NONLINEAR OP-AMP CIRCUITS: EXISTENCE AND

UNIQUENESS OF SOLUTION BY INSPECTION

by

T. Nishi and L. O. Chua

Memorandum No. UCB?ERL M83/32

1 June 1983

NONLINEAR OP-AMP CIRCUITS: EXISTENCE AND UNIQUENESS OF SOLUTION BY INSPECTION

by

T. Nishi and L. O. Chua

Memorandum No. UCB/ERL M83/32

1 June 1983

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

NONLINEAR OP-AMP CIRCUITS: EXISTENCE AND UNIQUENESS OF SOLUTION BY INSPECTION†

Tetsuo Nishi†† and Leon O. Chua

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

Even simple circuits containing only one op amp and linear resistors can have multiple dc operating points. Using a realistic nonlinear dc op-amp model which includes the saturation characteristics, this paper gives the necessary and sufficient conditions for an arbitrary op-amp circuit (containing op amps, linear resistors, strictly-increasing nonlinear resistors, and independent sources) to have a unique solution for all values of circuit parameters. These conditions are remarkable because they are couched strictly in topological terms. For many op-amp circuits (e.g., those containing only one op amp), the necessary and sufficient conditions can be checked by inspection.

---

†This research is supported in part by a fellowship of the Ministry of Education of Japan, by the Office of Naval Research under Contract N00014-76-C-0572, and by the Alexander von Humboldt Foundation under a Humboldt Award for U.S. Senior Scientists.

††T. Nishi is with the Department of Computer Science and Communication Engineering, Kyushu University. He is presently on leave at the University of California, Berkeley, California.

1. INTRODUCTION

This paper gives several criteria for testing a nonlinear op-amp circuit to have a unique solution for all circuit parameters. The following circuit elements are allowed: (1) dc voltage and/or current sources, (2) positive linear resistors, (3) nonlinear resistors characterized by strictly monotone increasing v-i curves, and (4) op-amps modelled by a realistic nonlinear dc model. In general such a circuit may have a unique solution, multiple solutions, or no solution depending on the circuit parameters. However, circuits having certain topological structures have a unique solution for all circuit parameters. For example, Nielsen and Willson gave a necessary and sufficient topological condition for a transistor circuit to have a unique solution [1]. According to [1], any circuit containing transistors, linear passive resistors and dc sources has a unique solution for all circuit parameters if and only if it contains no feedback structure. Our objective in this paper is to derive analogous (but different) topological criteria for op-amp circuits.

In this paper the op-amp in Fig. 1(a) is modelled by a nonlinear voltage-controlled voltage source (VCVS) in Fig. 1(b). Here the function $f(v_1)$ is described by the "saturation" characteristic shown in Fig. 2, or Fig. 3. In Fig. 2 we assume that

(1) f is a continuous and strictly monotone-increasing function.

(2) $f'(0)=\infty$ where the prime denotes the derivative of the function.

(3) $f(0)=0$.

In practice, we can replace (2) by

(2') $f'(0)$ is sufficiently large.

Furthermore we assume that:

Assumption: One of the two output terminals of each op amp is grounded. (1)

Although satisfied by most practical op-amp circuits, this assumption is nevertheless not essential. Indeed all theorems except for the last Corollary in Section 3 hold without this assumption.

Throughout this paper we refer to an op-amp modelled by Fig. 2 or Fig. 3 as Model C ("C" means "continuous") or Model D ("D" means "discontinuous"), respectively.

Our criteria are graph theoretic and can be applied by inspection for many simple circuits. For example, consider the circuit in Fig. 4(a).[1] The graph G associated with this circuit is shown in Fig. 4(b).[2] Here edges 1 and $\hat{1}$ represent the input and output port of the op-amp. Theorems 3 and 4 in Section 3 assert that this circuit has a unique solution for both Model C and Model D op-amps. For, G can be reduced by graph-theoretic operations (to be described in Section 2) to the graph shown in Fig. 5, but not Fig. 6. If the polarity of the input port of the op-amp is reversed, we will show the reduced graph is as shown in Fig. 6. Our theorems then assert that in this case the circuit does not have a unique solution.

Our main theorems are stated in terms of a new topological structure called a "cactus graph" to be defined in Section 2. For the moment, we simply state that the graphs in Figs. 5 and 6 are the simplest examples of cactus graphs.

Section 3 presents the main existence and uniqueness theorems along with many illustrative examples. Since the detailed proof of the theorems is rather long, we give only the main steps of the proof in Section 4. Additional details are given in the Appendix in terms of lemmas and their proofs. Reader interested only in applications may skip this section.

2. GRAPH REPRESENTATION, GRAPH OPERATION AND SPECIAL GRAPHS

In order to state the various topological criteria in this paper simply, and without ambiguity, it is essential that all notations, symbols, and graph operations be defined precisely. We will collect all of them here so that readers who have forgotten them can turn quickly to this short section for reference. To help the reader in remembering some of the more commonly used notations and terminologies, we have carefully chosen mnemonics for deciphering them.

A. Associated Graph of a Circuit

An associated graph, denoted by G, of a circuit is obtained from the circuit as follows:

---

[1] This example will be treated in Section 3 in more detail.

[2] Exact definition will be given in Section 2.

(1) Each independent voltage (resp., current) source is short-circuited (resp., open-circuited).

(2) Each 2-terminal resistor (linear or nonlinear) is represented by a non-directed edge.

(3) Each op amp in Fig. 1(a) is represented by a pair of directed edges whose directions are specified as shown in Fig. 1(c).[3] These two edges are labelled by the same number with a hat "^" added to that of the output edge. For example, edges 3 and $\hat{3}$ denote the input and output edges of op-amp #3, respectively.

Without loss of generality G is assumed to be connected.

B. Graph Operation

The topological criteria in Section 3 require the given graph G to be reduced into various simpler graphs via a combination of the following graph operations:

1. Open-circuit Operation $\mathcal{O}(\cdot)$.

Given an edge k, the operation $\mathcal{O}(k)$ deletes the connecting line but leaves the node intact as shown in Fig. 7(a).

2. Short-circuit Operation $\mathcal{S}(\cdot)$.

Given an edge k, the operation $\mathcal{S}(k)$ deletes the edge and coalesces the 2 nodes into one node as shown in Fig. 7(b).

3. Open/Short Operation $\mathcal{O}/\mathcal{S}(\cdot)$.

(a) Given a resistor edge R, the operation $\mathcal{O}/\mathcal{S}(R) \triangleq \mathcal{O}(R)$ or $\mathcal{S}(R)$, i.e., replace R by either Fig. 7(a) or Fig. 7(b).

(b) Given a pair of edges associated with an op-amp OA, the operation $\mathcal{O}/\mathcal{S}(OA)$ consists of open-circuiting one edge (either the input or output edge) and short-circuiting the second edge, as shown in Fig. 8.

4. Zero Operation $Z(\cdot)$.

This operation sets an op amp, OA, to zero in the usual way: Let a pair of edges associated with the op amp OA be $(k,\hat{k})$. Then $Z(OA)$ means $\mathcal{O}(k)$

---

[3]Note that the edge associated with a + and - sign is directed from + to -.

and $S(\hat{k})$ as shown in Fig. 9.

