$$\phi(h) \triangleq \bar{h} - \operatorname*{ess\ inf}_{x \in \Omega} h(x) . \tag{2.5}$$

Now it was shown in [H4] that when $\phi(h) \geq 0$, (i) the bilinear form a_h is elliptic; (ii) (2.3) has countably many eigenvalues; (iii) all the eigenvalues are real and positive, and each has finite multiplicity; (iv) the set of eigenvalues has no accummulation points. We number the eigenvalues of (2.3) in increasing order: $\lambda^1(h) \leq \lambda^2(h) \leq \lambda^3(h)$ _ ... . The $\lambda^i(h)$ are the squares of the natural frequencies of the beam. Consequently, the restriction on the natural frequencies can be expressed by means of the function $\psi : L^{\infty}(\Omega) \to \mathbb{R}$ defined by

$$\psi(h) \triangleq \bar{\lambda} - \lambda^1(h) , \tag{2.6}$$

with $\bar{\lambda} > 0$ given. Collecting all the pieces, we find that our simplest design problem has the form

$$\min\{f(h) | \phi(h) \leq 0, \psi(h) \leq 0\} . \tag{2.7}$$

Since neither $\phi(\cdot)$ nor $\psi(\cdot)$ are differentiable, we have to turn to nondifferentiable optimization techniques in constructing an algorithm for solving (2.7). As we shall see in Section 4, the main source of difficulty is the constraint $\psi(h) \leq 0$. Since $\phi(h)$ is a supremum of affine functions, it can be dealt with by means of a simple projection technique. However, first we must collect the various facts of nondifferentiable optimization that we need for our algorithm construction and analysis.

3. Nondifferentiable Optimization: A Summary

The first set of results are culled from [C4, C5] and [L1] and deal with certain properties of locally Lipschitz functions. In what follows, X will denote a real Banach space and X' its dual, i.e., the space of real valued bounded linear functionals on X. We shall denote the action of an $x' \in X'$ on an $x \in X$ either by (x',x) or by (x,x'). Furthermore, we shall assume that the domain $D(f)$, of any function $f : X \to \mathbb{R}$ that we discuss, is an open subset of X.

Definition 3.1: The function $f : X \to \mathbb{R}$ is said to be locally Lipschitz continuous if for every $\hat{x} \in D(f)$ there exist an open set $N \in D(f)$ and an $L \in (0,\infty)$ such that $\hat{x} \in N$ and for all $x,x' \in N$

$$|f(x) - f(x')| \leq L\|x-x'\| \; . \tag{3.1}$$

¤

Definition 3.2: Let $f : X \to \mathbb{R}$ be locally Lipschitz continuous. Then, given any $x \in D(f)$, $e \in X$ the generalized directional derivative of $f(\cdot)$ at x in the direction e is defined as

$$d_0 f(x,e) \triangleq \lim_{\substack{y \to 0 \\ s \searrow 0}} \frac{f(x+y+se)-f(x+y)}{s} \; . \tag{3.2}$$

¤

The generalized directional derivative of a locally Lipschitz continuous function always exists (see [C4]).

Definition 3.3: Let $f : X \in \mathbb{R}$ be locally Lipschitz continuous. Then given any $x \in D(f)$, the generalized gradient of $f(\cdot)$ at x is defined as

$$\partial f(x) \triangleq \{\xi \in X')|(\xi,e) \leq d_0 f(x,e) \quad \forall e \in X\} \; . \tag{3.3}$$

¤

Proposition 3.1: Let $f : X \to \mathbb{R}$ be locally Lipschitz continuous. Then

a) for every $x \in D(f)$, $\partial f(x)$ is a nonempty, convex, weak* compact subset of X';

b) for any $x \in D(f)$, $e \in X$,

$$d_0 f(x,e) = \max\{(\xi,e)|\xi \in \partial f(x)\} \; ; \tag{3.4}$$

c) the point to set map $x \to \partial f(x)$ is weak* upper semicontinuous (u.s.c); i.e., if $x_i \in D(f)$, $i = 1,2,3$, are such that $x_i \to \hat{x} \in D(f)$ and $\xi_i \in \partial f(x_i)$, $i = 1,2,3,\ldots$, are such that $\xi_i \xrightarrow{w^*} \hat{\xi}$, then $\hat{\xi} \in \partial f(\hat{x})$.

d) if $f(\cdot)$ is continuously Frechet differentiable on a neighborhood N of $x \in D(f)$, then for all $x' \in N$,

$$\partial f(x') = \{Df_{x'}\} \tag{3.15}$$

where $Df_{x'}$ denotes the Frechet derivative of f at x'. Conversely, if $\partial f(x')$ is a singleton for all x' in a neighborhood N of $x \in D(f)$ and if the map $y \to \partial f(y)$ is continuous, in the strong topology, on N, then $f(\cdot)$ is Frechet differentiable on N and (3.5) holds.

e) If $X = \mathbb{R}^n$, a finite dimensional Euclidean space, then $\partial f(x) = \text{co}\{\lim_{i\to\infty} \nabla f(x+h_i)\}$ where $\{h_i\}$ is any sequence such that $h_i \to 0$ $\nabla f(x+h_i)$ exists for all $i \in \mathbb{N}^+$, $\lim_{i\to\infty} \nabla f(x+h_i)$ exists, and co denotes the convex hull. ¤

Theorem 3.1 (Mean Value): Let $f : X \to \mathbb{R}$ be locally Lipschitz continuous. Then for any $x,y \in D(f)$, there exist an $s \in [0,1]$ and a $\xi \in \partial f(x+s(y-x))$ such that

$$f(y) - f(x) = (\xi, y-x) . \tag{3.6}$$

¤

Finally, consider the problem

$$\min\{f(x) | g^j(x) \leq 0, \quad j = 1,2,..m\} , \tag{3.7}$$

where $f : X \to \mathbb{R}$, $g^j : X \to \mathbb{R}$ are all locally Lipschitz continuous.

Theorem 3.2: If $\hat{x}$ is a local minimizer for (3.7), then $g^j(\hat{x}) \leq 0$ for $j = 1,2,\ldots,m$, and there exist $t_0,t_1,\ldots,t_m \in [0,1]$, such that $\sum_{j=0}^{m} t_j = 1$, $t_j g^j(\hat{x}) = 0$ for $j = 1,2,\ldots,m$, and

$$0 \in t_0 \partial f(\hat{x}) + \sum_{j=1}^{m} t_j \partial g^j(\hat{x}) . \tag{3.8}$$

¤

Next, we turn to algorithms for solving problems of the form (3.7). Our source is [P8]. Obviously, there are all kinds of algorithms for solving (3.7) when all the functions in (3.7) are continuously Frechet differentiable. The simplest idea in extending such a differentiable optimization algorithm to the nondifferentiable case consists of somehow replacing the Frechet derivatives, $Df(x)$, $Dg^j(x)$, by generalized gradients $\partial f(x)$, $Dg^j(x)$ in these algorithms. Consider the simplest case of (3.7), viz.

$$\min_{x\in X} f(x) \tag{3.9}$$

In this case, the above suggestion leads to the replacement of the steepest descent direction

$$\begin{aligned} h(x) &\triangleq \arg \min_{\|h\|\leq 1} df(x,h) \\ &= \arg \min_{\|h\|\leq 1} (h, Df(x)) \end{aligned} \tag{3.10a}$$

by the direction

$$\begin{aligned} h_f(x) &\triangleq \arg \min_{\|h\|\leq 1} d_0 f(x,h) \\ &\arg \min_{\|h\|\leq 1} \max_{\xi\in\partial f(x)} (h,\xi) \end{aligned} \tag{3.10b}$$

so that the Armijo gradient method [A3] becomes

$$x_{i+1} = x_i + s_i h_f(x_i), \quad i = 0,1,2,... \tag{3.11a}$$

where, for given $\alpha,\beta \in (0,1)$,

$$s_i = \beta^{k_i} \triangleq \arg \max_{k\in\mathbb{N}_+} \{\beta^k | f(x_i+\beta^k h_f(x_i)) - f(x_i) \leq -\beta^k\alpha\|h_f(x_i)\|\} \tag{3.11b}$$

and $\mathbb{N}_+ \triangleq \{0,1,2,3,....\}$.

The Armijo method, as well as others, demonstrably converge to stationary points, when $f(\cdot)$ is C^1. When $f(\cdot)$ is only locally Lipschitz continuous, its generalized gradient, usually is <u>not</u> even <u>locally</u> <u>uniformly</u> u.s.c.

and hence extensions such as the one defined by (3.10b) (3.11a,b) cannot be shown to be convergent in the above sense (for a counter example, see [P6]). Referring to [P8], we see that a general approach to dealing with this difficulty consists of replacing (in (3.10b) the generalized gradient $\partial f(x)$ by a family of u.s.c. maps $G_\varepsilon f(x)$, $\varepsilon > 0$, which are locally "uniformly u.s.c. with respect to $\partial f(x)$."

To be quite precise, the sets $G_\varepsilon f(x)$ are required to have the following properties (see Definition 2.1 in [P8]):

(i) For all $x \in X$, $\partial f(x) = G_0 f(x)$.

(ii) For all $\varepsilon \geq 0$, $x \in X$, the sets $G_\varepsilon f(x)$ are weak* compact and convex; for all $\varepsilon \geq 0$, the sets $G_\varepsilon f(x)$ are bounded on bounded sets in X; the maps $(\varepsilon,x) \to G_\varepsilon f(x)$ are u.s.c. in (ε,x) at $(0,x)$ for all $x \in X$.

(iii) For all $x \in X$, $\varepsilon < \varepsilon' \Rightarrow G_\varepsilon f(x) \subset G_{\varepsilon'} f(x)$.

(iv) For any $x \in X$, $\varepsilon > 0$, $\delta > 0$, there exists a $\rho > 0$ such that for any $x',x'' \in B(x,\hat{\rho}) \triangleq \{x' \mid \|x-x'\| \leq \rho\}$ and any $\eta' \in \partial f(x')$, there exists an $\eta'' \in G_\varepsilon f(x'')$ such that $\|\eta'-\eta''\| \leq \delta$.

