OPTIMIZATION-BASED DESIGN OF SISO CONTROL SYSTEMS WITH UNCERTAIN PLANT: PROBLEM FORMULATION

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Abstract

This paper explores the use of semi-infinite programming (SIP) algorithms for solving complex SISO control system design problems when the plant model contains both parametric and unstructured uncertainty. It is shown that to make such a design computationally tractable, it is necessary to replace the original performance-specifying semi-infinite inequalities by majorizations. The compatibility of these majorizations with certain SIP algorithms is established. Furthermore, tests for determining whether a controller structure is adequate are proposed.

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Introduction

Over the last decade, various optimization-based computer-aided design techniques have been introduced, see e.g., [D1, G1, K1, K2, K3, M2, P1, P4, P5, P7, Z1, Z2], in attempts to harness the computing power made available to the control engineer by modern digital computers. Natural generalizations of classical design requirements involving rise time, peak overshoot, bandwidth, gain margin, phase margin, etc., lead to <u>semi-infinite inequalities</u>, i.e., <u>infinite sets</u> of inequalities which must be satisfied by a <u>finite set</u> of design parameters. A specially designed new generation of semi-infinite optimization algorithms [G1, G2, M1, M2, P4, P5, P6, P7] is proving to be very effective in solving control system design problems.

A major goal in control system design is to ensure satisfactory system performance in the face of the inevitable uncertainty of the mathematical model of the plant, see e.g., [C2, D3, D4, D5, H3, H4, H5]. This leads to the concept of <u>worst case design</u>. The plant model uncertainty is caused by errors in plant identification, drift in plant characteristics, use of reduced order models in design, etc. The computational difficulty caused by model uncertainty in a worst case design situation depends largely on the form of uncertainty and on the performance requirements. For example, suppose, as in [C2, D3, D4, S1], that model uncertainty is of the form of a "small" multiplicative or additive perturbation of the plant transfer function. Stability robustness for the resulting closed loop system can then be ensured by a frequency domain test, see e.g. [C2, D3, D4, S1], which is expressible as a semi-infinite inequality of the form $g(x,\omega) \leq 0$ for all $\omega \geq 0$, where x is a vector consisting of the compensator parameters that must be designed and ω denotes frequency. Because the frequency parameter ω is one-dimensional, this inequality causes little computational difficulty. Now suppose that, as in [H3, H4], the uncertainty is assumed to be in terms of the parameters of the plant transfer function. Worst case stability is now ensured by satisfying an inequality of the form $h(x,\alpha,\omega) \leq 0$ for all $\alpha \in A$ and for all $\omega \geq 0$, where A is the set within which the plant coefficients are assumed to lie. Since, usually, A is multidimensional and h is not convex, this last inequality is extremely difficult to resolve computationally. In retrospect, one must admire early attempts (see [H4]), predating semi-infinite optimization, to resolve such inequalities by means of Nichols chart techniques and lots of intuition.

In this paper we restrict ourselves to the design of single-input single output (SISO) control systems, with both "structured" (parametric) (as in [H3, H4]) and "unstructured" (as in [C2, D3, D4, S1]) plant uncertainty. In Section 2 we introduce the plant model and a number of "natural" formulations of control system performance requirements in the form of semi-infinite inequalities. In Section 3 we develop some decomposition results and introduce the concept of majorization of inequalities. We then present some theorems which establish conditions for the replacement of intractable inequalities by simpler ones at the expense of tightening the design requirements. These results are then used to obtain majorizations for the frequency and time domain performance inequalities presented in Section 2. In Section 4, we show that the majorizing inequalities obtained in Section 3 involve only locally Lipschitz continuous functions, and hence that these inequalities are solvable by a number of semi-infinite optimization algorithms, such as those characterized in [P3]. In Section 5 we show that our decomposition results lead to tests for determining whether the proposed controller structure can possibly satisfy the design requirements. In sum, we present a set of techniques for formulating complex SISO control system design problems, involving plant uncertainty, in a computationally tractable form.

Notation

The following notation is used in this paper:

$*$ denotes complex conjugate

$\mathbb{C}^0_-$ denotes the open left half of the complex plane, $\mathbb{C}$.

$\mathbb{Z}$ denotes the set of integers.

$\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

The superscripts $\wedge$, $\vee$ denote maximizers and minimizers, respectively. For instance,

$$\hat{x} = \operatorname*{argmax}_{x \in X} f(x)$$

$$\check{\alpha}(x) = \operatorname*{argmin}_{\alpha \in A} \phi(x,\alpha)$$

$R \subset \mathbb{R}^2$ denotes a "rectangle" of the form,

$$R = \{(x,y) \mid x \in [\underline{x},\bar{x}],\ y \in [\underline{y},\bar{y}]\}$$

with $\underline{x}$, $\bar{x}$, $\underline{y}$, $\bar{y} \in \mathbb{R}$.

2. Design Problem Formulation

We consider the task of formulating the SISO control system design problem as consisting of three subtasks:

(i) The specification of the plant model and controller structures;

(ii) The division of the performance requirements into two classes, "hard" and "soft;"

(iii) The specification of the "hard" performance requirements in the form of, possibly semi-infinite, inequality constraints and of the "soft" performance requuirements in the form of a cost function.

We shall devote a separate subsection to the tasks (i) and (iii). Task (ii) is obviously subjective and is left to the discretion of the designer.

2.1. The Plant and Compensators

We shall be concerned with the design of SISO control systems of the form shown in Fig. 1, where $C(x,s)$ and $F(x,s)$ are the compensators to be designed. The components of the design vector, $x \in \mathbb{R}^{n_x}$, are the free parameters of the compensator, which need to be determined computationally.

The compensators can be specified in one of two forms: the first is

$$C(x,s) = \frac{K_C(s+a^0_{cn}) \prod_{i=1}^{k'_c} (s^2+2a^i_{cn}s+(b^i_{cn})^2)}{(s+a^0_{cd}) \prod_{i=1}^{k_c} (s^2+2a^i_{cd}s+(b^i_{cd})^2)} \qquad (2.1a)$$

$$F(x,s) = \frac{K_F(s+a^0_{Fn}) \prod_{i=1}^{K'_F} (s^2+2a^i_{Fn}s+(b^i_{Fn})^2)}{(s+a^0_{Fd}) \prod_{i=1}^{k_F} (s^2+2a^i_{Fd}s+(b^i_{Fd})^2)} \qquad (2.1b)$$

with $k_C' \leq k_C$, $k_F' \leq k_F$ integers, and x made up of the coefficients in (2.1a), (2.1b). Alternatively, we may set

$$C(x,s) = \frac{\sum_{i=1}^{k_C'} a_{cn}^i s^i}{\sum_{i=1}^{k_C} a_{cd}^i s^i} \tag{2.2a}$$

$$F(x,s) = K_F \frac{\sum_{i=1}^{k_F'} a_{Fn}^i s^i}{\sum_{i=1}^{k_F} a_{Fd}^i s^i}, \tag{2.2b}$$

again with $k_C' \leq k_C$, $k_F' \leq k_F$.

We shall see later that if there are bounds on the numerator and denominator coefficients of the compensators, then (2.2a,b) results in simple, affine inequalities, while (2.1a,b) results in less desirable polynomial inequalities. On the other hand, we shall see that (2.1a,b) leads to simpler expressions of stability requirements and to considerable simplifications in tests which help in determining whether the given controller complexity is adequate for coping with the stated performance requirements.

We shall assume that the plant model P is of the form†

$$P(s,\alpha,\ell(s)) = P_0(s,\alpha)\ell(s) \tag{2.3}$$

where

†It is more common to see in the literature the form $P_0(1+\ell)$ rather than (2.3). However, now our point of view (2.3) is more convenient. It is obviously simple to go from one of these forms to the other.

$$P_0(s,\alpha) = \frac{K_p \prod_{i=1}^{k'_{pR}} (s+z_R^i) \prod_{i=1}^{k'_{pC}} (s+z_C^i)(s+z_C^{i*})}{\prod_{i=1}^{k_{pR}} (s+p_R^i) \prod_{i=1}^{k_{pC}} (s+p_C^i)(s+p_C^{i*})} \tag{2.4}$$

where $k'_{pR} + k'_{pC} \leq k_{pR} + k_{pC}$ and α is a vector consisting of the plant d.c. gain, K_p, and the zero and pole coefficients of $P_0(s,\alpha)$; i.e., $\alpha = (K_p,z,p)^T$ with $z = (z_R^1,\ldots,z_R^{k'_{pR}},z_C^1,\ldots,z_C^{k'_{pC}})^T$ and $p = (p_R^1,\ldots,p_R^{k_{pR}}, p_C^1,\ldots,p_C^{k_{pC}})$, $k'_p \triangleq k'_{pR} + k'_{pC}$ and $k_p \triangleq k_{pR} + k_{pC}$. The superscript * is used to denote complex conjugates. For the real zeros and poles, z_R^i, p_R^i, we assume that they are contained in intervals:

$$z_R^i \in I_{zR}^i \triangleq [\underline{z}_R^i,\bar{z}_R^i],\ i = 1, 2, \ldots, k'_{pR} \tag{2.5a}$$

$$p_R^i \in I_{pR}^i \triangleq [\underline{p}_R^i,\bar{p}_R^i],\ i = 1, 2, \ldots, k_{pR}. \tag{2.5b}$$

Next, we impose a partial order on $\mathbb{C}$ defined by $z' \leq z$ if $\mathrm{Re}(z') \leq \mathrm{Re}(z)$, $\mathrm{Im}(z') \leq \mathrm{Im}(z)$. Then for the complex zeros and poles, z_C^i, p_C^i, we require that they be contained in rectangles in $\mathbb{C}$:

$$z_C^i \in I_{zC}^i \triangleq \{z \mid \underline{z}_C^i \leq z \leq \bar{z}_C^i\},\ i = 1, 2, \ldots, k'_{pC} \tag{2.5c}$$

$$p_C^i \in I_{pC}^i \triangleq \{p \mid \underline{p}_C^i \leq p \leq \bar{p}_C^i\},\ i = 1, 2, \ldots, k_{pC} \tag{2.5d}$$

where $\underline{z}_C^i \leq \bar{z}_C^i$ and $\underline{p}_C^i \leq \bar{p}_C^i$ are complex numbers. Finally, the d.c. gain satisfies

$$K_p \in [\underline{K}_p,\bar{K}_p]. \tag{2.5e}$$

We assume that perturbation functions, $\ell(s)$, are known to the following extent:

(a) The functions ℓ: $\mathbb{C} \rightarrow \mathbb{C}$ are locally Lipschitz continuous

(b) The functions, $\ell(j\omega)$, are bounded both in magnitude and phase

as follows:

$$\underline{\ell}_M(\omega) \leq |\ell(j\omega)| \leq \bar{\ell}_M(\omega)| \qquad \forall\ \omega \in [0,\infty) \tag{2.6a}$$

$$\underline{\ell}_A(\omega) \leq \arg \ell(j\omega) \leq \bar{\ell}_A(\omega) \qquad \forall\ \omega \in [0,\infty) \tag{2.6b}$$

where the bound functions satisfy the following assumptions:

(i) $\underline{\ell}_M(\cdot)$, $\bar{\ell}_M(\cdot)$, $\underline{\ell}_A(\cdot)$, $\bar{\ell}_A(\cdot)$ are locally Lipschitz continuous

(ii) $0 < \underline{\ell}_M(\omega) \leq 1 \leq \bar{\ell}_M(\omega) \quad \forall\ \omega \in [0,\infty)$

(iii) $\underline{\ell}_A(\omega) \leq 0$, $\bar{\ell}_A(\omega) \geq 0 \quad \forall\ \omega \in [0,\infty)$.

The set of <u>admissible perturbations</u>, ℓ, satisfying (a) and (b) will be denoted by L. The bounds described by (2.6a,b) are illustrated in Fig. 3.

Referring to Fig. 4, we see that it is easy to extract from (2.6a) and (2.6b) slightly conservative corresponding bounds on $(\ell(j\omega)-1)$ (see Appendix 1), viz,

$$\check{\ell}_M(j\omega) \leq |\ell(j\omega)-1| \leq \hat{\ell}_M(j\omega) \tag{2.7a}$$

$$\check{\ell}_A(j\omega) \leq \arg\{\ell(j\omega)-1\} \leq \hat{\ell}_A(j\omega). \tag{2.7b}$$

Model (2.3) allows the designer to account for a number of phenomena such as: (i) variations in the plant due to the manufacturing process, (ii) high frequency measurement errors, (iii) errors in fitting a mathematical model to experimental data, and (iv) obtaining a low order linear model from possibly nonlinear differential equations. The "structured" part, $P_0(s,\alpha)$, of model (2.3) is intended to represent the system accurately at low to medium frequencies. The "unstructured" perturbations, $\ell(s)$, will typically become significant at medium to high frequencies and may be used to account for errors in model order, high frequency measurement errors, unmodeled nonlinearities [D4], etc.

For example $P_0(s,\alpha)$ may be the result of fitting a model of given order ($k_p=k_{pR}+k_{pC}$) to experimental data by means of a process such as weighted least squares [D2] or the instrumental variable method [P8], etc., with the weights adjusted so as to get a better fit over low, rather than high frequencies. Because of the computational properties of such curve fitting schemes, the parameters of $P_0(s,\alpha)$ can only be assumed to be determined in the form of confidence intervals (see Fig. 2 for an example). The multiplicative perturbations, $\ell(s)$, may now be used to account for the low frequency bias in obtaining the structured model, $P_0(s,\alpha)$.

It is possible to give a probabilistic interpretation to our approach: our design will satisfy all the design requirements with the same probability as that the plant zeros and poles belong to the assumed confidnece intervals.

In the literature (see, for example, [C2,D3,D4,S1]) we find robustness tests which make use only of the magnitude of the perturbations, $\ell(s)$. When phase information for $\ell(s)$ is also available, then these magnitude - only robustness tests may be unduly conservative, see [B1,H5]. Even if delays are present in a system so that as $\omega \to \infty$, the system phase is unknown, useful phase information may nevertheless be estimable up to a given frequency of interest, see [H5]. We shall show that the Nyquist criterion may be modified to ensure not only stability/robustness for the closed-loop system with plant $P_0(s,\alpha)$, $\alpha \in A$, but also with respect to stable multiplicative perturbations for which magnitude and phase bounds of the type (2.6a,b) are known. However, other frequency and time domain requirements cannot be dealt with unless bounds on phase are also postulated, at least, for $s = j\omega$.

We shall see later that requirements of worst case closed loop pole placement inside a set in $\mathbb{C}_-^0$ can only be dealt with if the bounds (2.7a), (2.7b) can be generalized to contours in $\mathbb{C}_-^0$. This imposes further restrictions on the type of perturbation that one can consider within our framework of optimization-based SISO control system design.

2.2. Performance Requirements and Design Constraints

We begin by discussing a few performance requirements which are often treated as "hard" constraints and hence are expressed as inequalities. These include stability robustness, pole placement, output noise rejection and avoidance of saturation caused by output noise or the command input signals.

BIBO stability of the closed loop system is the most important performance requirement. Given the level of uncertainty in the plant model (2.3), it is quite difficult to ensure that the closed loop system will be BIBO stable. We propose two approaches to ensuring BIBO stability. The first, and less conservative one, is based on the modified Nyquist criterion [P2]. It applies when the bounds (2.6b) on the phase $\ell \in L$ are reliable, while the second one, based on stability robustness results, is to be used when only the amplitude constraint (2.6a) is available.

(i) BIBO Stability

Let d(s) be a polynomial of degree $k_p + k_c$ (i.e. the degree of the loop gain denominator for $\ell(s) \equiv 1$) whose zeros are all in $\mathbb{C}_-^0$. Let n_p, d_p, n_C, d_C denote the numerator and denominator polynomial of the transfer functions of the "nominal" plant $P_0(s,\alpha)$ and compensator $C(x,s)$, i.e.,

$$P(s,\alpha,\ell) = \left(\frac{n_p(s,\alpha)}{d_p(s,\alpha)}\right)\ell(s) \tag{2.8a}$$

$$C(x,s) = \frac{n_C(x,s)}{d_C(x,s)} . \tag{2.8b}$$

Let $d(s)$ be a polynomial of the same degree as $d_p d_C$ (i.e. k_p+k_C), and suppose that $\ell(\cdot)$ is a proper, BIBO stable rational function. Then, according to the modified Nyquist stability criterion [P2], the closed loop system in Fig. 1 is BIBO stable if and only if $F(x,s)$ is BIBO stable and the locus of

$$T(x,j\omega,\alpha,\ell(j\omega)) \overset{\Delta}{=} \frac{n_C(x,j\omega)n_p(j\omega,\alpha)(\ell(j\omega))+d_C(x,j\omega)d_p(j\omega,\alpha)}{d(j\omega)} \tag{2.9}$$

traced out for $-\infty < \omega < \infty$ does not encircle the origin for all $\alpha \in A$ and and for all $\ell \in L$.

When F is specified as in (2.1b), the BIBO stability requirement on F leads to the inequalities,

$$a_{Fd}^j \geq \varepsilon > 0 \text{ for } j = 0, 1, 2, \ldots, k_F \tag{2.10a}$$

$$b_{Fd}^i \geq \varepsilon > 0 \text{ for } i = 1, 2, \ldots, k_F. \tag{2.10b}$$

Next, the encirclement requirement can be replaced by the requirement that the locus of $T(x,j\omega,\alpha,\ell(j\omega))$ stay out of a parabolic region enclosing the origin, for all $\omega \in (-\infty,\infty)$, for all $\alpha \in A$ and for all $\ell \in L$ (see

Fig. 5). This leads to the following quite formidable, semi-infinite inequality:

$$\mathrm{Im}(T(x,j\omega,\alpha,\ell(j\omega))-k_1(\mathrm{Re}(T(x,j\omega,\alpha,\ell(j\omega))^2 + k_2 \leq 0$$

$$\forall\ \alpha \in A,\ \forall\ \ell \in L,\ \forall\ \omega \in [0,\infty) \qquad (2.11)$$

where k_1, $k_2 > 0$. The requirement $\omega \in [0,\infty)$ in (2.11) can usually be relaxed to $\omega \in (\omega',\omega'')$, with $0 < \omega' < \omega' < \infty$. However, even with this simplification, (2.11) remains totally intractable unless one resorts to the type of "majorization," which replaces (2.11) with a more conservative, but simpler inequality, that we will present in the next section. Finally, note that a judicious selection of the polynomial d(s) and the constants k_1, k_2 makes the test (2.11) "almost necessary;" i.e., it reduces the conservatism of this, basically sufficient condition.

(ii) Stability Robustness

Now suppose that nothing is known about the phase of the perturbation functions $\ell(s)$. In that case the expression, (2.11) cannot be evaluated and BIBO stability must be ensured by a two stage process. First we set $\ell(s) \equiv 1$ and require that the "structured" part of the system be BIBO stable, i.e., from (2.11), that

$$\mathrm{Im}(T(x,j\omega,\alpha,1)) - k_1(\mathrm{Re}(T(x,j\omega,\alpha,1)))^2 + k_2 \leq 0$$

$$\forall\ \alpha \in A,\ \forall\ \omega \in [0,\infty). \qquad (2.12a)$$

Then, we ensure that the high frequency effects represented by the allowed $\ell(s) \in L$ do not destroy the BIBO stability of the "structured" part of the closed loop system, by requiring, as in [C2, D3, D4, S1], that

$$|P_0(j\omega,\alpha)\ C(x,j\omega)[1+P_0(j\omega,\alpha)\ C(x,j\omega)]^{-1}| \leq \frac{1}{\hat{\ell}_M(j\omega)}$$

$$\forall\ \alpha \in A,\ \forall\ \omega \in [0,\infty). \qquad (2.12b)$$

Although (2.12a,b) are substantially simpler inequalities than (2.11), they are still quite forbidding because of the dimensionality of A. Fortunately, as we will see in the next section, this obstacle can be overcome by decomposition techniques.

Finally, note that an equivalent expression to (2.12b) is

$$|H^0_{yu}(x,j\omega;\alpha)| \leq \frac{1}{\hat{\ell}_M(j\omega)} \quad \forall\ \alpha \in A,\ \forall\ \omega \in [0,\infty) \qquad (2.12c)$$

where H^0_{yu} is the "structured" closed loop transfer function from u to y.

(iii) Pole Placement

Referring to [C1], we find that closed loop pole placement, specified only to the extent that the closed loop poles be confined to a region in the s-plane, is closely related to the task of ensuring BIBO stability.

Thus, let S be a subset of $\mathbb{C}^0_-$, which is symmetrical about the real axis, with boundary defined by $\sigma = f(\omega)$, where $f: \mathbb{R} \to \mathbb{R}$ is continuous, $f(-\omega) = f(\omega) \leq 0$ and $f(\omega) \nearrow \infty$ as $\omega \to \infty$; i.e.

$$S = \{s \in C | s=\sigma+j\omega, \sigma-f(\omega)\leq 0\}. \qquad (2.13)$$

For example, as illustrated in Fig. 6, S could be the hyperbolic region defined by $\sigma = -\sqrt{k_1+k_2\omega^2}$, with k_1, $k_2 > 0$.

Definition 2.1. Let $n_C(x,s)$, $n_F(x,s)$, $d_C(x,s)$, $d_F(x,s)$ denote the numerator and denominator polynomials, respectively, of the compensator blocks C(x,s) and F(x,s). We say that a given realization of the closed loop feedback system in Fig. 1 (characterized by a specific $\alpha \in A$, $\ell \in L$) is S-stable if (i) $d_F(x,s)$ has no zeros in S^c, the complement of S in $\mathbb{C}$, (ii) $n_C(x,s)$ has no zeros in S^c which cancel poles of $P(s,\alpha,\ell(j\omega))$ in S^c and (iii) the transfer function from r to y,

$$H_{yr}(x,s,\alpha,\ell(s)) \triangleq F(x,s)\frac{P(s,\alpha,\ell(s))C(x,s)}{1+P(s,\alpha,\ell(s))C(x,s)} \tag{2.14}$$

has no poles in S^c. ¤

First we shall show that pole placement, to the extent of ensuring S-stability, is possible for the "structured" part of the system. Referring to [P2] we see that the S-stability analog of the test (2.12a) which ensures exponential stability is

$$\mathrm{Im}(T(x,f(\omega)+j\omega,\alpha,1)) - k_1(\mathrm{Re}(T(x,f(\omega)+j\omega,\alpha,1))^2 + k_2 \leq 0, \ \forall\, \alpha \in A, \ \forall\, \omega \in [0,\infty) \tag{2.15}$$

To ensure S-stability for the precompensator $F(x,j\omega)$ specified by (2.16), we require that

$$- a^0_{Fd} \leq f(0) \tag{2.16a}$$

and, for $i = 1, 2, \ldots, k_F$,

$$\mathrm{Re}(-a^i_{Fd} \pm \sqrt{(b^i_{Fd})^2-(a^i_{Fd})^2}) \leq f(I_m(-a^i_{Fd} \pm \sqrt{(b^i_{Fd})^2-(a^i_{Fd})^2} \ . \tag{2.16b}$$

Ensuring S-stability in the face of unstructured stability is much more problematical not so much because of the added computational complexity, but because of the difficulty in obtaining bounds on $\ell(f(\omega)+j\omega)$, i.e., on $\ell(s)$ off the $j\omega$ axis. In any event, it now becomes necessary to restrict ourselves to perturbations $\ell \in L_S$, the subset of S-stable perturbations in L. Tentatively, suppose that it is valid to extend the bounds in (2.6a,b), by means of global Lipschitz constants L_1, L_2 as follows:

$$\underline{\ell}_M(\omega) - L_1|f(\omega)| \leq |\ell(f(\omega)+j\omega)| \leq \bar{\ell}_M(\omega) + L_2|f(\omega)| \qquad (2.17a)$$

$$\underline{\ell}_A(\omega) - L_2|f(\omega)| \leq \arg \ell(f(\omega)+j\omega) \leq \bar{\ell}_A(\omega) + L_2|f(\omega)|. \qquad (2.17b)$$

We denote the extended set of Lipschitz continuous functions, ℓ, which satisfy (2.17a) and (2.17b) (c.f. (2.6a) and (2.6b)) by L_e. The subset of S-stable functions in L_e is denoted $L_{e,S}$. We now have two options in dealing with the perturbations, $\ell(s)$. We can add an S-stability robustness condition to (2.15), viz.,

$$H^0_{yu}(x,f(\omega)+j\omega,\alpha)| \leq \frac{1}{\hat{\ell}_M(j\omega)+L_1|f(\omega)|} \quad \forall\ \alpha \in A,\ \forall\ \omega \in [0,\infty) \quad (2.18)$$

which is useful when (2.17b) is unreliable. Alternatively, we can enlarge (2.15) by choosing d(s) to have no zeros in S^c and then requiring that

$$\mathrm{Im}(T(x,f(\omega)+j\omega,\alpha,\ell(f(\omega)+j\omega)) - k_1(\mathrm{Re}(T(x,f(\omega)+j\omega,\ell(f(\omega)+j\omega)))^2 + k_2 \leq 0$$

$$\forall\ \alpha \in A,\ \forall\ \omega \in [0,\infty),\ \forall\ \ell \in L_{e,S}. \qquad (2.19)$$

Again, as we will show in the next section, (2.19) can be replaced by a somewhat more conservative, but computationally tractable inequality.

(iv) Noise Rejection

The need to reduce the effect of the output disturbance on the output can be expressed in the form

$$|H_{yd}(x,j\omega,\alpha,\ell(j\omega)|^2 \le \bar{\ell}_d(\omega)^2,\ \forall\ \omega \in [\omega_d',\omega_d''],\ \forall\ \alpha \in A,\ \forall\ \ell \in L \tag{2.20a}$$

where H_{yd} is the transfer function from d to y and $[\omega_d',\omega_d'']$ is a critical frequency interval, or, equivalently,

$$|(1+P(j\omega,\alpha,\ell(j\omega))C(x,j\omega))^{-1}|^2 \le \bar{\ell}_d(\omega)^2$$

$$\forall\ \omega \in [\omega_d',\omega_d''],\ \forall\ \alpha \in A,\ \forall\ \ell \in L. \tag{2.20b}$$

(v) Saturation Avoidance

It is desirable to prevent the disturbance, d(·), and command input, r(·), from saturating the plant. For this purpose we require that

$$|H_{vd}(x,j\omega,\alpha,\ell(j\omega))|^2 \le \bar{\ell}_s(\omega)^2$$

$$\forall\ \omega \in [\omega_s',\omega_s''],\ \forall\ \alpha \in A,\ \forall\ \ell \in L \tag{2.21a}$$

where H_{vd} is the transfer function from d to v, i.e., equivalently,

$$|C(x,j\omega)(1+P(j\omega,\alpha,\ell(j\omega))C(x,j\omega))^{-1}|^2 \le \bar{\ell}_s(\omega)^2$$

$$\forall\ \omega \in [\omega_s',\omega_s''],\ \forall\ \alpha \in A,\ \forall\ \ell \in L. \tag{2.21b}$$

For command input signals, the requirement is

$$|H_{vr}(x,j\omega,\alpha,\ell(j\omega))|^2 \le \tilde{\ell}_r(\omega)^2$$

$$\forall\ \omega \in [\tilde{\omega}_s',\tilde{\omega}_s''],\ \forall\ \ \in A,\ \forall\ \ell \in L, \tag{2.21c}$$

where

$$H_{vr}(x,j\omega,\alpha,\ell(j\omega)) = F(x,j\omega)C(x,j\omega)[1+P(j\omega,\alpha,\ell(j\omega))C(x,j\omega)]^{-1}. \tag{2.21d}$$

(vi) Time Domain Constraints

Let $\{r_i(\cdot)\}_{k=1}^{k}$ be a given set of inputs. First, we may impose input following requirements in the form (see Fig. 7)

$$\underline{\ell}_t^i(t) \leq y(t,x,\alpha,\ell,r_i) \leq \bar{\ell}_t^i(t), \quad \forall\, \alpha \in A,\ \forall\, \ell \in L,\ \forall\, t \geq 0,\ i=1,2,\ldots,k \tag{2.22}$$

where the bound functions $\underline{\ell}_t^i$, $\bar{\ell}_t^i$ are piecewise continuous and $y(t,x,\alpha,\ell,r_i)$ denotes the closed loop system zero state output corresponding to input $r_i(t)$ and $d(t) \equiv 0$.

Next, we may impose power constraints on the plant input (or output) in the form

$$\int_0^T u(t,x,\alpha,\ell,r_i)^2 dt \leq \bar{\ell}_p^i, \quad \forall\, \alpha \in A,\ \forall\, \ell \in \mathcal{L},\ i = 1, 2, \ldots, k. \tag{2.23a}$$

A major source of difficulty with the constraints (2.22), (2.23a) is the fact that there is no obvious way of evaluating a response such as $y(t,x,\alpha,\ell,r_i)$ for a given x, α, r_i, with ℓ specified only as a bounded Laplace transform. We shall deal with this difficulty to some extent in the next section. (One way out is to impose (2.22) and (2.23a) only for $\ell = 1$).

(vii) Cost Functions

Within our philosophy of design, a "hard" performance requirement is expressed as an inequality constraint, while a "soft" performance requirement is added to the cost function. Since semi-infinite optimization algorithms such as [G1] require that the cost function is differentiable, it is necessary to perform a simple transformation when converting a constraint such as (2.20a) into a cost function. Thus, if we wish to minimize the effect of disturbances on the output, subject to some of the other constraints described in this section, we enlarge the design vector by one component to (x^0,x) and solve

$$\underset{(x^0,x)}{\text{minimize}}\ x^0 \tag{2.24a}$$

subject to

$$|H_{yd}(x,j\omega,\alpha,\ell(j\omega))|^2 - x^0 \leq 0,$$

$$\forall\ \omega \in [\omega_d',\omega_d''],\ \forall\ \alpha \in A,\ \forall\ \ell \in L \tag{2.24b}$$

and other constraints. Of course the designer may select other cost functions which do not affect system performance, such as cost functions to minimize manufacturing cost.