C. Cactus Graph and Graph with a Complementary Tree Structure

These two graphs contain only the input and output edges of the op-amps and play a very important role in this paper. Denote the n pairs of edges associated with the op-amps by $(k,\hat{k})$ $(k=1,2,\ldots,n)$ and denote a connected graph (to be defined later) containing these edges by G_0.

1. Cactus Graph

To help visualize a cactus graph, consider a typical cactus plant shown in Fig. 10(a), consisting of leaves (shaded area) "hinged" between the top and the bottom only. The graph G_0 made up of the boundaries of the leaves, as shown in Fig. 10(b), is called a cactus graph iff it satisfies the following properties:

1) every loop is made of exactly 2 edges, k and $\widehat{k+1}$ $(k=1,2,\ldots,n;$ $\widehat{n+1}=\hat{1})$.[4]

2) every cutset is made of exactly 2 edges.

Formally, a cactus graph is defined by a fundamental loop matrix having the following structure.

$$B = \begin{array}{c} \\ \hat{1} \\ \hat{2} \\ \hat{3} \\ \vdots \\ \hat{n} \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & \ldots & n & \hat{1} & \hat{2} & \hat{3} & \ldots & \hat{n} \end{array} \\ \left[\begin{array}{ccccc|ccccc} 0 & & & & \varepsilon_n & 1 & & & & \\ \varepsilon_1 & 0 & & & & & 1 & & & \\ & \varepsilon_2 & 0 & & & & & 1 & & \\ & & & \ddots & & & & & \ddots & \\ & & & \varepsilon_{n-1} & 0 & & & & & 1 \end{array}\right] \end{array} \qquad (2)$$

where $\varepsilon_i = \pm 1$. Several cactus graphs are shown in Figs. 10(c)-(e). It follows from Assumption (1) (i.e., one output terminal of each op amp is grounded) that all cactus graphs encountered in this paper are of the form shown in Fig. 10(c). However, if we don't assume (1), we would encounter other cactus graphs, such as those shown in Figs. 10(b) and (e). Note that each leaf of

[4] The edges should be labelled as shown in Fig. 10(b). Note that the 2 edges k and $\widehat{k+1}$ pertain to 2 different op amps: edge k denotes the input edge of the kth op amp, whereas edge $\widehat{k+1}$ denotes the output edge of the (k+1)th op amp.

a cactus graph consists of 2 edges labelled consecutively (except the last number or when the graph has only 2 edges), one pertaining to an input edge of one op-amp, the other to the output edge of another op amp. These 2 edges form a loop. In the following topological criteria, each loop associated with a leaf of a cactus graph is said to be similarly directed iff the 2 edges are directed in the same direction (clockwise or counter-clockwise).

2. Graph with a Complementary Tree Structure

The graph G_0 alluded to above is said to have a complementary tree structure if and only if both the input edges $\{k|k=1,2,\ldots,n\}$ and the output edges $\{\hat{k}|k=1,2,\ldots,n\}$ form a tree of G_0. For example, the graph shown in Fig. 11 has a complementary tree structure. For, both input edges 1,2,3 and output edges $\hat{1},\hat{2},\hat{3}$ form a tree of the graph.

3. TOPOLOGICAL CRITERIA BY INSPECTION

In this section we present several toplogical criteria for determining, by inspection, whether a given op-amp circuit N has a unique solution for all circuit parameters. The circuit N may contain dc voltage and/or current sources, positive linear resistors, nonlinear resistors whose v-i characteristics are represented by strictly monotone-increasing onto functions g_μ ($\mu=1,2,\ldots$), and op amps (Model C or Model D). Throughout this paper, the phrase "for all circuit parameters" means for any choice of positive resistance for the linear resistors, any value of dc sources, any strictly-increasing onto function g_μ satisfying $g_\mu(0)=0$ for the nonlinear resistors, and any output saturation voltage for the op amps. Furthermore dc sources are allowed to be connected at any location in the circuit.

The following criteria are applied to one or more simplified graphs obtained from the associated graph G by various graph operations described in Section 2.

A. Circuits Containing One Op Amp

We consider two cases where the op amp is described by either Model C or Model D. Consider first the Model C case.

Theorem 1. (One Model C Op Amp)

Let N contain one Model C op amp. Then N has a unique solution for all circuit parameters if and only if the associated graph G does not contain any loop that includes both the input and output edges (associated with the op amp) in opposite direction.

Example 1. Consider the circuit in Fig. 4(a) again. The associated graph G in Fig. 4(b) contains only one loop L which includes both edges 1 and $\hat{1}$ (op amp edges). However, L includes them in the same direction. It therefore follows from Theorem 1 that the circuit has a unique solution (for the Model C case).

We can also verify this conclusion analytically as follows. From Fig. 4(a) we have the circuit equations:

$$v_1 = -g_1(i) - E \tag{3a}$$

$$\frac{f(v_1) + v_1}{R_2} = i \tag{3b}$$

where the function $f(\cdot)$ as defined in Fig. 2. Substituting (3b) into (3a) we obtain

$$g_1\left(\frac{f(v_1)+v_1}{R_2}\right) + v_1 = -E \tag{4}$$

Since both $f(v_1)+v_1$ and g_1 are strictly increasing continuous onto functions, (4) has a unique solution for any value E, as predicted by Theorem 1.

Next consider the circuit shown in Fig. 12(a), which is obtained from Fig. 4(a) by reversing the polarity of the input port of the op amp. The associated graph G is shown in Fig. 12(b). The graph G contains a loop which includes both edges 1 and $\hat{1}$ in opposite direction. It therefore follows from Theorem 1 that the circuit does not have a unique solution (for some circuit parameters).[5]

Example 2. Consider the circuit in Fig. 13(a) and its associated graph G in Fig. 13(b). The graph G contains a loop L (consisting of edges 1, R_1, $\hat{1}$, R_3, R_2) which includes edges 1 and $\hat{1}$ in opposite direction. Therefore we conclude that the circuit does not have a unique solution.

Note that Theorem 1 tells us that the circuit will have a unique solution

---

[5]Hereafter the phrase "for some circuit parameters" will be omitted for simplicity.

if either of the following two conditions is satisfied.

(i) At least one of the edges R_1, R_2, R_3 is open-circuited.

(ii) Either R_6 or R_7 is short-circuited.

Example 3. Consider the circuit shown in Fig. 14(a) and its associated graph in Fig. 14(b). There exists no loop containing both edges 1 and $\hat{1}$ (op amp edges) in the associated graph. Thus we conclude from Theorem 1 that if the op amp in the circuit is described by Model C, then this circuit has a unique solution. We can verify this conclusion analytically as follows:

We have the following circuit equations:

$$\begin{aligned} v_1 &= -g_3(i) \\ g_2(i) + g_3(i) &= E \\ v_2 &= f(v_1) \end{aligned} \tag{5}$$

where the function $f(\cdot)$ is as defined in Fig. 2. Since g_2 and g_3 are strictly monotone-increasing onto functions, and f is a bounded continuous function, it follows from these equations that, for each E, all voltages and currents in the circuit are uniquely determined.