The introduction of the sets $G_\varepsilon f(x)$ leads to the (nonunique) definition of ε-generalized directional derivatives of $f(\cdot)$, defined for any $(x,e) \in X$ as follows:

$$d_\varepsilon f(x,e) \triangleq \max_{\xi \in G_\varepsilon f(x)} (\xi,e) \, . \tag{3.12}$$

Since $\partial f(x) \subset G_\varepsilon f(x)$ for all $\varepsilon \geq 0$, we must have

$$d_0 f(x,e) \leq d_\varepsilon f(x,e) \qquad \forall x,e \in X \, , \tag{3.13}$$

so that whenever $d_\varepsilon f(x,e) < 0$, e is clearly a descent direction for $f(\cdot)$ at x. Consequently an "ε-steepest descent" direction is given by

$$g = \arg \min_{\|e\| \leq 1} d_\varepsilon f(x,e) = \arg \min_{\|e\| \leq 1} \max_{\xi \in G_\varepsilon f(x)} (\xi,e) \tag{3.14}$$

We shall see in the next section how the use of ε-generalized directional derivatives leads to a convergent algorithm. However, before we proceed to that section, it may be useful to show how appropriate sets $G_\varepsilon f(\cdot)$ can be constructed for a special class of functions $f(\cdot)$. Thus suppose that $f(\cdot)$ is a composition map of the form

$$f(x) = \phi(g(x))$$

with $g : X \to Y$ continuously Frechet differentiable, $\phi : Y \to \mathbb{R}$ locally Lipschitz continuous and Y a real Banach space. The maps $G_\varepsilon f(x)$ can be constructed to have the form

$$G_\varepsilon f(x) \triangleq \partial\phi(g(x) + v_\varepsilon(x)) \tag{3.12}$$

where, for some norm, $\|v_\varepsilon(x)\| \leq \varepsilon$ and, in some sense, "maximizes" the set $\partial\phi(g(x)+v)$, for $\|v\| \leq \varepsilon$.

The simplest example in $\mathbb{R}^n$ which is analogous to the problem we wish to solve in this paper is

$$\max_{x\in\mathbb{R}^n} \lambda^1(x) \tag{3.13}$$

where $\lambda^1(x) \triangleq f(x)$ is the minimum eigenvalue of a componentwise analytic mxm real symmetric, positive definite matrix $Q(x)$. In this case, $g(x) = Q(x) \in \mathbb{R}^{nxn}$ and $\phi(Q)$ = min eigenvalue of Q. Now, for any $x \in \mathbb{R}^n$, $\Lambda(x) = U(x)^T Q(x) U(x)$, where $U(x)^T U(x) = I$ and $\Lambda(x) = \text{diag}(\lambda^1(x),\lambda^2(x)\ldots {}^m(x))$, with the eigenvalues arranged according to increasing magnitude. Let the multiplicity of $\lambda^1(x)$ be $m^1(x)$, then (see [P3])

$$\partial f(x) = \text{co}\{v \in \mathbb{R}^n | v^i = \langle \bar{U}(x)z, \frac{\partial Q(x)}{\partial x^i} \bar{U}(x)z \rangle \quad i = 1,2,\ldots,n, \|z\| = 1\} \tag{3.14}$$

where $\bar{U}(x)$ is an $m \times m^1(x)$ matrix consisting of the first $\mu^1(x)$ columns of $U(x)$ and co denotes the convex hull. For this case we define $V_\varepsilon(x)$ to be the $m \times m$ matrix $U(x)^T \delta\Lambda_\varepsilon(x)\ U(x)$, with $\delta\Lambda_\varepsilon(x) = \mathrm{diag}(\delta\lambda^1(x), \delta\lambda^2(x), \ldots, \delta\lambda^m(x))$ such that $|\delta\lambda^i(x)| \leq \varepsilon$, which maximizes the multiplicity of the smallest eigenvalue of $Q(x) + V_\varepsilon(x)$. We note that if define $\bar{k}_\varepsilon(x)$ by

$$\bar{k}_\varepsilon(x) \triangleq \max\{k \in \underline{m} | \lambda^k(x) - \lambda^1(x) \leq \varepsilon\} \tag{3.15}$$

Then $\delta\lambda^j(x) \triangleq \lambda^1(x) - \lambda^j(x)$ for $j = 1,2,\ldots,\bar{k}_\varepsilon(x)$ and $\delta\lambda^j(x) \triangleq 0$ otherwise defines such a matrix $\delta\Lambda_\varepsilon(x)$. This yields a formula for $G_\varepsilon f(x)$ which is identical to (3.14) except that $\bar{U}(x)$ now contains the first $\bar{k}_\varepsilon(x)$ columns of $U(x)$. In practice, when ε is small, it may be difficult to tell whether two eigenvalues which are <u>numerically</u> ε apart are not, in fact, the same eigenvalue. Because of this, in [P3], a somewhat larger set $G_\varepsilon f(x)$ is used. It is defined as follows. Let

$$k_\varepsilon(x) \triangleq \min\{k \in \underline{m} | \lambda^{k+1}(x) - \lambda^k(x) > \varepsilon\} \tag{3.16}$$

with $\lambda^{m+1}(x) \triangleq \infty$. Then $k_\varepsilon(x) \geq \bar{k}_\varepsilon(x)$ and we define the $m \times m$ diagonal matrix $\delta\Lambda_\varepsilon(x)$ by $\delta\lambda^j(x) = \lambda^1(x) - \lambda^j(x)$ for $j = 1,2,\ldots,k_\varepsilon(x)$ and $\delta\lambda^j(x) = 0$ for $j = k_\varepsilon(x) + 1,\ldots,m$. Again the formula for $G_\varepsilon f(x)$ is obtained by replacing in (3.14) the $m \times m^1(x)$ matrix $\bar{U}(x)$ with the $m \times k_\varepsilon(x)$ matrix $\tilde{U}_\varepsilon(x)$ consisting of the first $k_\varepsilon(x)$ columns of $U(x)$.

We shall see that the construction in the next section is entirely analogous to the one in [P3], i.e., to the one we have just described.

4. The Algorithm

Before we can state the algorithm for solving the problem (2.7), with the functions defined by (2.4)-(2.6), we need to develop some of the properties of the function $\psi(\cdot)$ in (2.6) on the set

$$L_+^\infty(\Omega) \triangleq \{h \in L^\infty(\Omega) \,|\, \phi(h) \leq 0\} . \tag{4.1}$$

Let $\mathcal{B}$ denote the real Banach space of symmetric, bicontinuous, bilinear forms on $H_0^2(\Omega)$, with the norm of $a \in \mathcal{B}$ defined by

$$\|a\| \triangleq \sup\{|a(y,v)| \,\big|\, \|y\| = \|v\| = 1\} \tag{4.2}$$

Next, for any $h \in L_+^\infty(\Omega)$, let $m^1(h)$ denote the multiplicity of $\lambda^1(h)$, the smallest eigenvalue of (2.3). The following result can be found in [H4].

Proposition 4.1: a) The maps $h \to a_h$, $h \to b_h$, mapping $L_+^\infty(\Omega)$ into $\mathcal{B}$, with a_h, b_h defined by (2.2a), (2.2b) respectively, are continuously Fréchet differentiable.

b) Let the Frechet derivatives of a_h, b_h be denoted by a_h', b_h' respectively. Then for any $g \in L^\infty(\Omega)$, $a_h'g$, $b_h'g$ are the bilinear forms on $\mathcal{B}$ defined by

$$a_h'g(y,v) = \int_\Omega 3h^2 g y_{xx} v_{xx} d\mu \tag{4.3}$$

$$b_h'g(y,v) = \int_\Omega gyvd\mu . \tag{4.4}$$

¤

The next result is a modification of Theorems 1 and 2 in [H4]; for a proof see the Appendix.

Proposition 4.2: a) For every $k \in L_+^\infty(\Omega)$ and every $g \in L^\infty(\Omega)$ satisfying $\|g\|_\infty \leq \bar{h}/4$, there exists an $s > 0$ such that

(i) The function $\lambda_g^1(t) \triangleq \lambda^1(h+tg)$ is real analytic on $[0,s)$;

(ii) the orthogonal projection P_t, onto the direct sum of the eigenspaces of

$$\lambda^1(h+tg),\ \lambda^2(h+tg),\ \dots,\ \lambda^{m^1(h)}(h+tg)$$

is a well defined analytic function on $[0,s)$.

b) The function $h \to \lambda^1(h)$, from $L_+^\infty(\Omega)$ into $\mathbb{R}$, is uniformly locally Lipschitz continuous on bounded sets, i.e., for every $M > 0$ there exist a $\rho > 0$ and an $L < \infty$ such that for all $h \in L_+^\infty(\Omega) \cap B(0,M)$ and every $h' \in B(h,\rho)$,

$$|\lambda^1(h) - \lambda'(h')| \leq L \, \|h-h'\|_\infty \tag{4.5}$$

(with $B(z,v) \triangleq \{z' \in L^\infty(\Omega) \,|\, \|z-z'\|_\infty \leq v\}$). ¤

Next we proceed with the construction of the sets $G_\varepsilon\psi(h)$, whose role in search direction computation was somewhat discussed in the preceding section.

a) For every $v \in H_0^2(\Omega)$ and every $h \in L_+^\infty(\Omega)$ we define $\xi_h^v \in L^1(\Omega)$ by

$$\xi_h^v \triangleq \frac{1}{b_h(v,v)} \left(3h^2 v_{xx}^2 - \lambda^1(h) v^2\right) . \tag{4.6}$$

b) For any $k \in \mathbb{N}^+$ and $h \in L_+^\infty(\Omega)$ such that $\lambda^{k+1}(h) \neq \lambda^k(h)$, we define the set $\hat{G}^k\psi(h) \subset L^1(\Omega)$ by

$$\hat{G}^k\psi(h) \triangleq -\overline{co}\{\xi_h^v \,|\, v \text{ is an eigenvector of } (a_h, b_h) \text{ corresponding to } \lambda^i(h),\ i = 1,2,\ldots k\} , \tag{4.7}$$

where $\overline{co}$ denotes the closure of the convex hull in the L^1 topology.

<u>Remark</u>: When $m^1(h) = 1$, $\hat{G}^1\phi(h)$ is the Fréchet derivative of $\lambda^1(h)$, as was shown in [H4]. ¤

The following result will be proved in the Appendix.