3. Decomposition and Majorization of Performance Inequalities

3.1. General Results

Referring to (2.11), (2.12a), (2.12b), (2.15), (2.18), (2.19), (2.20a), (2.21a), (2.21d), (2.22), assuming that $\ell(s) \equiv 1$, we find that our performance requirements lead to inequalities of the form

$$\phi(x,\nu) \leq 0 \quad \forall\ \nu \in N \tag{3.1}$$

where $:\mathbb{R}^{n_x} \times \mathbb{R}^{n_\nu} \to \mathbb{R}$ is continuous in (x,ν) and continuously differentiable in x for each ν; $\nabla_x\phi(x,\nu)$ is continuous in (x,ν).† In (3.1),

†If $\phi(\cdot,\cdot)$ corresponds to a closed loop magnitude, it is shown in Appendix 3 that $\nabla_x\phi(x,\nu)$ fails to exist at points (x,ν) at which the loop gain is -1.

x is the design vector, of dimension n_x, and $\nu = (\alpha,\omega)$ is the variations vector, of dimension n_ν. An alternative way of writing (3.19) is

$$\max_{\nu \in N} \phi(x,\nu) \leq 0 \tag{3.1b}$$

In solving an optimization problem with constraints such as (3.1b), a semi-infinite optimization algorithm (see e.g. [G1, M1, P4, P5, P7]) must evaluate the function

$$\psi(x) \stackrel{\Delta}{=} \max_{\nu \in N} \phi(x,\nu) \tag{3.2}$$

at least once during each iteration. Since in our case N is multidimensional, the evaluation of $\psi(x)$ is, potentially, a source of extreme difficulty. Fortunately, the structure of the design problem allows two kinds of simplifications. The next two theorems are decomposition results.

<u>Theorem 3.1</u>. Suppose that in (3.2),

$$\phi(x,\nu) = \prod_{i=1}^{n_\nu - 1} \phi^i(x,\alpha^i,\omega) - b(\omega) \tag{3.3a}$$

with $\phi^i : \mathbb{R}^{n_x} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$, and that

$$N = A^1 \times A^2 \times \ldots \times A^{n_\nu - 1} \times \Omega \tag{3.3b}$$

where the A^i and Ω are compact intervals. Then

$$\psi(x) = \max_{\omega \in \Omega} \{ \prod_{i=1}^{n_\nu - 1} \psi^i(x,\omega) - b(\omega) \} \tag{3.4a}$$

where

$$\psi^i(x,\omega) = \max_{\alpha^i \in A^i} \phi^i(x,\alpha^i,\omega). \tag{3.4b}$$

Proof: We can write (3.2) as

$$\psi(x) = \max_{\omega\in\Omega}(\max_{\alpha\in A} \phi(x,\alpha,\omega)) \tag{3.5}$$

Let $\hat{\alpha}(\omega)$ be a maximizing function for $\max_{\alpha\in A} \phi(x,\alpha,\omega)$ and let $\bar{\alpha}^i$ be a maximizer for $\max_{\alpha^i\in A^i} \phi^i(x,\alpha^i,\omega)$. Clearly, we must have $\phi^i(x,\hat{\alpha},\omega) = \phi^i(x,\bar{\alpha}^i,\omega)$ and hence the desired result follows. ¤

Theorem 3.2. Suppose that in (3.2)

$$\phi(x,\nu) = \sum_{i=1}^{n_\nu-1} \phi^i(x,\alpha^i,\omega)-b(\omega) \tag{3.6}$$

and that N is as in (3.3b). Then

$$\psi(x) = \max_{\omega\in\Omega}\{\sum_{i=1}^{n_\nu-1} \max_{\alpha^i\in A^i} \phi^i(x,\alpha^i,\omega)-b(\omega)\}. \tag{3.7}$$

¤

We now open up the possibility of replacing the very hard problem (3.2) by a much easier one, provided we can find suitable majorizing sets that are not too big.

Theorem 3.3. Suppose that in (3.2)

$$\phi(x,\omega,\alpha) = \Phi(f(x,\omega,\alpha)) \tag{3.11}$$

where $f:\mathbb{R}^{n_x} \times \mathbb{R} \times \mathbb{R}^{n_\alpha} \to \mathbb{C}$ and $\Phi:\mathbb{C}\to\mathbb{R}$. For any $x \in \mathbb{R}^{n_x}$, $\omega \geq 0$, let

$$A^-(x,\omega) \stackrel{\Delta}{=} \{z \in \mathbb{C} | z = f(x,\omega,\alpha), \alpha \in A\} \quad (3.12)$$

and let $M(x,\omega) \subset \mathbb{C}$ be such that $A^-(x,\omega) \subset M(x,\omega)$. Then

$$\psi(x) \leq \max_{\omega \in \Omega} \max_{z \in M(x,\omega)} \Phi(z). \quad (3.13)$$

Proof: Clearly,

$$\psi(x) = \max_{\omega \in \Omega} \max_{z \in A^-(x,\omega)} \Phi(z) \quad (3.14)$$

Since $A^-(x,\omega) \subset M(x,\omega)$, the desired result follows immediately. ¤

To conclude, we introduce the following terminology.

Definition 3.1. Let ψ, $\bar{\psi} : \mathbb{R}^n \times \mathbb{C} \to \mathbb{R}$ and $b : \mathbb{C} \to \mathbb{R}$ be continuous. If $\bar{\psi}(x,s) \geq \psi(x,s)$ for all $x \in \mathbb{R}^n$ and for all $s \in B \subset \mathbb{C}$, then we say that the inequality

$$\bar{\psi}(x,s) - b(s) \leq 0 \ \forall \ s \in B$$

majorizes the inequality

$$\psi(x,s) - b(x) \leq 0 \ \forall \ s \in B.$$

3.2. Bounds on Open Loop Gain and Phase

We now make use of Theorems 3.1 and 3.2 to obtain bounds on the open loop gain and phase, as a function of the complex variable $s \in \mathbb{C}$. These bounds will be used in the following subsections to obtain majorizations for the intractable performance inequalities introduced in Section 2.

Consider the loop transfer function, $P(s,\alpha,\ell(s))C(x,s)$.

We need the following notation. For any $s \in \mathbb{C}$,

$$\hat{\ell}_M(s) \stackrel{\Delta}{=} \max_{\ell \in L_e} |\ell(s)-1| \quad (3.15a)$$

$$\check{\ell}_M(s) \stackrel{\Delta}{=} \min_{\ell \in L_e} |\ell(s)-1| \quad (3.15b)$$

$$\hat{\ell}_A(s) \stackrel{\Delta}{=} \max_{\ell \in L_e} \arg(\ell(s)-1) \quad (3.15c)$$

$$\check{\ell}_A(s) \stackrel{\Delta}{=} \min_{\ell \in L_e} \arg(\ell(s)-1) \quad (3.15d)$$

The above maxima and minima are computed in Appendix 1.

Next, for any $s \in \mathbb{C}$, let

$$\hat{z}^i_{MR}(s) \stackrel{\Delta}{=} \arg\max\{|(s+z)|^2 \,\big|\, z \in I^i_{zR}\},\ i = 1, 2, \ldots, k'_{pR} \quad (3.16a)$$

$$\hat{z}^i_{MC} \stackrel{\Delta}{=} \arg\max\{|(s+z)(s+z^*)|^2 \,\big|\, z \in I^i_{zC}\},\ i = 1, 2, \ldots, k'_{pC} \quad (3.16b)$$

$$\check{z}^i_{MR}(s) \stackrel{\Delta}{=} \arg\min\{|(s+z)|^2 \,\big|\, z \in I^i_{zR}\},\ i = 1, 2, \ldots, k'_{pR} \quad (3.16c)$$

$$\check{z}^i_{MC}(s) \stackrel{\Delta}{=} \arg\min\{|(s+z)(j\omega+z^*)|^2 \,\big|\, z \in I^i_{zC}\},\ i = 1, ., \ldots, k'_{pC} \quad (3.16d)$$

$$\hat{z}^i_{AR}(s) = \arg\max\{\arg[s+z] \,|\, z \in I^i_{zR}\},\ i = 1, 2, \ldots, k'_{pR} \quad (3.17a)$$

$$\hat{z}^i_{AC}(s) = \arg\max\{\arg[(s+z)(s+z^*)] \,|\, z \in I^i_{zC}\},\ i = 1, 2, \ldots, k'_{pC} \quad (3.17b)$$

$$\check{z}_{AR}(s) = \arg\min\{\arg[s+z] \,|\, z \in I^i_{zR}\}, i = 1, 2, \ldots, k'_{pR} \quad (3.17c)$$

$$\check{z}_{AC}(s) = \arg\min\{\arg[(s+z)(s+z^*)] \,|\, z \in I^i_{zC}\},\ i = 1, 2, \ldots, k'_{pC} \quad (3.17d)$$

The maximizers and minimizers of the amplitudes and angles of the denominator terms in (2.4) are defined analogously, and will be denoted by $\hat{p}^i_{MR}$, $\hat{p}^i_{MC}$, $\check{p}^i_{MR}$, $\check{p}^i_{MC}$, $\hat{p}^i_{AR}$, $\hat{p}^i_{AC}$, $\check{p}^i_{AR}$, $\check{p}^i_{AC}$. Referring to Appendix 2, we see that the $\hat{z}^i_{MR}(s)$, $\check{z}^i_{MR}(s)$, $\hat{z}^i_{AR}(s)$, $\check{z}^i_{AR}(s)$ and $\hat{p}^i_{MR}(s)$, $\check{p}^i_{MR}(s)$, $\hat{p}^i_{AR}(s)$, $\check{p}^i_{AR}(s)$ are ω-independent, while the other maximizers/minimizers are simple, piecewise continuous functions of ω and are given in Table A2.1 and A2.2. The extremizers for real poles and zeros are given by Theorem A2.1 whilst for complex perturbations, the extremizers are given by Theorem A2.2. Let

$$\hat{\alpha}_M(s) \triangleq (\bar{K}_p,(\hat{z}^i_{MR}(s))_{i\in\{1,\dots,k'_{pR}\}},(\hat{z}^i_{MC}(s))_{i\in\{1,\dots,k'_{pC}\}},$$
$$(\hat{p}^i_{MR}(s))_{i\in\{1,\dots,k_{pR}\}},(\hat{p}^i_{MC}(s))_{i\in\{1,\dots,k_{pC}\}})^T \tag{3.18a}$$

$$\check{\alpha}_M(s) \triangleq (\underline{K}_p,(\check{z}^i_{MR}(s))_{i\in\{1,\dots,k'_{pR}\}},(\check{z}^i_{MR}(s)_{i\in\{1,\dots,k'_{pC}\}},$$
$$(\check{p}^i_{MR}(s))_{i\in\{1,\dots,k_{pR}\}},(\check{p}^i_{MC}(s))_{i\in\{1,\dots,k_{pC}\}})^T \tag{3.18b}$$

$$\hat{\alpha}_A(s) \triangleq ((\hat{z}^i_{A\mathbb{R}}(s))_{i\in\{1,\dots,k'_{pR}\}},(\hat{z}^i_{AC}(s))_{i\in\{1,\dots,k'_{pC}\}},$$
$$(\hat{p}^i_{AC}(s))_{i\in\{1,\dots,k'_{pR}\}},(\check{p}^i_{AC}(s))_{i\in\{1,\dots,k_{pC}\}})^T \tag{3.18c}$$

$$\check{\alpha}_A(s) \triangleq ((\check{z}^i_{AR}(s))_{i\in\{1,\dots,k'_{pR}\}},(\check{z}^i_{AC}(s))_{i\in\{1,\dots,k'_{pC}\}},$$
$$(\check{p}^i_{AR}(s))_{i\in\{1,\dots,k'_{pR}\}},(\check{p}^i_{AC}(s))_{i\in\{1,\dots,k'_{pC}\}})^T. \tag{3.18d}$$

Applying Theorems 3.1 and 3.2 we obtain immediately the following result:

<u>Theorem 3.4</u>: For any $x \in \mathbb{R}^{n_x}$, $s \in \mathbb{C}$, let

$$\hat{M}_0(x,s) \triangleq \max_{\alpha\in A} |P_0(s,\alpha)C(x,s)| \tag{3.19a}$$

$$\check{M}_0(x,s) \triangleq \min_{\alpha\in A} |P_0(s,\alpha)C(x,s)| \tag{3.19b}$$

$$\hat{\Phi}_0(x,s) \triangleq \max_{\alpha\in A} \arg[P_0(s,\alpha)C(x,s)] \tag{3.19c}$$

$$\check{\Phi}_0(x,s) \triangleq \min_{\alpha\in A} \arg[P_0(s,\alpha)C(x,s)] \tag{3.19d}$$

$$\hat{M}(x,s) \triangleq \max_{\substack{\alpha\in A\\ \ell\in L}} |P(s,\alpha,\ell(s))C(x,s)| \tag{3.19e}$$

$$\check{M}(x,s) \triangleq \min_{\substack{\alpha\in A\\ \ell\in L}} |P(s,\alpha,\ell(s))C(x,s)| \tag{3.19f}$$

$$\hat{\Phi}(x,s) \triangleq \max_{\substack{\alpha\in A\\ \ell\in L}} \arg[P(s,\alpha,\ell(s))C(x,s)] \tag{3.19g}$$

$$\check{\Phi}(x,s) \triangleq \min_{\substack{\alpha\in A\\ \ell\in L}} \arg[P(s,\alpha,\ell(s))C(x,s)]. \tag{3.19h}$$

Then $\forall\ s \in \mathbb{C}$,

$$\hat{M}(x,s) = |P_0(s,\hat{\alpha}_M(s))C(x,s)| \tag{3.20a}$$

$$\check{M}(x,s) = |P_0(s,\check{\alpha}_M(s))C(x,s)| \tag{3.20b}$$

$$\hat{\Phi}(x,s) = \arg[P_0(s,\alpha_A(s))C(x,s)] \tag{3.20c}$$

$$\check{\Phi}(x,s) = \arg[P_0(s,\check{\alpha}_A(s))C(x,s)] \tag{3.20d}$$

and for $s = \sigma + j\omega$,

$$\hat{M}(x,s) = |P_0(s,\hat{\alpha}_M(s))|(\bar{\ell}_M(\omega)+L_1|\sigma|)|C(x,s)| \tag{3.20e}$$

$$\check{M}(x,s) = |P_0(s,\check{\alpha}_M(s))|(\underline{\ell}_M(\omega)-L_1|\sigma|)|C(x,s)| \tag{3.20f}$$

$$\hat{\Phi}(x,s) = \arg[P_0(s,\hat{\alpha}_A(s))] + \bar{\ell}_A(\omega) + L_2|\sigma| + \arg[C(x,s)] \tag{3.20g}$$

$$\check{\Phi}(x,s) = \arg[P_0(s,\check{\alpha}_A(s))] + \underline{\ell}_A(\omega)-L_2|\sigma| + \arg[C(x,s)]. \tag{3.20h}$$

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To illustrate the use of Theorem 3.4 in an optimization-based design scheme, consider the inequality on the loop gain:

Find an $x \in \mathbb{R}^{n_x}$ such that

$$|P(j\omega,\alpha,\ell(j\omega))C(x,j\omega)| \geq \underline{\ell}_g(\omega) \quad \forall\ \alpha \in A,\ \ell \in L,\ \omega \geq 0 \tag{3.21a}$$

or equivalently,

$$\max_{\substack{\alpha\in A \\ \ell\in L}} |P(j\omega,\alpha,\ell(j\omega))C(x,j\omega)| \geq \underline{\ell}_g(\omega) \quad \forall\ \omega \geq 0. \tag{3.21b}$$

By (3.20), this reduces to

$$|P_0(j\omega,\hat{\alpha}_M(j\omega))\bar{\ell}_M(\omega)C(x,j\omega)| \geq \underline{\ell}_g(\omega) \quad \forall\ \omega \geq 0 \tag{3.21c}$$

where for each $\omega \geq 0$, the vector $\hat{\alpha}_M(j\omega)$ and $\bar{\ell}_M(\omega)$ are known.

3.3. Majorization of Robustness, Noise Rejection and Saturation Avoidance Constraints

We shall now make use of the results of the preceding section to obtain majorizations of the inequalities (2.18), (2.20a) and (2.21a); i.e., we shall replace them with slightly tighter constraints which are computationally more tractable.

To simplify notation, we assume that $f(\omega) \equiv 0$ in (2.18). To obtain a more general result, the reader should replace $j\omega$ by $(f(\omega)+j\omega)$ in the appropriate expressions below. It is shown in Appendix 2 that our results for optimizing functions of $j\omega$ may be used to optimize functions of $(f(\omega)+j\omega)$ so that there is no loss of generality in assuming $f(\omega) \equiv 0$. For any $x \in \mathbb{R}^n$, $\omega \in [0,\infty)$ let

$$\psi_{yu}^0(x,j\omega) \triangleq \max_{\alpha \in A} |H_{yu}^0(x,j\omega,\alpha)|^2. \tag{3.22}$$

Then (2.18) becomes

$$\psi_{yu}^0(x,j\omega) - \frac{1}{\hat{\ell}_M(j\omega)^2} \leq 0 \quad \forall\ \omega \in [0,\infty) \tag{3.23}$$

Next, writing P_0, C in polar co-ordinates, we obtain

$$P_0(j\omega,\alpha) \triangleq m_{P_0}(j\omega,\alpha)e^{j\phi_{P_0}(j\omega,\alpha)} \tag{3.24a}$$

$$C(x,j\omega) \triangleq m_C(x,j\ \ )e^{j\phi_C(x,j\omega)} \tag{3.24b}$$

which define the magnitude and phase functions m_{P_0}, m_C, ϕ_{P_0}, ϕ_C. Let $P_0(j\omega,A)$ denote the set of all possible plant magnitude and phase variations, with respect to A, i.e.,

$$P_0(j\omega,A) \triangleq \{(m,\phi) \in \mathbb{R}^2 | m = m_{P_0}(j\omega,\alpha),\ \phi = \phi_{P_0}(j\omega,\alpha),\ \alpha \in A\} \tag{3.25}$$

Next, by substituting from (3.24a,b) into the formula for H_{yu}^0, we obtain

$$H_{yu}^0(x,j\omega,\alpha) = \frac{P_0(j\omega,\alpha)C(x,j\omega)}{1+P_0(j\omega,\alpha)C(x,j\omega)}$$

$$= \frac{m_{p_0}(j\omega,\alpha)m_c(x,j\omega)e^{j(\phi_{p_0}(j\omega,\alpha)+\phi_c(x,j\omega))}}{1+m_{p_0}(j\omega,\alpha)m_c(x,j\omega)e^{j(\phi_{p_0}(j\omega,\alpha)+\phi_c(x,j\omega))}} \cdot \tag{3.26}$$

Hence, $\psi^0_{yu}(x,j\omega)$ can be seen to be given by

$$\psi^0_{yu}(x,j\omega) = \max_{(m,\phi)\in P_0(j\omega,A)} \frac{m^2 m_c(x,j\omega)^2}{1+2mm_c(x,j\omega)\cos(\phi+\phi_c(x,j\omega))+m^2m_c(x,j\omega)^2} \cdot \tag{3.27}$$

Now, for any $\omega \in [0,\infty)$, let

$$R_{p_0}(x,j\omega) \triangleq \{(m,\phi)\in \mathbb{R}^2 | \check{M}_0(x,j\omega) \le m \le \hat{M}_0(x,j\omega), \check{\Phi}_0(x,j\omega \le \Phi \le \hat{\Phi}_0(x,j\omega)\}. \tag{3.28}$$

The set $R_{p_0}(x,j\omega)$ is a rectangular approximation to $P_0(j\omega,A)$ in $\mathbb{R}^2$, in the sense that $P_0(j\omega,A) \subset R_{p_0}(x,j\omega)$ and $P_0(j\omega,A)$ has points on each of the four sides of $R_{p_0}(x,j\omega)$ (see Fig. 12).

Now, let

$$\bar{\psi}^0_{yu}(x,j\omega) \triangleq \max_{(m,\phi)\in R_{p_0}(j\omega)} \frac{m^2 m_c(x,j\omega)^2}{1+2mm_c(x,j\omega)\cos(\phi+\phi_c(x,j\omega))+m^2m_c(x,j\omega)^2} \tag{3.29}$$

Since the maximization in (3.29) is over a larger set than in (3.27), we must have

$$\psi^0_{yu}(x,j\omega) \le \bar{\psi}^0_{yu}(x,j\omega) \quad \forall\, x \in \mathbb{R}^{n_x}, \quad \forall\, \omega \in [0,\infty). \tag{3.30}$$

Consequently, any x which satisfies

$$\bar{\psi}^{0}_{yu}(x,j\omega) - \frac{1}{\hat{\ell}_M(j\omega)^2} \leq 0 \quad \forall\, \omega \in [0,\infty) \tag{3.31}$$

must satisfy (3.23), i.e., (3.31) <u>majorizes</u> (3.23).

Clearly, the set of design vectors x which satisfies (3.16) is smaller than the one that satisfies (3.8). This conservatism can be reduced by replacing $R_{p_0}(x,j\omega)$ by a smaller convex polyhedron containing $P_0(j\omega,A)$. The great advantage of (3.31) over (3.23) is that (3.31) is quite easy to evaluate while (3.23) is extremely difficult. The reason for this has to do with the fact that $R_{p_0}(x,j\omega)$ is a rectangle in $\mathbb{R}^2$, while A is a "rectangle" in a higher dimensional space. Clearly, the maximizers $(\hat{m}(x,j\omega),\hat{\phi}(x,j\omega))$ for (3.29) are either in the interior of $R_{p_0}(x,j\omega)$ or on its boundary. If they are in the interior, then they must be <u>unconstrained</u> maximizers and the gradient of the maximand in (3.29) must vanish at these points. Now it is shown in Appendix 3 that the gradient† of the maximand cannot vanish†† in the interior of $R_{p_0}(x,j\omega)$ and hence $(\hat{m},\hat{\phi})$ must be on the <u>boundary</u> of $R_{p_0}(x,j\omega)$. Since $R_{p_0}(x,j\omega)$ is a rectangle in $\mathbb{R}^2$, its boundary consists of four <u>line</u> segments. It is shown in Appendix 3 that the maximization over these four segments reduces to at most nine function evaluations. Since the dimension of the boundary††† of A is greater than one, no comparable simplification can be obtained for the evaluation of $\psi^{0}_{yu}(x,j\omega)$.

Next we turn to the noise reduction and saturation constraints (2.20a) and (2.21a). Letting

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†If $m_{p_0} m_C = 1$, $\cos(\phi_{p_0}+\phi_C) = -1$, the gradient fails to exist and the maximand is infinite.

††Unless the maximand is a constant in which case any point in $R_{p_0}(x,j\omega)$ is a maximizer.

†††Details of computing $\partial P_0(j\omega,A)$, the boundary of $P_0(j\omega,A)$, are given in Appendix 4.

$$\ell(j\omega) = m_\ell(j\omega)e^{j\phi_\ell(j\omega)} \tag{3.32}$$

with $\ell \in L$ arbitrary, we obtain

$$P(j\omega,\alpha,\ell(j\omega)) = m_{p_0}(j\omega,\alpha)m_\ell(j\omega)e^{j(\phi_{p_0}(j\omega,\alpha)+\phi_\ell(j\omega))}$$

$$= m_p(j\omega,\alpha,\ell(j\omega))e^{j\phi_p(j\omega,\alpha,\ell(j\omega))} \tag{3.33}$$

with $m_p \triangleq m_{p_0}\, m_\ell$, $\phi_p \triangleq \phi_{p_0} + \phi_\ell$. Let

$$L^-(j\omega) \triangleq \{(m,\phi) \in \mathbb{R}^2 | m = |\ell(j\omega)|,\ \phi = \arg(\ell(j\omega)), \ell \in L\} \tag{3.34}$$

$$\psi_{yd}(x,j\omega) \triangleq \max_{\substack{\alpha\in A\\ \ell\in L}} |H_{yd}(x,j\omega,\alpha,\ell(j\omega))|^2 \tag{3.34a}$$

Then, expanding (3.34a), we get

$$\psi_{yd}(x,j\omega) = \max_{\substack{\alpha\in A\\ \ell\in L}}\{|1+m_{p_0}(j\omega,\alpha)m_\ell(j\omega)m_C(x,j\omega)e^{j(\phi_{p_0}(j\omega,\alpha)+\phi_\ell(j\omega) + \phi_C(x,j\omega)}|^2\}^{-1}$$

$$= \max_{\substack{(m',\phi')\in P_0(j\omega,A)\\ (m'',\phi'')\in L^-(j\omega)}} \{|1+m'm''m_C(x,j\omega)e^{j(\phi'+\phi''+\phi_C(x,j\omega)}|^2\}^{-1} \tag{3.34b}$$

Let

$$\hat{M}(x,j\omega) \triangleq \hat{M}_0(x,j\omega)\bar{\ell}_M(\omega) \tag{3.35a}$$

$$\hat{\Phi}(x,j\omega) \triangleq \hat{\Phi}_0(x,j\omega) + \bar{\ell}_A(\omega) \tag{3.35b}$$

with similar notation for minimizers. Define

$$R_p(x,j\omega) \triangleq \{(m,\phi) \in \mathbb{R}^2 | \check{M}(x,j\omega) \le m \le \hat{M}(x,j\omega), \check{\Phi}(x,j\omega) \le \phi \le \hat{\Phi}(x,j\omega)\}. \tag{3.35c}$$

Then it is easy to see that if $m = m'm''$ and $\phi = \phi' + \phi''$, with $(m',\phi') \in P_0(j\omega,A)$ $(m'',\phi'') \in L^-(j\omega)$ then $(m,\phi) \in R_p(x,j\omega)$, i.e., the approximating rectangle $R_p(x,j\omega)$ contains the set of actual plant variations,

$$P(j\omega,A,L) \triangleq \{(m,\phi) \in \mathbb{R}^2 | m = m'm'', \ \phi = \phi' + \phi'',$$

$$(m',\phi') \in P_0(j\omega,A), \ (m'',\phi'') \in L^-(j\omega)\}. \tag{3.36}$$

Furthermore, there are points $(m,\phi) \in P(j\omega,A,L)$ which lie on each of the four sides of $R_p(x,j\omega)$. Consequently, the noise suppression constraint (2.20a), which can be written as

$$\psi_{yd}(x,j\omega) - \bar{\ell}_d(j\omega)^2 \le 0 \quad \forall\ \omega \in [0,\infty) \tag{3.37a}$$

can be majorized by the constraint

$$\bar{\psi}_{yd}(x,j\omega)-\bar{\ell}_d(j\omega)^2 \le 0 \quad \forall\ \omega \in [0,\infty) \tag{3.37b}$$

where

$$\bar{\psi}_{yd}(x,j\omega) \triangleq \max_{(m,\phi)\in R_p(x,j\omega)} \{|1+mm_c(x,j\omega)e^{j(\phi+\phi_c(x,j\omega))}|^2\}^{-1}. \tag{3.37c}$$

As we shall show in Appendix 3, the evaluation of $\bar{\psi}_{yd}(x,j\omega)$ is again quite simple. Finally, let

$$\psi_{vd}(x,j\omega) \triangleq \max_{\substack{\alpha\in A \\ \ell\in L}} |H_{vd}(x,\alpha,j\omega,\ell(j\omega))|^2 \tag{3.38}$$

Since $H_{vd} = - CH_{yd}$, and C does not depend on (α,ℓ), we must have

$$\psi_{vd}(x,j\omega) = m_C(x,j\omega)^2\psi_{yd}(x,j\omega) \tag{3.39}$$

and hence, if we define

$$\bar{\psi}_{vd}(x,j\omega) \triangleq m_C(x,j\omega)^2\bar{\psi}_{yd}(x,j\omega) \tag{3.40}$$

we see that the saturation constraint (2.21a) can be majorized by

$$\bar{\psi}_{vd}(x,j\omega) - \bar{\ell}_s(j\omega)^2 \leq 0 \quad \forall\ \omega \in [0,\infty) \tag{3.41}$$

with $\bar{\psi}_{vd}$ computable with the same ease as $\bar{\psi}_{yd}$. Similarly we may majorize (2.21c).