Our next result applies to the case where the op amp is described by Model D.

Theorem 2. (One Model D Op Amp)

Let N contain one Model D op amp. Then N has a unique solution for all circuit parameters if and only if the associated graph G satisfies the following two conditions (let the input and output edges associated with the op amp be denoted by $(1,\hat{1})$):

1) G does not contain any loop that includes both edges 1 and $\hat{1}$ in opposite direction.
2) G contains at least one loop that includes both edges 1 and $\hat{1}$ in the same direction.

Example 4. Consider the circuit in Fig. 4(a) again. The associated graph G obviously satisfies conditions 1) and 2) above. It therefore follows from Theorem 2 that even in the case where the op amp is described by Model D, the circuit has a unique solution.

Comparing Theorem 1 with Theorem 2, we conclude that if the circuit in which the op amp is described by Model C does not have a unique solution, then the circuit in which the op amp is described by Model D also does not have a unique solution. Therefore the circuits in Figs. 12(a) and 13(a) do not have a unique solution for the case where op amp is described by Model D.

Example 5. Consider the circuit shown in Fig. 14(a). The associated graph in Fig. 14(b) does not satisfy condition 2) in Theorem 2. It therefore follows that the circuit does not have a unique solution for the case where the op amp is described by Model D. To verify this, it suffices to observe (5). If we set $E=0$ in (5), we would obtain $i=0$ and $v_1=0$. However, the value of f at the origin cannot be determined uniquely. Hence, this circuit does not have a unique solution, as predicted by Theorem 2.

B. Circuits Containing Any Number of Op Amps

For simplicity a circuit is said to be Model C (resp., Model D) if each op amp in the circuit is described by Model C (resp., Model D).

Theorem 3. (Model C Op-Amp Circuit)

Let N be a Model C op-amp circuit. Then N has a unique solution for all circuit parameters if and only if the associated graph G satisfies the following two conditions:

(I) G contains neither loop made exclusively of op-amp output edges nor cutset made exclusively of op-amp input edges.

(II) G cannot be reduced to a cactus graph with an even number (including zero) of similarly-directed loops by applying the following three graph theoretic operations.

(a) Apply $O/S(\cdot)$ to each resistor edge

(b) Apply $Z(\cdot)$ to some (possibly none) op amps

(c) For each reduced graph after operations (a) and (b) having a complementary tree structure, apply $O/S(\cdot)$ to some (possibly none) op amps.

Remark 1.

Let

k = number of op amps

n_0 = number of resistor edges

n_1 = number of op amps whose input (resp., output) edges are open-circuited (resp., short-circuited) in operations (b) and (c).

n_2 = number of op amps whose input (resp., output) edges are short-circuited (resp., open-circuited) in operation (c).

n_3 = number of remaining op amps $(= k-n_1-n_2)$

As the result of operations (a)-(c) in Theorem 3, we obtain n(not necessarily distinct) graphs, in general, where

$$n = \sum_{\substack{n_1=0 \\ n_1+n_2<k}}^{k} \sum_{n_2=0}^{k} 2^{n_0} \binom{k}{n_1} \binom{k-n_1}{n_2} \tag{6}$$

The number n is extremely large even for a small-size circuit. It appears that it is rather tedious to verify whether the conditions in Theorem 3 are satisfied or not. Note however that we are concerned only with cactus graphs and that the number of cactus graphs among those n graphs are usually very few compared with n. As seen from the following examples, we can easily apply this theorem to many practical circuits.

To see this, consider the circuit in Fig. 4(a) again. For this circuit, we have $n = 4$. If we apply operation (a) to Fig. 4(b) in all possible combinations[6], we would obtain 4 graphs in Figs. 15(a)-(d). However, among these 4 graphs only the one in Fig. 15(c) is a cactus graph. We notice immediately that operation $\mathcal{S}(R_1)$ or $\mathcal{O}(R_2)$ need not be applied.

As another example, consider the circuit in Fig. 13(a) again. For this circuit, n is given by $n = 2^7 = 128$. Among these 128 combinations of operation (a), it suffices for us to consider the following two combinations only:

(i) $\mathcal{S}(R_1)$, $\mathcal{S}(R_2)$, $\mathcal{S}(R_3)$, $\mathcal{O}(R_4)$, $\mathcal{O}(R_5)$, $\mathcal{O}(R_6)$ and $\mathcal{O}(R_7)$
(ii) $\mathcal{O}(R_1)$, $\mathcal{S}(R_2)$, $\mathcal{O}(R_3)$, $\mathcal{O}(R_4)$, $\mathcal{O}(R_5)$, $\mathcal{S}(R_6)$ and $\mathcal{S}(R_7)$.

For, we can verify by inspection that the other combinations do not give rise to any other cactus graph. Combination (i) (resp., (ii)) gives rise to the cactus graph in Fig. 6 (resp., Fig. 5). Since the graph in Fig. 6 has zero (hence even) similarly-directed loops, the same conclusion as before follows from Theorem 3.

[6] Note that if there are m resistor edges, then operation (a) has 2^m combinations.

Remark 2.

To show a circuit does not have a unique solution is usually much easier than to show it has a unique solution: all we need to do is to exhibit one disallowed cactus graph, and this can often be found by inspection.

We will now illustrate the conditions in Theorem 3 in more detail with the help of some examples. Note that when we apply Theorem 3, our first step is to find all graphs with a complementary tree structure by applying operations (a) and (b). Some of them may be cactus graphs. For each reduced graphs having a complementary tree structure which is not a cactus graph, we must apply operation (c).

Example 6. Consider the circuit shown in Fig. 16(a) and its associated graph in Fig. 16(b). Applying operations (a) and (b), we obtain many reduced graphs. Among them only three graphs, shown in Figs. 16(c)-(e), have a complementary tree structure. For example, the graph in Fig. 16(c) is obtained by applying operations $\mathcal{O}(R_1)$, $\mathcal{S}(R_2)$, $\mathcal{O}(R_3)$ and $\mathcal{S}(R_4)$. Similarly the graph in Fig. 16(d) is obtained by operations $\mathcal{O}(R_1)$, $\mathcal{S}(R_2)$, $\mathcal{O}(R_3)$, $\mathcal{S}(R_4)$, $\mathcal{O}(2)$ and $\mathcal{S}(\hat{2})$. Note that we need not consider other graphs, since they do not exhibit a complementary tree structure. Note also that the graph in Fig. 16(c) is not a cactus graph because the edges are not labelled in accordance with Fig. 10. Applying operation (c) to Fig. 16(c), we obtain the cactus graphs in Figs. 16(d) and (e). No more cactus graph other than those in Figs. 16(d) and (e) can be obtained. Since the cactus graphs obtained above are allowed (they have an odd number of similarly directed loops, namely, one), it follows from Theorem 3 that this "Model C" circuit has a unique solution.

