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Memorandum No. UCB/ERL M83/11

13 January 1983

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A Modified Nyquist Stability Test for Use in Computer Aided Design*.

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ABSTRACT

This note shows that the Nyquist stability criterion is not a convenient tool for use in computer-aided design of feedback systems. A substitute graphical test is proposed which is more suitable for use in CAD.

* This research was supported by the National Science Foundation under grant ECS-79-13148 and the Joint Services Electronics Program under grant F-49620-79-C-0178.

1. Introduction.

One of the interesting observations that has emerged in recent years is that procedures that are efficient for "hand" computations are frequently either inefficient or inappropriate for use on a digital computer. This observation obviously applies to such well known "manual" techniques as the inversion of a matrix by Cramer's rule and the determination of complete controllability of a single-input system by constructing the controllability matrix and attempting to determine if it is nonsingular.

The Nyquist stability criterion has served for years as a principal "manual" tool for determining linear system stability. However, it turns out to be quite incompatible with modern design techniques which make use of semi-infinite optimization, because it cannot be transcribed into a semi-infinite inequality. Nevertheless, it does lead directly to an alternative graphical stability test which is totally compatible with the requirements of semi-infinite optimization.

2. A New Graphical Stability Test.

Consider the design of the simple, single-input single-output (SISO) closed loop system shown in Fig. 1, where $P(s) = n_p(s)/d_p(s)$, $C(x,s) = n_c(s)/d_c(s)$ and $F(x,s) = n_F(s)/d_F(s)$ are real, proper rational functions in s, with the vector $x \in \mathbb{R}^n$ denoting the compensator coefficients to be determined by the designer. Let S be an open unbounded set in the s-plane, symmetrical with respect to the real axis, e.g., as in Fig. 2, to which the closed loop poles are required to be confined. We shall assume that S has a right boundary ∂S which is given by an expression of the form

$$\partial S = \{ s \in \mathbb{C} \mid s = \sigma + jw,\ \sigma = f(w),\ -\infty < w < \infty \}, \qquad (1)$$

where $f: \mathbb{R} - \mathbb{R}$ is a negative, piecewise continuously differentiable function which monotonically decreases as $|w|$ increases, with $f(w) \to -\infty$ as $|w| \to \infty$. Consequently, the set S has the characterization

$$S = \{ s \in \mathbb{C} \mid s = \sigma + jw,\ \sigma - f(w) < 0,\ -\infty < w < \infty \}. \qquad (2)$$

For example, suppose that the set S is defined by

$$S = \{ s \in \mathbb{C} \mid s = \sigma + jw,\ \sigma < -k_1 |w| - k_2,\ -\infty < w < \infty \}, \qquad (3)$$

where $k_1, k_2 > 0$.

<u>Definition 1</u>: Let $n(x,s) = n_p(s) n_c(x,s) n_r(x,s)$ and $d(x,s) = d_p(s) d_c(x,s) d_r(x,s)$. We shall say that the closed loop system in Fig. 1 is S-stable if all the zeros of its characteristic polynomial

$$c(x,s) \triangleq n(x,s) + d(x,s) \qquad (4)$$

are in S. #

We begin by generalizing the Nyquist stability criterion so that it can be used as a test of S-stability for the SISO system given in Fig. 1. We need to define an equation for an indented boundary of S.

Let x^* be a given set of compensator coeficients. Suppose that the polynomial $d(x^*,s)$ has k zeros, $p_1, \ldots, p_k$, on the boundary of S. Let $\varepsilon > 0$. For $i = 1,2,\ldots,k$, let I_i be an open interval defined by

$$I_i = (\mathrm{im}(p_i) - \varepsilon, \mathrm{im}(p_i) + \varepsilon),\ i = 1,2,\ldots,k, \qquad (5)$$

and let $\tilde{f}: \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\tilde{f}(w) = f(w)$ for all $w \notin UI_i$ and $\tilde{f}(w) < f(w)$ for all $w \in UI_i$. Furthermore, suppose that $d(x^*, \tilde{f}(w)+jw) \neq 0$ for all $w \geq 0$. Then

$$\ni \tilde{S} = \{ s \in \mathbb{C} \mid s = \sigma + jw \ , \ \sigma = \tilde{f}(w), \ -\infty < w < \infty \} \tag{6}$$

is an <u>indented boundary</u> of S (indented so as to include, in the resulting enlarged complement of S, all the zeros $\sigma' + jw'$ of $d(x^*,s)$ which satisfy $\sigma' = f(w')$).

The following result is obvious in view of the ordinary Nyquist stability criterion [1], see also [2].