3.4. Majorization of S-Stability Constraints

We now obtain a majorization for the most complex of the constraints described in Section 2.2, namely (2.11) and (2.19). Without loss of generality, it suffices to consider (2.11) only: the corresponding results for (2.19) are obtained by replacing $j\omega$ by $(f(\omega)+j\omega)$, as appropriate. Referring to (2.9), let m_1, m_2, ϕ_1, ϕ_2 be defined by

$$m_1(x,j\omega,\alpha,\ell(j\omega))e^{j\phi_1(x,j\omega,\alpha,\ell(j\omega))} \triangleq \frac{n_C(x,j\omega)n_p(j\omega,\alpha)\ell(j\omega)}{d(j\omega)} \tag{3.42a}$$

$$m_2(x,j\omega,\alpha)e^{j\phi_2(x,j\omega,\alpha)} \triangleq \frac{d_C(x,j\omega)d_p(j\omega,\alpha)}{d(j\omega)} \tag{3.42b}$$

Then (2.9) can be written (with the arguments suppressed) as

$$T = m_1 e^{j\phi_1} + m_2 e^{j\phi_2}. \tag{3.42c}$$

Making use of the decomposition results in Section 3.1 we can obtain continuous bound functions $\hat{M}_i(x,j\omega)$, $\check{M}_i(x,j\omega)$, $\check{\Phi}_i(x,j\omega)$, $\hat{\phi}_i(x,j\omega)$, $i = 1, 2$ such that

$$\check{M}_i(x,j\omega) \leq m_i(x,j\omega,\alpha,\ell(j\omega)) \leq \hat{M}_i(x,j\omega)$$

$$\forall\ \alpha \in A,\ \forall\ \ell \in L,\ i = 1, 2, \tag{3.43a}$$

$$\check{\Phi}_i(x,j\omega) \leq \phi_i(x,j\omega,\alpha,\ell(j\omega)) \leq \hat{\Phi}_i(x,j\omega)$$

$$\forall\ \alpha \in A,\ \forall\ \ell \in L,\ i = 1, 2. \tag{3.43b}$$

In terms of the polar notation (3.42a,b), (2.11) becomes (with the arguments x, $j\omega$, α, ℓ suppressed),

$$\max_{\substack{\alpha\in A \\ \ell\in L_S}} \{(m_1 \sin \phi_1 + m_2 \sin \phi_2) - k_1(m_1 \cos \phi_1 + m_2 \cos \phi_2)^2 + k_2\} \leq 0$$

$$\forall\ \omega \in [0,\infty). \tag{3.44}$$

Clearly, (3.44) is majorized by

$$\max\{(m_1 \sin \phi_1 + m_2 \sin \phi_2) - k_1(m_1 \cos \phi_1 + m_2 \cos \phi_2)^2$$

$$+ k_2 | \check{M}_i(x,j\omega) \leq m_i \leq \hat{M}_i(x,j\omega), \check{\Phi}_i(x,j\omega) \leq \phi_i \leq \hat{\Phi}_i(x,j\omega),$$

$$i = 1,2\} \leq 0 \qquad \forall\ \omega \in [0,\infty). \tag{3.45}$$

We note that (3.45) differs from (3.44) in that the max in (3.45) is only over four variables while the one in (3.44) is much more complex. Now, the constraint set in (3.45) is a hype rectangle, R^4, in $\mathbb{R}^4$. Clearly, the maximizing quadruplet $(\hat{m}_1,\hat{\phi}_1,\hat{m}_2,\hat{\phi}_2)$ is either in the interior or on one of its three dimensional faces. We shall show in Appendix 5 that $(\hat{m}_1,\hat{\phi}_1,\hat{m}_2,\hat{\phi}_2)$ is, in fact, on some two dimensional face of one of these three dimensional faces. To be precise, we shall prove the following result.

Theorem 3.4: Let $\xi:\mathbb{R}^4 \to \mathbb{R}^1$ be defined by

$$\xi(m_1,\phi_1,m_2,\phi_2) \triangleq (m_1 \sin \phi_1+m_2 \sin \phi_2) - k_1(m_1 \cos \phi_1+m_2 \cos \phi_2)^2 + k_2 \tag{3.46a}$$

and let $\underline{M}_i$, $\bar{M}_i$, $\underline{\Phi}_i$, $\bar{\Phi}_i \in \mathbb{R}$ be arbitrary. Then

$$\max\{\xi(m_1,\phi_1,m_2,\phi_2) \mid \underline{M}_i \le m_i \le \bar{M}_i,\ \underline{\Phi}_i \le \phi_i \le \bar{\Phi}_i,\ i = 1, 2\}$$

$$= \max[\max_{(\phi_1,\phi_2)} \{\xi(\bar{M}_1,\phi_1,\bar{M}_2,\phi_2) \mid \underline{\Phi}_i \le \phi_i \le \bar{\Phi}_i,\ i = 1, 2\},$$

$$\max_{(\phi_1,\phi_2)} \{\xi(\underline{M}_1,\phi_1,\bar{M}_2,\phi_2 \mid \underline{\Phi}_i \le \phi_i \le \bar{\Phi}_i,\ i = 1, 2\}$$

$$\max_{(\phi_1,\phi_2)} \{\xi(\underline{M}_1,\phi_1,\underline{M}_2,\phi_2) \mid \underline{\Phi}_i \le \phi_i \le \bar{\Phi}_i,\ i = 1, 2\}$$

$$\max_{(\phi_1,\phi_2)} \{\xi(\bar{M}_1,\phi_1,\underline{M}_2,\phi_2) \mid \underline{\Phi}_i \le \phi_i \le \bar{\Phi}_i,\ i = 1,2\}]. \tag{3.46b}$$

¤

Thus we see from Theorem 3.4 that for every (x,ω), the inequality (3.45) can be checked by carrying out four 2-dimensional maximizations. We shall show in Appendix 5 that these two 2-dimensional maximizations can be reduced to a small number of one dimensional maximizations, and evaluations of derivative zeros, as for the cases considered in the preceding section.

3.5. Majorization of Time Domain Constraints

We now turn to the most difficult constraints to majorize: those in the time domain, viz those given by (2.22), i.e., constraints of the form

$$\underline{\ell}_t(t) \leq y(t,x,\alpha,\ell,r) \leq \bar{\ell}_t(t) \quad \forall\, \alpha \in A,\ \forall\, \ell \in L,\ \forall\, t \geq 0 \tag{3.47}$$

where r(t) is an input to be followed. Referring to Theorem 3.3, and the majorizations in Sections 3.2 - 3.4, we see that we relied heavily on the fact that we dealt with constraint functions $\phi(x,j\omega,\alpha,\ell(j\omega))$ which were of the form $\phi(x,j\omega,\alpha,\ell(j\omega)) = \Phi(f(x,j\omega,\alpha,\ell(j\omega)))$ with $f:\mathbb{R}^{n_x} \times \mathbb{R} \times \mathbb{R}^{n_\alpha} \times L \to \mathbb{C}$ (or $\to \mathbb{R}^2$) so that the problem $\max_{\alpha\in A} \phi(x,j\omega,\alpha,\ell(j\omega))$ was easily reducible to a maximization problem over a rectangle in $\mathbb{R}^2$. Now there appears to be no obvious way of expressing the response $y(t,x,\alpha,\ell,r)$ in the form $\Phi(f(t,x,\alpha,(j\omega),r)$ with $f \in \mathbb{R}^2$ and hence there seems to be no way of easily majorizing (3.47) in the time domain directly (furthermore, we are not given any bounds on $\ell(t)$). Thus, one may well have to resort to simulation utilizing randomly generated $\alpha \in A$ to get an estimate of $\max_{\alpha\in A} y^0(t,x,\alpha,r)$ ($\min_{\alpha\in A} y^0(t,x,\alpha,r)$) for the response of the structured part and ignore the contribution of the unstructured perturbation $\ell(t)$.

Alternatively, one may try to verify (3.47) by frequency response methods. First, we observe that it does not seem possible to replace (3.47) with an equivalent inequality in the frequency domain involving bounds on magnitude and phase [H4, K4]. The reason for this is that functions which are close in the L_∞ sense in the time domain may have Laplace transforms which are not at all close in the L_∞ sense in the frequency domain, and vice versa. Of course, Parseval's identity can only be used for L_2 constraints and hence is of no use with the L_∞ constraint (3.47).

Since a simple substitution of bounds in the frequency domain for bounds in the time domain fails, one must turn to Fourier series as a measure of last resort, since, as we shall see, it leads to rather conservative majorizations. For this purpose, we replace the original input $r(t)$ by a periodic input $r_p(t)$, with period T, such that

$$r_p(t) = \begin{cases} r(t) & \text{for } 0 \leq t \leq T/2 \\ 0 & \text{for } T/2 < t < T \end{cases} \tag{3.48}$$

where T/2 is sufficiently large to allow the system transients to die out. At the same time, we replace the requirement of $t \in [0,\infty)$ in (3.47) by the requirement of $t \in [0,T/2]$. Assuming that $r_p(t) = \sum_{-\infty}^{\infty} r_k e^{jk\omega_0 t}$, with $\omega_0 = 2\pi/T$, the corresponding periodic output is

$$y(t,x,\alpha,\ell,r) = \sum_{-\infty}^{\infty} H_{yr}(x,jk\omega_0,\alpha,\ell(jk\omega_0))r_k e^{jk\omega_0 t}$$

and hence, truncating the sum at N, (3.47) becomes replaced by

$$\underline{\ell}_t(t) \leq \sum_{-N}^{N} H_{yr}(x,jk\omega_0,\alpha,\ell(jk\omega_0))r_i e^{jk\omega_0 t} \leq \bar{\ell}_t(t)$$

$$\forall\ \alpha \in A,\ \forall\ \ell \in L,\ \forall\ t \in [0,T/2]. \tag{3.48}$$

Clearly, each term of the form

$$[r_k e^{jk\omega_0 t} H_{yr}(x,jk\omega_0,\alpha,\ell(jk\omega_0)) + r_{-k} e^{-jk\omega_0 t} H_{yr}(x,-jk\omega_0,\alpha,\ell(-jk\omega_0))], \tag{3.49}$$

in the sum (3.48), can be majorized (minorized) independently by the techniques used in the preceding sections, to yield an upper (lower) bound $\bar{b}_k(t,x)$ ($\underline{b}_k(t,x)$). Hence (3.48) is obviously satisfied if

$$\sum_{k=0}^{N} \bar{b}_k(t,k) \leq \bar{\ell}_t(t) \quad \forall\, t \in [0,T/2] \tag{3.50a}$$

$$\underline{\ell}_t(t) \leq \sum_{k=0}^{N} \underline{b}_k(t,x) \quad \forall\, t \in [0,T/2]. \tag{3.50b}$$

The main drawback to this procedure is that the <u>same</u> α must be used for <u>all</u> k in (3.49), while in (3.50a,b) a <u>different</u> α may well have been used for <u>each</u> k. Consequently, the requirement (3.50a,b) may be much too stringent for practical purposes.

4. Properties of the Majorizing Functions

In the preceding section we have shown that a good number of computationally intractable constraints can be replaced by somewhat tighter ones which are quite easy to evaluate. Before leaving this subject, we must show that the majorizing constraint functions which we created are compatible with current semi-infinite optimization algorithms. Referring to [P3], we see that we only need to prove that they are locally Lipschitz continuous. Examining (3.16), (3.17), (3.22b), (3.26), (3.30) and (3.35) we see that our majorizing functions are of the form

$$\psi(x,\omega) \stackrel{\Delta}{=} \max_{(m,\phi)\in R(x,\omega)} \zeta(x,\omega,m,\phi). \tag{4.1}$$

<u>Definition 4.1</u>. a) We say that $\zeta : \mathbb{R}^{n_x} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is <u>locally Lipschitz continuous</u> if for every bounded set $B \subset \mathbb{R}^{n_x} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ there exists $L \in (0,\infty)$ such that for all $(x,',\omega',m',\phi')$, $(x'',\omega'',m'',\phi'')$ in B,

$$|\zeta(x',\omega',m',\phi') - \zeta(x'',\omega'',m'',\phi'')| \leq L\{\|x'-x''\|+|\omega'-\omega''|+|m'-m''|+|\phi'-\phi''|\} \tag{4.2a}$$

b) We say that the set valued function $R : \mathbb{R}^{n_x} \times \mathbb{R} \to 2^{\mathbb{R}^2}$ is <u>locally Lipschitz continuous</u> if for every bounded set $B \subset \mathbb{R}^{n_x} \times \mathbb{R}$ there exists $L \in (0,\infty)$ such that for all (x',ω'), (x'',ω'') in B, given that $(m',\phi') \in R(x',\omega')$ there exist $(m'',\phi'') \in R(x'',\omega'')$ such that

$$|m'-m''| + |\phi'-\phi''| \leq L\{\|x'-x''\|+|\omega'-\omega''|\} \tag{4.2b}$$

¤

<u>Theorem 4.1</u>: Suppose that ζ and R in (4.1) are both locally Lipschitz continuous. Then ψ is locally Lipschitz continuous.

<u>Proof</u>: To simplify notation, let $z = (x,\omega)$ and let $\nu = (m,\phi)$, with $\|z\|_e \triangleq \|x\| + |\omega|$, $\|\nu\|_1 = |m| + |\phi|$. Let B_z be a bounded set in $\mathbb{R}^{n_x} \times \mathbb{R}$. Then, by the continuity of $R(\cdot)$, there exists a bounded set B' such that $R(z) \subset B'$ for all $z \in B_z$. Hence there exists a Lipschitz constant $L \in (0,\infty)$ such that for all z', $z'' \in B_z$ and $\nu' \in R(z')$ there exists a $\nu'' \in R(z'')$ such that

$$\|\nu'-\nu''\|_1 \leq L\|z'-z''\|_e \tag{4.3a}$$

and, for the same ν', ν'',

$$|\zeta(z',\nu') - \zeta(z'',\nu'')| \leq L\{\|z'-z''\|_e + \|\nu'-\nu''\|_1\} \tag{4.3b}$$

Now suppose that with $z' \in B_z$, $\psi(z') = \zeta(z',\hat{\nu}')$, with $\hat{\nu}' \in R(z')$. By

Lipschitz continuity, for any $z'' \in B_z$, there exists a $\nu'' \in R(z'')$ such that

$$\|\hat{\nu}'-\nu''\|_1 \leq L\|z'-z''\|_e \tag{4.4a}$$

Hence

$$\psi(z') = \zeta(z',\hat{\nu}') \leq \zeta(z'',\nu'') + L\|z'-z''\|_e + L\|\hat{\nu}'-\nu''\|_1$$

$$\leq \zeta(z'',\nu'') + L(1+L)\|z'-z''\|_e$$

$$\leq \psi(z'') + L(1+L)\|z'-z''\|_e. \tag{4.4b}$$

Hence, since z' and z" are interchangeable (using $\hat{\nu}'',\nu'$) in (4.4a,b), we see that $\psi(\cdot)$ is locally Lipschitz continuous. ¤

The use of the above theorem in application to the majorizing functions in the preceeding section is facilitated by the following.

<u>Proposition 4.2</u>. Suppose that $R(z) = \text{co}\{\nu_i(z),\ i = 1,2,\ldots,k\}$, with $\nu_i : \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}^2$ locally Lipschitz (and co denoting the convex hull). Then $R(z)$ is locally Lipschitz.

<u>Proof</u>: Let B be a bounded set in $\mathbb{R}^{n_x} \times \mathbb{R}$. Then there exists $L \in (0,\infty)$ such that

$$\|\nu_i(z')-\nu_i(z'')\| \leq L\|z'-z''\|_e \quad \forall z',z'' \in B,\ i = 1, 2, \ldots, k. \tag{4.5}$$

Let $\nu' \in R(z')$. Then there exist $\mu_i' \geq 0$, $i = 1, 2, \ldots, k$ such that $\sum_{i=1}^{k} \mu_i' = 1$ and

$$\nu' = \sum_{i=1}^{k} \mu_i'\nu_i(z'). \tag{4.6a}$$

Let $z'' \in B$ and

$$\nu'' \stackrel{\Delta}{=} \sum_{i=1}^{k}{}' \mu_i' \nu_i(z'') \in \mathcal{R}(z'') \tag{4.6b}$$

Then

$$\|\nu'-\nu''\| = \|\sum_{i=1}^{k} \mu_i'(\nu_i(z')-\nu_i(z''))\|$$

$$\leq \sum_{i=1}^{k} \mu_i'\|\nu_i(z')-\nu_i(z'')\|$$

$$\leq L\|z'-z''\|_e. \tag{4.7}$$

This completes our proof. ¤

We show in Appendix 6 that the extremizers $\hat{M}(\cdot,\cdot)$ and $\hat{M}_0(\cdot,\cdot)$ ($\check{M}(\cdot,\cdot)$ and $\check{M}_0(\cdot,\cdot)$) are locally Lipschitz continuous for all $(x,\omega) \in \mathbb{R}^{n_x} \times \mathbb{R}_+$ such that $j\omega$ is not a pole of $P_0(s,\hat{\alpha}_M(s))C(x,s)$ ($P_0(s,\check{\alpha}_M(s))C(x,s)$). Further, we show that the extremizers of phase, $\hat{\Phi}(\cdot,\cdot)$, $\hat{\Phi}_0(\cdot,\cdot)$, $\check{\Phi}(\cdot,\cdot)$ and $\check{\Phi}_0(\cdot,\cdot)$, are locally Lipschitz continuous on $\mathbb{R}^{n_x} \times \mathbb{R}_+$. In the next result we establish local Lipschitz continuity of functions of the form of (4.1) when the maximization is over $\mathcal{R}_p(x,\omega)$ or $\mathcal{R}_{p_0}(x,\omega)$.

<u>Theorem 4.3</u>: Let $\zeta: \mathbb{R}^{n_x} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and let

$$\psi_p(x,\omega) \stackrel{\Delta}{=} \max_{(m,\phi)\in\mathcal{R}_p(x,\omega)} \zeta(x,\omega,m,\phi) \tag{4.8}$$

$$\psi_{p_0}(x,\omega) \stackrel{\Delta}{=} \max_{(m,\phi)\in\mathcal{R}_{p_0}(x,\omega)} \zeta(x,\omega,m,\phi) \tag{4.9}$$

where $\mathcal{R}_p(\cdot,\cdot)$ is defined by (3.35c) and $\mathcal{R}_{p_0}(\cdot,\cdot)$ is defined by (3.28). Then $\psi_p(\cdot,\cdot)$ and $\psi_{p_0}(\cdot,\cdot)$ are locally Lipschitz continuous on $\mathbb{R}^{n_x} \times \mathbb{R}_+$.

Proof: There are three cases to consider:

(i) Let (x_1,ω_1) be such that $\hat{M}(x_1,\omega_1) < \infty$. Then there exists a neighborhood, N_1, of (x_1,ω_1) such that $\hat{M}(x_1,\omega_1) < \infty$ for all $(x,\omega) \in N_1$. It follows from Lemma A6.5 that the vertices of $R_p(\cdot,\cdot)$ are locally Lipschitz continuous at (x_1,ω_1). From Proposition 4.2, we conclude that $R_p(\cdot,\cdot)$ is locally Lipschitz continuous at (x_1,ω_1). Hence, by Theorem 4.1, $\psi_p(\cdot,\cdot)$ is locally Lipschitz continuous at (x_1,ω_1).

(ii) Let (x_2,ω_2) be such that $\check{M}(x_2,\omega_2) < \hat{M}(x_2,\omega_2) = \infty$. Then there exists a neighborhood, N_2, of (x_2,ω_2) and a $b \in (0,\infty)$ such that $\check{M}(x,\omega) < b < \hat{M}(x,\omega)$ for all $(x,\omega) \in N_2$. For $(x,\omega) \in N_2$, let

$$R_p^b(x,\omega) \triangleq [\check{M}(x,\omega),b] \times [\check{\Phi}(x,\omega),\hat{\Phi}(x,\omega)]$$

$$R_p^\infty(x,\omega) \triangleq [b,\hat{M}(x,\omega)] \times [\check{\Phi}(x,\omega),\hat{\Phi}(x,\omega)]$$

$$R_p^-(x,\omega) \triangleq [\frac{1}{\hat{M}(x,\omega)}, \frac{1}{b}] \times [\check{\Phi}(x,\omega),\hat{\Phi}(x,\omega)].$$

Now $R_p^b(\cdot,\cdot)$ is non-empty on N_2 and its vertices are locally Lipschitz continuous on N_2 so it follows from Proposition 4.2 that $R_p^b(\cdot,\cdot)$ is locally Lipschitz continuous on N_2. Since $\frac{1}{\hat{M}(x,\omega)} < \frac{1}{b}$ for all $(x,\omega) \in N_2$, it follows from Lemma A6.5 that $\frac{1}{\hat{M}(\cdot,\cdot)}$ is locally Lipschitz continuous on N_2. Hence, $R_p^-(\cdot,\cdot)$ is non-empty and locally Lipschitz continuous on N_2. Since $R_p(x,\omega) = R_p^b(x,\omega) \cup R_p(x,\omega)$ for all $(x,\omega) \in N_2$, it follows that

$$\psi_p(x,\omega) = \max\{\max_{(m,\phi)\in R_p^b(x,\omega)} \zeta(x,\omega,m,\phi), \max_{(\mu,\phi)\in R_p^-(x,\omega)} (x,\omega,\frac{1}{\mu},\phi)\}$$

for all $(x,\omega) \in N_2$. By Theorem 4.1, it follows that $\psi_p(\cdot,\cdot)$ is the maximum of two locally Lipschitz continuous functions at (x_2,ω_2). Hence, $\psi_p(\cdot,\cdot)$ is locally Lipschitz continuous at (x_2,ω_2).

(iii) Let (x_3,ω_3) be such that $\check{M}(x_3,\omega_3) = \hat{M}(x_3,\omega_3) = \infty$. Then there exists a neighborhood, N_3, of (x_3,ω_3) and $b \in (0,\infty)$ such that $b < \check{M}(x,\omega)$ for all $(x,\omega) \in N_3$. For $(x,\omega) \in N_3$, let

$$R_p^-(x,\omega) \stackrel{\Delta}{=} \left[\frac{1}{\hat{M}(x,\omega)}, \frac{1}{\check{M}(x,\omega)}\right] \times [\check{\Phi}(x,\omega),\hat{\Phi}(x,\omega)].$$

$R_p^-(\cdot,\cdot)$ is locally Lipschitz continuous and non-empty on N_3 and

$$\psi_p(x,\omega) = \max_{(\mu,\phi)\in R_p^-(x,\omega)} \zeta(x,\omega,\tfrac{1}{\mu},\phi)$$

so that $\psi_p(\cdot,\cdot)$ is locally Lipschitz continuous at (x_3,ω_3).

Now $\hat{M}_0(x,\omega) = \dfrac{\hat{M}(x,\omega)}{\bar{\ell}_M(\omega)}$, $\check{M}_0(x,\omega) = \dfrac{\check{M}(x,\omega)}{\underline{\ell}_M(\omega)}$, $\hat{\Phi}_0(x,\omega) = \hat{\Phi}(x,\omega) - \bar{\ell}_A(\omega)$ and $\check{\Phi}_0(x,\omega) = \check{\Phi}(x,\omega) + \underline{\ell}_A(\omega)$ where $\bar{\ell}_M(\cdot)$, $\underline{\ell}_M(\cdot)$, $\bar{\ell}_A(\cdot)$ and $\underline{\ell}_A(\cdot)$ are locally Lipschitz continuous and $\bar{\ell}_M(\omega) \geq 1 \geq \underline{\ell}_M(\omega) > 0$ and $\bar{\ell}_A(\omega) \geq 0 \geq \underline{\ell}_A(\omega)$ for all $\omega \geq 0$. Hence by taking $\bar{\ell}_M(\omega) = \underline{\ell}_M(\omega) = 1$ and $\bar{\ell}_A(\omega) = \underline{\ell}_A(\omega) = 0$ for all $\omega \geq 0$, the desired result follows for $\psi_{p_0}(\cdot,\cdot)$. ¤

Finally we note that if $\zeta(\cdot,\cdot,\cdot,\cdot)$ corresponds to a closed loop magnitude, then $\zeta(\cdot,\cdot,\cdot,\cdot)$ fails to be locally Lipschitz continuous at those $(x,\omega) \in \mathbb{R}^{n_x} \times \mathbb{R}_+$ at which $P(j\omega,\alpha,\ell(j\omega))C(x,j\omega) = -1$ for some $\alpha \in A$. This requires that we consider piecewise locally Lipschitz continuous functions.

5. Tests for Infeasibility of the Compensator

In general, given a set of inequalities $f^i(x) \leq 0$, $j = 1, 2, \ldots, m$, there is no simple way of telling whether this set admits a solution or not. In the case of the inequalities presented in Section 2, it is possible to construct sufficient conditions for a single inequality not to have a solution. When this is the case, the controller structure must be augmented. Of course, even if the inequalities admit a solution one at a time, there is no guarantee that there is a set of compensator coefficients which satisfies them all. Nevertheless, the tests we are about to present are helpful in eliminating grossly under-structured controllers.

We shall consider two typical constraints (3.21a) and (2.20a). Let $X \subset \mathbb{R}^{n_x}$ denote the set of allowed designs. Inequality (3.21a) fails to have a solution $x \in X$ if and only if for some $\omega \geq 0$,

$$\max_{x\in X} \min_{\substack{\alpha\in A \\ \ell\in L}} |P(j\omega,\alpha,\ell(j\omega))C(x,j\omega)| < \underline{\ell}_g(\omega) \tag{5.1}$$

Similarly, inequality (2.20a) fails to have a solution $x \in X$ if for some $\omega \geq 0$,

$$\min_{x\in X} \max_{\substack{\alpha\in A \\ \ell\in L}} |H_{yd}(x,j\omega,\alpha,\ell(j\omega))|^2 > \bar{\ell}_d(\omega)^2 \tag{5.2}$$

The ease with which one can determine whether (5.1) or (5.2) hold depends very much on the specification of the set X. Suppose that C(x,s) is as in (2.1a), viz

$$C(x,s) = \frac{K_c(s+a_{cn}^0)\prod_{i=1}^{k_c'}(s^2+2a_{cn}^i s+(b_{cn}^i)^2)}{(s+a_{cd}^0)\prod_{i=1}^{k_c}(s^2+2a_{cd}^i s+(b_{cd}^i)^2)}$$

with the components of x being the a^i_{cn}, a^i_{cd}, b^i , b^i_{cd} and the set X defined by the constraints $a^i_{cn} \in [\underline{a}^i_{cn}, a^i_{cn}]$, $i = 0, 1, \ldots, k_c'$; $b^i_{cn} \in [\underline{b}^i_{cn}, \bar{b}^i_{cn}]$, $i = 1, 2, \ldots, k_c'$; $a^i_{cd} \in [\underline{a}^i_{cd}, \bar{a}^i_{cd}]$, $i = 0, 1, 2, \ldots, k_c$; $b^i_{cd} \in [\underline{b}_{cd}, \bar{b}^i_{cd}]$, $i = 1, 2, \ldots, k_c$.

First, (5.1) can be rewritten as

$$\max_{x \in X} |C(x,j\omega)| [\min_{\substack{\alpha \in A \\ \ell \in L}} P(j\omega,\alpha,\ell(j\omega))|] < \underline{\ell}_g(\omega). \tag{5.3a}$$

Making use of (3.20b) we conclude that (3.21a) fails to have a solution if and only if for some $\omega \geq 0$

$$\max_{x \in X} |C(x,j\omega)| < \underline{\ell}_g(\omega)/|P_0(j\omega,\check{\alpha}_M(j\omega))|\underline{\ell}_M(\omega). \tag{5.3b}$$

Referring to Section 3.2, we see that a maximizing $\hat{x}(\omega)$ for (5.3b) can be computed quite easily by the techniques presented in Appendix 2 for the structured plant, $P_0(j\omega,\alpha)$, $\alpha \in A$ so that (5.3b) is easily verified (see Fig. 14).

Next we turn to (5.2). Writing $C(x,j\omega) = m_C(x,j\omega)e^{j\phi_C(x,j\omega)}$, $P(j\omega,\alpha,\ell(j\omega)) = m_p(j\omega,\alpha,\ell(j\omega))e^{j\phi_p(j\omega,\alpha,\ell(j\omega))}$ we obtain that

$$|H_{yd}(x,j\omega,\alpha,\ell(j\omega))|^2$$

$$= \frac{1}{1+2m_p(j\omega,\alpha)m_C(x,j\omega)\cos[\phi_p(j\omega,\alpha)+\phi_C(x,j\omega)]+m_p(j\omega,\alpha)^2 m_C(x,j\omega)^2} \tag{5.4}$$

Proceeding for the compensator $C(x,j\omega)$ as we have done for the structured part of the plant $P_0(j\omega,\alpha)$ (see Sec. 3.3), we can easily compute a majorizing rectangle $R_C(j\omega) \subset \mathbb{R}^2$ such that $(m_C(x,j\omega),\phi_C(x,j\omega)) \in R_C(j\omega)$

for all $x \in X$. Hence, a sufficient condition for (5.3b) to hold is that for some $\omega \geq 0$,

$$\min_{(m_C,\phi_C)\in R_C(j\omega)} \max_{(m_p,\phi_p)\in R_p(x,j\omega)} \frac{1}{1+2m_p m_C \cos(\phi_p+\phi_C)+m_p^2 m_C^2} < \underline{\ell}_g(\omega). \tag{5.5}$$

We show in Appendix 7 that (5.5) is fairly easy to verify by making use of the fact that the denominator in (5.5) is a quadratic form. ¤

6. Conclusion

Early attempts, such as those described in [K1, P1, P4, Z1, Z2], to solve complex SISO design problems with uncertain plant, have yielded very limited results because the available computing tools were inadequate. The recent development of a new, very powerful tool in the form of semi-infinite programming (SIP) algorithms, has prompted us to reexamine the problem of designing SISO control system with uncertain plant.

In this paper, we have shown that a naive approach to SISO control system design via semi-infinite optimization leads to overwhelming computational difficulties when the plant model contains both structured and unstructured uncertainty. Fortunately, the structure of the design problem that we considered enabled us to replace the original "naive" specifications with slightly tighter ones which are locally Lipschitz continuous and simple to evaluate. Although at present there is no specific SIP algorithm which accepts our majorizing constraints, the theory in [P3] shows that such an algorithm can be constructed by a straightforward modification of the one in [G1]. The construction of this algorithm will be undertaken in the near future.

References

B1: Barrett, M. F., "Conservatism with Robustness Tests for Linear Feedback Control Systems," Ph.D. Thesis, Univ. of Minnesota, June 1980.

C1: Callier, F. M., and Desoer, C. A., Multivariable Feedback Systems, Springer-Verlag, New York, 1982.

C2: Chen, M. J., and Desoer, C. A., "Necessary and Sufficient Conditions for Robust Stability of Linear Distributed Feedback Systems," Int. J. Contr., Vol. 35, No. 2, pp. 255-267, 1982.

D1: Davison, E. J. and Ferguson, I. J., "The Design of Controllers for the Multivariable Robust Servomechanism Problem Using Parameter Optimization Methods," Systems Control Report No. 8002, University of Toronto, February 1980.

D2: Dennis, J. E., "Nonlinear Least Squares and Equations," pp. 269-312, in State of the Art in Numerical Analysis, D. A. Jacobs, Ed., Academis Press, London, N. Y., 1977.

D3: Doyle, J. C., "Robustness of Multiloop Linear Feedback Systems," Proc. IEEE Conf. on Dec. and Contr., San Diego, CA, 1979.

D4: Doyle, J. C., and Stein, G., "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," IEEE Trans. on Control, Vol. AC-26, No. 1, pp. 4-16, 1981.