Example 7. Consider the circuit in Fig. 17(a) and its associated graph G in Fig. 17(b). Among $2^3(=8)$ reduced graphs obtained by applying operation (a), only one graph, shown in Fig. 17(c), has a complementary tree structure. It is a 2-leaves cactus graph with one similarly-directed loop, and is obtained from G by applying operation $\mathcal{O}(R_1)$, $\mathcal{S}(R_2)$, and $\mathcal{O}(R_3)$. Furthermore applying operations (a) and (b) to G, we obtain another complementary tree structure graph (1-leaf cactus graph) shown in Fig. 17(d). Since both reduced graphs are already cactus graphs, we don't need to apply operation (c) in this example. Since both cactus graphs obtained above are allowed, it follows from Theorem 3 that this "Model C" circuit has a unique solution.

Note that we can obtain the disallowed graph shown in Fig. 17(e) by applying operations $S(2)$, $O(\hat{2})$, $O(R_1)$, $O(R_2)$, and $S(R_3)$ to G. However, this particular simplification is <u>not allowed in Theorem 3</u> because operation (c) can be applied <u>only to a graph with a complementary tree structure</u>.[7]

<u>Example 8</u>. Consider the circuit shown in Fig. 18(a) and its associated graph G in Fig. 18(b). By applying operations (a) and (b) to G, 2 reduced graphs with a complementary tree structure are obtained. They are shown in Figs. 18(c) and (d). Applying operation (c) to Fig. 18(d), we obtain the "disallowed" cactus graph in Fig. 18(e). Hence, we conclude that this "Model C" circuit does not have a unique solution.

Note that if the polarity of the input port of op amp #1 is reversed, then the circuit has a unique solution.

<u>Example 9</u>. Consider the circuit shown in Fig. 19(a) and its associated graph G in Fig. 19(b). Applying operations (a) and (b) to G, we obtain 4 graphs with a complementary tree structure. They are shown in Figs. 19(c)-(f). Since Fig. 19(f) is <u>not</u> a cactus graph, we apply operation (c) to obtain a new cactus graph in Fig. 19(e). Since all of the above cactus graphs are allowed, it follows that the "Model C" circuit in Fig. 19(a) has a unique solution.

Similarly, we can show that the "Model C" circuit in Fig. 20(a) has a unique solution. The graphs obtained by applying operations (a)-(c) are shown in Figs. 20(c)-(e).

<u>Theorem 4</u>. (Model D Op-Amp Circuit)

Let N be a Model D op-amp circuit. Then N has a <u>unique</u> solution for all circuit parameters <u>if and only if</u> the associated graph G satisfies the following condition (III) in addition to conditions (I) and (II) in Theorem 3:

(III) Let K_0 denote <u>any proper</u> subset of op amps in the circuit and let G_1 be a graph obtained from G by applying operation $Z(\cdot)$ to each op amp belonging to K_0 (henceforth abbreviated as operation $Z(K_0)$). Then there exists a combination of operation (a) such that applying the operation (a) to G_1 gives rise to a graph with a complementary tree structure.

---

[7] Note that the graph obtained by applying operations $O(R_1)$, $O(R_2)$ and $S(R_3)$ does not have a complementary tree structure (see Fig. 17(f)).

We will illustrate condition (III) by some examples.

Example 10. Consider first the circuit shown in Fig. 21(a) and its associated graph G in Fig. 21(b). Since the circuit contains 3 op amps, we have to consider $7(=2^3-1)$ cases for K_0. If $K_0 = \{\text{op amp \#1}\}$, then we choose as operation (a) $\mathcal{S}(R_1)$ and $\mathcal{O}(R_2)$. Applying $Z(K_0)$ and the above operation (a) to G, we obtain the graph in Fig. 21(c), which has a complementary tree structure. Similarly, if $K_0 = \emptyset$, {op amp #2}, {op amp #3}, or {op amps #1, #3}, then we choose as operation (a) $\mathcal{S}(R_1)$ and $\mathcal{O}(R_2)$ again. If $K_0 = \{\text{op amps \#1, \#2}\}$ or {op amps #2, #3}, then we choose as operation (a) $\mathcal{O}(R_1)$ and $\mathcal{S}(R_2)$. In any case we obtain a graph with a complementary tree structure by applying operation $Z(K_0)$ and operation (a) chosen above. We therefore conclude that the circuit in Fig. 21(a) satisfies condition (III).

Example 11. Consider the circuit shown in Fig. 17(a) again. First we consider the case where $K_0 = \emptyset$. In this case we choose as operation (a) $\mathcal{O}(R_1)$, $\mathcal{S}(R_2)$ and $\mathcal{O}(R_3)$. Applying $Z(K_0)$ and the above operation (a) to G, we obtain a complementary-tree-structure graph in Fig. 17(c). Next consider the case where $K_0 = \{\text{op amp \#1}\}$. Applying operation $z(K_0)$ and operations $\mathcal{S}(R_1)$, $\theta(R_2)$, $\theta(R_3)$, we obtain another complementary-tree-structure graph in Fig. 17(d). Thirdly we consider the case where $K_0 = \{\text{op amp \#2}\}$. In this case, however, the application of operation $z(K_0)$ gives rise to a cutset consisting of only one edge $\hat{1}$ (see Fig. 17(g)). Therefore any combination of operation (a) does not give rise to a graph with a complementary tree structure. That is, the circuit in Fig. 17(a) does not satisfy condition (III). It follows from Theorem 4 that the "Model D" circuit in Fig. 17(a) does not have a unique solution (cf. Example 7).

Example 12. Consider the graph shown in Fig. 20(b). Let $K_0 = \{\text{op amps \#2, \#3}\}$. Since the application of operation $z(K_0)$ results in a self-loop (edge 1), we cannot obtain a complementary tree structure in this case. We therefore conclude that the graph in Fig. 20(b) does not satisfy condition (III). This means that the "Model D" circuit in Fig. 20(a) does not have a unique solution (cf. Example 9).

Similarly, we can verify that the graphs in Figs. 19(b) and 14(b) do not satisfy condition (III). This corresponds to the previous result in Example 5.

Finally, note that the graphs in Figs. 4(b) and 16(b) satisfy condition (III). Since they also satisfy conditions (i) and (ii), we conclude that the circuits in Figs. 4(a) and 16(a) have a unique solution even if the op amps are described by Model D.

Corollary 1.

If a "Model C" circuit does not have a unique solution, then the corresponding "Model D" circuit also does not have a unique solution.

In many practical applications the following condition is satisfied.

Assumption: One of two input terminals of each op amp is grounded. (7)

Corollary 2.

If Assumption (7) is satisifed, then, in Theorems 3 and 4, we don't need to apply operation (c).

Proof: See Appendix 1.

This corollary greatly simplifies the application of Theorems 3 and 4 in practice. In many cases, the conclusions can be obtained by inspection.

4. PROOFS OF THEOREMS

Since Theorems 1 and 2 are special cases of Theorems 3 and 4, respectively, we will give the proofs of Theorems 3 and 4 only. Since the proofs are rather long and involved, we give only the major steps of the proof so that the reader can separate the trees from the forest. This is achieved with the help of many lemmas. Proofs of some of the non-trivial lemmas are given in the Appendix.

We start with the following lemma:

Lemma 1. Condition (I) in Theorems 3 and 4 is necessary for the solution to be unique.

Proof: See Appendix 2.