<u>Proposition 4.3</u>: Let $h \in L_+^\infty(\Omega)$ and $k \in \mathbb{N}^+$ be such that $\lambda^k(h) \neq \lambda^{k+1}(h)$. Then there exists a $\rho > 0$ such that (i) $\hat{G}^k\psi(h')$ is well defined for all $h' \in B(h,\rho) \cap L_+^\infty(\Omega)$, and (ii) the point to set map $h' \to \hat{G}^k\psi(h')$, from $B(h,\rho) \cap L_+^\infty(\Omega)$ into the class of subsets of $L^1(\Omega)$, is continuous in the

sense of Berge [B1]. If $S \subset B(h,\rho) \cap L_+^\infty(\Omega)$ is compact, then $\bigcup_{h' \in S} \hat{G}^k(h')$ is compact in $L^1(\Omega)$. ¤

Now, for any $h \in L_+^\infty(\Omega)$ and $\varepsilon \geq 0$, we define

$$k(h,\varepsilon) \triangleq \min\{k \in \mathbb{N}^+ \cup \{\infty\} \mid \lambda^{k+1}(h) - \lambda^k(h) > \varepsilon\} \tag{4.8}$$

and

$$G_\varepsilon(h) \triangleq \hat{G}^{k(h,\varepsilon)}(h) . \tag{4.9}$$

The following assumption is dictated both by theoretical and computational considerations.

Assumption 4.1: There exists an $\varepsilon_0 > 0$ and a $j_0 \in \mathbb{N}^+$, $j_0 > 0$, such that for all $h_i \in L_+^\infty(\Omega)$ constructed by the algorithm $k(h_i,\varepsilon_0) < j_0$. ¤

The following result will be proved in the Appendix.

Proposition 4.4: Suppose that Assumption 4.1 holds. Then for every $h \in L_+^\infty(\Omega)$, $G_0\psi(h) = \partial\psi(h)$. ¤

The algorithm which we will shortly present is a semi phase-I-Phase-II algorithm with projection. It constructs sequences $\{h_i\}$ which may violate the constraint $\psi(h) \leq 0$, but they will always satisfy the constraint $\phi(h) \leq 0$. Since $\phi(\cdot)$ is the supremum of affine functions, it is easily handled by a projection mechanism in the search direction computation, as we shall shortly see. First, for every $h \in L_+^\infty(\Omega)$ we define

$$F^\phi(h) \triangleq \{g \in L^\infty(\Omega) \mid \|g\|_\infty \leq 1,\ (h+g) \in L_+^\infty(\Omega)\} . \tag{4.10}$$

Note that if $h \in L_+^\infty(\Omega)$ and $(h+g) \in L_+^\infty(\Omega)$, then we must have $(h+sg) \in L_+^\infty(\Omega)$ for all $s \in [0,1]$.

Next, for every $h \in L_+^\infty(\Omega)$ and $\varepsilon \geq 0$, we define the sets

$$U_{\psi,\varepsilon}(h) \triangleq \arg \min_{g \in F^\phi(h)} \left(\max_{\xi \in G_\varepsilon \psi(h)} (g,\xi) \right) \tag{4.11a}$$

and

$$U_{f,\varepsilon}(h) \triangleq \arg \min_{g \in F^\phi(h)} \left(\max_{\xi \in \mathrm{co}\{f'(h), G_\varepsilon^+\psi(h)\}} (g,\xi) \right), \tag{4.11b}$$

where

$$G_\varepsilon^+\psi(h) \triangleq \begin{cases} G_\varepsilon\psi(h) \text{ if } \psi(h) \geq -\varepsilon \\ \text{empty set otherwise} \end{cases} \tag{4.12}$$

and $f'(h)$ is the Fréchet derivative of $f(\cdot)$ at h. We note that a $g_\psi \in U_{\psi,\varepsilon}(h)$ is a projected phase-I descent direction for $\psi(\cdot)$ when $\psi(h) > 0$ and the min max < 0, in (4.11a) and that a $g_f \in U_{F,\varepsilon}(h)$ is a projected phase-II feasible descent direction for $f(\cdot)$ when $\psi(h) \leq 0$ and the min max < 0 in (4.11b). In both cases the projections are into the set $F^\phi(h)$. We shall also need the values of the programs in (4.11a) (4.11b), which we define as follows:

$$\theta_{\psi,\varepsilon}(h) \triangleq \min_{g \in F^\phi(h)} \max_{\xi \in G_\varepsilon \psi(h)} (g,\xi) \tag{4.13a}$$

$$\theta_{F,\varepsilon}(h) \triangleq \min_{g \in F^\phi(h)} \max_{\xi \in \mathrm{co}\{f'(h), G_\varepsilon\psi(h)\}} (g,\xi). \tag{4.13b}$$

Next, with $\gamma \geq 1$ given, we define the <u>crossover</u> function $\Gamma : L_+^\infty(\Omega) \to [0,1]$ by

$$\Gamma(h) \triangleq \exp(-\gamma\psi(h)_+) \tag{4.14}$$

where $\psi(h)_+ \triangleq \max\{\psi(h), 0\}$. The crossover function is used to construct a search direction which switches from a descent direction for $\psi(\cdot)$ when $\psi(h) > 0$ to a feasible descent direction for $F(\cdot)$ when $\psi(h) \leq 0$, as follows:

$$U_\varepsilon(h) \triangleq \Gamma(h)U_{f,\varepsilon}(h) + (1-\Gamma(h))U_{\psi,\varepsilon}(h) \tag{4.15a}$$

and the corresponding <u>optimality</u> function

$$\theta_\varepsilon(h) \triangleq \min\{\Gamma(h)\theta_{f,\varepsilon}(h), (1-\Gamma(h))\theta_{\psi,\varepsilon}(h)\} . \tag{4.15b}$$

As we have mentioned earlier, the function of $\varepsilon > 0$ is to produce a kind of local uniform upper semi-continuity in the sets $G_\varepsilon\psi(h)$. However, ε must eventually be driven to zero. We do this by making use of a standard device in feasible directions methods (see [P7]). Let $\varepsilon_0 > 0$ and $\nu \in (0,1)$ be given. We define the set

$$E \triangleq \{0, \varepsilon_0, \nu\varepsilon_0, \nu^2\varepsilon_0, \nu^3\varepsilon_0, \ldots\} \tag{4.16a}$$

and the <u>optimality</u> function $E : L_+^\infty(\Omega) \to E$ by

$$\varepsilon(h) \triangleq \max\{\varepsilon \in E \,|\, \theta_\varepsilon(h) \leq -\varepsilon\} \cdot \tag{4.16b}$$

As we shall shortly see, $\varepsilon(h) = 0$ iff $\theta_0(h) = 0$ and $\varepsilon(\cdot)$ and $\theta_0(\cdot)$ are optimality functions in the sense that if $\hat{h}$ is optimal for our problem, then $\varepsilon(\hat{h}) = 0$ and $\theta_0(\hat{h}) = 0$, so that $\varepsilon(\hat{h}) > 0$ $(\theta_0(\hat{h}) < 0)$ implies that $\hat{h}$ cannot be optimal. However, $\varepsilon(\cdot)$ is a much nicer optimality function than $\theta_0(\cdot)$ because given a sequence $\{h_i\}$ such that $h_i \to \hat{h}$ as $i \to \infty$ with $\theta_0(\hat{h}) = 0$, it is possible to have $\theta_0(h_i) \leq -1$ for all i, but we always have $\varepsilon(h_i) \to 0$ as $i \to \infty$. We use $\varepsilon(\cdot)$ in the following two definitions. For all $h \in L_+^\infty(\Omega)$,

$$U(h) \triangleq U_{\varepsilon(h)}(h) , \tag{4.17a}$$

$$\theta(h) \triangleq \theta_{\varepsilon(h)}(h) . \tag{4.17b}$$

In order to ensure that we can, at least, compute feasible points h, we need the following commonly made hypothesis.

Assumption 4.2: $\theta_{\psi,0}(h) < 0$ for all $h_i \in L_+^\infty(\Omega)$ such that $\psi(h_i) \geq 0$, constructed by the algorithm. ¤

Before starting the algorithm, we summarize the properties of the functions $\theta_\varepsilon(h)$, $\theta(h)$ and $\varepsilon(h)$. The proofs will be given in the Appendix.

Proposition 4.5:

a) For every $\varepsilon \geq 0$ and $h \in L_+^\infty(\Omega)$ the sets $U_{\psi,\varepsilon}(h)$ and $U_{f,\varepsilon}(h)$ are well defined.

b) For any $h \in L_+^\infty(\Omega)$, $0 \leq \varepsilon < \varepsilon' \Rightarrow \theta_\varepsilon(h) \leq \theta_{\varepsilon'}(h)$.

c) For all $\varepsilon \geq 0$, the map $\theta_\varepsilon : L_+^\infty(\Omega) \to \mathbb{R}$ is u.s.c.

d) For all $h \in L_+^\infty(\Omega)$, $\theta(h) = 0 \Leftrightarrow \varepsilon(h) = 0$.

e) If $\hat{h} \in L_+^\infty(\Omega)$ is a local minimizer for P then $\theta(\hat{h}) = \varepsilon(\hat{h}) = 0$.

f) If $\hat{h} \in L_+^\infty(\Omega)$ is such that $\theta(\hat{h}) < 0$, then there exists a $\hat{\rho} > 0$ such that $\varepsilon(h) \geq \nu\varepsilon(\hat{h}) > 0$ for all $h \in B(\hat{h},\hat{\rho}) \cap L_+^\infty(\Omega)$. ¤

We are finally ready to state our algorith.

Algorithm 4.1

Parameters: $\alpha \in (0,1)$, $\beta \in (0,1)$, $\varepsilon_0 > 0$, $\nu \in (0,1)$, $\gamma \geq 1$.

Data: $h_0 \in L_+^\infty(\Omega)$.

Step 0: Set $i = 0$.

Step 1: Compute $\theta(h_i)$ and a $g_i \in U(h_i)$. Stop if $\theta(h_i) = 0$.

Step 2: Compute the largest stepsize $s_i = \beta^{k_i}$, with $k_i \in \mathbb{N}^+$, such that

if $\psi(h_i) > 0$ then

$$\psi(h_i+s_ig_i) - \psi(h_i) \leq s_i\alpha\theta(h_i) \tag{4.18}$$

if $\psi(h_i) \leq 0$, then

$$f(h_i+s_ig_i) - f(h_i) \leq s_i\alpha\theta(h_i) \tag{4.19a}$$

and

$$\psi(h_i+s_ig_i) \leq 0 . \tag{4.19b}$$

Step 3: Set $h_{i+1} = h_i + s_ig_i$, set $i = i + 1$ and go to Step 1. ¤

Remark: The algorithm above is conceptual, since it assumes that we can compute quantities such as $\theta(h_i)$ and g_i in Step 2. When the design problem is reduced to a finite dimensional problem, e.g., via a finite element method), a fully implementable version of Algorithm 4.1 does exist, see [P3]. ¤

The following results will be proved in the Appendix.