D5: Doyle, J. C., "Analysis of Feedback Systems with Structured Uncertainties," Proc. IEE, Vol. 129, Part D, No. 6, pp. 242-250, 1982.

G1: Gonzaga, C., Polak, E., and Trahan, R., "An Improved Algorithm for Optimization Problems with Functional Inequality Constraints," IEEE Trans. on Automatic Control, Vol. AC-25, No. 1, 1980.

G2: Gonzaga, C., Polak, E., "On Constraint Dropping Schemes and Optimality Functions for a Class of Outer Approximations Algorithms," J. SIAM Control and Optimization, Vol. 17, 1979.

H1: Harvey, C. A. and Pope, R. E., "Study of Synthesis Techniques for Insensitive Aircraft Control Systems," NASA Contractor Report CR-2803, Langley Research Center, April 1977.

H2: Hettich, R., "Semi-infinite Programming," Springer-Verlag Lecture Notes in Control and Information Sciences, Vol. 15, 1979.

H3: Horowitz, I. M., Synthesis of Feedback Systems, Academic Press, N. Y., 1963.

H4: Horowitz, I. M., and Sidi, M., "Synthesis of Feedback Systems with Large Plant Ignorance for Prescribed Time Domain Tolerances," Int. J. Contr., Vol. 16, pp. 287-309, 1972.

H5: Horowitz, I. M., "Quantitative Feedback Theory," Proc. IEE, Vol. 129, Part D, No. 6, pp. 215-226, 1982.

K1: Karmakar, J. S. and Siljak, D. D., "A Computer-Aided Regulator Design," Proc. Allerton Conf. on Circuits and Systems, Monticello, IL, 1971.

K2: Karmarkar, J. S., "Multiparameter Design of Linear Optimal Regulators with Prescribed Degree of Exponential Stability," Proc. Asilomar Conf. on Circuits and Systems, Pacific Grove, CA, 1970.

K3: Karmarkar, J. S., Siljak, D. D., "Maximization of Absolute Stability Regions by Mathematical Programming Methods," Regelungtechnik, No. 2, 1975.

K4: Krishnan, K. R. and Cruickshanks, A., "Frequency-Domain Design of Feedback Systems for Specified Insensitivity of Time Domain Response to Parameter Variations," Int. J. Contr., Vol. 25, pp. 609-620, 1977.

M1: Mayne, D. Q., Polak, E., "A Quadratically Convergent Algorithm for Solving Infinite-Dimensional Inequalities," Memo No. UCB/ERL M80/11, University of California, Berkeley, 1980.

M2: D. Q. Mayne and E. Polak, "Algorithms for the Design of Control Systems Subject to Singular Value Inequalities," in Mathematical Programming Study 18, Algorithms and Theory in Filtering and Control, D. C. Sorensen and R. J.-B. Wets, ed., North Holland, New York, pp. 112-135, 1982.

P1: Polak, E., "On the Use of Optimization Algorithms in the Design of Linear Systems," University of California, Electronics Research Laboratory, Report No. UCB/ERL - M377, Berkeley, 1973.

P2: E. Polak, "A Modified Nyquist Stability Criterion for Use in Computer-Aided Design," Memo No. UCB/ERL M82/92, University of California, Berkeley, 1983.

P3: E. Polak and D. Q. Mayne, "Algorithm Models for Nondifferentiable Optimization," Memo No. UCB/ERL No. M82/34, University of California, Berkeley, 1982.

P4: Polak, E. and Mayne, D. Q., "An Algorithm for Optimization Problems with Functional Inequality Constraints," IEEE Trans. on Automatic Control, Vol. AC-21, No. 2, 1976.

P5: Polak, E., and Mayne, D. Q., "On the Solution of Singular Value Inequalities over a Continuum of Frequencies," IEEE Trans. on Automatic Control, AC-26, No. 3, pp. 690-695, 1981.

P6: E. Polak and A. Tits, "A Recursive Quadratic Programming Algorithm for Semi-Infinite Optimization Problems," J. Appl. Math. and

Optimization, Vol. 8, pp. 325-349, 1982.

P7: Polak, E., and Wardi, Y., "A Nondifferentiable Optimization Algorithm for the Design of Control Systems Subject to Singular Value Inequalities over a Frequency Range," Automatica, Vol. 18, No. 3, pp. 267-283, 1982.

P8: E. Polak and K. Y. Wong, "Identification of Linear Discrete Time Systems Using the Instrumental Variable Method," IEEE Trans. on Automatic Control, Vol. AC-12, No. 6, pp. 707-718, 1967.

S1: Sandel, N. R., "Robust Stability of Systems with Applications to Singular Value Perturbations," Automatica, Vol. 15, p. 467, 1979.

Z1: Zakian, V. and Al-Naib, L., "Design of Dynamical and Control Systems by the Method of Inequalities," Proc. IEE, 120 (11), 1973.

Z2: Zakian, V., "New Formulation for the Method of Inequalities," Proc. IEE, Vol. 126, No. 6, 1979.

Appendix 1: Computation of Bounds on $(\ell(s)-1)$, $\ell \in L$.

Putting $\ell(s) = m_\ell(s)e^{j\phi_\ell(s)}$, we have,

$$|\ell(s)-1|^2 = m_\ell^2(s) - 2m_\ell(s)\cos\phi_\ell(s) + 1 \qquad (A1.1)$$

$$\arg\{\ell(s)-1\} = \pm k\pi \pm \tan^{-1}\left\{\frac{m_\ell(s)\sin\phi_\ell(s)}{m_\ell(s)\cos\phi_\ell(s)-1}\right\} \qquad (A1.2)$$

where $k \in \mathbb{Z}$ is chosen to account for angles outside the range $[-\pi/2, \pi/2]$. Differentiating with respect to m_ℓ, ϕ_ℓ and suppressing the s-dependence,

$$\nabla\{|\ell-1|^2\} = 2\begin{pmatrix} m_\ell - \cos\phi_\ell \\ m_\ell \sin\phi_\ell \end{pmatrix} \qquad (A1.3)$$

$$\nabla\{\arg(\ell-1)\} = \frac{1}{|\ell-1|^2}\begin{pmatrix} -\sin\phi_\ell \\ m_\ell^2 - m\ \cos\phi_\ell \end{pmatrix} \qquad (A1.4)$$

It follows that the only stationary points of $|\ell-1|^2$ are $(1,(2n\pi))$ and $(0,(2n+1)\pi/2)$, $n \in \mathbb{Z}$, whilst $\arg(\ell-1)$ has stationary points, $(0,n\pi)$, $n \in \mathbb{Z}$. By considering the respective Hessian matrices, we deduce that $(1,(2n\pi))$ is a local minimum and $(0,(2n+1)\pi/2)$ is a local maximum of $|\ell-1|^2$ $\forall\, n \in \mathbb{Z}$. Similarly, for $\arg(\ell-1)$, $(0,n\pi)$, $n \in \mathbb{Z}$ is a local max. We construct an approximating set, $R_{(\ell-1)}(j\omega)$ which contains $\{\ell(j\omega)-1 \mid \ell \in L\}$ by computing

$$\hat{\ell}_M(j\omega) \triangleq \max_{\substack{m_\ell \in [\underline{\ell}_M, \bar{\ell}_M] \\ \phi_\ell \in [\underline{\ell}_A, \bar{\ell}_A]}} \{m_\ell^2 - 2m_\ell \cos\phi_\ell + 1\}^{1/2} \qquad (A1.5)$$

$$\hat{\ell}_A(j\omega) \stackrel{\Delta}{=} \max_{\substack{m_\ell \in [\underline{\ell}_M, \bar{\ell}_M] \\ \phi_\ell \in [\underline{\ell}_A, \bar{\ell}_A]}} \arg\{(m_\ell \cos \phi_\ell - 1) + jm_\ell \sin \phi_\ell\} \tag{A1.6}$$

and similar definitions for the minimizers $\check{\ell}_M(j\omega)$, $\check{\ell}_A(j\omega)$. Since the stationary points of $|\ell(j\omega)-1|^2 + \arg(\ell(j\omega)-1)$ are known, the above optimization merely requires that we check if a stationary point is the extremizer and if not, search the boundary of $[\underline{\ell}_M(\omega), \bar{\ell}_M(\omega)] \times [\underline{\ell}_A(\omega), \bar{\ell}_A(\omega)]$ which is a simple one dimensional problem solved by checking stationary points of the reduced gradient and the endpoints of the intervals. Then, defining

$$R_{(\ell-1)}(j\omega) \stackrel{\Delta}{=} \{(m,\phi) \;|m \in [\check{\ell}_M(j\omega), \hat{\ell}_M(j\omega)], \phi \in [\check{\phi}_A(j\omega), \hat{\phi}_A(j\omega)]\} \tag{A1.7}$$

we clearly have,

$$\{\ell(j\omega)-1 | \ell \in L\} \subset \{R_{(\ell-1)}(j\omega) | \omega \geq 0\}. \tag{A1.8}$$

Finally, note if the following condition holds,

$$\omega \in \Omega_\ell \stackrel{\Delta}{=} \{\omega \geq 0 | \bar{\ell}_M(\omega) > 1, \; \bar{\ell}_A(\omega) \geq \pi \text{ and } \; \underline{\ell}_A(\omega) \leq -\pi\} \tag{A1.9}$$

then $R_{(\ell-1)}(j\omega)$ must be a circle and so $\hat{\ell}_A(j\omega)$ and $\check{\ell}_A(j\omega)$ as defined by (A1.6) are infinite. Hence, without loss of generality, we define

$$\hat{\ell}_A(j\omega) = \pi \;\; \forall \; \omega \in \Omega_\ell \tag{A1.10a}$$

$$\check{\ell}_A(j\omega) = -\pi \;\; \forall \; \omega \in \Omega_\ell. \tag{A1.11a}$$

¤

Appendix 2: Extremizers of Open Loop Gain and Phase

Now

$$M_0(x,s,\alpha) \stackrel{\Delta}{=} |P_0(s,\alpha)C(x,s)| \tag{A2.1}$$

$$\Phi_0(x,s,\alpha) \stackrel{\Delta}{=} \arg\{P_0(s,\alpha)C(x,s)\} \tag{A2.2}$$

From Theorem 3.1, it follows that,

$$\max_{\alpha\in A} M_0(x,s,\alpha) = C(x,s)\,\frac{\bar{K}_p \prod_{i=1}^{k'_{pR}} \max_{z_R^i\in I_{zR}^i} |s+z_R^i| \prod_{i=1}^{k'_{pC}} \max_{z_C^i\in I_{zC}^i} |(s+z_C^i)(s+z_C^{i^*})|}{\prod_{i=1}^{K_{pR}} \min_{p_R^i\in I_{pR}^i} |s+p_R^i| \prod_{i=1}^{K_{pC}} \min_{p_C^i\in I_{pC}^i} |(s+p_C^i)(s+p_C^{i^*})|} \tag{A2.3}$$

and from Theorem 3.2 that,

$$\max_{\alpha\in A} \Phi_0(x,s,\alpha) = \arg C(x,s) + \sum_{i=1}^{k'_{pR}} \max_{z_R^i\in I_{zR}^i} \arg(s+z_R^i)$$

$$+ \sum_{i=1}^{k'_{pC}} \max_{z_C^i\in I_{zC}^i} \arg\,(s+z_C^i)(s+z_C^{i^*}) - \sum_{i=1}^{k_{pR}} \min_{p_R^i\in I_{pR}^i} \arg(s+p_R^i)$$

$$- \sum_{i=1}^{k_{pC}} \min_{p_C^i\in I_{pC}^i} \arg\{(s+p_C^i)(s+p_C^{i^*})\} \tag{A2.4}$$

with similar expressions for the minimizers. Observe that evaluation of the right hand sides of (A2.3) and (A2.4) requires that for each $s \in C$, we solve a one-dimensional problem for real poles and zeros and a two-dimensional problem for complex poles and zeros.

By considering real and complex perturbations separately, we develop two propositions which yield the solutions of the one and two dimensional optimization problems required for computing $\hat{M}_0(x,s)$, $\check{M}_0(x,s)$, $\hat{\Phi}_0(x,s)$ and $\check{\Phi}_0(x,s)$ defined in Theorem 3.4.

Real Perturbations

Clearly,

$$\bar{K}_p = \underset{K_p \in [\underline{K}_p, \bar{K}_p]}{\operatorname{argmax}} \log K_p \tag{A2.5a}$$

and

$$\underline{K}_p = \underset{K_p \in [\underline{K}_p, \bar{K}_p]}{\operatorname{argmin}} \log K_p \tag{A2.5b}$$

by monotonicity.

For real pole or zero perturbations, it will suffice to consider

$$\text{max or} \min_{z^i \in [\underline{z}^i, \bar{z}^i]} |j\omega + z^i| \tag{A2.6a}$$

$$\text{max or} \min_{z^i \in [\underline{z}^i, \bar{z}^i]} \arg(j\omega + z^i). \tag{A2.6b}$$

The geometric interpretations of (A2.6a) and (A2.6b) are shown in Fig. 9. Now

$$|j\omega + z^i| = \sqrt{\omega^2 + (z^i)^2}$$

$$= \sqrt{\omega^2 + |z^i|^2}$$

which is monotone in $|z^i|$. Hence,

$$\hat{z}^i_M(j\omega) \triangleq \underset{z^i\in[\underline{z}^i,\bar{z}^i]}{\operatorname{argmax}} |j\omega+z^i| \quad \forall\ \omega \geq 0$$

$$= \underset{z^i\in\{\underline{z}_i,\bar{z}^i\}}{\operatorname{argmax}} |z^i| \quad \forall\ \omega \geq 0 \tag{A2.7}$$

$$\check{z}^i_M(j\omega) \triangleq \underset{z^i\in[\underline{z}^i,\bar{z}^i]}{\operatorname{argmin}} |j\omega+z^i| \quad \forall\ \omega \geq 0.$$

$$= \begin{cases} 0 & \text{if} \quad 0 \in [\underline{z}^i,\bar{z}^i] \quad \forall\ \omega \geq 0. \\ \underset{z^i\in\{\underline{z}^i,\bar{z}^i\}}{\operatorname{argmin}} |\underline{z}^i| & \text{else} \end{cases} \tag{A2.8}$$

For phase, we have $\arg(j\omega+z^i) = \tan^{-1}(\frac{\omega}{z^i})$, so that

$$\hat{z}^i_A(j\omega) \triangleq \underset{z^i\in[\underline{z}^i,\bar{z}^i]}{\operatorname{argmax}} \{\arg(j\omega+z^i)\} \quad \forall\ \omega \geq 0$$

$$= \begin{cases} 0 & \text{if} \quad 0 \in [\underline{z}^i,\bar{z}^i] \quad \forall\ \omega \geq 0 \\ \underline{z}^i & \text{else} \end{cases} \tag{A2.9}$$

$$\check{z}^i_A(j\omega) \triangleq \underset{z^i\in[\underline{z}^i,\bar{z}^i]}{\operatorname{argmin}} \{\arg(j\omega+z^i)\} \quad \forall\ \omega \geq 0$$

$$= \bar{z}^i \quad \forall\ \omega \geq 0. \tag{A2.10}$$

As the only difference between the case of poles and zeros is a negative sign, we have that the same results with analogous notation hold for poles but with max and min reversed. We summarize the results for real perturbations in the following result:

<u>Theorem A2.1</u>. If $[\underline{z}^i,\bar{z}^i]$, $[\underline{p}^i,\bar{p}^i] \subset \mathbb{R}^1$, then $\forall\ \omega \geq 0$,

$$\hat{z}^i_M(j\omega) = \underset{z^i \in \{\underline{z}^i, \bar{z}^i\}}{\operatorname{argmax}} |z^i| \tag{A2.11a}$$

$$\check{z}^i_M(j\omega) = \begin{cases} 0 & \text{if} \quad 0 \in [\underline{z}^i, \bar{z}^i] \\ \underset{z^i \in \{\underline{z}^i, \bar{z}^i\}}{\operatorname{argmin}} |z^i| & \end{cases} \tag{A2.11b}$$

$$\hat{p}^i_M(j\omega) = \begin{cases} 0 & \text{if} \quad 0 \in [\underline{p}^i, \bar{p}^i] \\ \underset{p^i \in \{\underline{p}^i, \bar{p}^i\}}{\operatorname{argmin}} |p^i| & \end{cases} \tag{A2.11c}$$

$$\check{p}^i_M(j\omega) = \left\{ \underset{p^i \in \{\underline{p}^i, \bar{p}^i\}}{\operatorname{argmin}} |p^i| \right. \tag{A2.11d}$$

$$\hat{z}^i_A(j\omega) = \begin{cases} 0 & \text{if} \quad 0 \in [\underline{z}^i, \bar{z}^i] \\ \underline{z}^i & \text{else} \end{cases} \tag{A2.11e}$$

$$\check{z}^i_A(j\omega) = \bar{z}^i \tag{A2.11f}$$

$$\hat{p}^i_A(j\omega) = \bar{p}^i \tag{A2.11g}$$

$$\check{p}^i_A(j\omega) = \begin{cases} 0 & \text{if} \quad 0 \in [\underline{p}^i, \bar{p}^i] \\ \underline{p}^i & \text{else.} \end{cases} \tag{A2.11h}$$

⌑

Introduce the notation,

$$\hat{\alpha}_M(j\omega) \triangleq \underset{\alpha \in A}{\operatorname{argmax}}\, M_0(x, j\omega, \alpha)$$

$$= (\bar{K}_p, \hat{z}_M(j\omega), \hat{p}_M(j\omega)) \tag{A2.12a}$$

$$\hat{\alpha}_M(j\omega) \triangleq \underset{\alpha\in A}{\text{argmin}}\ M_0(x,j\omega,\alpha)$$

$$= (\underline{K}_p, \check{z}_M(j\omega), \check{p}_M(j\omega)) \quad \text{(A2.12b)}$$

$$\hat{\alpha}_A(j\omega) \triangleq \underset{\alpha\in A}{\text{argmax}}\ \Phi_0(x,j\omega,\alpha)$$

$$= (\hat{z}_A(j\omega), \hat{p}_A(j\omega)) \quad \text{(A2.12c)}$$

$$\check{\alpha}_A(j\omega) \triangleq \underset{\alpha\in A}{\text{argmin}}\ \Phi_0(x,j\omega,\alpha)$$

$$= (\check{z}_A(j\omega), \check{p}_A(j\omega)). \quad \text{(A2.12d)}$$

From Theorem A2.1, it follows that for real pole-zero variations, the extremizers of magnitude and phase are ω-irvariant. For the special case of real variations in the left half plane, $\mathbb{C}_-^0$, we have

$$\hat{\alpha}_M(j\omega) = (\bar{K}_p, \check{\alpha}_A(j\omega)) \qquad \forall\ \omega \geq 0 \quad \text{(A2.13a)}$$

$$\check{\alpha}_M(j\omega) = (\underline{K}_p, \hat{\alpha}_A(j\omega)) \qquad \forall\ \omega \geq 0. \quad \text{(A2.13b)}$$

<u>Complex Perturbations</u>

Consider a conjugate pair of complex zeros,

$$z = \beta + j\gamma, z^* \quad \text{for} \quad \beta \in [\underline{\beta},\bar{\beta}],\ \gamma \in [\underline{\gamma},\bar{\gamma}] \quad \text{(A2.14)}$$

and let

$$M_C(j\omega,\beta,\gamma) \triangleq \log|j\omega+z|^2 + \log|j\omega+z^*|^2$$

$$= \log\{(\beta^2+\gamma^2-\omega^2)^2+4\omega^2\beta^2\} \quad \text{(A2.15)}$$

$$\phi_C(j\omega,\beta,\gamma) \triangleq \arg(j\omega+z) + \arg(j\omega+z^*)$$

$$= \tan^{-1}\left(\frac{2\beta\omega}{\beta+\gamma^2-\omega^2}\right). \qquad \text{(A2.16)}$$

Further,

$$\frac{\partial M_C}{\partial \beta}(j\omega,\beta,\gamma) = \frac{4\beta(\beta^2+\gamma^2-\omega^2)}{(\beta^2+\gamma^2-\omega^2)^2+4\omega^2\beta^2} \qquad \text{(A2.17a)}$$

$$\frac{\partial M_C}{\partial \gamma}(j\omega,\beta,\gamma) = \frac{4\gamma(\beta^2+\gamma^2-\omega^2)}{(\beta^2+\gamma^2-\omega^2)^2+4\omega^2\beta^2} \qquad \text{(A2.17b)}$$

so that

$$\frac{\partial M_C}{\partial \beta}(j\omega,\beta,\gamma) = 0 \quad \text{iff} \quad \beta = 0 \qquad \text{(A2.18a)}$$

$$\frac{\partial M_C}{\partial \gamma}(j\omega,\beta,\gamma) = 0 \quad \text{iff} \quad \gamma = 0 \text{ or } \omega^2 = \beta^2 + \gamma^2 \qquad \text{(A2.18b)}$$

$$\frac{\partial \phi_C}{\partial \beta}(j\omega,\beta,\gamma) = \frac{-2\omega[\beta^2+(\omega^2-\gamma^2)]}{(\beta^2+\gamma^2-\omega^2)^2+4\omega^2\beta^2} \qquad \text{(A2.19a)}$$

$$\frac{\partial \phi_C}{\partial \gamma}(j\omega,\beta,\gamma) = \frac{-4\omega\beta\gamma}{(\beta^2+\gamma^2-\omega^2)^2+4\omega^2\beta^2} \qquad \text{(A2.19b)}$$

Hence,

$$\frac{\partial \phi_C}{\partial \beta}(j\omega,\beta,\gamma) = 0 \quad \text{iff} \quad \omega = 0 \text{ or } \omega^2 = \gamma^2 - \beta^2 \qquad \text{(A2.20a)}$$

$$\frac{\partial \phi_C}{\partial \gamma}(j\omega,\beta,\gamma) = 0 \quad \text{iff} \quad \omega \text{ or } \beta \text{ or } \gamma = 0 \qquad \text{(A2.20b)}$$

By using (A2.18a,b) and (A2.20a,b), we determine if an extremizer is on the boundary or interior to the confidence interval.

Consider the magnitude function, $M_C(\omega,\beta,\gamma)$ (see Fig. 9b). Now,

$$M_C(j\omega,\beta,\gamma) = \log\{\beta^4+2\beta^2(\omega^2+\gamma^2) + \gamma^4-2\gamma^2\omega^2+\omega^4\} \tag{A2.21}$$

and so for extremization with respect to $\beta \in [\underline{\beta},\bar{\beta}]$ it suffices to consider, $\{|\beta|^4+2|\beta|^2(\omega^2+\gamma^2)\}$, which is monotone in $|\beta|$. Letting

$$\hat{\beta}_M(j\omega) \triangleq \underset{\beta\in[\underline{\beta},\bar{\beta}]}{\text{argmax}}\ M_C(j\omega,\beta,\gamma) \tag{A2.22a}$$

$$\hat{\gamma}_M(j\omega) \triangleq \underset{\gamma\in[\underline{\gamma},\bar{\gamma}]}{\text{argmax}}\ M_C(j\omega,\beta,\gamma) \tag{A2.22b}$$

$$\check{\beta}_M(j\omega) \triangleq \underset{\beta\in[\underline{\beta},\bar{\beta}]}{\text{argmin}}\ M_C(j\omega,\beta,\gamma) \tag{A2.22c}$$

$$\check{\gamma}_M(j\omega) \triangleq \underset{\gamma\in[\underline{\gamma},\bar{\gamma}]}{\text{argmin}}\ M_C(j\omega,\beta,\gamma) \tag{A2.22d}$$

it is clear that,

$$\hat{\beta}_M(j\omega) = \underset{\beta\in\{\underline{\beta},\bar{\beta}\}}{\text{argmax}}\ |\beta|$$

$$\check{\beta}_M(j\omega) = \begin{cases} 0 & \text{if} \quad 0 \in [\underline{\beta},\bar{\beta}] \\ \min\{|\underline{\beta}|,|\bar{\beta}|\}, & \text{else.} \end{cases}$$

To extremize with respect to γ, consider $\gamma^4 + 2\gamma^2(\beta^2-\omega^2)$ which represents a pair of parabolas in γ^2, one corresponding to $(\beta^2-\omega^2) \geq 0$, the other to $(\beta^2-\omega^2) \leq 0$, as shown in Fig. 10.

(i) For $(\beta^2-\omega^2) \geq 0$, $\{\gamma^4+2\gamma^2(\beta^2-\omega^2)\}$ is monotone in $\gamma^2 > 0$ so

$$\hat{\gamma}_M(j\omega) = \underset{\gamma\in\{\underline{\gamma},\bar{\gamma}\}}{\operatorname{argmax}} |\gamma|$$

$$\check{\gamma}_M(j\omega) = \underset{\gamma\in\{\underline{\gamma},\bar{\gamma}\}}{\operatorname{argmin}} |\gamma|$$

(ii) For $(\beta^2-\omega^2) < 0$, we refer to the appropriate parabola in Fig. 10 and consider separately the cases

(a) $0 \le \underline{\gamma} < \bar{\gamma}$

(b) $\underline{\gamma} < 0 \le \bar{\gamma}$

(c) $\underline{\gamma} < \bar{\gamma} \le 0$

(a) In this case, we have $0 \le \underline{\gamma}^2 < \bar{\gamma}^2$ and so $\frac{1}{2}(\underline{\gamma}^2+\bar{\gamma}^2)$ is the midpoint of $[\underline{\gamma}^2,\bar{\gamma}^2]$.

The parabola, $\gamma^4 + 2\gamma^2(\beta^2-\omega^2)$, is symmetric about $\gamma^2 = -(\beta^2-\omega^2)$ at which point it achieves its minimum and so to determine the maximizers, it suffices to consider if the midpoint of $[\underline{\gamma}^2,\bar{\gamma}^2]$ is to the left or right of $-(\beta^2-\omega^2)$. So,

$$\tfrac{1}{2}(\underline{\gamma}^2+\bar{\gamma}^2) \le -(\beta^2-\omega^2) \Rightarrow \hat{\gamma}_M(\omega) = \underline{\gamma} \qquad \forall\ \omega^2 \ge \beta^2 + \tfrac{1}{2}(\underline{\gamma}^2+\bar{\gamma}^2)$$

$$\tfrac{1}{2}(\underline{\gamma}^2+\bar{\gamma}^2) > -(\beta^2-\omega^2) \Rightarrow \hat{\gamma}_M(\omega) = \bar{\gamma} \qquad \forall\ \omega^2 < \beta^2 + \tfrac{1}{2}(\underline{\gamma}^2+\bar{\gamma}^2)$$

For minimizers, note that from (A2.17b) we have that $\omega^2 = \beta^2 + \gamma^2$, $\gamma \in [\underline{\gamma},\bar{\gamma}]$, is a local minimum of $M_C(j\omega,\beta,\gamma)$. Hence,

$$\bar{\gamma}^2 < \omega^2 - \beta^2 \qquad \Rightarrow \check{\gamma}_M(j\omega) = \bar{\gamma} \qquad \forall\ \omega^2 > \bar{\gamma}^2 + \beta^2$$

$$\underline{\gamma}^2 \le \omega^2 - \beta^2 \le \bar{\gamma}^2 \Rightarrow \check{\gamma}_M(j\omega) = \gamma \qquad \text{for } \omega^2 = \beta^2 + \gamma^2$$

$$\underline{\gamma}^2 > \omega^2 - \beta^2 \qquad \Rightarrow \check{\gamma}_M(j\omega) = \underline{\gamma} \qquad \forall\ \omega^2 < \beta^2 + \underline{\gamma}^2$$

(b) Now consider the interval $[0,\max\{\underline{\gamma}^2,\bar{\gamma}^2\}]$ which has midpoint $\frac{1}{2}\max\{\underline{\gamma}^2,\bar{\gamma}^2\}$. To determine the maximizers and minimizers we proceed as in case (a).

$$\frac{1}{2}\max\{\underline{\gamma}^2,\bar{\gamma}^2\} < -(\beta^2-\omega^2) \quad \Rightarrow \hat{\gamma}_M(j\omega) = 0$$

$$\forall\ \omega^2 > \beta^2 + \frac{1}{2}\max\{\underline{\gamma}^2,\bar{\gamma}^2\}$$

$$\frac{1}{2}\max\{\underline{\gamma}^2,\bar{\gamma}^2\} \geq -(\beta^2-\omega^2) \quad \Rightarrow \hat{\gamma}_M(j\omega) = \underset{\gamma\in\{\underline{\gamma},\bar{\gamma}\}}{\operatorname{argmax}} |\gamma|$$

$$\forall\ \omega^2 \leq \beta^2 + \frac{1}{2}\max\{\underline{\gamma}^2,\bar{\gamma}^2\}$$

$$\max\{\underline{\gamma}^2,\bar{\gamma}^2\} < \omega^2 - \beta^2 \quad \Rightarrow \check{\gamma}_M(j\omega) = \underset{\gamma\in\{\underline{\gamma},\bar{\gamma}\}}{\operatorname{argmax}} |\gamma|$$

$$\forall\ \omega^2 > \beta^2 + \frac{1}{2}\max\{\underline{\gamma}^2,\bar{\gamma}^2\}$$

$$\min\{\underline{\gamma}^2,\bar{\gamma}^2\} \leq \omega^2 - \beta^2 \leq \max\{\underline{\gamma}^2,\bar{\gamma}^2\} \Rightarrow \check{\gamma}_M(j\omega) = \gamma \text{ for } \omega^2 = \beta^2 + \gamma^2,$$

$$\gamma \in [\underline{\gamma},\bar{\gamma}]$$

$$\min\{\underline{\gamma}^2,\bar{\gamma}^2\} > \omega^2 - \beta^2 \quad \Rightarrow \check{\gamma}_M(j\omega) = \underset{\gamma\in\{\underline{\gamma},\bar{\gamma}\}}{\operatorname{argmin}} |\gamma|$$

$$\forall\ \omega^2 < \beta^2 + \frac{1}{2}\min\{\underline{\gamma}^2,\bar{\gamma}^2\}$$

(c) Since $M_C(j\omega,\beta,\gamma)$ is an even function of γ, we obtain the results for $\underline{\gamma} < \bar{\gamma} \leq 0$ by substituting $\bar{\gamma}$ for $\underline{\gamma}$ and $\underline{\gamma}$ for $\bar{\gamma}$ in the results obtained in (a).