4.1 Analytical Condition for the Solution to be Unique

Consider a circuit N with k op amps and m nonlinear resistors. Suppose for the moment that each op amp is replaced by a nonlinear VCVS defined by the function in Fig. 22(a) or Fig. 22(b). Here, the function f in Fig. 22(a) satisfies the following conditions:

1) f is a continuous and strictly-increasing function.

2) $f'(0) = \alpha$ and $0 < f'(v) < \alpha$ for $v \neq 0$.

3) $f(0) = 0$.

Let this modified circuit be denoted by N_{VCVS}. Let the VCVS's and the nonlinear resistors in N_{VCVS} be extracted across a linear passive resistive (2k+m)-port N_0, as shown in Fig. 23. Here N_0 includes all dc sources. Denote the port-currents and the port-voltages by i_μ and v_μ ($\mu = 1,2,\cdots,2k+m$), respectively, and let

$$I_a = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_k \end{bmatrix} \qquad V_a = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}$$

$$I_b = \begin{bmatrix} i_{k+1} \\ \vdots \\ i_{2k} \end{bmatrix} \qquad V_b = \begin{bmatrix} v_{k+1} \\ \vdots \\ v_{2k} \end{bmatrix}$$

$$I_c = \begin{bmatrix} i_{2k+1} \\ \vdots \\ i_{2k+m} \end{bmatrix} \qquad V_c = \begin{bmatrix} v_{2k+1} \\ \vdots \\ v_{2k+m} \end{bmatrix}$$

The characteristics of the VCVS's and the nonlinear resistors are represented by

$$I_a = 0 \tag{8a}$$

$$V_b = F(V_a) \tag{8b}$$

$$-I_c = G(V_c) \tag{8c}$$

where

$$F(V_a) = \begin{bmatrix} f_1(v_1) \\ f_2(v_2) \\ \vdots \\ f_k(v_k) \end{bmatrix} \quad \text{and} \quad G(V_c) = \begin{bmatrix} g_1(v_{2k+1}) \\ g_2(v_{2k+2}) \\ \vdots \\ g_m(v_{2k+m}) \end{bmatrix}$$

Here, f_μ ($\mu = 1,\cdots,k$) are functions represented by Fig. 22(a) or 22(b), and g_μ ($\mu = 1,\cdots,m$) are continuous and strictly-increasing function mapping R' onto R'.

Suppose for the moment the following assumption is satisfied.

Assumption 1. The (2k+m)-port N_0 has an admittance representation.

The case where Assumption 1 does not hold will be treated in Section 4.4. Then N_0 can be represented by

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} Y_{aa} & Y_{ab} & Y_{ac} \\ Y_{ba} & Y_{bb} & Y_{bc} \\ Y_{ca} & Y_{cb} & Y_{cc} \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} + \begin{bmatrix} J_a \\ J_b \\ J_c \end{bmatrix} \tag{9}$$

Substituting (8) into (9) we obtain

$$\begin{bmatrix} 0 \\ I_b \\ -G(V_c) \end{bmatrix} = \begin{bmatrix} Y_{aa} & Y_{ab} & Y_{ac} \\ Y_{ba} & Y_{bb} & Y_{bc} \\ Y_{ca} & Y_{cb} & Y_{cc} \end{bmatrix} \begin{bmatrix} V_a \\ F(V_a) \\ V_c \end{bmatrix} + \begin{bmatrix} J_a \\ J_b \\ J_c \end{bmatrix} \tag{10}$$

The second equation of (10) can be regarded as the equation for determining I_b. Therefore it is unnecessary to consider the second equation of (9) in our subsequent discussions.

Let

$$A = \begin{bmatrix} Y_{aa} & Y_{ac} \\ Y_{ca} & Y_{cc} \end{bmatrix} \qquad B = \begin{bmatrix} Y_{ab} & 0 \\ Y_{cb} & \mathbb{1} \end{bmatrix}$$

$$J = \begin{bmatrix} J_a \\ J_c \end{bmatrix}$$

Then (10) can be rewritten as

$$A\begin{bmatrix} V_a \\ V_c \end{bmatrix} + B\begin{bmatrix} F(V_a) \\ G(V_c) \end{bmatrix} + J = 0 \tag{11}$$

Let

$$\left.\begin{aligned} D_F &= \operatorname{diag}[d_{f1}, d_{f2}, \cdots, d_{fk}] \\ D_g &= \operatorname{diag}[d_{g1}, d_{g2}, \cdots, d_{gm}] \\ D &= \begin{bmatrix} D_F & 0 \\ 0 & D_G \end{bmatrix} \end{aligned}\right\} \tag{12}$$

$$\Delta \triangleq |A + BD|$$

$$= \begin{vmatrix} Y_{aa} + Y_{ab}D_F, & Y_{ac} \\ Y_{ca} + Y_{cb}D_F, & Y_{cc} + D_G \end{vmatrix} \tag{13}$$

<u>Lemma 2</u>. Suppose the circuit N_{VCVS} satisfies condition (I) in Theorem 3. Then, for any given values of linear resistors, Equation (11) has a unique solution for all circuit parameters if and only if

$$\Delta \neq 0 \text{ for all D satisfying (15) or (16).} \tag{14}$$

1) In the case where f is represented by Fig. 22(a):

$$0 < d_{f\mu} < \alpha_\mu \quad (\mu = 1,2,\cdots,k) \tag{15a}$$

$$0 < d_{g\mu} < \gamma_\mu (=\infty) \quad (\mu = 1,2,\cdots,m) \tag{15b}$$

2) In the case where f_μ is represented by Fig. 22(b):

$$0 \leqq d_{f\mu} \leqq \alpha_\mu \quad (\mu = 1,2,\ldots,k) \tag{16a}$$

$$0 < d_{g\mu} < \gamma_\mu \; (=\infty) \quad (\mu = 1,2,\cdots,m) \tag{16b}$$

<u>Proof</u>: See Appendix 3.

Until now we have assumed the function f_μ are described by Fig. 22(a) or 22(b). Hereafter we will use the limiting characteristics shown in Figs. 2 and 3. To do so, it suffices to set

$$\alpha_\mu = \infty \quad (\mu = 1,2,\ldots,k) \tag{17}$$

in the equations (15) and (16).

Let K be a set of numbers $\{1,2,\cdots,k\}$ and let K_1 and K_2 be a partition of K. That is, $K = K_1 \cup K_2$ and $K_1 \cap K_2 = \phi$. Either K_1 or K_2 may possibly be the null set. Set

$$\begin{bmatrix} Y_{aa} & Y_{ab} & Y_{ac} \\ Y_{ca} & Y_{cb} & Y_{cc}+D_G \end{bmatrix} = [p_1 p_2 \cdots p_k \vdots q_1 \cdots\cdots q_k \vdots r_1 \cdots r_m] \tag{18}$$

and

$$\Delta_\infty = |t_1 \; t_2 \cdots t_k | r_1 \cdots r_m| \tag{19}$$

where

$$t_\mu = \begin{cases} p_\mu & \text{for } \mu \in K_1 \\ q_\mu & \text{for } \mu \in K_2 \end{cases} \tag{20}$$

<u>Lemma 3</u>. Condition (14) is equivalent to the following condition (21) (resp.,(22)) when (15) (resp.,(16)) is satisfied.

$$\begin{cases} \Delta_\infty \geqq 0 \text{ for } \underline{\text{all}} \; D_G \text{ and for } \underline{\text{any}} \text{ partition of K} & \text{(21a)} \\ \Delta_\infty \neq 0 \text{ for } \underline{\text{some}} \; D_G \text{ and for } \underline{\text{at least one}} \text{ partition of K} & \text{(21b)} \end{cases}$$

$$\begin{cases} \Delta_\infty \geqq 0 \text{ for } \underline{\text{all}} \; D_G \text{ and for } \underline{\text{any}} \text{ partition of K} & \text{(22a)} \\ \Delta_\infty \neq 0 \text{ for } \underline{\text{some}} \; D_G \text{ and for } \underline{\text{any}} \text{ partition of K} & \text{(22b)} \end{cases}$$

<u>Proof</u>: See Appendix 4.

