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STOCHASTICITY AND RECONNECTION IN  
TWO-DIMENSIONAL MAPPING

by

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# STOCHASTICITY AND RECONNECTION IN TWO-DIMENSIONAL MAPPINGS

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## *ABSTRACT*

Reconnection is studied by means of a model mapping in which the island centers are shifted in phase. This leads to reconnection of KAM surfaces and necessitates a reexamination of the island overlap criterion for the breakdown of adiabatic barriers between island chains. An analytic reconnection threshold is derived from an averaged Hamiltonian and found to agree well with numerical surfaces of section.

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## STOCHASTICITY AND RECONNECTION IN TWO-DIMENSIONAL MAPPINGS

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Reconnection plays an important role in a variety of physical problems, including RF acceleration in particle accelerators <sup>1</sup>, motion of magnetic field lines <sup>2</sup>, particle motion in two-dimensional potentials <sup>3</sup>, wave-particle interactions <sup>4,5</sup>, and possibly the free-electron laser <sup>6</sup>. In many of these cases involving nonintegrable systems one is concerned with the extent to which the motion is stochastic or regular. For example, in ion or electron cyclotron resonance heating, regular phase space curves, called Kolmogorov- Arnold-Moser (KAM) curves, can present barriers to stochastic heating. Stochasticity of magnetic field lines can lead to rapid particle and energy loss in fusion devices. In this Letter we show that there is an intimate relationship between stochasticity and reconnection, with the result that reconnection can effectively destroy an adiabatic barrier. Reconnection thresholds are derived for a model problem and are found to agree well with numerically computed barriers.

Many dynamical systems of current interest are particular cases of the radial twist mapping

$$x' = x - K \sin \theta$$

$$\theta' = \theta + f(x) \tag{1}$$

where  $K$  is a constant and  $f(x)$  is analytic in some domain. For example,  $f(x) = x$  gives the Taylor-Chirikov map <sup>7</sup>, while  $f(x) = 1/x$  yields the Fermi map <sup>8</sup>. The general case  $f(x) = x^n$  has also been studied <sup>9</sup>. Now consider a function  $f(x)$  whose inverse is multivalued, so that  $f(x) - 2\pi n = 0$  has multiple roots, corresponding to families of island chains. If  $f(x;\alpha)$  depends continuously on the parameter  $\alpha$  such

that  $f' = \partial f / \partial x$  changes sign as  $\alpha$  is varied, then pairs of island chains merge whenever  $f' = 0$ . From the tangent map

$$L = \begin{bmatrix} 1 & -K \cos \theta \\ f' & 1 - K f' \cos \theta \end{bmatrix} \quad (2)$$

it may be shown that the rotational sense of islands of any order is  $- \text{sgn}(f')$ . The merging of such counter-rotating islands is necessarily accompanied by the reconnection of their separatrices, which can happen in one of two ways. Since the period-one islands lie only at  $\theta = 0$  or  $\pi$ , the island centers are either aligned or shifted by  $\pi$ ; similar restrictions apply approximately to higher order islands.

When aligned counter-rotating islands merge, a vortex, similar to Hill's vortex, is formed. When staggered islands merge, their separatrices form a chain of loops, as depicted in Fig. 1. The latter mode of reconnection is the only possible one for period-one islands, since  $\text{Tr}L - 2 = -K f' \cos \theta$  changes sign with  $f'$ . This scenario was first observed by Symon and Sessler<sup>1</sup> in calculating beam stacking in particle accelerators; their mapping can in fact be put in the form (1). Mappings of this form also occur in our studies of multifrequency ECRH<sup>10</sup> where they arise in calculating resonance overlap in a four-dimensional phase space.

In general, reconnection might be defined as a topological rearrangement of level curves in which critical points do not change their type (as opposed to bifurcations, in which critical points may be created or destroyed, or change type.) Examination of a number of occurrences of reconnection in two-dimensional potentials  $V_i = V(x, y; \alpha_i)$  suggests that this always involves the merging of two separatrices. Thus, a necessary condition for reconnection is that  $V_1 = V_2$ . For example, it may be shown that the Hénon-Heiles potential<sup>11</sup> is a reconnection point of the more general Hall-McNamara potential<sup>3</sup>. In the case of the radial twist map (1) a reconnection threshold may be derived using the averaged Hamiltonian<sup>12</sup>

$$\bar{H}(x, \theta) = \int_{x_1}^x [f(\xi) - 2\pi n] d\xi - K \cos \theta \quad (3)$$

which yields a continuous approximation to the mapping in the vicinity of a fixed point. Reconnection occurs when the upper and lower separatrices have the same value of  $\bar{H}$ . With no loss of generality, suppose that the upper island chain in Fig. 1. has x-points at  $\pm\pi$ , so that the lower separatrix passes through x-points at  $\theta = 0$  and  $2\pi$ . From Eq.(3) the lower and upper separatrices are given by  $\bar{H}_{ls} = +K$  and

$$\bar{H}_{us} = \int_{x_1}^{x_2} [f(\xi) - 2\pi n] d\xi - K \quad (4)$$

Equating  $\bar{H}_{us} = \bar{H}_{ls}$  then gives the reconnection threshold

$$K(\alpha) = \frac{1}{2} \int_{x_1(\alpha)}^{x_2(\alpha)} [f(\xi; \alpha) - 2\pi n] d\xi. \quad (5)$$

As we shall see, this simple formula often gives a useful estimate for the breakdown of an adiabatic barrier.