<u>Theorem 1</u> (Extended Nyquist stability criterion): Let x^* be a given set of compensator coeeficients. Suppose that the polynomial $d(x^*,s)$ has m zeros, $p_1, \ldots, p_m$, in S^c, the complement of S. Let the indented boundary of S be defined as in (6). Then all the zeros of $c(x^*,s)$ are in S if and only if

(i) the zeros of $F(x^*,s)$ and $C(x^*,s)$ which are in S^c do not cancel any poles of $d(x^*,s)$ which are in S , and

(ii) the locus of

$$t(x^*,w) = \frac{n(x^*,\tilde{f}(w)+jw) + d(x^*,\tilde{f}(w)+jw)}{d(x^*,\tilde{f}(w)+jw)} , \tag{7}$$

traced out for w taking the values from $-\infty$ to $+\infty$, encircles the origin counterclockwise m times. #

Let us consider how we might attempt to verify by computer whether the locus of $t(x^*,w)$ encircles the origin exactly m times.

<u>Method 1</u>: Define the integer valued function N(x) by

$$N(x) = \lim_{w \to \infty} [\arg[t(x,w)] - \arg[t(x,0)]/2\pi \qquad (8)$$

where $\arg[z]$ denotes the argument of the complex number z. Then the number of encirclements of the origin by the locus of $t(x^*,w)$ is given by $N(x^*)$ and hence all zeros of $c(x^*,s)$ are in S if $N(x^*) = m$. #

The evaluation of $N(x^*)$ requires the evaluation of $\arg[t(x^*,w)]$ for a large number of values of w in $[0,\infty)$, which is needed so as not to lose any 2π increment. There may be some numerical difficulty in the vicinity of zeros of $d(x^*,s)$ which are on the boundary of S. However, the major objection to the use of the function $N(x)$ for counting encirclement on a digital computer stems from the fact that $N(x)$ is a discontinuous function of x and therefore incompatible with semi-infinite optimization techniques which may be required for adjusting the compensator coefficient vector x.

Method 2: Let $q: \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $q(0) < 0$ and $q(u) \to \infty$ as $|u| \to \infty$, e.g., $q(u) = k_1 u^2 - k_2$, with k_1, $k_2 > 0$, see Fig. 3. Consider the set Q in the complex plane $\mathbb{C}$, defined by

$$Q = \{z \in \mathbb{C} \mid z = u + jv,\ v - q(u) > 0\}. \qquad (9)$$

Clearly, Q contains the origin in its interior and is unbounded in the v-direction. Consequently, the locus of $t(x^*,w)$ cannot encircle the origin if it does not penetrate Q. The locus of $t(x^*,w)$ does not penetrate Q if and only if

$$\mathrm{im}(t(x^*,w)) - q(\mathrm{re}(t(x^*,w)) \leq 0 \text{ for all } w \in [0,\infty). \qquad (10)$$

The geometric interpretation of (10) is given in Fig. 3.

This leads us to a <u>special</u> <u>case</u>. Suppose that $d(x^*,s)$ has no zeros in S^c and let $\varepsilon = 0$, i.e., $\tilde{f}(w) = f(w)$ for all w. Then the closed loop system is S-stable if (10) holds. #

Clearly, the narrower the region in the complex plane defined by the inequality $-q(u) + v > 0$, the less conservative the test (10) becomes. The advantage of Method 2, assuming that $d(x^*,s)$ has no zeros in S for all x values to be considered, is that the function $\phi: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(x,w) \triangleq \mathrm{im}(t(x,w)) - q(\mathrm{re}(t(x,w))) \tag{11}$$

is differentiable in x and hence compatible with the use of semi-infinite optimization algorithms for compensator parameter adjustment. The disadvantages of Method 2 are (i) that it can only be used when $d(x,s)$ has no zeros in S^c for all x of interest and (ii) that it results in a sufficient, rather than both necessary and sufficient condition of S stability. The first disadvantage can be totally removed and the second one considerably mitigated by replacing Theorem 1 with the following obvious equivalent:

<u>Theorem</u> <u>2</u>: (Modified extended Nyquist criterion). Let D(s) be any monic polynomial of the same degree as $d(x^*,s)$, such that all the zeros of D(s) are in S. Let

$$T(x^*,w) = \frac{n(x^*,f(w)+jw) + d(x^*,f(w)+jw)}{D(f(w)+jw)} . \tag{12}$$

Then all the zeros of $c(x^*,s)$ are in S if and only if the locus of

$T(x^*,w)$ traced out for w taking values from $-\infty$ to ∞ does not encircle the origin. #

Corollary: Let $q(u)$ be defined as in Method 2. Then all the zeros of $c(x^*,s)$ are in S if

$$\mathrm{im}(T(x^*,w)) - q(\mathrm{re}(T(x^*,w)) \leq 0 \text{ for all } w \in [0,\infty). \qquad (13)$$

#

The polynomial $D(s)$ can be chosen in such a way as to make the test (13) not only sufficient but also necessary. For example, when $P(s)$ is strictly proper, a reasonable choice for $D(s)$ is a monic polynomial such that $D(0)$ is close to $c(x^*,0)/2$ for the range of x^* being considered, so that the plot of $T(x^*,w)$ starts at a value of around 2 for $w = 0$ and goes to 1 as w goes to ∞ , minimizing the chances that a stable system would violate the test (13).