Using Condition (I), we can easily verify that

$$\Delta_\infty > 0 \text{ for } K_1 = K, \; K_2 = \phi \text{ and } D_G \to \infty.^8 \tag{23}$$

Hence, condition (21b) is always satisfied. Therefore it remains to investigate only conditions (22a) and (22b). (Note that (21a) is the same as (22a)).

In the following we investigate the topological conditions for $\Delta_\infty > 0$, $\Delta_\infty = 0$ or $\Delta_\infty < 0$.

---

[8] $D_G \to \infty$ means that each diagonal element of D_G is sufficiently large.

4.2. Analysis of the Linear Network

We will now investigate when condition (21) or (22) holds for all values of linear resistors and for all D_G. Let

$$\tilde{Y} = \begin{bmatrix} Y_{aa} & Y_{ab} & Y_{ac} \\ Y_{ba} & Y_{bb} & Y_{bc} \\ Y_{ca} & Y_{cb} & Y_{cc}+D_G \end{bmatrix} \tag{24}$$

Equation (24) is the admittance matrix of the (2k+m)-port in Fig. 24 where $\tilde{N}_0$ is the network obtained from N_0 by short-circuiting the dc voltage sources and open-circuiting the dc current sources, and γ_μ ($\mu = 1,\cdots,m$) denotes linear (positive) resistors. In order to investigate conditions (21) and (22), it suffices to consider $\tilde{Y}$.

The associated graph $\tilde{G}$ of the network in Fig. 24 is defined as the graph obtained by replacing each resistor (including γ_μ), each port μ ($\mu = 1,\cdots,k$), each port $k+\mu$ ($\mu = 1,\cdots,k$) and each port $2k+\mu$ ($\mu = 1,\cdots,m$), respectively, by oriented edges R_μ, a_μ, b_μ and c_μ. The direction of R_μ is arbitrarily chosen. Edges a_μ, b_μ, and c_μ are directed from the + sign to the - sign in Fig. 24 (See Fig. 25), and are called R-, a-, b-, and c-edge, respectively. The graph $\tilde{G}$ is <u>connected</u>, by assumption. We further stipulate that

<u>Assumption 2</u>. $\tilde{G}$ has no cutset consisting exclusively of a-, b-, and c-edges. The case where Assumption 2 does not hold will be treated in <u>Section 4.4</u>. Let

$$m_0 = \text{nullity of } \tilde{G} - \text{total number of a-, b-, and c-edges} \tag{25}$$

From Assumption 2 it follows that $m_0 \geq 0$.

We can modify $\tilde{G}$ by adding m_0 d-edges d_μ ($\mu = 1,2,\cdots,m_0$) so that all the a-, b-, c-, and d-edges form a cotree, $\bar{T}$, of $\tilde{G}$. For simplicity, we denote hereafter the modified graph by the same symbol $\tilde{G}$. Let the fundamental loop matrix of $\tilde{G}$ with respect to the cotree $\bar{T}$ be

$$B = \begin{array}{c} \quad T \quad \bar{T} \\ [B_T \,\vdots\, \mathbb{1}] \end{array}$$

and let the rows of B be arranged in the order of a-, b-, c- and d-edges. The submatrix B_T will henceforth be referred to as the <u>main part</u> of the fundamental loop matrix B. Without loss of generality we will choose

$$K_1 = \{1,2,\cdots,k_1\}$$
and $\qquad (0 \leq k_1 \leq k) \qquad (27)$
$$K_2 = \{k_1+1,\cdots,k\} \quad ,$$

for Δ_∞ in (19) and (20).

Set

$$k_2 = k-k_1 \quad . \tag{28}$$

Then B_T can be written as in Fig. 26 where $M = \{1,2,\cdots,m\}$ and $M_0 = \{1,2,\cdots,m_0\}$. In addition a_{K1} means the set of a-edges a_μ $(\mu \in K_1)$, and a_{K_2}, c_M, d_{M_0} are defined in a similar way. Let

$$H = B_T \, Ⓗ \, B_T' \tag{29}$$

where the prime means the transpose of a matrix and Ⓗ is a diagonal matrix whose diagonal elements are the values of the linear resistors (including γ_μ in Fig. 24).

<u>Lemma 4</u>

$$\Delta_\infty = (-1)^{kk_2}|H|^{-1}\delta_0 \tag{30}$$

where δ_0 is the determinant of the submatrix shaded by oblique lines in Fig. 27.

<u>Proof</u>: See Appendix 5.

Since $|H| > 0$, it is sufficient for us to consider the sign of $(-1)^{kk_2}\delta_0$. By using (29), we can rewrite δ_0 as

$$\delta_0 = |B_{T1} \, Ⓗ \, B_{T2}'| \tag{31}$$

where B_{T1} (resp., B_{T2}) is the submatrix of B_T in Fig. 26, shaded by oblique (resp., vertical) lines.

Let $Ⓗ_0$ denote an arbitrary set of k+m R-edges and let δ_1 (resp.,δ_2) be the determinant of the submatrix of B_{T1} (resp., B_{T2}) consisting of all the rows of B_{T1} (resp., B_{T2}) and the columns corresponding to $Ⓗ_0$ (See Figs. 26 and 28). Let

$$\delta = (-1)^{kk_2}\delta_1\delta_2 \quad . \tag{32}$$

Then we have:

<u>Lemma 5</u>. We can choose the values of resistors so that

$$\Delta_\infty < 0 \tag{33}$$

if and only if there exists a $Ⓗ_0$ such that

$$\delta < 0 \; . \tag{34}$$

Proof: See Appendix 6.