The simplest mapping which exhibits reconnection is the "logistic twist map," for which

$$f(x) = x - \alpha x^2 \quad (6)$$

where  $\alpha > 0$ . The period-one fixed points are located at

$$x_n^\pm = \frac{1}{2\alpha} (1 \pm \sqrt{1 - 8\pi n \alpha}). \quad (7)$$

For positive  $n$  both roots are positive real for  $0 \leq 8\pi n \alpha \leq 1$ , coalescing when  $\alpha^* = (8\pi n)^{-1}$  at  $x_n^* = 4\pi n$ . The  $x_n^-$  reduce to the standard mapping island centers as  $\alpha \rightarrow 0$ ; the  $x_n^+$  islands are born at  $x = \infty$ , descending and merging sequentially with the rising  $x_n^-$  as  $\alpha$  is increased (with the exception of the  $n = 0^-$  island, which is

unperturbed). Identifying  $x_1 = x_n^-$  and  $x_2 = x_n^+$  and evaluating the integral in (7) gives the reconnection threshold

$$K_r(\alpha) = \frac{(1 - 8\pi n\alpha)^{3/2}}{12\alpha^2}. \quad (8)$$

Equation (8) has been verified by visual inspection of numerical surfaces of section for the case  $n = 1$  and  $K \leq 2.5$ ; above this value the stochastic layers surrounding the islands obscure the reconnection process. Figure 2 shows typical phase plots in the vicinity of the  $n = 1$  islands before and after reconnection. Notice the band of KAM curves separating the upper and lower islands in Fig. 2a, which form a ("type I") barrier to orbits initialized in the stochastic region near  $x = 0$ . After reconnection a second ("type II") barrier exists, with the upper island now topologically below the previously lower island. There is at present no theoretical method for predicting the existence or destruction of KAM barriers between staggered islands.

The second mode of reconnection, vortex formation, is shown in Fig. 3 for the period-two aligned islands. The upper and lower elliptic fixed points are located at

$$x_m^\pm = \frac{1}{2\alpha}(1 \pm \sqrt{1 - 4\pi m\alpha}) \quad (9)$$

where  $m$  is an odd integer. As  $\alpha$  is increased, the x-points move together nearly vertically, joining at the reconnection point, after which they move apart horizontally. As  $\alpha$  is increased further, each vortex shrinks as a unit, vanishing when  $\alpha^* = (4\pi m)^{-1}$ , an apparently previously unobserved kind of "multifurcation." While a reconnection threshold may be obtained by deriving an averaged Hamiltonian for the period-two islands, it is easier in this case to work directly with the period-two mapping equations. Requiring that the x-points merge then gives

$$K_r(\alpha) = \frac{1}{\alpha}\sqrt{1 - 4\pi m\alpha}. \quad (10)$$

This result has been verified visually to more than five significant figures. We have also observed reconnection of higher order islands up to period six, similar to those seen by Fukuyama <sup>4</sup>. In general, even-order islands form vortices and odd-order islands make loops. A reconnection threshold may also be calculated for the third-order islands, which play a significant role in barrier formation.

Extensive numerical calculations of the type I barrier have been carried out for the  $n = 1$  islands by fixing  $\alpha$  and following single orbits for increasing  $K$  until a breakthrough was observed. The results are shown in Fig. 4 for  $0.020 \leq \alpha \leq 0.040$ , along with the reconnection threshold (8). The dashed line is the bifurcation threshold  $K_{bif} = 4(1 - 8\pi n\alpha)^{-1/2}$  (found by setting  $Tr L = -2$ ), above which the notion of reconnection becomes meaningless. The most striking feature of this figure is the convergence of the barrier data to the reconnection curve for  $\alpha \geq 0.031$ . While one would expect approximate agreement as the stochastic layers diminish with decreasing  $K$ , the convergence is rapid, even in the presence of a thick stochastic layer. For example, at  $\alpha = 0.035$ , where Fig. 5 reveals a very thick stochastic layer near the barrier ( $K = 2.82$ ), the agreement with  $K_r$  is better than one part in  $10^3$ ; at  $\alpha = 0.0365$  the relative difference is only  $3 \times 10^{-6}$ ! The sharp minima in the barrier data at  $\alpha \approx 0.0235$  and  $0.0290$  are due to reconnection of period-three islands, while the dip near  $\alpha = 0.0255$  is a consequence of the period-two vortices depicted in Fig. 3. It may be shown that the barrier data also lie near the reconnection thresholds in these regions. The close agreement between  $K_b$  and  $K_r$  is surprising both because  $K_r$  was calculated from an approximate Hamiltonian and because of the apparent lack of influence of the stochastic layers. These and other questions leave considerable scope for future work.