Proposition 4.6: If $h_i \in L_+^\infty(\Omega)$ is such that $\theta(h_i) < 0$, then (a) s_i, as defined in Step 2 of Algorithm 4.1, satisfies $s_i > 0$, i.e., the algorithm is well defined, and (b) $h_i + s_ig_i \in L_+^\infty(\Omega)$. ¤

Theorem 4.1: If $\{h_i\}_{i=0}^\infty$ is a sequence generated by Algorithm 4.1, then any accummulation point $\hat{h}$ of $\{h_i\}_{i=0}^\infty$ satisfies $\theta(\hat{h}) = 0$. ¤

Since the design parameter space is infinite dimensional, it is not clear, a priori, that a sequence $\{h_i\}_{i=0}^\infty$, constructed by Algorithm 4.1, will have accummulation points, even if this sequence is bounded. In the case of optimal control, a similar phenomenon has led to the extension of the design space to that of relaxed controls, in which accummulation points always exist. In the present case it seems simpler to use an extension based on the topology of minimizing sequences, as defined in [P4]. The extended norm for that topology is defined by

$\|\{h_i\}_{i=0}^{\infty}\| = \overline{\lim}_{i\to\infty} \|h_i\|$. We reproduce two definitions from [P4].

<u>Definition 4.1</u>: A bounded sequence $\{h_i\}_{i=0}^{\infty}$ in $L_+^{\infty}(\Omega)$ is said to be <u>eventually feasible</u> if

$$\overline{\lim}\ \psi(h_i) \leq 0 \ . \tag{4.20}$$

¤

<u>Definition 4.2</u>: An eventually feasible, bounded sequence $\{\hat{h}_i\}_{i=0}^{\infty}$ in $L_+^{\infty}(\Omega)$ is said to be a <u>local minimizing</u> sequence for P if there exists a $\hat{\rho} > 0$ such that for any eventually feasible bounded sequence $\{h_i\}_{i=0}^{\infty}$ satisfying

$$\overline{\lim}\ \|h_i - \hat{h}_i\| \leq \rho \tag{4.21}$$

we have

$$\overline{\lim_{\substack{i\in K\\ i\to\infty}}}\ f(\hat{h}_i) \leq \overline{\lim_{\substack{i\in K\\ i\to\infty}}}\ f(h_i) \tag{4.22}$$

for any infinite subset $K \subset \mathbb{N}^+$. ¤

Definition 4.2 is so constructed that if $\{\hat{h}_i\}_{i=0}^{\infty}$ is a local minimizing sequence, then any subsequence $\{h_i\}_{i\in K}$, $K \subset \mathbb{N}^+$ is also a local minimizing sequence. Referring to [P4] we find the following optimality condition for minimizing sequences.

<u>Proposition 4.7 [P4]</u>: Suppose that $\{\hat{h}_i\}_{i=0}^{\infty}$ is a minimizing sequence for P. Then

$$\overline{\lim}\ \varepsilon(\hat{h}_i) = \lim\ \varepsilon(\hat{h}_i) = 0. \qquad ¤$$

As far as Algorithm 4.1 is concerned, the following result holds in the topology of minimizing sequences.

Theorem 4.2: Let $\{h_i\}_{i=0}^{\infty}$ be a bounded sequence generated by Algorithm 4.1. Let $K \triangleq \{i \in \mathbb{N}^+ | \psi(h_i) \geq 0\}$. Suppose that $\overline{\lim}_{i \in K} \theta_{\psi,0}(h_i) < 0$. Then $\overline{\lim}\, \psi(h_i) \leq 0$ and either $\underline{\lim}\, f(h_i) = -\infty$ or $\lim \varepsilon(h_i) = 0$. ¤

The proof of Theorem 4.2 is obtained by modifying the proof of Theorem 4.1, in the manner discussed in [P4]. It will not be presented in this paper.

5. Conclusion:

The algorithm presented in this paper is conceptual in the sense that it contains no instructions for approximating the various vectors and function values that are used in the search direction and step size computations. Fortunately, this is not a seriour drawback since in [P3,P2,P7] we find appropriate theoretical results which allow one to convert a conceptual algorithm into an implementable one (i.e. into an algorithm that can be programmed on a digital computer) that retains the convergence properties of the conceptual algorithm from which it is derived. The main reason for presenting the algorithm in conceptual form is that it allowed us to simplify the exposition to a considerable degree without impairing the reader's ability to convert our results into a practical algorithm.

References

[A1] Adams, R.A., Sobolev Spaces, Academic Press, New York, 1975.

[A2] Ahlfors, L.V., Complex Analysis, McGraw-Hill, New York, 1953.

[A3] Armijo, L., "Minimization of Functions Having Lipschitz Continuous First Partial Derivatives," Pacific Journal of Mathematics, 16, pp. 1-3, 1966.

[A4] Aubin, J-P., Approximation of Elliptic Boundary-Value Problems, Wiley-Interscience, Pure and Applied Mathematics, Ed. R. Courant, Vol. 26, 1972.

[B1] Berge, C., Topological Spaces, McMillan Co., New York, 1963.

[C1] Choi, K.K., and Haug, E.J., "Optimization of Structures with Repeated Eigenvalues," Optimization of Distributed Parameter Structures, Eds. E.J. Haug and Cea, J. Sijtoff & Noordhoff, Alpen van den Rijn, Netherlands, pp. 223-281, 1981.

[C2] Choi, K.K. and Haug, E.J., "Repeated Eigenvalues in Mechanical Optimization Problems," Contemporary Mathematics, Vol. 4, 1981, pp. 61-86.

[C3] Choi, K.K., Haug, E.J. and Lam, H.L., "A Numerical Method for Distributed Parameter Structural Optimization Problems with Repeated Eigenvalues," to appear.

[C4] Clarke, F.H., "Generalized Gradients and Applications," Trans. Amer. Math. Soc., Vol. 205, pp. 247-262, 1975.

[C5] Clarke, F.H., "A New Approach to Lagrange Multipliers," Math. of Oper. Res., Vol. 1, No. 2, 1976.

[C6] Clarke, F.H., Nonsmooth Analysis, to appear.

[C7] Cullum, J., Donath, W.E., and Wolfe, P., "Minimization of Certain Nondifferentiable sums of Eigenvalues of Symmetric Matrices," Nondifferentiable Optimization, Math. Programming Study No. 3, M.L. Balinski and P. Wolfe, eds. North Holland, Amsterdam, pp. 35-55, 1975.

[F1] Fichera, G., "Existence Theorems in Elasticity," Handbuch Der Physik, Vol. VIa/2, 1972, pp. 347-389.

[F2] Folland, G.B., Introduction to Partial Differential Equations, Princeton University Press, 1976.

[H1] Haug, E.J., "A Review of Distributed Parameter Structural Optimization Literature," Optimization of Distributed Parameter Structures, Eds. E.J. Haug and J. Cea, Sijtoff & Noordhoff, Alpen van den Rijn, Netherlands, 1980.

[H2] Haug, E.J., and Choi, K.K., "Systematic Occurence of Repeated Eigenvalues in Structured Optimization," to appear.

[H3] Haug, E.J., and Rousselet, B., "Design Sensitivity Analysis in Structural Mechanics I: Static Response Variations," Journal of Structural Mechanics, Vol. 8, pp. 17-41, 1980.

[H4] Haug, E.J., and Rousselet, B., "Design Sensitivity Analysis in Structural Mechanics II: Eigenvalue Variations," Journal of Structural Mechanics, Vol. 8, pp. 161-186, 1980.

[J1] Jouron, C., "Sur un Problème d'Optimization ou la Contrainte Porte sur la Fréquence Fondamentale," R.A.I.R.O. Analyse Numerique/Numerical Analysis, Vol. 12, No. 4, pp. 349-374, 1978.

[K1] Kato, T., "Perturbation Theory for Linear Operators," Second Edition, Springer-Verlag, New York, 1980.

[L1] Lebourg, G., "Valeur Moyenne pour Gradient Generalizé," C.R. Acad. Sci., Paris, Vol. 281, pp. 795-797, 1975.

[L2] Lemarechal C., "An Extension of Davidon Methods to Nondifferentiable Problems," in Mathematical Programming Studies, 3, M.L. Balinski and R. Wolfe, Eds., North Holland, Amsterdam, pp. 95-109, 1975.

[L3] Lions, J.L., and Margenes, E., Non-Homogenious Boundary Value Problems and Applications, Vol. 1, Springer-Verlag, New York, 1972.

[M1] Masur, E.F., and Mroz, Z., "Singular Solutions in Structural Optimization Problems," Proceedings IUTAM Symposium on Variational Methods in Mechanics of Solids, Ed. S. Nermat-Nasser, Pergamon Press, New York, pp. 337-343, 1980.

[M2] Mansfield, E.H., The Bending and Stretching of Plates, International Series of Monographs on Aeronautics and Astronautics, Vol. 6, Pergamon Press, Macmillan Co., 1964.

[O1] Olhoff, N., and Rasmussen, S.H., "On Single and Bimodal Optimum Buckling Loads of Clamped Columns," International Journal of Solids and Structures, Vol. 13, pp. 605-614, 1977.

[P1] Polak, E., and Mayne, D.Q., "On the Solution of Singular Value Inequalities over a Continuum of Frequencies," Memorandum No. UCB/ERL M80/8, January 1980; 20th IEEE Conference on Decision and Control, Albuquerque, New Mexico, December 12-14, 1980.

[P2] Polak, E., Mayne, D.Q., and Wardi, Y., "On the Extension of Constrained Optimization Algorithms from Differentiable to Nondifferentiable Problems," SIAM J. Control and Optimization, in press.

[P3] Polak, E., and Wardi, Y., "Nondifferentiable Optimization Algorithm for Designing Control Systems Subject to Sincular Value Inequalities," Automatica, Vol. 18, No. 3, pp. 267-283, 1982.

[P4] Polak, E., and Wardi, Y., "A Study of Minimizing Sequences," University of California, Electronics Research Laboratory Memo No. UCB/ERL M82/22, March 1982.