To determine the extremizers of phase, the discussion will be simplified by making two observations. Firstly, we observe that $\tan^{-1}(\cdot)$

is a monotone function on any interval,

$$I_k \triangleq [(2k-1)\pi/2,(2k+1)\pi/2],\ k \in \mathbb{Z} \tag{A2.23}$$

and assuming we may account for transitions between intervals, we need only consider extremizers of $(\frac{2\beta\omega}{\beta^2+\gamma^2-\omega^2})$. Secondly, observe

$$\phi_C(j\omega,\beta_+,\gamma) \geq \phi_C(j\omega,\beta_-,\gamma) \qquad \forall\ \beta_+ \geq 0 \geq \beta_-, \qquad \forall\ \omega \geq 0.$$

Hence, we may divide any interval $B \triangleq [\underline{\beta},\bar{\beta}]$ into $B_+ \triangleq [\max\{0,\underline{\beta}\},\bar{\beta}]$ and $\underline{B} \triangleq [\underline{\beta},\min 0,\bar{\beta}\}]$ and so to maximize $\phi_C(j\omega,\beta,\gamma)$ over B, it suffices to consider B_+ whilst for minimization we consider only B_-.

Graphs of $(\frac{2\beta\omega}{\beta^2+\gamma^2-\omega^2})$ as functions of β and γ are given in Fig. 11 and it is seen that the extremizers are frequency dependent. Suppose, initially, that $\frac{\partial\phi_C}{\partial\beta}(j\omega,\beta,\gamma) \neq 0$ and the extremizing $\beta \in [\underline{\beta},\bar{\beta}]$ is sought. This can be obtained from the graphs in Fig. 11 by considering the possible orderings, $\{0\leq\underline{\beta}<\bar{\beta},\underline{\beta}<0\leq\bar{\beta},\underline{\beta}<\bar{\beta}\leq 0\}$, and the location of the interval $[\underline{\beta},\bar{\beta}]$ on the graphs to decide whether $\underline{\beta}$ or $\bar{\beta}$ is the extremizer.

Referring to (A2.20a,b), we have those values of $\omega \geq 0$, $\beta \in [\underline{\beta},\bar{\beta}]$ and $\gamma \in [\underline{\gamma},\bar{\gamma}]$ at which stationary points of $\phi_C(j\omega,\beta,\gamma)$ may occur and so we need ammend the previous procedure to account for the possibility of local maxima and minima. The results are summarized by the Tables A2.2. Observe that as $\gamma^2 \to \omega^2 - \beta^2$ or $\beta^2 \to \omega^2 - \gamma^2$, the tangent function, $\frac{2\beta\omega}{\beta^2+\gamma^2-\omega^2} \to \pm\infty$ so that the inverse tangent function, $\phi_C(j\omega,\beta,\gamma)$ moves from the interval I_0 to $I_{\pm 1}$. Hence we may use the monotonicity of $\tan^{-1}(\cdot)$ over the I_k and simply determine when the tangent, $\frac{2\beta\omega}{\beta^2+\gamma^2-\omega^2}$, goes outside of the range of I_0. We summarize the results for complex perturbations in the following theorem.

Theorem A2.2. If $z^i = \beta^i + j\gamma^i$, $\beta^i \in [\underline{\beta}^i, \bar{\beta}^i]$, $\gamma^i \in [\underline{\gamma}^i, \bar{\gamma}^i]$, then $\hat{\beta}^i_M(j\omega)$, $\check{\beta}^i_M(j\omega)$, $\hat{\gamma}^i_M(j\omega)$ and $\check{\gamma}^i_M(j\omega)$ are given by Table A2.1 whilst $\hat{\beta}^i_A(j\omega)$, $\check{\beta}^i_A(j\omega)$, $\hat{\gamma}^i_A(j\omega)$ and $\check{\gamma}^i_A(j\omega)$ are given by Table A2.2, $\forall\ \omega \geq 0$. For $p^i = \beta^i + j\gamma^i$, a complex pole occurring in conjugate pairs, the extremizers of magnitude and phase are obtained from those for z^i by reversing the role of the maximizers and minizers. ¤

For a pair of complex zeros, $z = \beta + j\gamma$, z^* with $\underline{\beta}, \underline{\gamma} \geq 0$, $\hat{\beta}_M(j\omega)$, $\check{\beta}_M(j\omega)$, $\hat{\gamma}_M(j\omega)$, $\check{\gamma}_M(j\omega)$ are illustrated in Fig. 8.

Remark

The results of Theorems A2.1 and A2.2 may be used to determine the solutions to

$$\max_{\alpha \in A} \text{(or min)} \, |P_0(\sigma + j\omega, \alpha)|$$

and

$$\max_{\alpha \in A} \text{(or min)} \arg P_0(\sigma + j\omega, \alpha)$$

for $\sigma \neq 0$. If z^i is a real zero varying in $[\underline{z}^i, \bar{z}^i]$ then,

$$\max_{z^i \in [\underline{z}^i, \bar{z}^i]} |\sigma + j\omega + z^i| = \max_{z^i_\sigma \in [\underline{z}^i_\sigma, \bar{z}^i_\sigma]} |j\omega + z^i_\sigma| \tag{A2.24}$$

$$\max_{z^i \in [\underline{z}^i, \bar{z}^i]} \arg(\sigma + j\omega + z^i) = \max_{z^i \in [\underline{z}^i_\sigma, \bar{z}^i_\sigma]} \arg(j\omega + z^i_\sigma) \tag{A2.25}$$

and similarly for the respective minimizers, where

$$z^i_\sigma = z^i + \sigma. \tag{A2.26}$$

Since the extremization problems on the right hand sides of (A2.24) and

(A2.25) are evaluated along the $j\omega$-axis, our previous theorems apply with the new confidence interval, $[\underline{z}_\sigma^i, \bar{z}_\sigma^i]$. Analogously, for complex perturbations, say

$$z^i = \beta^i + j\gamma^i \tag{A2.27}$$

with

$$\beta^i \in [\underline{\beta}^i, \bar{\beta}^i],\ \gamma^i \in [\underline{\gamma}^i, \bar{\gamma}^i],$$

we put,

$$\beta_\sigma^i = \beta^i + \sigma \tag{A2.28}$$

and consider the new constraints,

$$(\beta_\sigma^i, \gamma^i) \in [\underline{\beta}_\sigma^i, \bar{\beta}_\sigma^i] \times [\underline{\gamma}^i, \bar{\gamma}^i].$$

Table A2.1

Maximizers and Minimizers of Magnitude for Complex Perturbations

$$\hat{\beta}^i_M(j\omega) = \underset{\beta^i \in \{\underline{\beta}^i, \bar{\beta}^i\}}{\text{argmax}} |\beta^i|$$

$$\check{\beta}^i_M(j\omega) = \begin{cases} 0 & \text{if} \quad 0 \quad [\underline{\beta}^i, \bar{\beta}^i] \\ \underset{\beta^i \in \{\underline{\beta}^i, \bar{\beta}^i\}}{\text{argmin}} |\beta^i| & \end{cases}$$

Condition for Applicability	frequency, ω	$\hat{\gamma}^i_M(j\omega)$	frequency, ω	$\check{\gamma}^i_M(j\omega)$
(a) $(\beta^i)^2-\omega^2 \geq 0$	$\beta^i \geq \omega \geq 0$	$\underset{\gamma^i \in \{\underline{\gamma}^i, \bar{\gamma}^i\}}{\text{argmax}} \lvert\gamma^i\rvert$	$\beta^i \geq \omega \geq 0$	$\underset{\gamma^i \in \{\underline{\gamma}, \bar{\gamma}^i\}}{\text{argmin}} \lvert\gamma^i\rvert$
(b) $(\beta^i)^2-\omega^2 < 0$	$\omega < \sqrt{(\beta^i)^2 + \frac{1}{2}[(\underline{\gamma}^i)^2 + (\bar{\gamma}^i)^2]}$	$\underset{\gamma^i \in \{\underline{\gamma}, \bar{\gamma}^i\}}{\text{argmax}} \lvert\gamma^i\rvert$	$\omega < \sqrt{(\beta^i)^2 + \min\{(\underline{\gamma}^i)^2, (\bar{\gamma}^i)^2\}}$	$\underset{\gamma^i \in \{\underline{\gamma}, \bar{\gamma}^i\}}{\text{argmin}} \lvert\gamma^i\rvert$
$0 \leq \underline{\gamma}^i < \bar{\gamma}^i$	$\omega \geq \sqrt{(\beta^i)^2 + \frac{1}{2}[(\underline{\gamma}^i)^2 + (\bar{\gamma}^i)^2}$	$\underset{\gamma^i \in \{\underline{\gamma}^i, \bar{\gamma}^i\}}{\text{argmin}} \lvert\gamma^i\rvert$	$\omega = \sqrt{(\beta^i)^2 + (\gamma^i)^2}$	γ^i
$\underline{\gamma}^i < \bar{\gamma}^i \leq 0$	"	"	$\omega > \sqrt{(\beta^i)^2 + \max\{(\underline{\gamma}^i), (\bar{\gamma}^i)^2\}}$	$\underset{\gamma^i \in \{\underline{\gamma}^i, \bar{\gamma}^i\}}{\text{argmax}} \lvert\gamma^i\rvert$
(c) $(\beta^i)^2-\omega^2 < 0$	$\omega < \sqrt{(\beta^i)^2 + \frac{1}{2}\max\{(\underline{\gamma}^i)^2, (\bar{\gamma}^i)^2\}}$	0	$\omega < \sqrt{(\beta^i)^2 + \frac{1}{2}\min\{(\underline{\gamma}^i)^2, (\bar{\gamma}^i)^2\}}$	$\underset{\gamma^i \in \{\underline{\gamma}, \bar{\gamma}^i\}}{\text{argmin}} \lvert\gamma^i\rvert$
$\underline{\gamma}^i < 0 \leq \bar{\gamma}^i$	$\omega > \sqrt{(\beta^i)^2 + \frac{1}{2}\max\{(\underline{\gamma}^i)^2, (\bar{\gamma}^i)^2\}}$	$\underset{\gamma^i \in \{\underline{\gamma}^i, \bar{\gamma}^i\}}{\text{argmax}} \lvert\gamma^i\rvert$	$\omega = \sqrt{(\beta^i)^2 + (\gamma^i)^2}$	γ^i
			$\omega > \sqrt{(\beta^i)^2 + \frac{1}{2}\max\{(\underline{\gamma}^i)^2, (\bar{\gamma}^i)^2\}}$	$\underset{\gamma^i \in \{\underline{\gamma}, \bar{\gamma}\}}{\text{argmax}} \lvert\gamma^i\rvert$

Table A2.2

Maximizers and Minimizers of Phase for Complex Perturbations

$\underline{\beta} \geq 0$			$\underline{\gamma} \geq 0$		
frequency, ω	$\hat{\beta}_A(j\omega)$	$\check{\beta}_A(j\omega)$	frequency, ω	$\hat{\gamma}_A(j\omega)$	$\check{\gamma}_A(j\omega)$
$0 \leq \omega^2 \leq \gamma^2 - \bar{\beta}^2$	$\bar{\beta}$	$\underline{\beta}$	$\omega \geq 0$	$\underline{\gamma}$	$\bar{\gamma}$
$\gamma^2 - \bar{\beta}^2 \leq \omega^2 \leq \gamma^2 - \underline{\beta}\bar{\beta}$	$(\gamma^2-\omega^2)^{1/2}$	$\underline{\beta}$			
$\gamma^2 - \underline{\beta}\bar{\beta} \leq \omega^2 \leq \gamma^2 - \underline{\beta}^2$	$(\gamma^2-\omega^2)^{1/2}$	$\bar{\beta}$			
$\gamma^2 - \underline{\beta}^2 \leq \omega^2 \leq \gamma^2$	$\underline{\beta}$	$\bar{\beta}$			
$\gamma^2 \leq \omega^2 \leq \underline{\beta}^2 + \gamma^2$	$\underline{\beta}$	$\bar{\beta}$			
$\omega^2 \geq \underline{\beta}^2 + \gamma^2$	$\underline{\beta}$	$\bar{\beta}$			

$\bar{\beta} \geq 0$			$\underline{\gamma} \leq 0$		
frequency, ω	$\hat{\beta}_A(j\omega)$	$\check{\beta}_A(j\omega)$	frequency, ω	$\hat{\gamma}_A(j\omega)$	$\check{\gamma}_A(j\omega)$
$0 \leq \omega^2 \leq \gamma^2 - \bar{\beta}^2$	$\underline{\beta}$	$\bar{\beta}$	$\omega \geq 0$	$\bar{\gamma}$	$\underline{\gamma}$
$\gamma^2 - \bar{\beta}^2 \leq \omega^2 \leq \gamma^2 - \underline{\beta}\bar{\beta}$	$(\gamma^2-\omega^2)^{1/2}$	$\bar{\beta}$			
$\gamma^2 - \underline{\beta}\bar{\beta} \leq \omega^2 \leq \gamma^2 - \underline{\beta}^2$	$(\gamma^2-\omega^2)^{1/2}$	$\underline{\beta}$			
$\gamma^2 - \underline{\beta}^2 \leq \omega^2 \leq \gamma^2$	$\bar{\beta}$	$\underline{\beta}$			
$\gamma^2 \leq \omega^2 \leq \underline{\beta}^2 + \gamma^2$	$\bar{\beta}$	$\underline{\beta}$			
$\omega^2 \geq \underline{\beta}^2 + \gamma^2$	$\bar{\beta}$	$\underline{\beta}$			

Appendix 3: Extremizers of Closed Loop Gain and Phase

Suppose we wish to extremize a C^1 function, $g: \mathbb{R}^2 \to \mathbb{R}$ over the rectangle, $R = \{(m,\phi)\ m \in [\underline{m},\bar{m}], \phi \in [\underline{\phi},\bar{\phi}]\}$, and the stationary points of $g(\cdot,\cdot)$ are known. Along each of the line segments constituting the boundary of R, ∂R, g is a function of a single variable. Optimization over ∂R then reduces to comparing the values of the corners of the rectangles with those of any stationary points of the reduced gradient evaluated along each line segment. Hence, optimization over R merely requires comparing the extremum over ∂R with the values of any stationary points in R^0.

For the special case when g is either the magnitude or phase of one of the closed loop transfer functions, H_{ab}, ($a = y$ or v, $b = r$, u or d) we show that there are no non-trivial† stationary points. Observe, firstly, that the compensators $C(x,s)$ and $F(x,s)$ play no role in optimization over $A \times L$ and so it will suffice to consider just the transfer functions, $H_{yu}(x,j\omega,\alpha,\ell(j\omega))$ and $H_{yd}(x,j\omega,\alpha,\ell(j\omega))$. Put

$$P(j\omega,\alpha,\ell(j\omega)) \triangleq m_p(j\omega,\alpha,\ell(j\omega))e^{j\phi_p(j\omega,\alpha,\ell(j\omega))} \tag{A3.1}$$

and since

$$H_{yu}(x,j\omega,\alpha,\ell(j\omega)) = P(j\omega,\alpha,\ell(j\omega))C(x,j\omega)[1+P(j\omega,\alpha,\ell(j\omega))C(x,j\omega)]^{-1} \tag{A3.2}$$

$$H_{yd}(x,j\omega,\alpha,\ell(j\omega)) = [1+P(j\omega,\alpha,\ell(j\omega))]^{-1} \tag{A3.3}$$

it follows that (suppresing the arguments on the right hand sides for brevity)

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†We consider the stationary points of constant functions as trivial.

$$H_{yu}(x,j\omega,\alpha,\ell(j\omega)) = \frac{(m_p m_C + e^{j(\phi_p+\phi_C)})m_p m_C}{1+2m_p m_C \cos(\phi_p+\phi_C)+m_p^2 m_C^2} \quad \text{(A3.4)}$$

$$H_{yd}(x,j\omega,\alpha,\ell(j\omega)) = \frac{1+m_p m_C e^{-j(\phi_p+\phi_C)}}{1+2m_p m_C \cos(\phi_p+\phi_C)+m_p^2 m_C^2} \quad \text{(A3.5)}$$

and that† (again suppressing the arguments),

$$M_{yu}(m_p,\phi_p) = [1+\frac{2}{m_p m_C}\cos(\phi_p+\phi_C)+\frac{1}{m_p^2 m_C^2}]^{-1/2} \quad \text{(A3.6)}$$

$$\phi_{yu}(m_p,\phi_p) = \tan^{-1}[\frac{\sin(\phi_p+\phi_C)}{m_p m_C+\cos(\phi_p+\phi_C)}] \quad \text{(A3.7)}$$

$$M_{yd}(m_p,\phi_p) = [1+2m_p m_C \cos(\phi_p+\phi_C)+m_p^2 m_C^2]^{-1/2} \quad \text{(A3.8)}$$

$$\phi_{yd}(m_p,\phi_p) = \tan^{-1}[\frac{m_p m_C \sin(\phi_p+\phi_C)}{1+m_p m_C \cos(\phi_p+\phi_C)}]. \quad \text{(A3.9)}$$

Let $\tilde{\mathbb{R}}^2 \triangleq \mathbb{R}^2|_{(1/m_C,(2n+1)\pi-\phi_C)}$, $n \in \mathbb{Z}$. The following proposition establishes that $M_{yu}(\cdot,\cdot)$ and $\phi_{yu}(\cdot,\cdot)$ have no stationary points on $\tilde{\mathbb{R}}^2$.

Proposition A3.1: (i) ∇M_{yu}, $\nabla\phi_{yu}$ exist everywhere on $\tilde{\mathbb{R}}^2$ and fail to exist at $(m_p,\phi_p) = (\frac{1}{m_C},(2n+1)\pi)-\phi_C)$, $n \in \mathbb{Z}$.
(ii) $\nabla M_{yu} \neq 0$ except at $j\omega$-axis zeros of $C(x,j\omega)$ for which $\nabla M_{yu} \equiv 0$.
(iii) $\nabla\phi_{yu} \neq 0$.

†We assume, for simplicity, that $m_p m_C + \cos(\phi_p+\phi_C) \geq 0$ for all $(m_p,\phi_p) \in R_p(x,j\omega)$ so that $\phi_{yu}, \phi_{yd} \in [-\pi/2,\pi/2]$. If this condition is not satisfied, additional conditions will need to be imposed to determine the correct quadrant for the angles.

<u>Proof</u>. (i) Now $M_{yu}(m_p,\phi_p) = \dfrac{m_p m_C}{(m_p^2 m_C^2 + 2m_p m_C \cos(\phi_p+\phi_C)+1)^{1/2}}$ is a quotient of C^1 functions $\forall\ (m_p,\phi_p) \in \tilde{\mathbb{R}}^2$ so the partials,

$$\frac{\partial M_{yu}}{\partial m_p}(m_p,\phi_p) = \frac{m_p m_C^2 \cos(\phi_p+\phi_C)+m_C}{(m_p^2 m_C^2+2m_p m_C \cos(\phi_p+\phi_C)+1)^{3/2}}$$

$$\frac{M_{yu}}{\partial \phi_p}(m_p,\phi_p) = \frac{m_p^2 m_C^2 \sin(\phi_p+\phi_C)}{(m_p^2 m_C^2+2m_p m_C \cos(\phi_p+\phi_C)+1)^{3/2}}$$

exist and are continuous on $\tilde{\mathbb{R}}^2$. It follows that

$$\nabla M_{yu}(m_p,\phi_p) = \left(\frac{\partial M_{yu}}{\partial m_p}(m_p,\phi_p), \frac{\partial M_{yu}}{\partial \phi_p}(m_p,\phi_p)\right)^T$$

exists on $\tilde{\mathbb{R}}^2$. However,

$$\lim_{m_p \to 1/m_C} \nabla M_{yu}(m_p,(2k+1)\pi) = \begin{pmatrix} -\infty \\ 0 \end{pmatrix}$$

whilst,

$$\lim_{\phi_p \to \{(2k+1)\pi - \phi_C\}} \nabla M_{yu}(1,\phi_p) = \begin{pmatrix} \infty \\ \infty \end{pmatrix}$$

by application of L'Hopital's Rule, so that the gradient fails to exist at $(1/m_C,(2k+1)\pi-\phi_C)$, $n \in \mathbb{Z}$. Similar analysis establishes the result for $\nabla\phi_{yu}$.

(ii) From the expressions for the partials, it is clear that if $m_C = 0$ then $M_{yu} \equiv 0$ and so $\nabla M_{yu} \equiv 0$. Now suppose $\nabla M_{yu} = 0$ and $m_C > 0$. From $\frac{\partial M_{yu}}{\partial m_p} = 0$, we require that $\cos(\phi_p+\phi_C) = -\frac{1}{m_p m_C}$ so the condition on the other partial becomes,

$$\frac{\partial M_{yu}}{\partial \phi_p} = m_p m_C (m_p^2 m_C^2 - 1)^{1/2} = 0$$

Hence the only solution requires $m_p m_C = 1$, $\cos(\phi_p+\phi_C) = -1$. But then $(m_p,\phi_p) \notin \tilde{\mathbb{R}}^2$ so we conclude $\nabla M_{yu} \neq 0$ except if $m_C = 0$.

(iii) Now

$$\nabla\phi_{yu}(m_p,\phi_p) = \frac{1}{m_p^2 m_C^2+2m_p m_C \cos(\phi_p+\phi_C)+1}\begin{pmatrix}-m_C \sin(\phi_p+\phi_C) \\ m_p m_C \cos(\phi_p+\phi_C)+1\end{pmatrix}$$

and so for this gradient to vanish,

$$\sin(\phi_p+\phi_C) = 0$$

and

$$\cos(\phi_p+\phi_C) = -1/m_p m_C.$$

This again implies $(m_p,\phi_p) \notin \tilde{\mathbb{R}}^2$. ¤

Because of its similarity to Proposition A3.1, the proof of the next result is omitted.

<u>Proposition A3.2</u>: (i) ∇M_{yd}, $\nabla\phi_{yd}$ exist everywhere on $\tilde{\mathbb{R}}^2$ and fail to exist at $(m_p,\phi_p) = (\frac{1}{m_C}, (2n+1)\pi-\phi_C)$, $n \in \mathbb{Z}$.

(ii) $\nabla M_{yd} \neq 0$ except at $j\omega$-axis zeros of $C(x,j\omega)$ for which $\nabla M_{yd} \equiv 0$.

(iii) $\nabla\phi_{yd} \neq 0$ except at $j\omega$-axis zeros of $C(x,j\omega)$ for which $\nabla\phi_{yd} \equiv 0$. ¤

Because of the possibility of $j\omega$-axis poles of $P_0(j\omega,\alpha)$, we need to consider the case of m_p infinite so that the rectangle, $\mathcal{R}$, over which we extremize may be unbounded. In particular, let

$$\mathcal{R}_\infty \stackrel{\Delta}{=} \{(m,\phi)\,|\,m\in[\underline{m},\infty],\phi\in[\underline{\phi},\bar{\phi}]\}$$

where $\underline{m} \geq 0$. The following results indicate that we may reduce certain optimization problems over $\mathcal{R}_\infty$ to ones over bounded rectangles.

Proposition A3.3: There exists a finite $\bar{m} > 0$ such that

$$\sup_{(m_p,\phi_p)\in\mathcal{R}_\infty} M_{yu}(m_p,\phi_p) = \max\{1, \max_{(m_p,\phi_p)\in\mathcal{R}} M_{yu}(m_p,\phi_p)\}$$

where $\mathcal{R} = [\underline{m},\bar{m}] \times [\underline{\phi},\bar{\phi}]$.

Proof: By Proposition A3.1, we have that $\nabla M_{yu} \neq 0$ for all $(m_p,\phi_p) \in \mathcal{R}_\infty^0$. It then follows that,

$$\sup_{(m_p,\phi_p)\in\mathcal{R}_\infty} M_{yu}(m_p,\phi_p) = \max\{ \max_{\phi_p\in[\underline{\phi},\bar{\phi}]} M_{yu}(\underline{m},\phi), \sup_{m_p\in[\underline{m},\infty]} M_{yu}(m_p,\underline{\phi}),$$

$$\sup_{m_p\in[\bar{m},\infty]} M_{yu}(m_p,\bar{\phi}), \max_{\phi_p\in[\underline{\phi},\bar{\phi}]} M_{yu}(\infty,\phi)\}.$$

Since $M_{yu}(\infty,\cdot) \equiv 1$, $\max_{\phi_p\in[\underline{\phi},\bar{\phi}]} M_{yu}(\infty,\phi) = 1$.

Now $\sup_{m_p\in[\underline{m},\infty]} M_{yu}(m_p,\underline{\phi}) = \max\{M_{yu}(\underline{m},\underline{\phi}),M_{yu}(\infty,\underline{\phi}),M_{yu}(m_{\underline{\phi}},\underline{\phi})\}$

where $m_{\underline{\phi}}$ is given by

$$m_{\underline{\phi}} \triangleq \begin{cases} \dfrac{-1}{m_C\cos(\underline{\phi}+\phi_C)} & \text{if } \underline{m} \leq \dfrac{-1}{m_C\cos(\underline{\phi}+\phi_C)} < \infty \\ \underline{m} & \text{else} \end{cases}$$

i.e., $m_{\underline{\phi}}$ is a stationary point of $M_{yu}(\cdot,\underline{\phi})$ if this is finite and $m_{\underline{\phi}} = \underline{m}$, else.

Hence,

$$\sup_{m_p\in[\underline{m},\infty]} M_{yu}(m_p,\underline{\phi}) = \max\{M_{yu}(\underline{m},\underline{\phi}),1,M_{yu}(m_{\underline{\phi}},\underline{\phi})\}.$$

Similarly, we may show

$$\sup_{m_p \in [\underline{m},\infty]} M_{yu}(m_p,\bar{\phi}) = \max\{M_{yu}(\bar{m},\bar{\phi}),1,M_{yu}(m_{\bar{\phi}},\bar{\phi})\}$$

where

$$m_{\bar{\phi}} \overset{\Delta}{=} \begin{cases} \dfrac{-1}{m_C\cos(\bar{\phi}+\phi_C)} & \text{if} \quad \underline{m} \leq \dfrac{-1}{m_C\cos(\bar{\phi}+\phi_C)} < \infty \\ \underline{m} & \text{else.} \end{cases}$$

Now, define $\bar{m} = \max\{\underline{m}, m_{\underline{\phi}}, m_{\bar{\phi}}\}$ and the result follows. ¤

The proof of the following result is analogous to that of Proposition A3.3 and is deleted.

<u>Proposition A3.4</u>: There exists a finite $\bar{m} \geq 0$ such that

$$\sup_{(m_p,\phi_p)\in R_\infty} M_{yd}(m_p,\phi_p) = \max_{(m_p,\phi_p)\in R} M_{yd}(m_p,\phi_p)$$

where $R = [\underline{m},\bar{m}] \times [\underline{\phi},\bar{\phi}]$. ¤

As a consequence of Propositions A3.1-4, it follows that maximizing M_{ab} (for a = y or v; b = u, r or d) over any rectangle in $\mathbb{R}^2$ reduces to maximizing over a bounded rectangle. Moreover, Propositions A3.1-2 imply that maximizing these functions over a bounded rectangle, R, requires checking whether $(1/m_C,(2n+1)\pi-\phi_C) \in R$ for any $n \in \mathbb{Z}$ in which case this point is the solution and if not, maximizing over ∂R. Since ∂R consists of line-segments, this maximization is a one-dimensional problem for which any stationary points are known.

We now show that this optimization may be further simplified because it suffices to check only two of the four line segments constituting the boundary. Consider the examples of extremizing $M_{yu}(m_p,\phi_p)$ or $\phi_{yu}(m_p,\phi_p)$ over $R_p(x,j\omega)$ (defined by (3.28)).

Let

$$\phi_1 = \underset{\phi_P\in\{\phi_P(\omega,\check{\alpha}_A(j\omega))+\underline{\ell}_A(\omega),\phi_P(\omega,\hat{\alpha}_A(j\omega))+\bar{\ell}_A(\omega)\}}{\text{argmax}} \cos(\phi_P+\phi_C)\} \quad \text{(A3.10a)}$$

$$\phi_2 = \underset{\phi_P\in\{\phi_P(\omega,\check{\alpha}_A(j\omega))+\underline{\ell}_A(\omega),\phi_P(\omega,\hat{\alpha}_A(j\omega))+\bar{\ell}_A(\omega)\}}{\text{argmin}} \{\cos(\phi_P+\phi_C)\}. \quad \text{(A3.10b)}$$

It follows from (A3.6) that

$$M_{yu}(m_P,\phi_2) \geq M_{yu}(m_P,\phi_1) \qquad \forall\ m_P \geq 0 \qquad \text{(A3.11)}$$

and if

$$2m_C(\omega)\cos\phi \geq -[m_{P_0}(\omega,\check{\alpha}_M(j\omega))\underline{\ell}_M(\omega)]^{-1} - [m_{P_0}(\phi,\hat{\alpha}_M(j\omega))\bar{\ell}_M(\omega)]^{-1} \qquad \text{(A3.12)}$$

then

$$M_{yu}(m_{P_0}(\omega,\check{\alpha}_M(j\omega))\underline{\ell}_M(\omega)),\phi) \geq M_{yu}(m_{P_0}(\omega,\hat{\alpha}_M(j\omega))\bar{\ell}_M(\omega),\phi) \quad \forall\ \phi \in \mathbb{R}. \qquad \text{(A3.13)}$$

Hence to extremize M_{yu} over $\mathcal{R}_P(x,j\omega)$ only two sides of the rectangle need to be considered. For ϕ_{yu} it may be deduced from (A3.7) that

$$\phi_{yu}(m_{P_0}(\omega,\check{\alpha}_M(j\omega))\underline{\ell}_M(\omega),\phi) \geq \phi_{yu}(m_{P_0}(\omega,\hat{\alpha}_M(j\omega))\bar{\ell}_M(\omega),\phi) \qquad \text{(A3.14)}$$

and if

$$m \geq \frac{\sin[\phi_{P_0}(\omega,\check{\alpha}_A(j\omega))+\phi_{P_0}(\omega,\hat{\alpha}_A(j\omega))+\underline{\ell}_A(\omega)-\bar{\ell}_A(\omega)]}{\sin[\phi_{P_0}(\omega,\hat{\alpha}_A(j\omega))+\bar{\ell}_A(\omega)+\phi_C(\omega)]-\sin[\phi_{P_0}(\omega,\check{\alpha}_A(j\omega))+\underline{\ell}_A(\omega)+\phi_C(\omega)]} \qquad \text{(A3.15)}$$

then

$$\phi_{yu}(m,\phi_{P_0}(\omega,\hat{\alpha}_A(j\omega))+\bar{\ell}_A(\omega)) \geq \phi_{yu}(m,\phi_{P_0}(\omega,\check{\alpha}_A(j\omega))+\underline{\ell}_A(\omega) \quad \forall\ m \geq 0 \qquad \text{(A3.16)}$$

and so two sides of the rectangle may be deleted in extremizing ϕ_{yu}.