Suppose that (34) holds for some $Ⓗ_0$. Since δ_1 and δ_2 (and therefore δ) depend only on the rows a_{K_2}, b_K and d_{M_0} and the columns $Ⓗ_0$ of B_T, we define $B_T^{(0)}$ as shown in Fig. 28. Then δ_1 (resp. δ_2) is the determinant of the submatrix shaded by the oblique (resp. vertical) lines in Fig. 28. By carrying out the following operations (i)-(iii)

(i) Multiply some columns by ± 1

(ii) Add the above columns to other columns

(iii) Interchange columns.

appropriately, we can transform $B_T^{(0)}$ in Fig. 28 into $B_T^{(1)}$ in Fig. 29, where Γ_1 and Γ_2 are nonsingular diagonal matrices whose elements are ± 1 and

$$Q = -\mathbb{1} \; . \tag{35}$$

Since

$$\begin{aligned} \delta_1 &= \varepsilon|\Gamma_1||\Gamma_2||P| \\ \delta_2 &= (-1)^{k_1 k_2}\varepsilon|\Gamma_1||\Gamma_2||Q| \\ &= (-1)^{k_1 k_2 + k_2}\varepsilon|\Gamma_1||\Gamma_2| \\ \varepsilon &= \pm 1 \; , \end{aligned} \tag{36}$$

we have by (36) and (32)

$$\delta = |P| \quad . \tag{37}$$

Set

$$B^{(2)} = \begin{array}{c} \\ a_{K_2} \end{array} \begin{array}{c} \begin{array}{cc} b_{K_1} & a_{K_2} \end{array} \\ \left[\begin{array}{c:c} P & \mathbb{1} \end{array} \right] \end{array} \; . \tag{38}$$

4.3. Graph Theoretical Interpretation of $B^{(2)}$

Lemma 6. Let $G^{(2)}$ be the graph obtained from $\tilde{G}$ by the following operations:

(i) Apply $O(\cdot)$ to each c-edge and $S(\cdot)$ to each d-edge

(ii) Apply $O(\cdot)$ to all R-edges belonging to $Ⓗ_0$ and $S(\cdot)$ to the remaining R-edges

(iii) Apply $O(\cdot)$ to each a-edge belonging to a_{K_1} and $S(\cdot)$ to each b-edge belonging to b_{K_1}.

Then $G^{(2)}$ is a connected graph with a complementary tree structure and has a fundamental loop matrix $B^{(2)}$ in (38) if and only if $\delta \neq 0$.

Lemma 7. Let P_0 denote an arbitrary principal submatrix of P in (38). Then there exists an operation (b) in Theorem 3 which operates on $G^{(2)}$ to produce a $G^{(3)}$ having the following fundamental loop matrix:

$$B^{(3)} = [P_0 \vdots \mathbb{1}] \tag{39}$$

4.4. Further Considerations on $G^{(2)}$

We will derive the graph-theoretic conditions for

$$|P| < 0 \ . \tag{40}$$

Henceforth we assume (40). Then we can find a principal submatrix P_0 of P which satisfies the following conditions:

1) $|P_0| < 0$ (41)

2) Each principal minor (except for $|P_0|$) of P_0 is positive or zero.

Suppose that some principal minor of P_0 is positive. Then we can choose a principal submatrix P_1 such that:

1) $|P_1| > 0$ (42)

2) Each principal minor (except for $|P_1|$ and $|P_0|$) of P_0 which includes P_1 in it is zero.

Without loss of generality we can rewrite P_0 as

$$P_0 = \left[\begin{array}{c|c} P_1 & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] \ . \tag{43}$$

Set

$$P_2 = P_{22} - P_{21}P_1^{-1}P_{12} \ . \tag{44}$$

Lemma 8. P_2 in (44) has the following properties:

1) $|P_2| < 0$

2) Each principal minor (excluding $|P_2|$) is zero.

3) There exists a graph $G^{(4)}$ whose fundamental loop matrix is given by

$$B^{(4)} = [P_2 \vdots \mathbb{1}] \quad . \tag{45}$$

Lemma 9. $G^{(4)}$ in Lemma 8 is a cactus graph with an even number of similarly directed loops.

Lemma 10. The process of obtaining P_0 and P_1 corresponds to the operation (b) and that of obtaining P_2 corresponds to the operation (c) in Theorem 3.

Lemma 11. Theorems 3 and 4 hold under Assumptions 1 and 2.

Finally we get

Lemma 12. Theorems 3 and 4 holds without Assumptions 1 and 2.

The lengthy proofs of Lemmas 6-12 are omitted because they can be constructed using similar (but not identical) techniques given in [3].

5. Conclusion

Topological necessary and sufficient conditions for op-amp circuits to have a unique solution are given. The theorems given in this paper can be generalized to allow circuits containing all 4 kinds of nonlinear controlled sources described by Model C or Model D.

Appendix 1. Proof of Corollary 2

If a circuit satisfies Assumptions (1) and (7), then any reduced graph with a complementary tree structure can be drawn as shown in Fig. A.1. Here, one of two edges forming each loop in Fig. A.1 is an input edge of an op amp and another one is an output edge. We therefore conclude that if we can obtain a graph from the graph in Fig. A.1 by applying operation (c), then we can obtain it by operation (b), too.

Appendix 2. Proof of Lemma 1

Suppose that an associated graph G contains a loop $\mathcal{L}$ made exclusively of output edges of the op amp's. For example, see Fig. A.2(a), where edges $\hat{k}$ $(k = 1,\cdots,4)$ denote the output edges of the op amps. If we insert into $\mathcal{L}$ a voltage source whose value is sufficiently large, then the circuit obviously has no solution[9] because the output voltage of each op amp is bounded.

Next suppose that G contains a cutset C made exclusively of input edges of the op amps. See, for example, Fig. A.2(b) where edges 1, 2, 3 denote input edges of the op amps. In this case we cannot connect any dc current source J as shown by the dotted line in Fig. A.2(b). This contradicts the assumption.

[9] Remember that any dc source can be inserted at any location in the circuit.

Appendix 3. Proof of Lemma 2

Let

$$x = \begin{bmatrix} V_a \\ V_c \end{bmatrix}$$

$$\tilde{F}(x) = \begin{bmatrix} F(V_a) \\ G(V_c) \end{bmatrix} .$$

Then (11) can be written as

$$Q(x) \equiv Ax + B\tilde{F}(x) + J = 0 \tag{A2.1}$$

We will show that (14) is necessary and sufficient for (A2.1) to have a unique solution for each vector J.

Necessity: To show that (14) is necessary for uniqueness, we prove that if (14) is not satisfied, then (A2.1) has more than one solution for some vector J.

Suppose that

$$\Delta \equiv |A + BD_0| = 0 \tag{A2.2}$$

for some $D = D_0 = \begin{bmatrix} D_{F0} & 0 \\ 0 & D_{G0} \end{bmatrix}$ satisfying (15) or (16). Then there exists a nontrivial solution $x^{(0)}$ such that

$$(A+BD_0)x^{(0)} = 0 \tag{A2.3}$$

We can verify the following from the observation of Fig. A.3:[10]

There exist a point $x^{(1)}$ and a sufficiently small positive number δ such that

$$\tilde{F}(x^{(1)}+\delta x^{(0)}) - \tilde{F}(x^{(1)}) = D_0 \delta x^{(0)} \tag{A2.4}$$

Let $\tilde{J}$ satisfy

$$Ax^{(1)} + B\tilde{F}(x^{(1)}) + \tilde{J} = 0 \tag{A2.5}$$

[10]Figure A.3 is drawn only for the case of Fig. 22(b). We can draw similar figures for the case of Fig. 22(a).