[P5] Polak, E., Trahan, R., and Mayne, D.Q., "Combined Phase I-Phase II Methods of Feasible Directions," Math. Prog., Vol. 17, pp. 61-73, July 1979.

[P6] Polak, E., and Sangiovanni-Vincentelli, A., "Theoretical and Computational Aspects of Design Centering, Tolerancing and Tuning Problems," IEEE Transactions on Circuits and Systems, Vol. CAS-26, No. 9, pp. 295-318, 1979.

[P7] Polak, E., Computational Methods in Optimization: A Unified Approach, Academic Press, New York, 1971.

[P8] Polak, E., and Mayne, D.Q., "Algorithm Models for Nondifferentiable Optimization," University of California, Electronics Research Laboratory, Memo No. UCB/ERL M82/34.

[P9] Pshenichny, B.N., Necessary Conditions for an Extremum, Translation edited by L.W. Neustadt, translated by K. Makovski, Marcel Dekker, Inc. New York, 1971.

[R1] Rellich, F., Perturbation Theory of Eigenvalue Problems, Gordon and Breach, New York, London, Paris, 1969.

[R2] Rousselet, B., "Multiple Eigenvalues (Numerical Algorithms) and Supremum Norm Constraints," Proceedings of NATO Advanced Study Institute, University of Iowa, Iowa City, IA, May 1980, to appear.

[R3] Rudin, W., Functional Analysis, McGraw-Hill, New York, 1973.

[S1] Seong, H., Choi, K., and Haug, E., "A Study of Problems of Optimization of Structures with Repeated Eigenvalues," Technical Report No. 81-3, Center for Computer Aided Design, College of Engineering, University of Iowa, Iowa City, August 1981.

[T1] Tompson, J.M.T., and Hunt, G., A General Theory of Elastic Stability, John Wiley and Sons, New York, 1973.

[T2] Tompson, J.M.T., and Hunt G.W., "Degrees of Structural Optimization," Engineering Optimization, Vol. 1, pp. 99-110, 1974.

[T3] Treves, R., Basic Linear Partial Differential Equations, Academic Press, New York, 1975.

[Z1] Zolesio, J.P., "Semi Derivatives of Repeated Eigenvalues," Optimization of Distributed Parameter Structures, Eds. E.J. Haug and J. Cea, Sijthoff & Noordhoff, Alpen van den Rijn, Netherlands, pp. 223-281, 1981.

Appendix A: Proofs

We shall now provide the proofs that were omitted in Section 4. We devote a separate subsection to each proof.

A1. Proof of Proposition 4.2:

We begin with a). We recall from [H4] that λ is an eigenvalue of (a_h,b_h) if and only if $\delta = \frac{1}{\lambda}$ is a nonzero eigenvalue of $\bar{A}_h^{-1}\bar{B}_h$, where $\bar{A}_h$, $\bar{B}_h$ were defined in (4) of [H4]. We reproduce the definitions and discuss the properties of $\bar{A}_h$ and $\bar{B}_h$, since we shall need them in our analysis. For details see [H3,H4].

Consider the diagram in Fig. 1, which we use to define the map $\bar{A}_h$. In this diagram,

(a) i is the inclusion of $H_0^2(\Omega)$ into $L^2(\Omega)$.

(b) i' is the inclusion of $L^2(\Omega)'$ into $H_0^2(\Omega)'$.

(c) L^2 and $(L^2)'$ are brought into correspondence by means of the Riesz representation theorem.

(d) The map A_h is defined by

$$(A_h,y,v) \triangleq a_h(y,v) \quad \forall y,v \in H_0^2(\Omega) . \tag{A1.1}$$

Since a_h is elliptic, A_h is an isomorphism onto $H_0^2(\Omega)'$.

(e) The map $\bar{A}_h$: $H_0^2(\Omega) \to L^2$ is an unbounded operator whose domain, $D(\bar{A}_h)$, is given by

$$D(\bar{A}_h) \triangleq \{y \in H_0^2(\Omega) \,|\, A_h y \in L^2(\Omega)'\} \quad . \tag{A1.2}$$

The map $\bar{A}_h$ is defined by

$$\langle \bar{A}_h y,v \rangle_{L^2} \triangleq (A_h\; y,v) = a_h(y,v) \; . \tag{A1.3}$$

Note that $i'\bar{A}_h = A_h$, and hence that $\bar{A}_h^{-1} = A_h^{-1} i'$ maps $L^2(\Omega)$ into $H_0^2(\Omega)$, because A_h is an isomorphism and $\|i'\| \leq 1$. Clearly, it is a bounded operator.

(f). $G_h v \triangleq h^{3/2} v_{xx}$ (A1.4a)

$$(G_h' f, v) \triangleq \langle G_h v, f \rangle_{L^2} \tag{A1.4b}$$

Since G_h is an isomorphism onto its range, and $G_h'\big|_{R'(G_h)}$ is an isomorphism onto $H_0^2(\Omega)'$, where $R(G_h)$ denotes the range of G_h. We consider G_h as an operator from $H_0^2(\Omega)$ onto its range. Thus G_h^{-1} and $(G_h')^{-1}$ are well defined. Furthermore, we note that $G_h' G_h = A_h$.

Next, suppose that $\|h\|_{L^\infty} \leq M$, and let g be an element in the ball $B(h, \bar{h}/4)$. As was done in equation (45) of [H.3], we define three operators, C_1, C_2, C_3, from $R(G_h)$ into $R(G_h)$ (which is a closed subspace of $L^2(\Omega)$ because G_h is an isomorphism onto its range), as follows:

$$\langle C_1 G_h y, G_h v \rangle_{L^2} \triangleq \int 3h^2 g \; y_{xx} \; v_{xx} \, d\mu \, , \tag{A1.5a}$$

$$\langle C_2 G_h y, G_h v \rangle_{L^2} \triangleq \int 3hg^2 \; y_{xx} \; v_{xx} \, d\mu \, , \tag{A1.5b}$$

$$\langle C_3 G_h y, G_h v \rangle_{L^2} \triangleq \int g^3 y_{xx} \; v_{xx} \, d\mu \, . \tag{A1.5c}$$

It can be easily verified from equations (A1.5a)-(A1.5c) that

$$\|C_1\| \leq 3\|g\|_{L^\infty}/\|h\|_{L^\infty} \leq 3/4 \, , \tag{A1.6a}$$

$$\|C_2\| \leq 3(\|g\|_{L^\infty}/\|H\|_{L^\infty})^2 \leq 3/16 \, , \tag{A1.6b}$$

$$\|C_3\| \leq (\|g\|_{L^\infty}/\|h\|_{L^\infty})^3 \leq \frac{1}{64} \, . \tag{A1.6c}$$

Now, for $t \in [-1,1]$ we have that

$$(A_{h+tg}y,v) = a_{h+tg}(y,v) = \int h^3 y_{xx}v_{xx}d\mu + t\int 3h^2 g y_{xx}v_{xx}d\mu$$

$$+ t^2\int 3hg^2 y_{xx}v_{xx}d\mu + t^3\int g^3 y_{xx}v_{xx}d\mu = (G_h'G_h y,v) + t(G_h'C_1G_h y,v)$$

$$+ t^2(G_h'C_2G_h y,v) + t^3(G_h'C_3G_h y,v) \tag{A1.7}$$

From (A1.7) we conclude that

$$A_{h+tg} = G_h'(I+tC_1+t^2C_2+t^3C_3)G_h \ . \tag{A1.8}$$

Now, there exists a $\bar{t} > 1$ such that for all $t \in [-\bar{t},\bar{t}]$,

$$\frac{3}{4}t + \frac{3}{16}t^2 + \frac{1}{64}t^3 < \frac{63}{64}\ , \tag{A1.9}$$

and hence, from (A1.6)-(A1.8),

$$(I+tC_1+t^2C_2+t^3C_3)^{-1} = \sum_{n=0}^{\infty} (-1)^n(tC_1+t^2C_2+t^3C_3)^n,$$

so that

$$A_{h+tg}^{-1} = \sum_{n=0}^{\infty} (-1)^n\ G_h^{-1}(tC_1+t^2C_2+t^3C_3)^n(G_h')^{-1}\ . \tag{A1.10}$$

As $\|G_h^{-1}\| = \|(G_h')^{-1}\| \leq \dfrac{C^{1/2}}{\bar{h}^{3/2}}$, where C is the constant in the Pincaré inequality (see [F.1]. Equation (A1.10) implies that the map $t \to A_{h+tg}^{-1}$, from $\mathbb{R}$ into the Banach space of bounded operators from $H_0^2(\Omega)'$ into $H_0^2(\Omega)$, is analytic (i.e., it has an absolutely and uniformly convergent power series representation), and the radius of convergence at $t \in \mathbb{R}$ such that $\phi(h+tg) \leq 0$, is $\bar{t} > 1$.

It now follows that $\bar{A}_{h+tg}^{-1}$ is also analytic in t, with the same radius of convergence, because $\bar{A}_{h+tg}^{-1} = A_{h+tg}^{-1}\ i'$.

Now $\bar{B}_h$ is defined similarly to $\bar{A}_h$ in (A.3), i.e.,

$$\langle \bar{B}_h y,v\rangle_{L^2} \triangleq b_h(y,v)\ . \tag{A1.11}$$

It is easily seen that $\bar{B}_h$ is a bounded operator whose domain is the whole of $H_0^2(\Omega)$ and whose range is in $L^2(\Omega)$. Consequently, from (A.11)

$$\bar{B}_h y = h(iy) \; . \tag{A1.12}$$

It follows that the map $t \to \bar{B}_{h+tg}$ is analytic, with radius of convergence ∞.

We thus have the following: for all $h \in L_+^\infty(\Omega)$ and $g \in B(h,\bar{h}/4)$, the map $t \to \bar{A}_{h+tg}^{-1} \bar{B}_{h+tg}$ is analytic on $-(\bar{t},\bar{t})$, and the radius of convergence at t s.t. $\phi(h+tg) \leq 0$ is $\bar{t} > 1$. Therefore (a) is a consequence of the preceding statement and theorems VII 1.7-1.8 of [K1].