It may further be shown that

$$M_{yd}(m_P,\phi_2) \geq M_{yd}(m_P,\phi_1) \qquad \forall\, m_P \geq 0 \tag{A3.17}$$

and that if

$$m_{P_0}(\omega,\check{\alpha}_M(j\omega))\underline{\ell}_M(\omega) + m_{P_0}(\omega,\hat{\alpha}_M(j\omega))\bar{\ell}_M(\omega) \leq \frac{-2}{m_C(\omega)} \cos\phi \tag{A3.18}$$

then,

$$M_{yd}(m_{P_0}(\omega,\hat{\alpha}_M(j\omega))\bar{\ell}_M(\omega),\phi) \geq M_{yd}(m_{P_0}(\omega,\check{\alpha}_M(j\omega))\underline{\ell}_M(\omega),\phi) \qquad \forall\, \phi \in \mathbb{R} \tag{A3.19}$$

For phase,

$$\phi_{yd}(m_{P_0}(\omega,\hat{\alpha}_M(j\omega))\bar{\ell}_M(\omega),\phi) \geq \phi_{yd}(m_{P_0}(\omega,\check{\alpha}_M(j\omega))\underline{\ell}_M(\omega),\phi) \qquad \forall\, \phi \in \mathbb{R} \tag{A3.20}$$

whilst

$$m \geq \frac{\sin[\phi_{P_0}(\omega,\check{\alpha}_A(j\omega))+\phi_{P_0}(\omega,\hat{\alpha}_A(j\omega))+\bar{\ell}_A(\omega)-\underline{\ell}_A(\omega)]}{\sin[\phi_{P_0}(\omega,\hat{\alpha}_A(j\omega))+\bar{\ell}_A(\omega)+\phi_C(\omega)]-\sin[\phi_{P_0}(\omega,\check{\alpha}_A(j\omega))+\underline{\ell}_A(\omega)+\phi_C(\omega)]} \tag{A3.21a}$$

implies

$$\phi_{yd}(m,\phi_{P_0}(\omega,\hat{\alpha}_A(j\omega)+\bar{\ell}_A(\omega)) \geq \phi_{yd}(m,\phi_{P_0}(\omega,\check{\alpha}_A(j\omega)+\underline{\ell}_A(\omega)) \qquad \forall\, m \geq 0. \tag{A3.21b}$$

From the results for H_{yu} and H_{yd}, results reducing the number of computations for extremizing any of the closed loop transfer functions follow easily. ¤

Appendix 4: Computation of the Boundary of $P_0(j\omega,A)$

Let

$$P_0(j\omega,\alpha) = m_{P_0}(j\omega,\alpha)e^{j\phi_{P_0}(j\omega,\alpha)} \tag{A4.1}$$

$$\phi_\lambda(j\omega) \triangleq \lambda\phi(j\omega,\check{\alpha}_A(j\omega)) + (1-\lambda)\phi(j\omega,\hat{\alpha}_A(j\omega)), \quad \lambda \in [0,1] \tag{A4.2}$$

Then the boundary, $\partial P_0(j\omega,A)$ (see Fig. 12) may be computed by solving for each $\lambda \in [0,1]$,

$$P_\lambda : \max_{\alpha\in A} \text{ (or } \min_{\alpha\in A}\text{) } m_{P_0}(j\omega,\alpha) \ni \phi_{P_0}(j\omega,\alpha) = \phi_\lambda(j\omega) \tag{A4.3}$$

Assuming that first order optimality conditions hold at an extremum for P_λ, and all poles and zeros of $P_0(j\omega,\alpha)$ are real, then a solution to P_λ must satisfy for some $\xi \in \mathbb{R}$,

$$z^i = \begin{cases} \xi\omega & \text{if} \quad \xi\omega \in [\underline{z}^i,\bar{z}^i] \\ \underline{z}^i \text{ or } \bar{z}^i & \end{cases} \qquad \forall\ i \in \underline{k}'_{pR} \tag{A4.4}$$

$$p^i = \begin{cases} \xi\omega & \text{if} \quad \xi\omega \in [\underline{p}^i,\bar{p}^i] \\ \underline{p}^i \text{ or } \bar{p}^i & \end{cases} \qquad \forall\ i \in \underline{K}_{pR} \tag{A4.5}$$

where

$$\sum_{i=1}^{k'_p} \tan^{-1}\left(\frac{\omega}{z^i}\right) - \sum_{i=1}^{k_p} \tan^{-1}\left(\frac{\omega}{p^i}\right) = \phi_\lambda(j\omega) \tag{A4.6}$$

From (A4.4-5), it may be seen that determination of boundary points for $\lambda \in (0,1)$ is computationally non-trivial.

Appendix 5: Decomposition of Modified Nyquist Criterion

Proof of Theorem 3.4: Let

$R_i \overset{\Delta}{=} \{(m_i,\phi_i) | \underline{M}_i \le m_i \le \bar{M}_i, \underline{\Phi}_i \le \phi_i \le \bar{\Phi}_i\}$, i = 1,2.

(i) We begin by showing the maximization of ξ over $R_1 \times R_2$ has no solutions in $(R_1 \times R_2)^0$.

Suppose

$$\begin{pmatrix} \frac{\partial \xi}{\partial m_i} \\ \frac{\partial \xi}{\partial \phi_i} \end{pmatrix} = 0 \quad i = 1,2.$$

Then,

$$\frac{\partial \xi}{\partial m_i} = 2k \cos \phi_i [m_1 \cos \phi_1 + m_2 \cos \phi_2] - \sin \phi_i, \; i = 1,2 \qquad \text{(A5.2)}$$

$$\frac{\partial \varepsilon}{\partial \phi_i} = -2km_i \sin \phi_i [m_1 \cos \phi_1 + m_2 \cos \phi_2] - m_i \cos \phi_i, \; i = 1,2 \qquad \text{(A5.3)}$$

so that

$$m_1 \cos \phi_1 + m_2 \cos \phi_2 = \frac{-1}{2k} \tan \phi_i \qquad i = 1,2$$

$$m_1 \cos \phi_1 + m_2 \cos \phi_2 = \frac{}{2k} \cot \phi_i \qquad i = 1,2$$

requiring that

$$\tan^2 \phi_i = -1 \qquad i = 1,2$$

which contradicts (A5.1). It follows that

$$\left(\frac{\partial \xi}{\partial m_i}, \frac{\partial \xi}{\partial \phi_i}\right) \ne 0 \quad i = 1,2. \qquad \text{(A5.4)}$$

Hence any solution of the max problem must be on $\partial(R_1 \times R_2)$.

(ii) $\partial(R_1 \times R_2)$ is the boundary obtained by setting one of $m_i = \underline{m}_i$ or $\bar{m}_i$ or $\phi_i = \underline{\phi}_i$ or $\bar{\phi}_i$, i = 1, 2 in turn. A stationary point of the reduced gradient on $\{\partial(R_1 \times R_2)\}^0$ must have three of the four partials $\frac{\partial\xi}{\partial m_i}$, $\frac{\partial\xi}{\partial\phi_i}$, i = 1, 2 (evaluated on the appropriate constant contour of the fourth variable) simultaneously zero. From (A5.4), we see that there are no stationary points of the reduced gradient so that no solution lies in $\{\partial(R_1 \times R_2)\}^0$.

(iii) It then suffices to consider stationary points on the two-dimensional surfaces of $\partial\{\partial(R_1 \times R_2)\}$. This boundary is obtained by setting in turn two of the four variables m_i, ϕ_i, i = 1,2 equal to their values at the endpoints of their confidence intervals. Hence we must consider zeros of the partials with respect to the pairs,

(a) (m_i, ϕ_i) i = 1,2 (b) (m_1, m_2) (c) (ϕ_1, ϕ_2) (d) (m_1, ϕ_2) (e) (m_2, ϕ_1)

(a) From (A5.4) it follows that we may discount this case.

(b) From (A5.2), it follows that the Hessian with respect to m_1, m_2 is

$$\left[\frac{\partial^2\xi}{\partial m_i m_j}\right]_{i,j=1,2} = 2k\begin{bmatrix} \cos^2\phi_1 & \cos\phi_1\cos\phi_2 \\ \cos\phi_1\cos\phi_2 & \cos^2\phi_2 \end{bmatrix} > 0. \tag{A5.5}$$

Hence ξ is convex in (m_1, m_2) and so a maximizer must be on the boundary of $[\underline{m}_1, \bar{m}] \times [\underline{m}_2, \bar{m}_2]$.

(c) From (A5.3) it follows that

$$\left(\frac{\partial\xi}{\partial\phi_i}\right)_{i=1,2} = 0 \quad \text{implies} \quad \tan\phi_1 = \tan\phi_2,$$

or, equivalently,

$$\phi_1 - \phi_2 = n\pi,\ n \in \mathbb{Z}. \tag{A5.6}$$

Along the contour defined by (A5.6),

$$\xi(m_1,\phi_1,m_2,\phi_1+n\pi) = k(m_1\pm m_2)^2\cos^2\phi_1 - (m_1\pm m_2)\sin\phi_1 - c \qquad (A5.7)$$

which has stationary points (with respect to ϕ_1) satisfying

$$\cos\phi_1 = 0 \quad \text{or} \quad \sin\phi_1 = \frac{-2}{k(m_1\pm m_2)}$$

i.e.

$$\phi_1 = (2n+1)\pi/2 \quad \text{or} \quad \phi_1 = \sin^{-1}\left(\frac{-2}{k(m_1\pm m_c)}\right) \qquad (A5.8)$$

(d) From (A5.2,3) we have that a necessary condition for $\left(\frac{\partial\xi}{\partial m_1}, \frac{\partial\xi}{\partial\phi_2}\right)^T = 0$ is

$$\phi_1 = \phi_2 + (2n+1)\pi/2,\ n \in \mathbb{Z}. \qquad (A5.9)$$

Along the contour defined by (A5.9),

$$\xi(m_1,\phi_2+(2n+1)\pi/2,m_2,\phi_2) = k(m_1\ m_2)^2\cos^2\phi_2-(m_1\mp m_2)\sin\phi_2 - c \qquad (A5.10)$$

and from (A5.10), it follows that for any $n \in \mathbb{Z}$

$$\left(\frac{\partial}{\partial m_1}\xi(m_1,\phi_2+(2n+1)\pi/2,m_2,\phi_2), \frac{\partial}{\partial\phi_2}\xi(m_1,\phi_2+(2n+1)\pi/2,m_2,\phi_2)\right)^T = 0$$

iff $(m_1\pm m_2)^2 = \dfrac{-1}{4k^2\cos^2\phi_2}$,

which is clearly impossible. Hence ξ has no stationary points with respect to (m_1,ϕ_2).

(e) By symmetry, we may draw the same conclusion as in (d).

Hence, we have shown that the only stationary points which may be local maxima of ξ on $\partial\{\partial(\mathcal{R}_1\times\mathcal{R}_2)\}$ are those given by (A5.8). This establishes the result. ¤

Computationally, it is necessary to check any feasible stationary point given by (A5.8) and to compare the values of ξ evaluated at these stationary points with the maximizers on the boundary, $\partial\{[\underline{\phi}_1,\bar{\phi}_2]\times[\underline{\phi}_2,\bar{\phi}_2]\}$. We characterize the maximizers of $\xi(m_1,\cdot,m_2,\cdot)$ on this boundary by the following result:

<u>Proposition A5.1</u>:
$$\max_{(\phi_1,\phi_2)\in\partial\{[\underline{\phi},\bar{\phi}_1]\times[\underline{\phi}_2,\bar{\phi}_2]\}} \xi(m_1,\phi_1,m_2,\phi_2)$$

$$= \max_{v\in V} \xi(m_1,v^1,m_2,v^2)$$

where

$$V \overset{\Delta}{=} \{(\underline{\phi}_1,\underline{\phi}_2),(\underline{\phi}_1,\bar{\phi}_2),(\bar{\phi}_1,\underline{\phi}_2),(\bar{\phi}_1,\bar{\phi}_2)\}$$

is the set of vertices of $[\underline{\phi}_1,\bar{\phi}_1]\times[\underline{\phi}_2,\bar{\phi}_2]$.

<u>Proof</u>. From (A5.3), we have that

$\frac{\partial\xi}{\partial\phi_i} = 0$ requires that

$$m_1 \cos\phi_1 + m_2 \cos\phi_2 = -\frac{1}{2k}\cot\phi_i,\ i = 1,2 \tag{A5.11}$$

Differentiating (A5.11) with respect to ϕ_i, we obtain that a further necessary condition for $\frac{\partial\xi}{\partial\phi_i} = 0$ is

$$-m_i \sin\phi_i = \frac{1}{2k}\operatorname{cosec}^2\phi_i \qquad i = 1,2 \tag{A5.12}$$

substituting from (A5.11,12), we obtain

$$\xi(m_1,\phi_1,m_2,\phi_2) = \frac{1}{4k}(3\operatorname{cosec}^2\phi_1-1) - m_2\sin\phi_2 - c$$

However,

$$\frac{\partial^2 \xi}{\partial \phi_1^2} = \frac{3}{2k} \left(\frac{\sin^2\phi_1 + 3\cos^2\phi_1}{\sin^4 \phi_1}\right) > 0 \qquad \forall\ \phi_1 \in \mathbb{R} \tag{A5.12}$$

so that any stationary point is a local minimum. The result now follows. ¤

The above proposition implies that the maximization over $(\phi_1,\phi_2) \in [\underline{\phi}_1,\bar{\phi}_1] \times [\underline{\phi}_2,\bar{\phi}_2]$ required by Theorem 3.4 requires merely that we compare the values of ξ at any feasible stationary point of those defined by (A5.8) with the four vertices, V.

Appendix 6: Local Lipschitz Continuity of Extremizers of Open Loop Gain an d Phase

We establish some properties of $\hat{M}_0(\cdot,\cdot)$, $\check{M}_0(\cdot,\cdot)$, $\hat{\Phi}_0(\cdot,\cdot)$ and $\check{\Phi}_0(\cdot,\cdot)$ defined by (3.19) of Section 3.2. From Theorem 3.1 of Section 3.1 we have that $\hat{M}_0(x,j\omega)$ ($\check{M}_0(x,j\omega)$) is the product of $|C(x,j\omega)|$ and the individual maximum (minimum) magnitudes due to the individual pole and zero variations. Similarly, for phases, we have from Theorem 3.2 of Section 3.1 a decomposition into a sum of [arg $C(x,j\omega)$] and the individual extremizers of phase corresponding to the individual pole and zero variations. We begin by developing some results for the individual pole and zero variations for the real and complex case. We abbreviate local Lipschitz continuity to ℓ.L.C.

Proposition A6.1: Suppose $z^i \in [\underline{z}^i,\bar{z}^i]$, $p^i \in [\underline{p}^i,\bar{p}^i]$, and that

$$\hat{M}_{z^i}(j\omega) \triangleq \max_{z^i\in[\underline{z}^i,\bar{z}^i]} |j\omega+z^i|$$

$$\hat{M}_{p^i}(j\omega) \triangleq \max_{p^i\in[\underline{p}^i,\bar{p}^i]} \frac{1}{|j\omega+p^i|},$$

with similar definitions for the minima, $\check{M}_{z^i}(j\omega)$ and $\check{M}_{p^i}(j\omega)$. Then, $\hat{M}_{z^i}(j\omega)$ and $\check{M}_{z^i}(j\omega)$ are C^1 functions of $\omega \in (0,\infty)$ and $\hat{M}_{p^i}(j\omega)$ and $\check{M}_{p^i}(j\omega)$ are continuously differentiable for $(\omega,p^i) \neq 0$.

Proof: Now $|j\omega+z^i| = [\omega^2+(z^i)^2]^{1/2}$

so

$$\frac{\partial}{\partial\omega}|j\omega+z^i| = \begin{cases} \omega[\omega^2+(z^i)^2]^{-1/2} & (\omega,z^i) \neq 0 \\ 1 & \omega \neq 0,\ z^i = 0 \\ 0 & \omega = 0 \end{cases}$$

Hence, $|j\omega+z^i|$ is C^1 in $\omega \geq 0$ for any fixed z^i. By Proposition A2.1, $\hat{z}_M^i(j\omega)$ and $\check{z}_M^i(j\omega)$ are constant so

$$\hat{M}_{z^i}(j\omega) = |j\omega+z_M^i(j\omega)|$$

$$\check{M}_{z^i}(j\omega) = |j\omega+z_M^i(j\omega)|$$

are C^1 functions of $\omega \geq 0$.

For poles, p^i, the above applies for all $(\omega,p^i) \neq 0$ since then the magnitude function is the non-zero reciprocal of that for zeros. ¤

Proposition A6.2: (a) Suppose $z^i = \beta^i + j\gamma^i$, $\beta^i \in [\underline{\beta}^i,\bar{\beta}^i]$, $\gamma^i \in [\underline{\gamma}^i,\bar{\gamma}^i]$. Then,

$$M_c(j\omega,\beta^i,\gamma^i) \triangleq |(j\omega+(\beta^i+j\gamma^i))(j\omega+(\beta^i-j\gamma^i))|$$

$$\hat{M}_{z^i}(j\omega) \triangleq \max_{(\beta^i,\gamma^i)\in[\underline{\beta}^i,\bar{\beta}^i]\times[\underline{\gamma}^i,\bar{\gamma}^i]} M_c(j\omega,\beta^i,\gamma^i)$$

$$\check{M}_{z^i}(j\omega) \triangleq \min_{(\beta^i,\gamma^i)\in[\underline{\beta}^i,\bar{\beta}^i]\times[\underline{\gamma}^i,\bar{\gamma}^i]} M_c(j\omega,\beta^i,\gamma^i)$$

are ℓ.L.C. for $\omega > 0$.

(b) If $p^i = \beta^i + j\gamma^i$, $\beta^i\in[\underline{\beta}^i,\bar{\beta}^i]$, $\gamma^i\in[\underline{\gamma}^i,\bar{\gamma}^i]$ the corresponding functions, $\hat{M}_{p^i}(j\omega)$ and $\check{M}_{p^i}(j\omega)$ are ℓ.L.C. $\forall\ \omega > 0$ excepting any $j\omega$-axis poles.

<u>Proof:</u> (a) For $(\omega,\beta^i,\gamma^i) \neq 0$, $M_c(j\omega,\beta^i,\gamma^i)$ is continuously differentiable in ω, since

$$\frac{\partial M_c}{\partial\omega}(j\omega,\beta^i,\gamma^i) = \begin{cases} \dfrac{2\,[(\omega^i)^2-(\beta^i)^2+\omega^2]}{M_c(\omega,\beta^i,\gamma^i)} & \beta^i \neq 0 \\ -\,2\omega & \beta^i = 0. \end{cases}$$

Now,

$$\hat{M}_{z^i}(j\omega) = M_c(\omega,\hat{\beta}^i_M(j\omega),\hat{\gamma}^i_M(j\omega))$$

and referring to Table A2.1, we see that $\hat{\beta}^i_M(j\omega)$ is constant and $\hat{\gamma}^i_M(j\omega)$ is piecewise constant. Hence it suffices to consider those $\omega \geq 0$ at which $\hat{\gamma}^i_M(j\omega)$ changes value:

(i) $\omega_1 = \sqrt{(\hat{\beta}^i_M(j\omega))^2+\frac{1}{2}((\underline{\gamma}^i)^2+(\bar{\gamma}^i)^2)}$ with $\underline{\gamma}^i \geq 0$ or $\bar{\gamma}^i \leq 0$

From Table A2.1,

$$\hat{M}_{z^i}(j\omega) = \begin{cases} M_c(j\omega,\hat{\beta}^i_M(j\omega)\ \underset{\gamma^i\in\{\underline{\gamma}^i,\bar{\gamma}^i\}}{\operatorname{argmax}}\ |\gamma^i|) & \omega \leq \omega_1 \\ M_c(j\omega,\hat{\beta}^i_M(j\omega)\ \underset{\gamma^i\in\{\underline{\gamma}^i,\bar{\gamma}^i\}}{\operatorname{argmin}}\ |\gamma^i|) & \omega \geq \omega_1 \end{cases}$$

since $\hat{M}_{z^i}(j\omega)$ is ℓ.L.C. on $(0,\omega_1]$ and on $[\omega_1,\infty)$ and since

$$M_c(j\omega_1,\hat{\beta}^i_M(j\omega_1),\ \underset{\gamma^i\in\{\underline{\gamma},\bar{\gamma}^i\}}{\operatorname{argmax}}\ |\gamma^i|) = M_c(j\omega_1,\hat{\beta}^i_M(j\omega_1),\ \underset{\gamma^i\in\{\underline{\gamma}^i,\bar{\gamma}^i\}}{\operatorname{argmin}}\ |\gamma^i|)$$

it follows that $\hat{M}_{z^i}(\cdot)$ is locally Lipschitz continuous on $(0,\infty)$.

(ii) $\omega_2 = \sqrt{(\hat{\beta}^i_M(j\omega))^2+\frac{1}{2}((\underline{\gamma}^i)^2+(\bar{\gamma}^i)^2)}$ with $\underline{\gamma}^i < 0 \leq \bar{\gamma}^i$.

From Table A2.1,

$$\hat{M}_{z^i}(j\omega) = \begin{cases} M_c(j\omega,\hat{\beta}_M^i(j\omega),0) & \omega \leq \omega_2 \\ M_c(j\omega,\hat{\beta}_M^i(j\omega), \underset{\gamma^i \in \{\underline{\gamma}^i,\bar{\gamma}^i\}}{\operatorname{argmax}} |\gamma^i|) & \omega \geq \omega_2 \end{cases}$$

and as

$$M_c(j\omega_2,\hat{\beta}_M^i(j\omega_2),0) = M_c(j\omega,\hat{\beta}_M^i(j\omega_2) \underset{\gamma^i \in \{\underline{\gamma},\bar{\gamma}^i\}}{\operatorname{argmax}} |\gamma^i|),$$

the result follows.

For the minimizing function, $\check{M}_{z^i}(j\omega)$, we need only consider $\omega \in [\omega_3,\omega_4]$ (for $(\check{\beta}_M^i(j\omega))^2-\omega^2 < 0$) where

$$\omega_3 = \sqrt{(\check{\beta}_M^i(j\omega))^2+\min\{(\underline{\gamma}^i)^2,(\bar{\gamma}^i)^2\}}$$

$$\omega_4 = \sqrt{(\check{\beta}_M^i(j\omega))^2+\max\{(\underline{\gamma}^i)^2,(\bar{\gamma}^i)^2\}}$$

since $\check{\beta}_M^i(j\omega)$ is a constant for all $\omega \in [0,\infty)$ and $\check{\gamma}_M^i(j\omega)$ is constant for $\omega \notin [\omega_3,\omega_4]$. Now

$$\check{M}_{z^i}(j\omega) = \begin{cases} M_c(j\omega,\check{\beta}_M^i(j\omega), \underset{\gamma^i \in \{\underline{\gamma},\bar{\gamma}^i\}}{\operatorname{argmin}} |\gamma^i|) & \omega \leq \omega_3 \\ M_c(j\omega,\check{\beta}_M^i(j\omega),\gamma^i) & \omega = \sqrt{(\beta_M^i(\omega))^2+(\gamma^i)^2}, \omega \in [\omega_3,\omega_4] \\ M_c(j\omega,\check{\beta}_M^i(j\omega), \underset{\gamma^i \in \{\underline{\gamma},\bar{\gamma}^i\}}{\operatorname{argmax}} |\gamma^i|) & \omega \geq \omega_4. \end{cases}$$

As $\check{M}_{z^i}(j\omega) = 2\omega\check{\beta}_M^i(j\omega)$ for $\omega \in [\omega_3,\omega_4]$ and

$$M_c(j\omega_k,\check{\beta}_M^i(j\omega_k), \underset{\gamma^i \in \{\underline{\gamma},\bar{\gamma}^i\}}{\operatorname{argmin}} |\gamma^i|) = 2\omega\check{\beta}_M^i(j\omega_k) \qquad k = 3,4$$

it follows that $\check{M}_{z^i}(j\omega)$ is ℓ.L.C. on (ω_3,ω_4) and at ω_k, $k = 3,4$.

(b) Since $\check{M}_{z^i}(j\omega) = M_c(j\omega,\hat{\beta}^i_M(j\omega),\hat{\gamma}^i_M(j\omega))$

$$\neq 0 \qquad\qquad \text{if } \hat{\beta}^i_M(j\omega) \neq 0,$$

$$\hat{M}_{p^i}(j\omega) = 1/\check{M}_{z^i}(j\omega),$$

it follows that $\hat{M}_{p^i}(j\omega)$ is ℓ.L.C., provided $\hat{\beta}^i_M(j\omega) \neq 0$. If $\hat{\beta}^i_M(j\omega) = 0$,

$$\hat{M}_{p^i}(j\omega) = \frac{1}{(\gamma^i_M(j\omega))^2-\omega^2}$$

which is ℓ.L.C. for $\omega \neq \hat{\gamma}^i_M(j\omega)$. Similarly, we may show that $\check{M}_{p^i}(\omega)$ is ℓ.L.C. except when $\omega = \check{\gamma}^i_M(j\omega)$ and $\check{\beta}^i_M(j\omega) = 0$. ⌑

<u>Proposition A6.3</u>: Suppose $z^i \in [\underline{z}^i,\bar{z}^i]$, $p^i \in [\underline{p}^i,\bar{p}^i]$. Then the functions

$$\hat{\phi}_{z^i}(j\omega) \triangleq \max_{z^i\in[\underline{z}^i,\bar{z}^i]} \arg(j\omega+z^i)$$

$$\hat{\phi}_{p^i}(j\omega) \triangleq \max_{p^i\in[\underline{p}^i,\bar{p}^i]} \{-\arg(j\omega+p^i)\}$$

and $\check{\phi}_{z^i}(j\omega)$, $\check{\phi}_{p^i}(j\omega)$ are continuously differentiable for $\omega \in (0,\infty)$ where $\check{\phi}_{z^i}(j\omega)$ and $\check{\phi}_{p^i}(j\omega)$ are the similarly defined minima.

<u>Proof</u>: Since $\arg(j\omega+z^i) = \tan^{-1} \frac{\omega}{z^i}$,

$$\frac{\partial}{\partial\omega} \arg(j\omega+z^i) = \frac{z^i}{\omega^2+(z^i)^2} .$$

Since $\check{\phi}_{z^i}(j\omega) = \arg(j\omega + \check{z}_A^i(j\omega))$

$$\hat{\phi}_{z^i}(j\omega) = \arg(j\omega + \hat{z}_A^i(j\omega))$$

with $\check{z}_A^i(j\omega)$ and $\hat{z}_A^i(j\omega)$ constant (Proposition A2.1), it follows that they are C^1. Similarly the result holds for $\hat{\phi}_{p^i}(j\omega)$ and $\check{\phi}_{p^i}(j\omega)$. ¤

<u>Proposition A6.4</u>: (a) Suppose z^i is a complex zero as in Proposition A6.2, and that

$$\phi_c(j\omega,\beta^i,\gamma^i) \stackrel{\Delta}{=} \arg\{[(\beta^i)^2+(\gamma^i)^2-\omega^2]+j(2\beta^i\omega)\}$$

$$\hat{\phi}_{z^i}(j\omega) \stackrel{\Delta}{=} \max_{(\beta^i,\gamma^i)\in[\underline{\beta}^i,\bar{\beta}^i]\times[\underline{\gamma}^i,\bar{\gamma}^i]} \phi_c(j\omega,\beta^i,\gamma^i)$$

$$\check{\phi}_{z^i}(j\omega) \stackrel{\Delta}{=} \min_{(\beta^i,\gamma^i)\in[\underline{\beta}^i,\bar{\beta}^i]\ [\underline{\gamma}^i,\bar{\gamma}^i]} \phi_c(j\omega,\beta^i,\gamma^i).$$

Then $\hat{\phi}_{z^i}(\cdot)$, $\check{\phi}_{z^i}(\cdot)$ are ℓ.L.C. on $(0,\infty)$.

(b) If p^i is a complex pole as in Proposition A6.2, then the functions

$$\hat{\phi}_{p^i}(j\omega) \stackrel{\Delta}{=} \max_{(\beta^i,\gamma^i)\in[\underline{\beta}^i,\bar{\beta}^i]\ [\underline{\gamma}^i,\bar{\gamma}^i]} \{-\phi_c(j\omega,\beta^i,\gamma^i)\}$$

$$\check{\phi}_{p^i}(j\omega) \stackrel{\Delta}{=} \min_{(\beta^i,\gamma^i)\in[\underline{\beta}^i,\bar{\beta}^i]\times[\underline{\gamma}^i,\bar{\gamma}^i]} \{-\phi_c(j\omega,\beta^i,\gamma^i)\}$$

are ℓ.L.C. on $(0,\infty)$.