3. Conclusion.

We have shown that it is possible to extend the classical Nyquist stability test to allow for the verification of S-stability in a numerically well conditioned manner. In the process, we have made it impossible to determine the usual gain and phase margins. However, this is not a great loss, since, by means of semi-infinite optimization, stability robustness can be ensured in a much more sophisticated manner, see [3].

4. References

[1] H. Nyquist, "Regeneration Theory", Bell Syst. Tech. J. vol. 2, pp. 126-147, Jan. 1932.

[2] C.T. Chen, Introduction to Linear System Theory, Holt Rinehart and Winston, New York, 1970, pp. 369-371.

[3] E. Polak and R. Trahan, "An Algorithm for Computer Aided Design of Control Systems", Proc. IEEE Conf. on Dec. and Control, 1976.

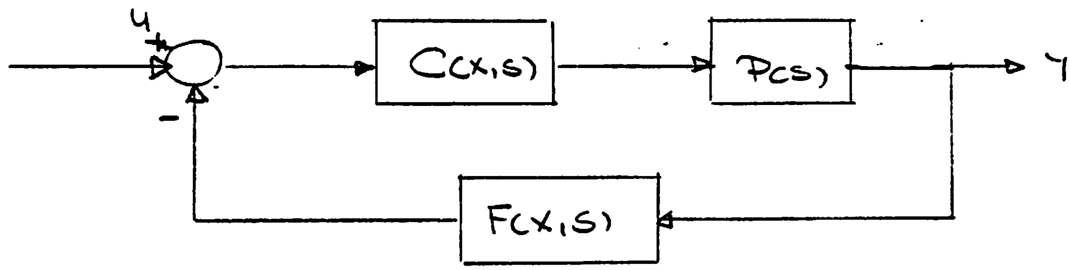

Fig. 1. Control System Configuration

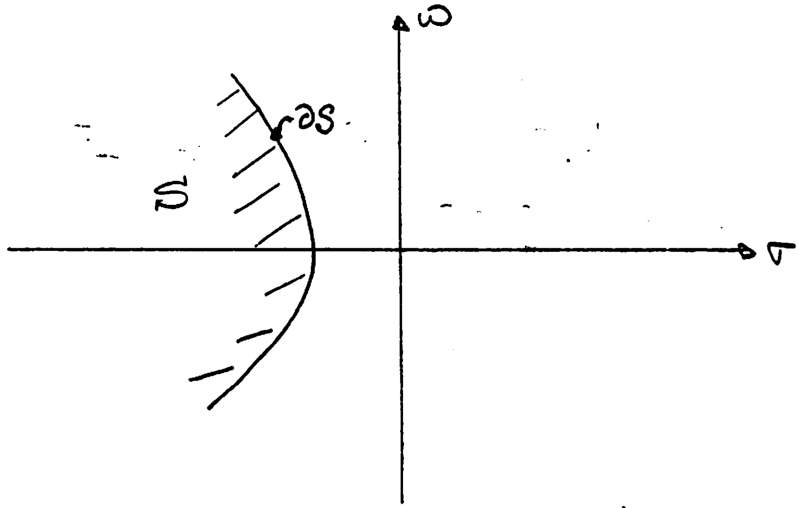

Fig 2. Stability Constraint in s-plane.

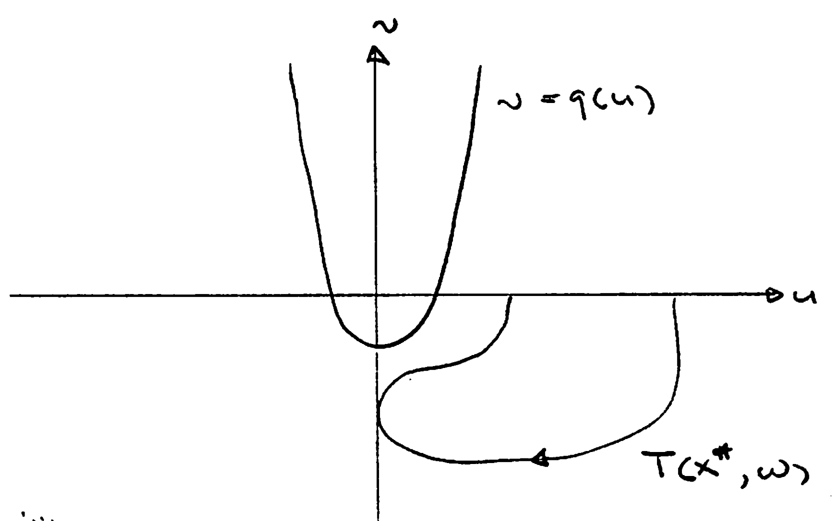

Fig 3. Stability Constraint in $T(x^*, \omega)$ plane.