To establish (b), we observe that, with $t = 1$,

$$\begin{aligned}\|\bar{A}_{h+tg}^{-1} \bar{B}_{h+tg} - \bar{A}_h^{-1}\bar{B}_h\| &\leq \|\bar{A}_{h+tg}^{-1}\|\|\bar{B}_{h+tg}-\bar{B}_h\| + \|\bar{B}_h\|\|\bar{A}_{h+tg}^{-1}-\bar{A}_h^{-1}\| \\ &\leq \frac{\sqrt{C}}{(\bar{h}/2)^{3/2}} \|g\|_{L^\infty} + \bar{h}\,\frac{C}{\bar{h}^3} \\ &\quad \sum_{n=1}^{\infty} (3\|g\|_{L^\infty}/\|h\|_{L^\infty} + 3\|g\|^2_{L^\infty}/(\|h\|_{L^\infty})^2 + \|g\|^3_{L^\infty}/\|h\|^3_{L^\infty})^n .\end{aligned} \tag{A1.13}$$

The last term in (A1.13) satisfies

$$\begin{aligned}&\sum_{n=1}^{\infty} (3\| \tfrac{g}{h} \|_{L^\infty} + 3\|\tfrac{g}{h}\|^2_{L^\infty} + \|\tfrac{g}{h}\|^3_{L^\infty})^n \\ &\qquad \leq \|g\|_{L^\infty}(3\,\frac{1}{\bar{h}} + 3\,\frac{1}{4\bar{h}} + \frac{1}{16}\,\frac{1}{\bar{h}}) \sum_{n=0}^{\infty} (\frac{3}{4} + \frac{3}{16} + \frac{1}{64})^n .\end{aligned} \tag{A1.14}$$

We see from (A1.14) and (A1.13) that there exists a $K > 0$, such that for all $h \in L_+^\infty(\Omega)$ and $g \in B(h,\bar{h}/4)$ such that $\phi(h+g) \leq 0$,

$$\|\bar{A}_{h+tg}^{-1} \bar{B}_{h+tg} - \bar{A}_h^{-1}\bar{B}_h\| \leq K\|g\|_{L^\infty} \; . \tag{A1.15}$$

Finally, let $\delta^1(h)$ be the largest eigenvalue of $\bar{A}_h^{-1}\,\bar{B}_h$. Then $\delta^1(h) = 1/\lambda^1(h)$, and hence

$$|\lambda^1(h+g) - \lambda^1(h)| = \left|\frac{1}{\delta^1(h+g)} - \frac{1}{\delta^1(h)}\right|$$

$$\leq K\|g\|_{L^\infty} \frac{1}{\delta^1(h+g)} \cdot \frac{1}{\delta^1(h)} = K\|g\|_\infty\, \lambda^1(h+g)\, \lambda^1(h). \tag{A1.16}$$

We now make use of the fact that $\|h\|_{L^\infty} \leq M$ (and hence that $\|h+g\|_{L^\infty} \leq M+\bar{h}$) as follows:

$$\lambda^1(h) = \inf_{\substack{y\in H_0^2(\Omega)\\ y\neq 0}} \frac{\int_\Omega h^3 y_{xx}^2 d\mu}{\int_\Omega h y^2 d\mu} \quad . \tag{A1.17}$$

Let $y \neq 0$ be arbitrary. Then

$$\lambda^1(h) \leq \frac{M^3\int y_{xx}^2 d\mu}{\bar{h}\int y^2 d\mu} \,. \tag{A1.18}$$

Similarly,

$$\lambda^1(h+g) \leq \frac{(M+\bar{h})^3}{\bar{h}} \frac{\int_\Omega y_{xx}^2 d\mu}{\int_\Omega y^2 d\mu} \,. \tag{A1.19}$$

Substituting for $\lambda^1(h)$, $\lambda^1(h+g)$ into (A1.16), we conclude that $\lambda^1(\cdot)$ is locally Lipschitz. This completes the proof of (b) and hence of Proposition 4.2. ◻

A2. Proof of Proposition 4.3

Let $h \in L_+^\infty(\Omega)$ and $k \in \mathbb{N}^+$ be such that $\lambda^k(h) \neq \lambda^{k+1}(h)$.

(i) It follows from the continuity of the functions $h \to \lambda^i(h)$ that there exists a $\rho > 0$ such that for all $h' \in B(h,\rho) \cap L_+^\infty(\Omega)$ $\hat{G}^k{}_\psi(\Omega)$ is well defined.

(ii) For all $h' \in B(h,\rho) \cap L_+^\infty(\Omega)$ let

$$Z^k(h') \triangleq \{z \in H_0^2(\Omega) | z \text{ is a unit eigenvector of } (a_{h'}, b_{h'}) \text{ corresponding to } {}^j(h'),\ j = 1,2,\dots k\} \tag{A2.1}$$

and let $P_{h'}^k$ denote the orthogonal projection operator from $H_0^2(\Omega)$ onto the subspace of $H_0^2(\Omega)$ spanned by $Z^k(h')$.

Let $\delta^1(h') \geq \delta^2(h') \geq \delta^3(h') \geq \dots$ be the eigenvalues of the operator $\bar{A}_{h'}^{-1}\, \bar{B}_{h'}$, defined in Section A1. Consider a closed curve Γ in the complex plane, which encircles $\delta^1(h)$, $\delta^2(h)$, $\dots\delta^k(h)$ and no other eigenvalue of $\bar{A}_h^{-1}\, \bar{B}_h$. If ρ is small enough, then for all $h' \in B(h,\rho) \cap L_+^\infty(\Omega)$, Γ encircles $\delta^j(h')$, $j = 1,2,\dots k$, and no other eigenvalue of $\bar{A}_{h'}^{-1}\, \bar{B}_{h'}$. In this case we have from [K.1] that

$$P_{h'}^k = \frac{1}{2\pi j} \int_\Gamma (\xi - \bar{A}_{h'}^{-1}\, \bar{B}_{h'})^{-1} d\xi \tag{A2.2}$$

Consequently, for every h', $h'' \in B(h,\rho) \cap L_+^\infty(\Omega)$ we have that

$$\begin{aligned} \|P_{h'}^k - P_{h''}^k\| &\leq \frac{1}{2\pi} \int_\Gamma \|(\xi - \bar{A}_{h''}^{-1}\bar{B}_{h''})^{-1}\|\, \|\bar{A}_{h''}^{-1}\, \bar{B}_{h''} - \bar{A}_{h'}^{-1}\, \bar{B}_{h'}\|\, \|(\xi - \bar{A}_{h'}^{-1}\bar{B}_{h'})^{-1}\| d\xi \\ &\leq \frac{1}{2}\, \ell(\Gamma)\, \text{Max}\{|\xi - \delta^j(h'')|^{-1} \big| \xi \in \Gamma,\ j \in \underline{k}\} \times \\ &\quad \text{Max}\{|\xi - \delta^j(j')|^{-1} \big| \xi \in \Gamma,\ j \in \underline{k}\}\, \|\bar{A}_{h'}^{-1}\, \bar{B}_{h'} - \bar{A}_{h''}^{-1}\, \bar{B}_{h'}\|, \end{aligned} \tag{A2.3}$$

where $\ell(\Gamma)$ is the length of the curve Γ. Now, it can be seen from (A1.13)-(A1.19) that the Lipschitz constant K and the convergence radius ρ in Proposition 4.2 depend only on $\|h\|_{L^\infty(\Omega)}$, but not on h or h'. Thus, if ρ is small enough, then there exists a $\bar{K} > 0$ such that

$$\|\bar{A}_{h'}^{-1}\, \bar{B}_{h'} - \bar{A}_{h''}^{-1}\, \bar{B}_{L''}\| \leq \bar{K}\|h' - h''\|,$$

with $\bar{K}$ depending on $\|h\|_{L^\infty(\Omega)}$ but not on h. Therefore it follows from (A2.3) that $P_{h'}^k$ is locally Lipschitz continuous in h'.

Next, consider a sequence $\{h_i'\}_{i=0}^{\infty}$ such that for all i, $h_i' \in L_+^\infty(\Omega) \cap B(h,\rho)$ and $h_i' \to h$. Let $z_i \in Z_{h_i'}^k$, $i \in \mathbb{N}^+$. Since $P_{h_i'}^k \to P_h^k$ and $\| P_{h_i'}^k z_i\| = \|z_i\| = 1$, there exists an $i_0 \in \mathbb{N}^+$ such that for all $i > i_0$, $\| P_h^k z_i\| > \frac{1}{2}$, and hence $\left\| \frac{z_i}{P_h^k z_i} \right\| < 2$. Thus we have that

$$\lim_{i\to\infty} \| P_h^k \frac{z_i}{\|P_h^k z_i\|} - z_i\| = \lim_{i\to\infty} \| P_h^k z_i - z_i\| = \lim_{i\to\infty} \| P_h^k z_i - P_{h_i'}^k z_i\| = 0 \quad \text{(A2.4)}$$

Since $P_h^k \frac{z_i}{\| P_h^k z_i\|} \in Z^k(h)$, it follows that $Z^k(h)$ is upper semi-continuous in h. On the other hand, if $z \in Z^k(h)$ then $z = P_h^k z = \lim_{i\to\infty} P_{h_i'}^k z$ $= \lim_{i\to\infty} \frac{P_{h_i'}^k z}{\| P_{h_i'}^k z\|_{H_0^2(\Omega)}}$. Since $\frac{P_{h_i'}^k z}{\| P_{h_i'}^k z\|_{H_0^2(\Omega)}} \in Z^k(h_i)$, it follows that $Z^k(h)$ is lower semi-continuous. Thus it is continuous in the sense of Berge ([B1]) The continuity of $\hat{G}^k\psi(h)$ follows directly.

Next, for all $h' \in L_+^\infty(\Omega) \cap B(h,\rho)$, $Z^k(h')$ is a compact set. The set $\{\xi_{h'}^z | z \in Z^k(h')\}$ is a continuous image of a compact set, and hence it is also compact. Finally, $\hat{G}^k\psi(h')$ is the closure of the convex hull of a compact set, hence it is also compact. The proof of the fact that $\bigcup_{h'\in S} \hat{G}^k(h')$ is compact, with S a compact set, is straightforward and is therefore omitted. This completes the proof of Proposition 4.3.

$\square$

A3. Proof of Proposition 4.4

We will have to make use of the following set of facts.