<u>Proof</u>: (a) Since $\frac{\partial}{\partial\omega}\phi_c(j\omega,\beta^i,\gamma^i) = \frac{\partial}{\partial\omega}\tan^{-1}\frac{2\beta^i\omega}{(\beta^i)^2+(\gamma^i)^2-\omega^2}$,

$\phi_c(\cdot,\beta^i,\gamma^i)$ is ℓ.L.C. for any $\omega > 0$, $\forall\ \beta^i, \gamma^i \in \mathbb{R}$ such that

$\omega^2 \neq (\beta^i)^2 + (\gamma^i)^2$. From Theorem 3.4,

$$\hat{\phi}_{z^i}(j\omega) = \phi_c(j\omega, \hat{\beta}_A^i(j\omega), \hat{\gamma}_A^i(j\omega))$$

$$\check{\phi}_{z^i}(j\omega) = \psi_c(j\omega), \check{\beta}_A^i(j\omega), \check{\gamma}_A^i(j\omega))$$

and from Table A2.2, if $\underline{\beta}^i, \underline{\gamma}^i \geq 0$ then

$$\hat{\beta}_A^i(j\omega) = \begin{cases} \bar{\beta}^i & 0 \leq \omega^2 \leq (\underline{\gamma}^i)^2 - (\bar{\beta}^i)^2 \\ ((\gamma^i)^2 - \omega^2)^{1/2} & \omega^2 = (\gamma^i)^2 - (\beta^i)^2, \quad \beta^i \in [\underline{\beta}^i, \bar{\beta}^i] \\ \underline{\beta}^i & \omega^2 \geq (\underline{\gamma}^i)^2 - (\underline{\beta}^i)^2 \end{cases}$$

$$\hat{\gamma}_A^i(j\omega) = \underline{\gamma}^i \qquad \omega \geq 0$$

while

$$\check{\beta}_A^i(j\omega) = \begin{cases} \underline{\beta}^i & 0 \leq \omega^2 \leq (\bar{\gamma}^i)^2 - \underline{\beta}^i \bar{\beta}^i \\ \bar{\beta}^i & (\bar{\gamma}^i)^2 - \underline{\beta}^i \bar{\beta}^i \leq \omega^2 \end{cases}$$

$$\check{\gamma}_A^i(j\omega) = \bar{\gamma}^i \qquad \omega \geq 0.$$

Since the functions $\hat{\phi}_{z^i}(j\omega)$ and $\check{\phi}_{z^i}(j\omega)$ are uniquely defined at $\omega = \{(\underline{\gamma}^i)^2 - (\bar{\beta}^i)\}^{1/2}$, $\{(\underline{\gamma}^i)^2 - (\underline{\beta}^i)^2\}^{1/2}$ and $\{(\bar{\gamma}^i)^2 - \underline{\beta}^i \bar{\beta}^i\}^{1/2}$, they are ℓ.L.C. at these ω provided that $\omega^2 \neq (\hat{\beta}_A^i(j\omega))^2 + (\hat{\gamma}_A^i(j\omega))^2$ or $\omega^2 \neq (\check{\beta}_A^i(j\omega))^2 + (\check{\gamma}_A^i(j\omega))^2$. For $\omega \in (\{(\underline{\gamma}^i)^2 - (\bar{\beta}^i)^2\}^{1/2}, \{(\bar{\gamma}^i)^2 - (\underline{\beta}^i)^2\}^{1/2})$,

$$\hat{\phi}_{z^i}(j\omega) = \tan^{-1} \frac{\omega}{[(\gamma^i)^2 - \omega^2]^{1/2}}$$

which is ℓ.L.C. for $\omega^2 \neq (\gamma^i)^2$. Now consider

$$\omega^2 = (\hat{\beta}_A^i(j\omega))^2 + (\hat{\gamma}_A^i(j\omega))^2, \; \hat{\beta}_A^i(j\omega) \neq 0.$$

We have that

$$\hat{\phi}_{z^i}(j\omega) = \begin{cases} \tan^{-1}\left[\dfrac{2\hat{\beta}_A^i(j\omega)\cdot\omega}{(\hat{\beta}_A^i(j\omega))^2+(\hat{\gamma}_A^i(j\omega))^2-\omega^2}\right] & \omega^2 < (\hat{\beta}_A^i(j\omega))^2 + (\hat{\gamma}_A^i(j\omega))^2 \\ \pm\,\pi/2 & \text{for } \omega^2 = (\hat{\beta}_A^i(j\omega))^2 + (\hat{\gamma}_A^i(j\omega))^2 \text{ and the sign is that of } \hat{\beta}_A^i(j\omega) \\ \pm\,\pi + \tan^{-1}\left[\dfrac{2\hat{\beta}_A^i(j\omega)\cdot}{(\hat{\beta}_A^i(j\omega))^2(\hat{\gamma}_A^i(j\omega))^2-\omega^2}\right] & \omega^2 > (\hat{\beta}_A^i(j\omega))^2 + (\hat{\gamma}_A^i(j\omega))^2 \text{ and the sign is that taken for } \omega^2 = (\hat{\beta}_A^i(j\omega))^2 + (\hat{\gamma}_A^i(j\omega))^2. \end{cases}$$

Also

$$\lim_{\omega \nearrow \sqrt{(\hat{\beta}_A^i(j\omega))^2+(\hat{\gamma}_A^i(j\omega))^2}} \{\hat{\phi}_{z^i}(j\omega)\} = \pi/2$$

$$= \lim_{\omega \searrow (\hat{\beta}_A^i(j\omega))^2+(\hat{\gamma}_A^i(j\omega))^2} \{\phi_{z^i}(j\omega)\}$$

and so $\hat{\phi}_{z^i}(j\omega)$ is ℓ.L.C. at $\omega = \sqrt{(\hat{\beta}_A^i(j\omega))^2+(\hat{\gamma}_A^i(j\omega))^2}$. If $\hat{\beta}_A^i(j\omega) = 0$, then $\hat{\phi}_{z^i}(j\omega)$ is a constant and hence is ℓ.L.C. The arguments for $\check{\phi}_{z^i}(j\omega)$ are similar. Hence for $\underline{\beta}^i \geq 0$, $\underline{\gamma}^i \geq 0$, $\hat{\phi}_{z^i}(\cdot)$, $\check{\phi}_{z^i}(\cdot)$ are ℓ.L.C. for $\omega > 0$. For $\bar{\beta}^i \leq 0$, $\bar{\gamma}^i \leq 0$, ℓ.L.C. is established by analogous arguments. Since from Appendix 2 we know that the above two cases establish the result for all possible orderings of $\underline{\beta}^i$, $\bar{\beta}^i$, $\underline{\gamma}^i$, $\bar{\gamma}^i$ with respect to the origin, (a) is proven.

(b) Since

$$\hat{\phi}_{p^i}(j\omega) = -\check{\phi}_{\bar{z}^i}(j\omega)$$

and

$$\check{\phi}_{p^i}(j\omega) = -\hat{\phi}_{z^i}(j\omega)$$

the result for p^i follows from (a). ¤

Lemma A6.5: (a) The functions $\hat{M}(\cdot,\cdot)$ and $\hat{M}_0(\cdot,\cdot)$ ($\check{M}(\cdot,\cdot)$ and $\check{M}_0(\cdot,\cdot)$) are ℓ.L.C. continuous for all $(x,\omega) \in \mathbb{R}^{n_x} \times \mathbb{R}_+$ such that $j\omega$ is not a pole of $P_0(s,\hat{\alpha}_M(s))C(x,s)$ ($P_0(s,\check{\alpha}_M(s))C(x,s)$).

(b) The functions $\hat{\phi}(\cdot,\cdot)$, $\hat{\phi}_0(\cdot,\cdot)$, $\check{\phi}(\cdot,\cdot)$ and $\check{\phi}_0(\cdot,\cdot)$ are locally Lipschitz continuous on $\mathbb{R}^{n_x} \times \mathbb{R}_+$.

Proof: (a) From Theorem 3.4, we have that

$$M(x,j\omega) = |P_0(j\omega,\hat{\alpha}_M(j\omega))\bar{\ell}_M(\omega)C(x,j\omega)|$$

$$= |P_0(j\omega,\hat{\alpha}_M(j\omega)|\cdot|C(x,j\omega)|\cdot\bar{\ell}_M(\omega).$$

Now $|C(x,j\omega)|$ is the quotient of the magnitudes of two polynomials, each evaluated along the $j\omega$-axis. As each of these magnitudes is ℓ.L.C., the quotient is ℓ.L.C. except when the demominator vanishes, viz, at $j\omega$-axis poles of $C(x,\cdot)$. From (A2.3), we have that $|P_0(j\omega,\hat{\alpha}_M(j\omega))|$ is the product of the maximum magnitudes of the zeros and the recrprocals of the minimum magnitudes of the poles. It follows from Propositions A6.1 and A6.2 that $|P_0(j\omega,\hat{\alpha}_M(j\omega))|$ is locally Lipschitz continuous except at $j\omega$-axis poles of $P_0(\cdot,\hat{\alpha}_M(\cdot))$. Hence, the product,

$$M_0(x,j\omega) = |P_0(j\omega,\alpha_M(j\omega))||C(x,j\omega)|$$

is ℓ.L.C. except at $j\omega$-axis poles of $P_0(\cdot,\hat{\alpha}_M(\cdot))C(x,\cdot)$. As $\bar{\ell}_M(\cdot)$ is by definition locally Lipschitz, we have that $\hat{M}(x,j\omega)$ is ℓ.L.C., except for pairs (x,ω) corresponding to $j\omega$-axis poles of $P_0(s,\hat{\alpha}_M(s))C(x,s)$.

Similar arguments hold for $\check{M}(\cdot,\cdot)$ and $\check{M}_0(\cdot,\cdot)$.

(b) From Theorem 3.4,

$$\hat{\Phi}(x,j\omega) = \arg P_0(j\omega,\hat{\alpha}_A(j\omega))C(x,j\omega) + \bar{\ell}_A(\omega)$$

$$= \arg P_0(j\omega,\hat{\alpha}_A(j\omega)) + \arg C(x,j\omega) + \bar{\ell}_A(\omega)$$

From A2.4, it follows that $\arg P_0(j\omega,\hat{\alpha}_A(j\omega))$ is the sum of the maximizing phases of the zeros and minimizing phases of the poles. By Propositions A6.3 and A6.4, these phases are locally Lipschitz. By definition, $\bar{\ell}_A(\cdot)$ is locally Lipschitz and since $\arg C(x,j\omega)$ is the phase of a rational function, it follows that $\hat{\Phi}(\cdot,\cdot)$ and $\hat{\Phi}_0(\cdot,\cdot)$ are ℓ.L.C.. Similarly we establish the desired result for $\check{\Phi}(\cdot,\cdot)$ and $\check{\Phi}_0(\cdot,\cdot)$. ¤

For determining stability in Section 3.4 we must construct the sets,

$$R_1(x,j\omega) = \left(\frac{n_C(x,j\omega)}{d(j\omega)}\right) \times R_{n_p}(j\omega) \times R_\ell(j\omega)$$

$$R_2(x,j\omega) = \frac{d_C(x,j\omega)}{d(j\omega)} \times R_{d_p}(j\omega)$$

where

$$R_{n_p}(x,j\omega) \triangleq \{(m,\phi)\,|\,m \in [|n_p(x,\check{z}_M(j\omega))|,|n_p(x,\hat{z}_M(j\omega))|],$$

$$\phi \in [\arg n_p(x,\check{z}_A(j\omega)),\arg n_p(x,\hat{z}_A(j\omega))]\}$$

$$\mathcal{R}_{d_p}(x,j\omega) \stackrel{\Delta}{=} \{(m,\phi)\,|\,m \in [|d_p(x,\check{p}_M(j\omega))|,|d_p(x,\hat{p}_M(j\omega))|],$$

$$\phi \in [\arg d_p(x,\check{p}_A(j\omega)),\ \arg d_p(x,\hat{p}_A(j\omega))]\}$$

$$\mathcal{R}_\ell(j\omega) \stackrel{\Delta}{=} \{(m_\ell,\phi_\ell)\ |\,m_\ell(j\omega) \in [\underline{\ell}_M(\omega),\bar{\ell}_M(\omega)],\phi_\ell(j\omega) \in [\underline{\ell}_A(\omega),\bar{\ell}_A(\omega)],$$

$$m_\ell(\cdot),\ \phi_\ell(\cdot)\ \text{locally Lipschitz continuous}\}$$

In view of Lemma A6.5, the next result is immediate:

<u>Proposition A6.6</u>: Let $\mathcal{R}_i : \mathbb{R}^{n_x} \times \mathbb{R} \to 2^{\mathbb{R}}$ be defined as above for i = 1,2. Then the $\mathcal{R}_i$, i = 1,2 are locally Lipschitz continuous. ¤

Appendix 7: Solution of $\min_{(m_C,\phi_C)\in R_C(j\omega)} \max_{(m_p,\phi_p)\in R_p(x,j\omega)} |H_{yd}(x,j\omega,\alpha,\ell(j\omega))|$

Suppose

$$|H_{yd}(x,j\omega,\alpha,\ell(j\omega)| \neq 0 \qquad \forall\ \omega \geq 0,\ \alpha \in A,\ \ell \in L,\ x \in X$$

and

$$R_p(j\omega) = \{(m_p,\phi_p)|\check{M}_C(j\omega) \leq m_p(j\omega) \leq \hat{M}(j\omega), \check{\phi}(j\omega) \leq \phi_p(j\omega) \leq \hat{\phi}(j\omega)\} \tag{A7.1}$$

then

$$\min_{(m_C,\phi_C)\in R_C(j\omega)} \max_{(m_p,\phi_p)\in R_p(j\omega)} |H_{yd}(x,j\omega,\alpha,\ell(j\omega))|$$

$$= \min_{(m_C,\phi_C)\in R_C(j\omega)} \{[\min_{(m_p,\phi_p)\in R_p(j\omega)} (m_p^2 m_C^2+2m_p m_C \cos(\phi_p+\phi_C)]+1\}^{-1} \tag{A7.2}$$

Let

$$\phi_p(m_C,\phi_C) \triangleq \operatorname*{argmin}_{\phi_p\in[\underline{\phi}_p,\bar{\phi}_p]} \cos(\phi_p+\phi_C) \tag{A7.3}$$

$$m_p(m_C,\phi_C) \triangleq \operatorname*{argmin}_{m_p\in[\underline{m}_p,\bar{m}_p]} \{m_C^2 m_p^2+2m_C m_p \cos(\phi_p+\phi_C)\} \tag{A7.4}$$

$$\xi(m_C,\phi_C) \triangleq \cos(\phi_C+\phi_p(m_C,\phi_C)) \tag{A7.5}$$

$$\psi(m_C,\phi_C) \triangleq m_C^2 m_p^2(m_C,\phi_C) + 2m_C m_p(m_C,\phi_C)\xi(m_C,\phi_C) + 1. \tag{A7.6}$$

Now $\xi(m_C,\phi_C) \in [-1,1]$, $m_p \geq 0$ so that $\psi(m_C,\phi_C)$ represents a family of curves as shown in Fig 13. It follows from Fig. 13, that

$$m_P(m_C,\phi_C) = \begin{cases} \text{If } (m_C,\phi_C) < 0, & \\ \text{(a) } -\dfrac{\xi(m_C,\phi_C)}{m_C} & \text{if } \exists\, m_C \in [\underline{m}_C,\bar{m}_C] \ni -\dfrac{\xi(m_C,\phi_C)}{m_C} \in [\underline{m}_P,\bar{m}_P] \\ \text{(b) } \bar{m}_P & \text{if } \bar{m}_P < -\dfrac{\xi(m_C,\phi_C)}{m_C} \;\; \forall\, m_C \in [\underline{m}_C,\bar{m}_C] \\ \text{(c) } \underline{m}_P & \text{if } \underline{m}_P > -\dfrac{\xi(m_C,\phi_C)}{m_C} \;\; \forall\, m_C \in [\underline{m}_C,\bar{m}_C] \\ \text{If } \xi(m_C,\phi_C) \geq 0, & \\ \text{(d) } \underline{m}_P & \end{cases}$$

$$= \begin{cases} \text{If } (\phi_P(\phi_C)+\phi_C) \in (\pm\pi/2,\pm 3\pi/2) & \\ \text{(a) } -\dfrac{\xi(m_C,\phi_C)}{m_C} & \text{if } -\xi(m_C,\phi_C) \in [\underline{m}_P\underline{m}_C,\bar{m}_P\bar{m}_C] \\ \text{(b) } \bar{m}_P & \text{if } \bar{m}_P\bar{m}_C < -\xi(m_C,\phi_C) \\ \text{(c) } \underline{m}_P & \text{if } \underline{m}_P\underline{m}_C > -\xi(m_C,\phi_C) \\ \text{If } (\phi_P(\phi_C)+\phi_C) \in [-\pi/2,\pi/2],\ [\pm 3\pi/2,\pm 2\pi] & \\ \text{(d) } \underline{m}_P & \end{cases} \qquad \text{(A7.7)}$$

For $\phi_P \in [-\pi,\pi]$, $\phi_C \in [-\pi,\pi]$,

$$\phi_P(\phi_C) = \underset{\phi_P \in [\underline{\phi}_P,\bar{\phi}_P]}{\text{argmin}} \min\{|\pi-\phi_C-\phi_P|,|-\pi-\phi_C-\phi_P|\}$$

$$= \underset{\phi_P \in [\underline{\phi}_P,\bar{\phi}_P]}{\text{argmin}} [\min\{|\pi-\phi_C-\phi_P|,|\pi+\phi_C+\phi_P|\}] \qquad \text{(A7.8)}$$

Hence,

$$
\psi(m_C,\phi_C) = \begin{cases} \text{If } (\phi_p(\phi_C)+\phi_C) \in (\pm\pi/2, \pm 3\pi/2) & \\ \text{(a)}\ -\xi^2(m_C,\phi_C) + 1 & \text{if } -\xi(m_C,\phi_C) \in [\underline{m}_p\underline{m}_C, \bar{m}_p\bar{m}_C] \\ \text{(b)}\ m_C^2\bar{m}_p^2 + 2m_C\bar{m}_p\xi(m_C,\phi_C) + 1 & \text{if } \bar{m}_p\bar{m}_C < -\xi(m_C,\phi_C) \\ \text{(c)}\ m_C^2\underline{m}_p^2 + 2m_C\underline{m}_p\xi(m_C,\phi_C) + 1 & \text{if } \underline{m}_p\underline{m}_C > -\xi(m_C,\phi_C) \\ \text{If } (\phi_p(\phi_C)+\phi_C) \in [-\pi/2,\pi/2] \text{ or } [\pm 3\pi/2, \pm 2\pi] & \\ \text{(d)}\ m_C^2\underline{m}_p^2 + 2m_C\underline{m}_p\xi(m_C,\phi_C) + 1 & \end{cases} \tag{A7.9}
$$

so that

$$
\min_{(m_C,\phi_C)\in\mathcal{R}_C(j\omega)} \ \max_{(m_p,\phi_p)\in\mathcal{R}_p(x,j\omega)} |H_{yd}(x,j\omega,\alpha,\ell(j\omega))|
$$

$$
= \min_{(m_C,\phi_C)\in\mathcal{R}_C(\omega)} \psi(m_C,\phi_C) \tag{A7.10}
$$

where the minimization on the right hand side is easily computed from (A7.9).

FIGURE CAPTIONS

Fig. 1. Control system to be designed.

Fig. 2. Regions of allowed pole-zero variation for the plant,

$$P_0(s,\alpha) = \frac{10}{s+5} \frac{s+z_R}{(s+p_C)(s+p_C^*)} \text{ with } z_R \in [\frac{1}{2}, \frac{3}{2}]$$

$$p_C \in [-\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$$

Fig. 3a. Plant magnitude uncertainty with respect to multiplicative perturbations.

Fig. 3b. Plant phase uncertainty with respect to multiplicative perturbations.

Fig. 4. Region of allowed perturbations, $\{\ell(j\omega) | \ell \in L\}$ and approximating bounds, $R_{\ell-1}(j\omega)$, for $(\ell(j\omega)-1)$, $\ell \in L$.

Fig. 5. Parabolic exclusion region for modified Nyquist criterion.

Fig. 6. Typical region for S-stability, $\sigma = -\sqrt{k_1+k_2\omega^2}$

Fig. 7. Envelope of acceptable time response.

Fig. 8a. $\hat{\beta}_M(j\omega)$, $\hat{\gamma}_M(j\omega)$ for $\mathbb{C}_-^0$ perturbations.

Fig. 8b. $\check{\beta}_M(j\omega)$, $\check{\gamma}_M(j\omega)$ for $\mathbb{C}_-^0$ perturbations.

Fig. 9a. Real perturbations.

Fig. 9b. Complex perturbations occurring in conjugate pairs.

Fig. 10. Parabolas corresponding to $\gamma^4 + 2\gamma^2(\beta^2-\omega^2)$.

Fig. 11. Graphs of $(\frac{2\beta\omega}{\beta^2+\gamma^2-\omega^2})$.

Fig. 12. Construction of rectangular approximation to $P_0(j\omega,A)$.

Fig. 13. Parabolas representing $\psi(m_c,\phi_c)$.

Fig. 14. Feasibility test for inequality (3.21a).

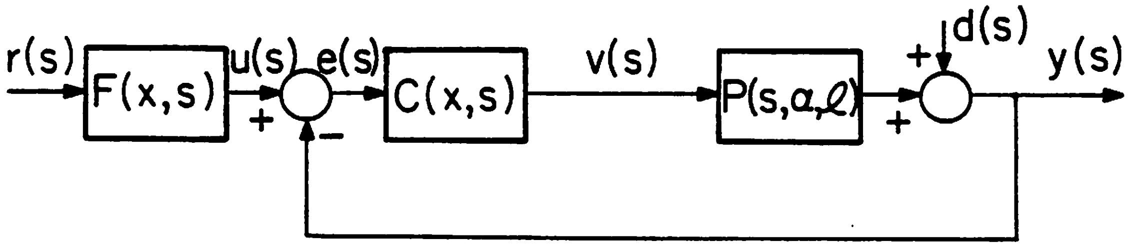


Fig. 1

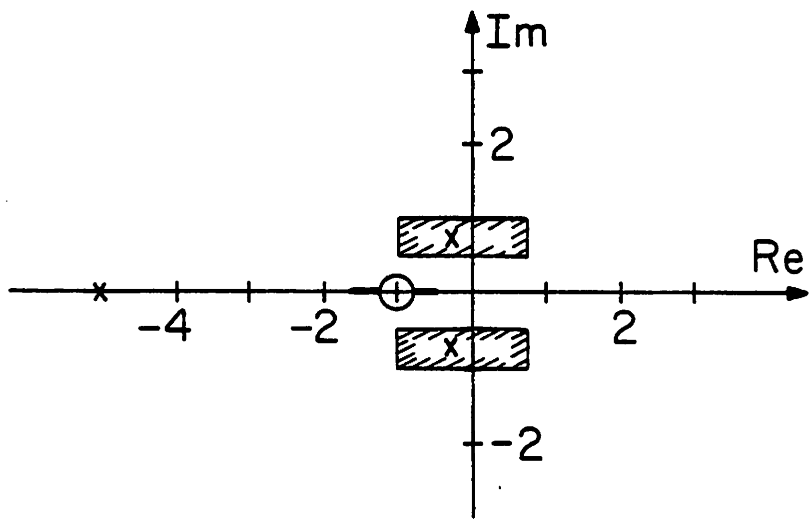


Fig. 2

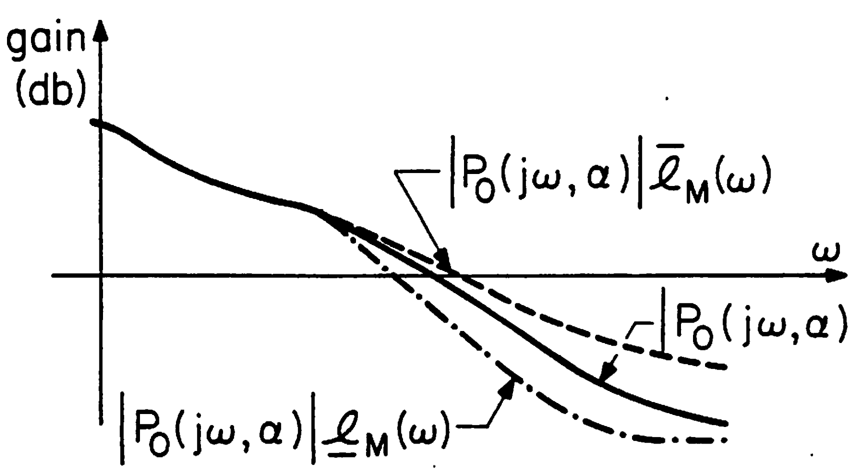


Fig. 3a

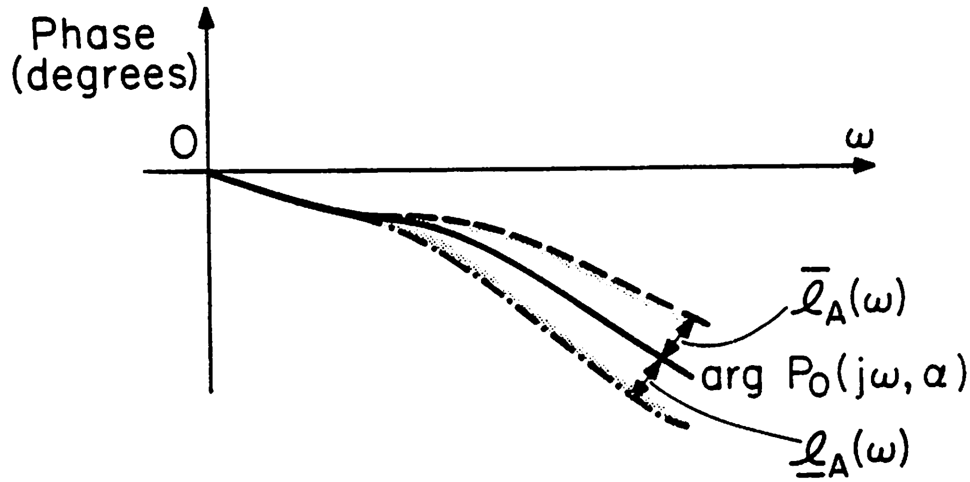

Phase
(degrees)
O
ω
$\bar{\ell}_A(\omega)$
arg $P_0(j\omega, \alpha)$
$\underline{\ell}_A(\omega)$

Fig. 3b

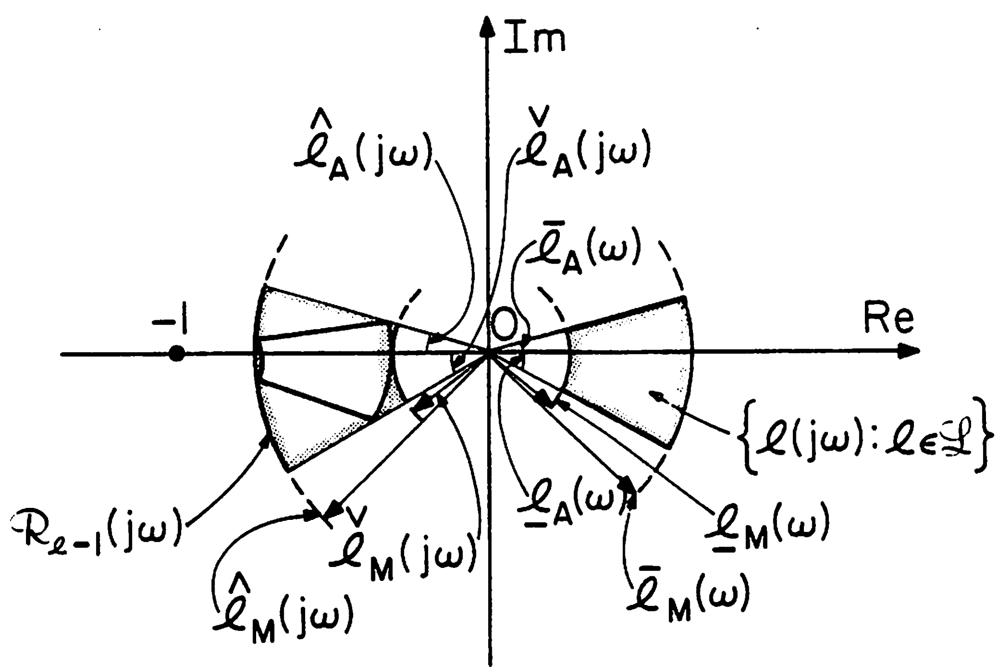

Im
$\hat{\ell}_A(j\omega)$
$\check{\ell}_A(j\omega)$
$\bar{\ell}_A(\omega)$
-1
O
Re
$\{\ell(j\omega):\ell\in\mathcal{L}\}$
$\mathcal{R}_{\ell-1}(j\omega)$
$\underline{\ell}_A(\omega)$
$\underline{\ell}_M(\omega)$
$\check{\ell}_M(j\omega)$
$\bar{\ell}_M(\omega)$
$\hat{\ell}_M(j\omega)$