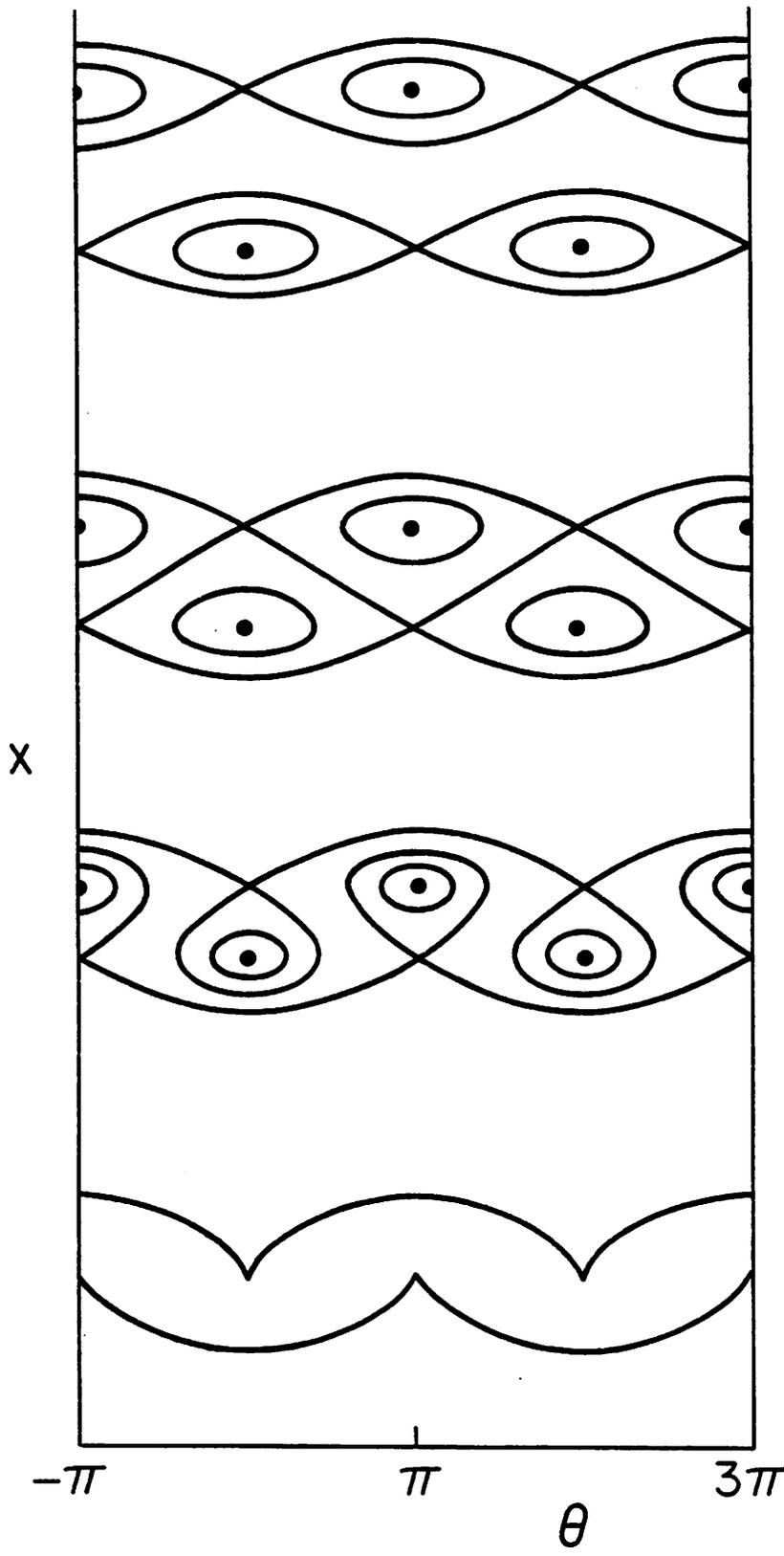
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FIGURE CAPTIONS

- FIG.1. Reconnection scenario for logistic twist map (two periods are shown for clarity).
- FIG.2. Surfaces of section for logistic twist map for  $K = 1.5$  (a) before reconnection ( $\alpha = 0.036$ ) and (b) after reconnection ( $\alpha = 0.038$ ). Both  $x$  and  $\theta$  are in degrees.
- FIG.3. Surfaces of section for  $K = 4$ , showing vortex formation. (a)  $\alpha = 0.0260$ ; (b)  $\alpha = 0.02635$ .
- FIG.4. Computed adiabatic barriers between  $n = 1$  islands compared with reconnection threshold  $K_r(\alpha)$ .
- FIG.5. Surface of section near barrier breakdown for  $\alpha = 0.035$  and  $K = 2.5$ . Note the very thick stochastic layer.



(a)  
 $\alpha < \alpha_r$

(b)  
 $\alpha = \alpha_r$

(c)  
 $\alpha_r < \alpha < \alpha^*$

(d)  
 $\alpha = \alpha^*$