Fact A3.1: Suppose that the design parameter space is one dimensional, with t being the design parameter, and that the bilinear forms a_t and b_t are analytic in t. Let a_t' and b_t' denote the derivatives of a_t and b_t (respectively) with respect to t. Let the eigenvalues of (a_t,b_t) be numbered in increasing order, $\lambda^1(t) \leq \lambda^2(t) \leq \lambda^3(t) \leq \ldots$. For the function $t \to \lambda^1(t)$, $\partial\lambda^1(t_0) = \mathrm{co}\{\frac{1}{b_{t_0}(h,u)} (a'_{t_0}(h,u) - \lambda^1(t_0)\, b'_{t_0}(h,u))|$ u is an eigenvector of (a_{t_0}, b_{t_0}) corresponding to $\lambda^1(t_0)$, and $\|h\|_{H_0^2(\Omega)} = 1\}$.

Proof: It is known from perturbation theory ([A2]) that the functions $\lambda^j(t)$, $j \in \underline{m}^1(t_0)$ (where $m^1(t_0)$ is the multiplicity of $\lambda^1(t_0)$) are branches of an analytic equation. It is readily seen that if $m^1(t) = 1$, then

$$\frac{d\lambda'(t)}{dt} = \frac{1}{b_t(u,u)} (a'_t(u,u) - \lambda^1(t)\, b'_t(u,u)),$$

with u an eigenvector of (a_t, b_t) corresponding to $\lambda^1(t)$, and $\|h\|_{H_0^2(\Omega)} = 1$. The proof now follows from analytic function theory and Proposition 3.1(e). ¤

Fact A3.2: Under the conditions of Fact A3.1, $d\psi(t_0,1) = d_0\psi(t_0,1)$, where $d\psi$ denotes directional derivative and $d_0\psi$ denotes generalized directional derivative of ψ.

Proof: This fact follows directly from the fact that the functions $\lambda^j(t)$, $j \in \underline{m}^1(t_0)$ are branches of an analytic equation on some neighborhood of t_0. ¤

Fact A3.3: Let $h \in L_+^\infty(\Omega)$ and $g \in L^\infty(\Omega)$ be such that $\|g\|_{L^\infty(\Omega)} \leq \frac{\bar{h}}{4}$. Then $d\psi(h,g) = \mathrm{Max}\{(\xi,g) | \xi \in G_0\psi(h)\}$.

Proof: Let $\bar{\psi}(t) \triangleq \psi(h+tg)$, then $d\psi(h,g) = d\bar{\psi}(0,1)$. From Fact A3.2 we conclude that $d\bar{\psi}(0,1) = d_0\bar{\psi}(0,1)$. From Fact A3.1 we have that $d_0\bar{\psi}(0,1) = \mathrm{Max}\{\frac{-1}{b_h(u,u)} (3h^2 g u_{xx}^2 - \lambda^1(h) g u^2)|$ u is an eigenvector of (a_h, b_h) corresponding to $\lambda^j(h)$, $j \in \underline{m}^1(h)\}$. Therefore $d\psi(h,g) = \mathrm{Max}\{(\xi,g) | \xi \in G_0\psi(h)\}$, and the proof is complete. ¤

We are now ready to prove Proposition 4.4. It follows from Fact A3.3 and the fact that $G_0\psi(h)$ is weak* compact that $\xi \in (L^\infty(\Omega))'$ satisfies $\xi \in G_0\psi(h)$ if and only if $(\xi,g) \leq d\psi(h,g)$ for all $g \in L^\infty(\Omega)$. Now, if $\xi \in G_0\psi(h)$ then $(\xi,g) \leq d_0\psi(h,g)$ for all $g \in L^\infty(\Omega)$, and hence $\xi \in \partial\psi(h)$. Thus we have shownthat $G_0\psi(h) \subset \partial\psi(h)$. To show the opposite inclusion, we need the fact that the point to set map $h \to G_0\psi(h)$ has the mean value property with respect to ψ, i.e., that for all $g,h \in L_+^\infty(\Omega)$ there exists a $t \in [0,1]$ and an $\eta \in G_0\psi(h+tg)$ such that $\psi(g+h) - \psi(h) = (\eta,g)$. We shall now prove this.

Let $\bar{\psi}(t)$ be defined as in the proof of Fact A3.3. We have from the mean value theorem of Lebourg ([L.1]) that $\psi(g+h) - \psi(h) = \bar{\psi}(1) - \bar{\psi}(0) = \bar{\eta}$, for some $\bar{\eta} \in \partial\bar{\psi}(t)$, and some $t \in [0,1]$. It follows from Fact A3.1 that $-\bar{\eta} \in \mathrm{co}\{\frac{1}{b_{h+tg}(u,u)} \int_\Omega 3(h+tg)^2\, gu_{xx}^2 - \lambda^1(h+tg)gu^2)d\mu$ with u an eigenvector of (a_{h+tg}, b_{h+tg}) corresponding to $\lambda^j(h+tg)$, $j \in m^1(h+tg)$, and $\|u\|_{H_0^2(\Omega)}^2 = 1\}$. This means that $\bar{\eta} = (g,\eta)$ for some $\eta \in G_0\psi(h+tg)$, which completes the proof that the point to set map $h \to G_0\psi(h)$ has the mean value property with respect to ψ.

Next, we proceed to show that $\partial\psi(h) \subset G_0\psi(h)$. Suppose, for the sake of contradiction, that some $\bar{\xi} \in \partial\psi(h)$ satisfies $\bar{\xi} \notin G_0\psi(h)$. Since $G_0\psi(h)$ is weak* compact, there exists a $g \in L^\infty(\Omega)$ and an $\alpha > 0$ such that for all $\eta \in G_0\psi(h)$

$$(g,\bar{\xi}) > \alpha > (g,\eta)\ . \tag{A3.1}$$

It now follows from Fact A3.3 that

$$d_0\psi(h,g) \geq \alpha > d\psi(h,g)\ . \tag{A3.2}$$

But $d_0\psi(h,g) = \overline{\lim_{\substack{h_i \to 0 \\ s_i \searrow 0}}}\ \dfrac{\psi(h+h_i+s_ig) - (h+h_i)}{s_i}$.

Now, for all $i \in \mathbb{N}^+$ there exists a $t_i \in [0,1]$ and an $\eta_i \in G_0\psi(h+h_i+t_is_ig)$ such that $\frac{\psi(h+h_i+s_ig) - (h+h_i)}{s_i} = (\eta_i, g)$. As $i \to \infty$, a subsequence of $\{\eta_i\}_{i=1}^{\infty}$ converges to some $\bar{\eta} \in G_0(h)$, and we get that $(\bar{\eta}, g) \geq \alpha > d\psi(h,g)$. This contradicts the fact that $d\psi(h,g) = \text{Max}\{(\eta,g) | \eta \not\subset G_0\psi(h)\}$. This contradicts our assumption that $\partial\psi(h) \notin G_0\psi(h)$, and the proof of Proposition 4.4 is complete. ¤

A.4. Proof of Proposition 4.5

a) We will show that $U_{\psi,\varepsilon}(h)$ is well defined. The proof that $U_{f,\varepsilon}(h)$ is well defined is similar and hence will be omitted.

$G_\varepsilon\psi(h)$ is a compact convex subset of $L^1(\Omega)$, and $F^\phi(h)$ is a weakly closed subset of $L^\infty(\Omega)$. For all $i \in \mathbb{N}^+$, let $\xi_i \in G_\varepsilon\psi(h)$ and $g_i \in F^\psi(h)$ be such that

$$(g_i, \xi_i) = \underset{\xi \in G_\varepsilon\psi(h)}{\text{Max}} (\xi, g_i), \tag{A4.1}$$

and

$$(g_i, \xi_i) \leq \inf_{g \in F^\phi(h)} \underset{\xi \in G_\varepsilon\psi(h)}{\text{Max}} (g, \xi) + \frac{1}{i}. \tag{A4.2}$$

Let $\bar{\xi} \in G_\varepsilon\psi(h)$, $\bar{g} \in F^\phi(h)$ and let K be an infinite subset of $\mathbb{N}^+$ such that $\xi_i \underset{\text{weakly}}{\overset{K}{\to}} \bar{\xi}$ and $g_i \overset{K}{\to} \bar{g}$. For any $\xi \in G_\varepsilon\psi(h)$ we have

$(\bar{g},\xi) = \lim_{i \in K} (g_i, \xi) \leq \lim_{i \in K} (g_i, \xi_i) = (\bar{g}, \bar{\xi})$, hence $\bar{\xi} \in \arg\max\{(\bar{g},\xi) | \xi \in G_\varepsilon\psi(h)\}$.
Next, for every $g \in F^\phi(h)$, $\underset{\xi \in G_\varepsilon\psi(h)}{\max} (g,\xi) \geq \overline{\lim_{i \in K}} (g, \xi_i) \geq \overline{\lim_{i \in K}} ((g_i, \xi_i) - \frac{1}{i})$
$= (\bar{g}, \bar{\xi})$. Thus, $\bar{g} \in \underset{g \in F^\phi(h)}{\text{argmin}} \underset{\xi \in G_\varepsilon\psi(h)}{\text{Max}} (g,\xi)$, and hence $U_{\psi,\varepsilon}$ is well defined.

b) This part follows directly from the facts that $\theta_{f,\varepsilon}(h) \leq \theta_{f,\varepsilon'}(h)$ and $\theta_{\psi,\varepsilon}(h) \leq \theta_{\psi,\varepsilon'}(h)$.

c) This part follows directly from the fact that both $\theta_{f,\varepsilon}(\cdot)$ and $\theta_{\psi,\varepsilon}(\cdot)$ are upper semi-continuous.

d) Since $\theta_{\varepsilon(h)}(h) \leq -\varepsilon(h) \leq 0$ by definition, $\theta_{\varepsilon(h)}(h) = 0$ implies that $\varepsilon(h) = 0$.

To establish the converse, suppose that $\varepsilon(h) = 0$. We will show that $\theta_{f,\varepsilon(h)}(h) = 0$; the proof that $(1-\Gamma(h))\ \theta_{\psi,\varepsilon(h)}(h)$ is identical, and hence omitted. Suppose for the sake of contradiction that $\varepsilon(h) = 0$ and that $\theta_{f,0}(h) < 0$. Let $\bar{\varepsilon} > 0$ be such that for all $\varepsilon \leq \bar{\varepsilon}$, $k(h,\varepsilon) = k(h,0)$, and hence $\text{co}\{f'(h),G_{\varepsilon}\psi(h)\} = \text{co}\{f'(h),G_0\psi(h)\}$. Then, for all ε satisfying $0 \leq \varepsilon \leq \bar{\varepsilon}$, $\theta_{f,\varepsilon}(h) = \theta_{f,0}(h) < 0$. We see that $\varepsilon(h) = 0 > -\nu\theta_{f,0}(h) > 0$, contradicting the definition of $\varepsilon(h)$. This completes the proof of (d).