Fig. 4

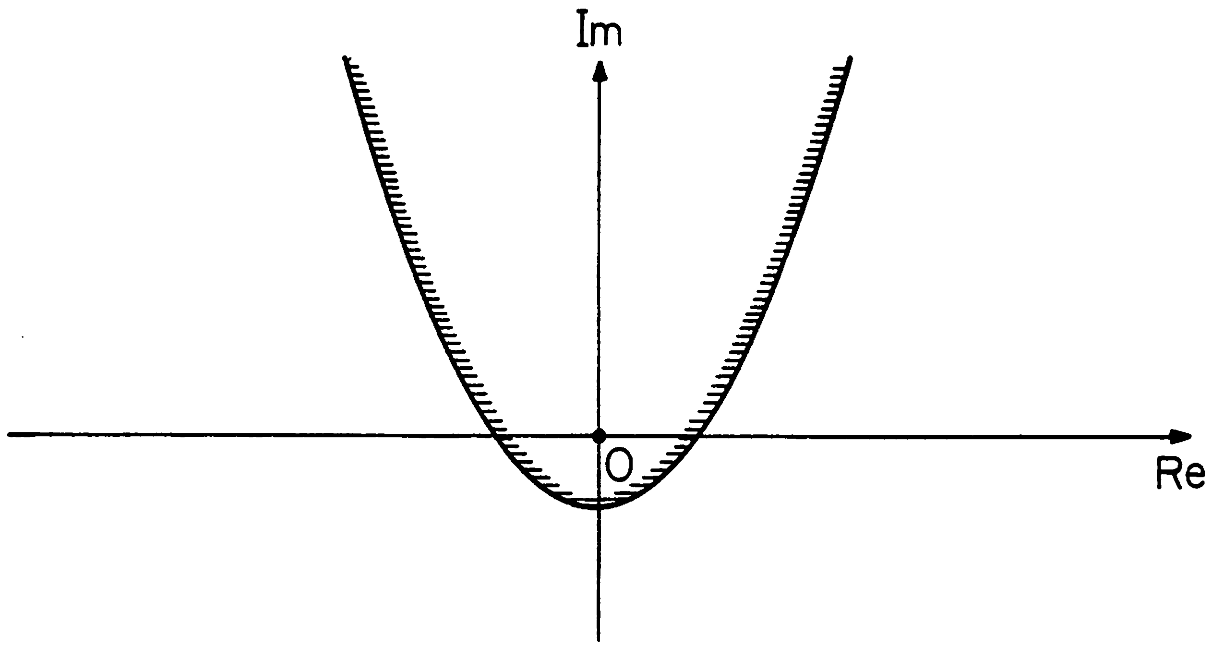


Fig. 5

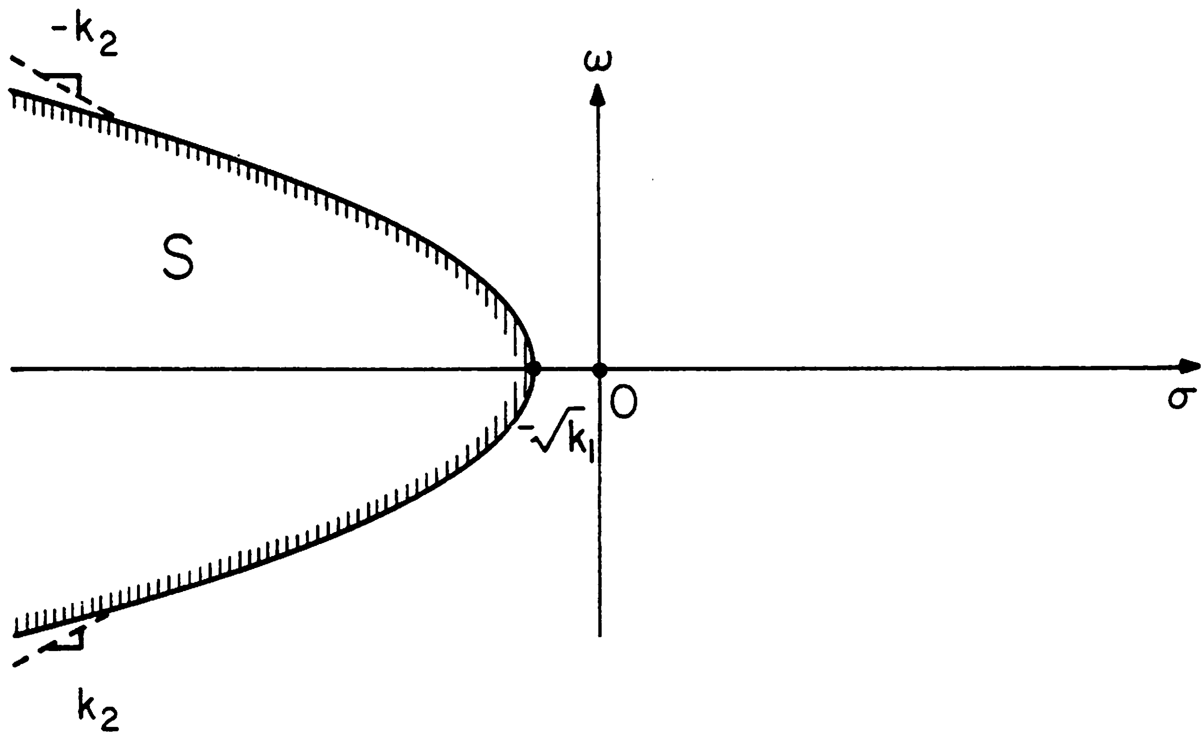


Fig. 6

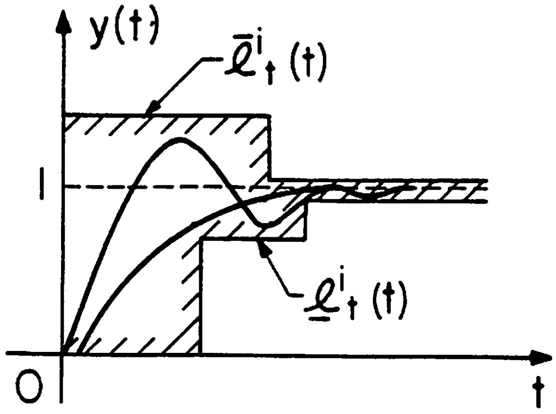


Fig. 7

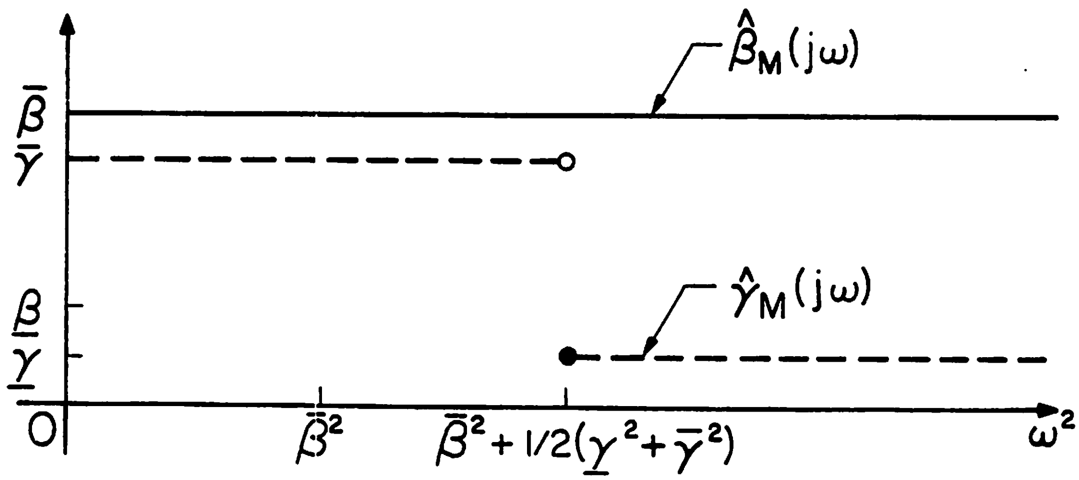


Fig. 8(a)

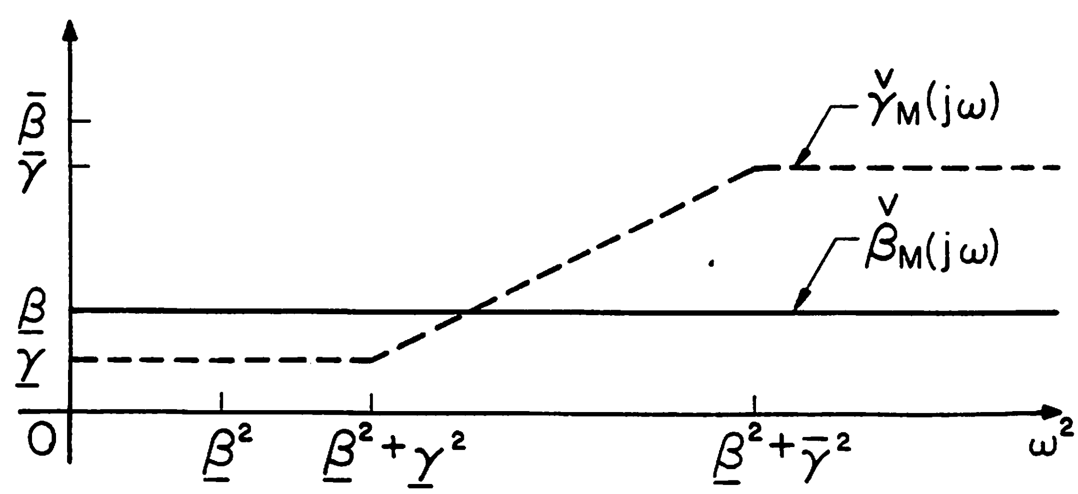


Fig. 8(b)

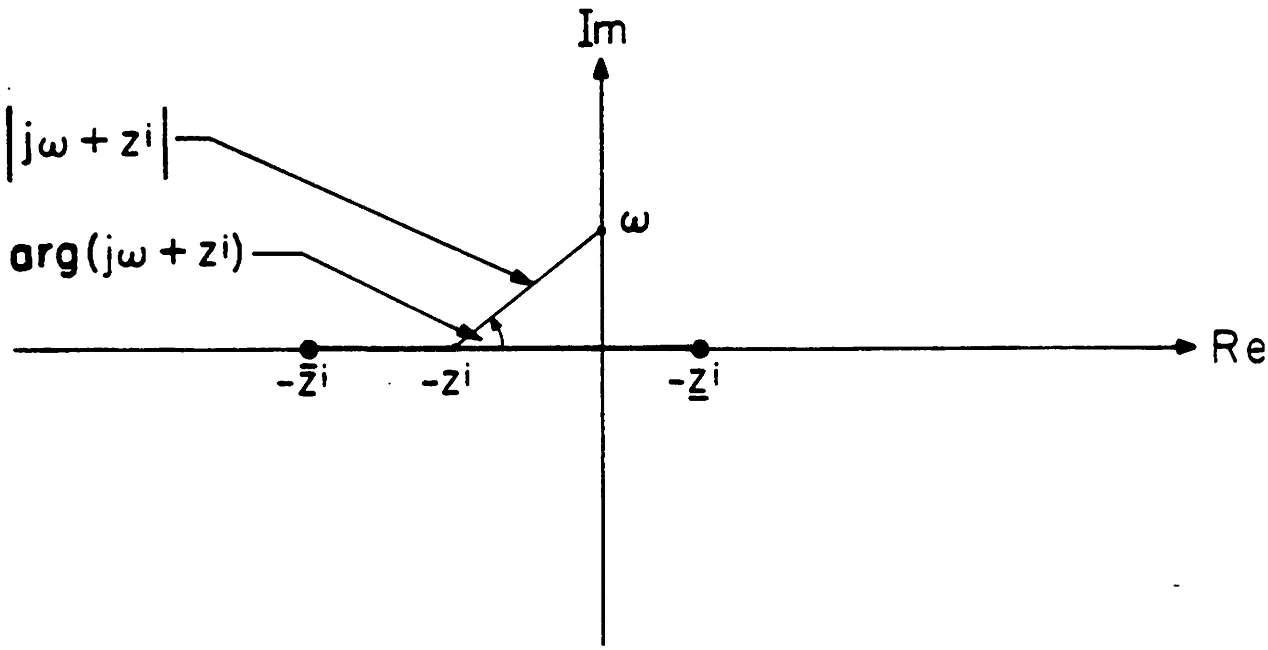

Im
|jω + zi|
arg(jω + zi)
ω
Re
-z̄i
-zi
-z̲i

Fig. 9(a)

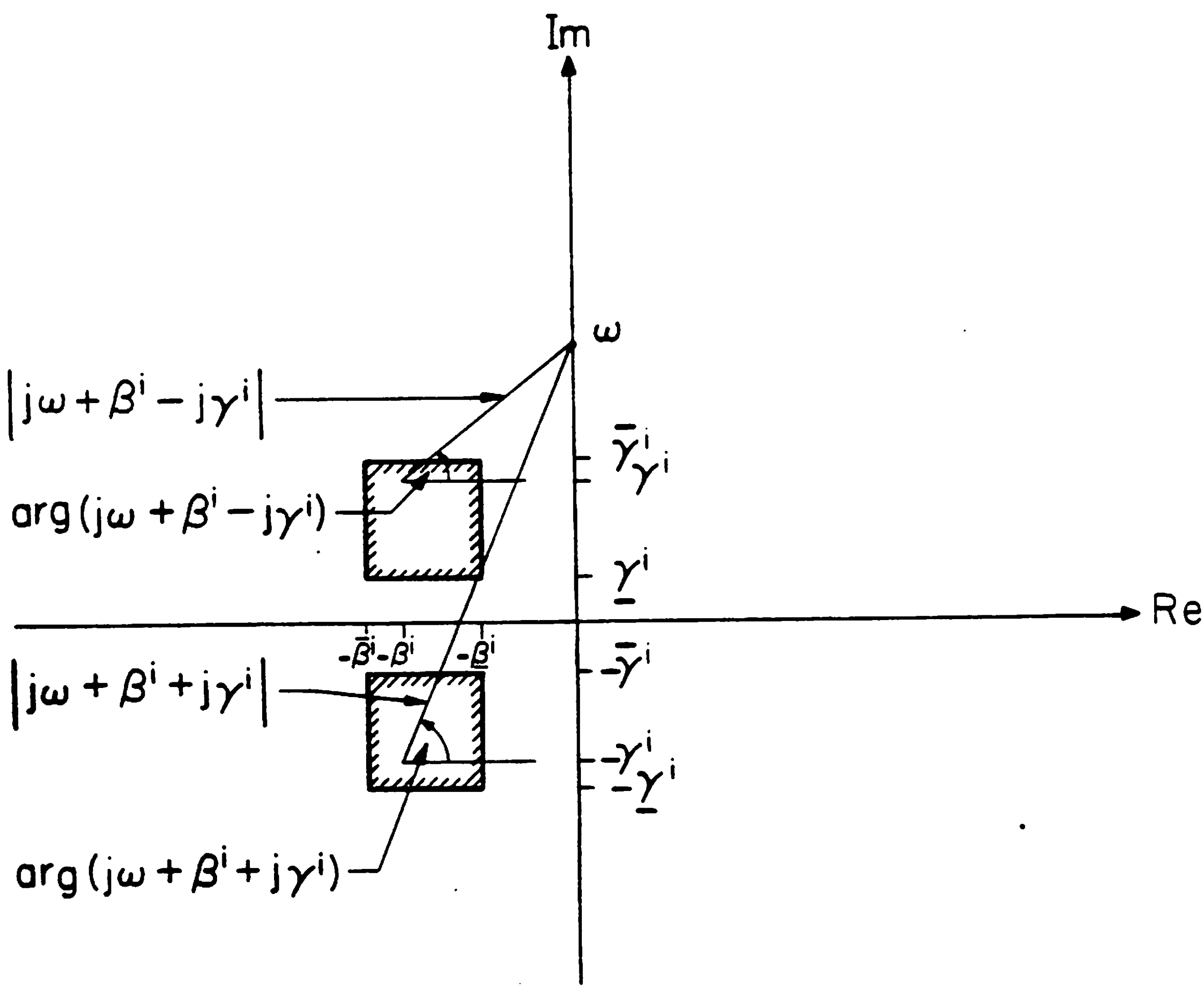

Im
ω
|jω + βi − jγi|
arg(jω + βi − jγi)
γ̄i
γi
γ̲i
Re
-β̄i
-βi
-β̲i
|jω + βi + jγi|
-γ̄i
-γi
-γ̲i
arg(jω + βi + jγi)

Fig. 9(b)

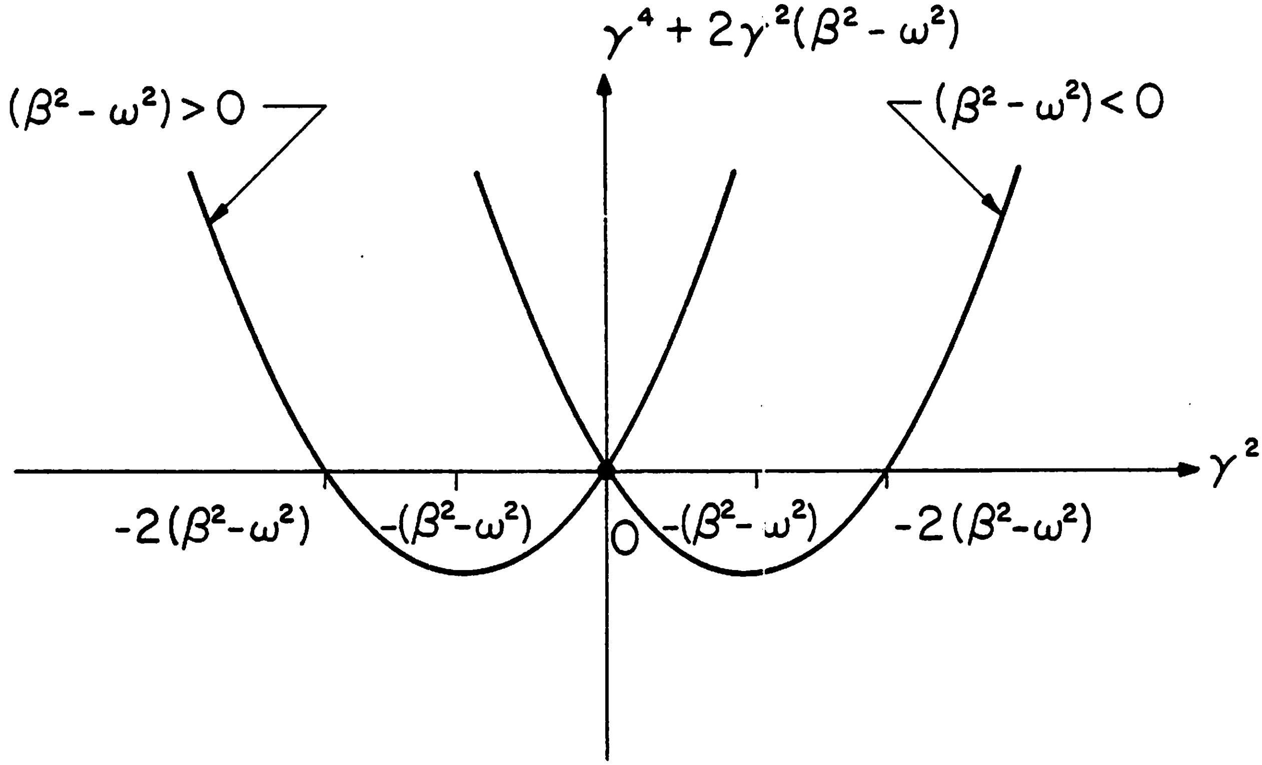
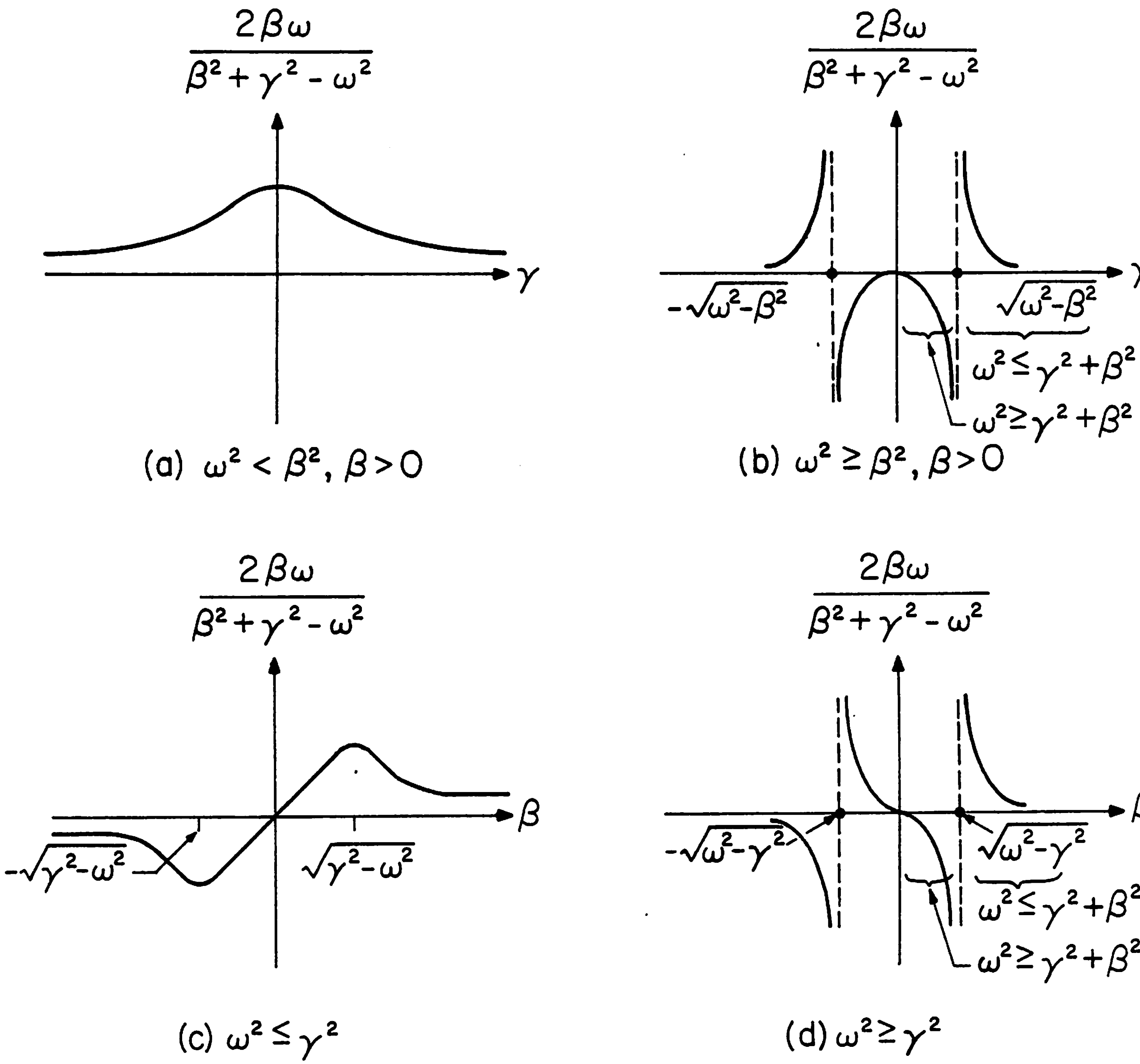


Fig. 10



$\frac{2\beta\omega}{\beta^2+\gamma^2-\omega^2}$
γ
(a) $\omega^2 < \beta^2, \beta > 0$
$\frac{2\beta\omega}{\beta^2+\gamma^2-\omega^2}$
γ
$-\sqrt{\omega^2-\beta^2}$
$\sqrt{\omega^2-\beta^2}$
$\omega^2 \leq \gamma^2+\beta^2$
$\omega^2 \geq \gamma^2+\beta^2$
(b) $\omega^2 \geq \beta^2, \beta > 0$
$\frac{2\beta\omega}{\beta^2+\gamma^2-\omega^2}$
β
$-\sqrt{\gamma^2-\omega^2}$
$\sqrt{\gamma^2-\omega^2}$
(c) $\omega^2 \leq \gamma^2$
$\frac{2\beta\omega}{\beta^2+\gamma^2-\omega^2}$
β
$-\sqrt{\omega^2-\gamma^2}$
$\sqrt{\omega^2-\gamma^2}$
$\omega^2 \leq \gamma^2+\beta^2$
$\omega^2 \geq \gamma^2+\beta^2$
(d) $\omega^2 \geq \gamma^2$

Fig. 11

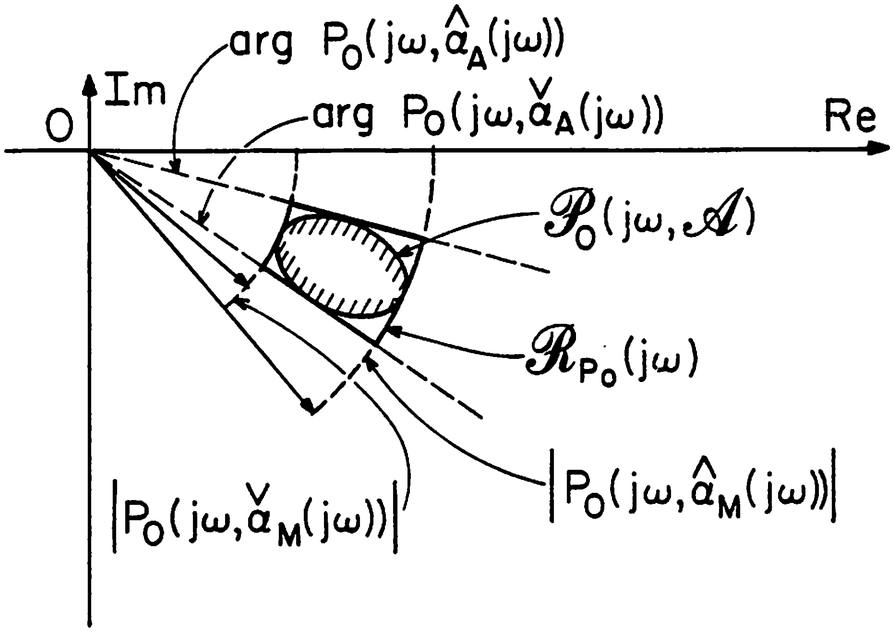

arg $P_O(j\omega, \hat{\alpha}_A(j\omega))$
Im
O
arg $P_O(j\omega, \check{\alpha}_A(j\omega))$
Re
$\mathscr{P}_O(j\omega, \mathscr{A})$
$\mathscr{R}_{P_O}(j\omega)$
$|P_O(j\omega, \hat{\alpha}_M(j\omega))|$
$|P_O(j\omega, \check{\alpha}_M(j\omega))|$

Fig. 12

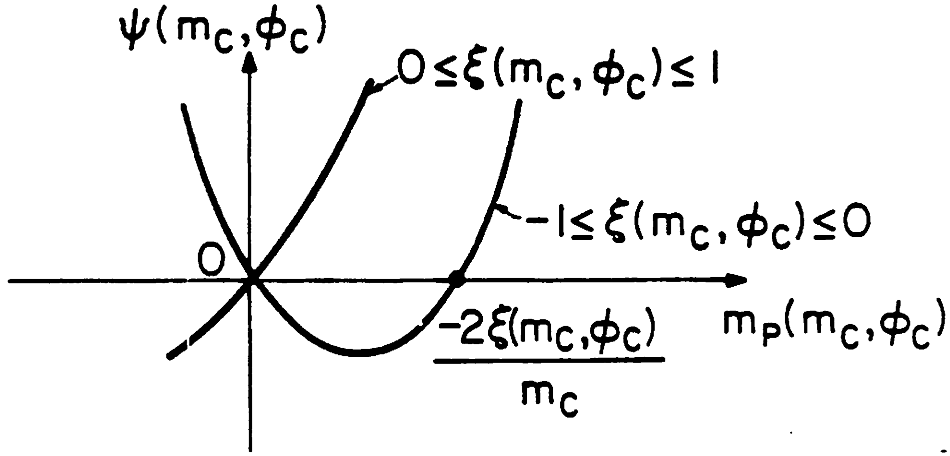

ψ(m_C, φ_C)
0≤ξ(m_C, φ_C)≤1
-1≤ξ(m_C, φ_C)≤0
O
-2ξ(m_C, φ_C) / m_C
m_P(m_C, φ_C)

Fig. 13

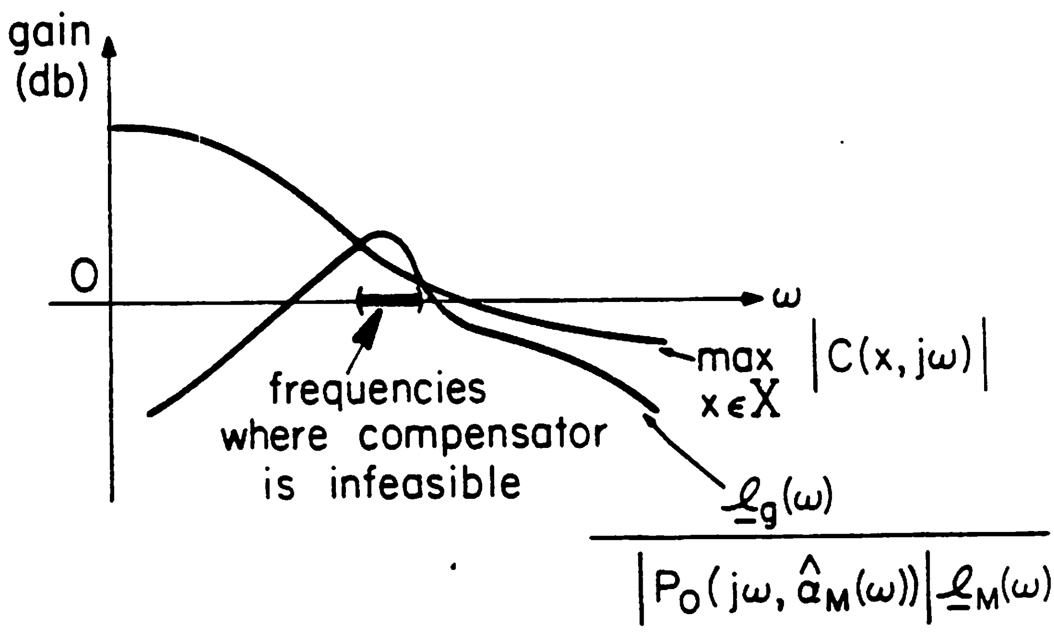

gain
(db)
O
ω
max |C(x, jω)|
x∈X
frequencies
where compensator
is infeasible
ℓ_g(ω) / |P_O(jω, α̂_M(ω))| ℓ_M(ω)

Fig. 14