(e) This part follows from the optimality condition (Theorem 3.2) and the fact that $\Gamma(\hat{h}) = 1$.

(f) Suppose that the statement in (f) is not true. Then there exists a sequence $\{h_i\}_{i=1}^{\infty}$, $h_i \in L_+^{\infty}(\Omega)$, such that $h_i \underset{i\to\infty}{\to} \hat{h}$, and $\varepsilon(h_i) < \nu\varepsilon(\hat{h})$ and $\varepsilon(\hat{h}) > 0$. For all $i \in \mathbb{N}^+$, we have $\theta_{\nu\varepsilon(h)}(h_i) > -\nu\varepsilon(\hat{h})$, and as $i \to \infty$ we get from (c) that

$$\theta_{\nu\varepsilon(h)}(\hat{h}) \geq -\nu\varepsilon(\hat{h}). \tag{A4.3}$$

$-\nu\theta(\hat{h}) > -\varepsilon(\hat{h}) \geq \theta_{\varepsilon(\hat{h})}(\hat{h}) \geq \theta_{\nu\varepsilon(\hat{h})}(\hat{h})$, which contradicts (A4.3). This completes the proof of (f) and hence of Proposition 4.5. ¤

A5. <u>Proof of Proposition 4.6 and Theorem 4.1</u>

<u>Proof of Proposition 4.6</u>: Let $h_i \in L_+^{\infty}(\Omega)$ be such that $\theta(h_i) < 0$. We prove the result for the case where $\psi(h_i) = 0$; the case where $\psi(h_i) \neq 0$ is simpler and the proof is omitted.

Suppose, for the sake of contradiction, that $s_i = 0$ for some $g_i \in U(h_i) = U_f(h_i)$, i.e., that for all $s = \beta^k > 0$ with $k \in \mathbb{N}^+$, either

$$f(h_i+sg_i) - f(h_i) > \alpha\ s\theta(h_i) \tag{A5.1a}$$

or

$$\psi(h_i+sg_i) - \psi(h_i) > \alpha\, s\theta(h_i) \tag{A5.1b}$$

Suppose that (A5.1b) holds for all $i \in K$, with K an infinite subset of $\mathbb{N}^+$ (the case where (A5.1a) holds infinitely often is similar, and hence will be omitted). By the mean value theorem of Lebourg ([L1]), for all $i \in K$ there exist a $t_s \in [0,1]$ and a $\xi_i \in \partial\psi(h_i+st_sg_i)$ such that

$$(\xi_i,g_i) > \alpha\theta(h_i) \tag{A5.2}$$

It follows from (A5.2), Proposition 4.5(c) and the fact that $G_{\varepsilon(h_i)}\psi(h_i)$ is compact, that there exists a $\bar{\xi} \in \partial\psi(h_i)$ such that $(\bar{\xi},g_i) \geq \alpha\theta(h_i)$. But $0 > \theta(h_i) \geq \theta_{\psi,\varepsilon(h_i)}(h_i) \geq (\bar{\xi},g_i)$, and hence we get a contradiction. This completes the proof of Proposition 4.6. ¤

<u>Proof of Theorem 4.1</u>:

Let $\{h_i\}_{i=1}^{\infty}$ be a sequence constructed by Algorithm 3.1, let $\hat{h} \in L_+^{\infty}(\Omega)$ and let $K \in \mathbb{N}^+$ be an infinite set such that $h_i \overset{K}{\to} \hat{h}$. Suppose, for the sake of contradiction, that $\theta(\hat{h}) < 0$. We shall consider only the case where $\psi(\hat{h}) = 0$, since the case where $\psi(\hat{h}) \neq 0$ is simpler.

First, it follows from Proposition 4.5(f) that there exists a $\hat{\rho} > 0$ such that for all $h \in B(\hat{h},\hat{\rho})$ $\varepsilon(h) \geq \nu\varepsilon(\hat{h}) > 0$. Hence there exists an $i_0 \in \mathbb{N}^+$ such that for all $i \in K$, $i > i_0, \varepsilon(h_i) \geq \nu\varepsilon(\hat{h})$.

Second, from the definition of $\varepsilon(\cdot)$, for all $i \in K$, $i > i_0$, one of the following two relations must hold, with g_i determined in Step 1:

$$f(h_i+\beta^{-1}s_ig_i) - f(h_i) > \alpha\beta^{-1}s_i\theta(h_i), \tag{A5.3a}$$

$$\psi(h_i+\beta^{-1}s_ig_i) - \psi(h_i) > \beta^{-1}s_i\theta(h_i). \tag{A5.3b}$$

Now, (A5.3a) can not hold for any i because f is linear. To complete the proof we will show that (A5.3b) leads to a contradiction. By the mean value theorem of Lebourg, for all $i \in K$ there exist a $t_i \in [0,1]$ and an $\eta_i \in \partial\psi(h_i+t_is_ig_i)$ such that

$$(\eta_i, g_i) > \alpha\theta(h_i). \tag{A5.4}$$

Now, because of Assumption 4.1, we can assume, without loss of generality, the existence of a $\bar{k} \in \mathbb{N}^+$, such that for all $i \in K$, $i > i_0$,

$$\partial\psi(h_i) = G^{\bar{k}}\psi(h_i) \tag{A5.5}$$

Now, we know that $G^{\bar{k}}\psi(h_i+t_is_ig_i)$ is well defined for $i \in K$, i large enough, for otherwise it follows from the continuity of the eigenvalues that $\lambda^{\bar{k}}(\hat{h}) = \lambda^{\bar{k}+1}(\hat{h})$, contradicting the fact that

$$0 < \nu^{-1}\theta(\hat{h}) \leq \lambda^k(h) - \lambda^{k+1}(\hat{h}) \leq \underline{\lim}_{i\in K} (\lambda^k(h_i)-\lambda^{k+1}(h_i)) .$$

We can thus assume, without loss of generality, that for all $i \in K$, $i > t_0$, $\partial\psi(h_i+t_is_ig_i) \subset \hat{G}^{\bar{k}}\psi(h_i+t_is_ig_i)$, the latter being well defined.
Next, by (A5.4) and the continuity of the point to set map $h \to \hat{G}^{\bar{k}}\psi(h)$, we get that $\underline{\lim}_{i\in K} \ \mathrm{Max}\{(\xi,g_i) | \xi \in \hat{G}^k\psi(h_i)\} \geq \overline{\lim}_{i\in K} \alpha\theta(h_i)$, and hence that

$$\underline{\lim}_{i\in K} \theta(h_i) \geq \alpha \ \overline{\lim}_{i\in K} \theta(h_i) . \tag{A5.6}$$

But (A5.6) is possible only if $\overline{\lim}_{i\in K} \theta(h_i) = 0$, which contradict the fact that for all $i \in K$, $i > i_0$, $\theta(h_i) \leq -\varepsilon(h_i) \leq -\nu\varepsilon(\hat{h}) < 0$.

This completes the proof of Theorem 4.1. ¤

$$\begin{tikzcd}
H_0^2(\Omega) \arrow[r, hook, "i"] \arrow[r, bend left, "G_h"] \arrow[rrrr, bend left=50, "A_h"] \arrow[rrr, bend right=50, "\bar{A}_h"'] & L^2(\Omega) \arrow[r, phantom, "="] & \; & L^2(\Omega)' \arrow[r, hook, "i'"] \arrow[r, bend right=50, "G_h'"'] & H_0^2(\Omega)'
\end{tikzcd}$$

COMPUTER-AIDED DESIGN OF STRUCTURES SUBJECT TO EIGENVALUE INEQUALITY CONSTRAINTS*.

by

Y. Y. WARDI and E. POLAK
Department of Electrical Engineering
and Computer Sciences,
and Electronics Research Laboratory,
University of California
Berkeley, California, 94720.

ABSTRACT.

A major goal in the design of elastic structures is the reduction of resonances. Since damping increases with frequency, resonances can be kept withing acceptable limits by ensuring that the lowest natural frequency of the structure being designed lies above a certain threshold. Optimization offers powerful tools for coping with this design constraint on the natural frequencies as well as with the constraints imposed by various other performance requirements.

This paper presents a demonstrably convergent, nondifferentiable optimization algorithm for the design of structures subject to inequality constraints on the lowest natural frequency, on the profile of the structure as well as on other factors. Although only the design of cross sections of vibrating strings, beams, membranes and plates is considered explicitely, the results are easily extended to any optimal design problem with inequality constraints involving L_∞ variables and the eigenvalues of an elliptic

boundary value problem.

The elliptic boundary value problems considered in this paper have an infinite number of eigenvalues which form a countable subset of R. Since it is known that strings, beams and plates can have multiple eigenvalues, we assume that multiple eigenvalues may exist. Although distinct eigenvalues are usually Frechet differentiable in the design parameter, multiple eigenvalues may or may not be differentiable. Thus the design problem considered in this paper must be treated as an infinite dimensional nondifferentiable optimization problem.

Infinite dimensional optimization problems present special difficulties partly because their analysis is mathematically very difficult and partly because boundedness does not imply compactness. Consequently, either an infinite dimensional optimization problem may fail to have a solution, or the sequences constructed by an algorithm in the process of its solution may fail to have accumulation points in the space in which the algorithm is defined. Consequently, to ensure that our convergence results are not vacuous, we present the convergence theorems for our algorithm in terms of a topology of sequences and we make use of recent results on the characterization of minimizing sequences.

General purpose nondifferentiable optimization algorithms are extremely cumbersome since they require the accumulation of bundles of generalized gradients in the search direction computation. This is a process which is too complex and too ill-conditioned numerically to implement in the solution of structural optimal design problems. Because of this, it was necessary to attempt the construction of an algorithm which exploits the structure of the eigenvalue constraint. Fortunately, a theory has recently been developed which enables one to approach such a task in a systematic way. Our earlier experience with optimization algorithms for solving optimization problems with constraints on the eigenvalues of a matrix provided guidance in the utilization of this theory. The result is a sophisticated nondifferentiable optimization algorithm for the solution of optimization problems with inequality constraints including some on the eigenvalues of an elliptic boundary value problem.

---

* This research was supported by the National Science Foundation under grants ECS-79-13148 and CEE-81-05790.