Obviously $x^{(1)}$ is a solution of an equation

$$Ax + B\tilde{F}(x) + \tilde{J} = 0 \quad . \tag{A2.6}$$

Furthermore we can see that $x^{(1)} + \delta x^{(0)}$ is also a solution of (A2.6). For,

$$\begin{aligned} & A(x^{(1)}+\delta x^{(0)}) + B\tilde{F}(x^{(1)}+\delta x^{(0)}) + \tilde{J} \\ & = Ax^{(1)} + A\delta x^{(0)} + B[\tilde{F}(x^{(1)}) + D_0]\delta x^{(0)} + \tilde{J} \\ & = [Ax^{(1)} + B\tilde{F}(x^{(1)}) + \tilde{J}] + (A+BD_0)\delta x^{(0)} \\ & = 0. \end{aligned} \tag{A2.7}$$

Thus for $J = \tilde{J}$ the equation (A2.1) has at least two solutions.

Sufficiency: Suppose that (14) is satisfied. We will prove the uniqueness and existence of the solution of (A2.1) as follows:

1) Uniqueness

Suppose that (A2.1) has two solutions $x^{(1)}$ and $x^{(2)}$. Then we have

$$A(x^{(1)}-x^{(2)}) + B(\tilde{F}(x^{(1)}) - \tilde{F}(x^{(2)})) = 0 \tag{A2.8}$$

It follows from the observation of Fig. 22 that there exist diagonal matrices D_F and D_G satisfying both (15) or (16) and

$$\tilde{F}(x^{(1)}) - \tilde{F}(x^{(2)}) = D(x^{(1)} - x^{(2)}) \tag{A2.9}$$

Substituting (A2.9) into (A2.8), we have

$$(A+BD)(x^{(1)}-x^{(2)}) = 0 \quad . \tag{A2.10}$$

From (A2.10) and (14) we conclude that $x^{(1)} = x^{(2)}$.

2) Existence of a solution

We will show that there exists an M_0 such that

$$x'Q(x) > 0 \text{ for all } x \text{ satisfying } \|x\| = M > M_0\text{[11]} \tag{A2.11}$$

If (A2.11) holds, then (A2.1) has a solution [2]. Let

$$\left.\begin{aligned} p &= x'Ax \\ q &= x'BF(x) \\ r &= x'J \end{aligned}\right\} \tag{A2.12}$$

[11] Throughout the Appendix, the "prime" means the "transpose" of a matrix.

Consider first the quadratic form p. It follows from condition (I) that Y_{aa} is a positive-definite matrix. Therefore all eigenvalues of Y_{aa} are positive. Let the smallest eigenvalue be λ_0 and let

$$x = \begin{bmatrix} x_a \\ x_b \end{bmatrix} \begin{matrix} \} \, k \\ \} \, m \end{matrix} \tag{A2.13}$$

Then since $Y_{cc}-Y_{ca}Y_{aa}^{-1}Y_{ac}$ is a positive semi-definite matrix, we have

$$P = x' \begin{bmatrix} \mathbb{1} & 0 \\ Y_{ca}Y_{aa}^{-1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ -Y_{ca}Y_{aa}^{-1} & \mathbb{1} \end{bmatrix} A \begin{bmatrix} \mathbb{1} & -Y_{aa}^{-1}Y_{ac} \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & Y_{aa}^{-1}Y_{ac} \\ 0 & \mathbb{1} \end{bmatrix} x$$

$$= \begin{bmatrix} x_a+Y_{aa}^{-1}Y_{ac}x_b \\ x_b \end{bmatrix}' \begin{bmatrix} Y_{aa} & 0 \\ 0 & Y_{cc}-Y_{ca}Y_{aa}^{-1}Y_{ac} \end{bmatrix} \begin{bmatrix} x_a+Y_{aa}^{-1}Y_{ac}x_a \\ x_b \end{bmatrix}$$

$$\geq \lambda_0 \, \| x_a + Y_{aa}^{-1}Y_{ac}x_b \|^2 \, . \tag{A2.14}$$

There exists a β such that

$$\| Y_{aa}^{-1}Y_{ab}x_b \| \leq \beta \, \| x_b \| \tag{A2.15}$$

Now we evaluate p, q, and r for x satisfying

$$\| x \| = M \, . \tag{A2.16}$$

We consider two cases:

(i) $\| x_a \| \geq 2\beta \, \| x_b \|$ (A2.17)

In this case we have

$$\| x_a + Y_{aa}^{-1}Y_{ab}x_b \| \geq \big| \| x_a \| - \| Y_{aa}^{-1}Y_{ab}x_b \| \big|$$

---

[11] Throughout the Appendix, the "prime" means the "transpose" of a matrix.

$$\geq \|x_a\| - \beta \|x_b\|$$

$$\geq \|x_a\| - \beta \frac{1}{2\beta} \|x_a\|$$

$$= \frac{1}{2} \|x_a\| \tag{A2.18}$$

Since

$$M^2 = \|x\|^2 = \|x_a\|^2 + \|x_b\|^2 \leq \|x_a\|^2 + \frac{1}{4\beta^2} \|x_a\|^2 , \tag{A2.19}$$

we have

$$\|x_a\|^2 \geq \frac{4\beta^2}{1+4\beta^2} M^2 . \tag{A2.20}$$

By (A2.14), (A2.18), and (A2.20) we have

$$p \geq \frac{\beta^2}{1+4\beta^2} \lambda_0 M^2 \tag{A2.21}$$

On the other hand we have

$$q = \begin{bmatrix} x_a \\ \\ x_b \end{bmatrix}' \begin{bmatrix} Y_{ab} & 0 \\ Y_{cb} & \mathbb{1} \end{bmatrix} \begin{bmatrix} F(x_a) \\ G(x_b) \end{bmatrix}$$

$$= x_a' Y_{ab} F(x_a) + x_b' Y_{cb} F(x_a) + x_b' G(x_b) \tag{A2.22}$$

Since each element of $F(\cdot)$ is bounded, the first and the second terms of the right-hand side of (A2.22) are of order $O(M)$. The third term is clearly non-negative. Furthermore, $r = x'J$ is of order $O(M)$.

From the above development and from (A2.21), we conclude that there exists a positive number M_0 such that for $M > M_0$

$$x' Q(x) > 0 \tag{A2.23}$$

(ii) $\|x_a\| < 2\beta \|x_b\|$ (A2.24)

Since A is nonnegative, we see that

$$p \geq 0 \ . \tag{A2.25}$$

Consider the quadratic form q. Since

$$M^2 = \|x\|^2 = \|x_a\|^2 + \|x_b\|^2 \leq 4\beta^2 \|x_b\|^2 + \|x_b\|^2 \tag{A2.26}$$

we have

$$\|x_b\|^2 > \frac{1}{1+4\beta^2} M^2 \ . \tag{A2.27}$$

Therefore the largest among the absolute values of the elements of x_b is equal to or greater than

$$\frac{1}{\sqrt{m(1+4\beta^2)}} M \ . \tag{A2.28}$$

Define a scalar function $g_{min}(y)$ by

$$g_{min}(y) = \min_i \left[|g_1(y)|,\ |g_1(-y)|,\cdots,\ |g_m(y)|,\ |g_m(-y)|\right] \tag{A2.29}$$

Then we can easily verify that

$$x_b' G(x_b) > \frac{M}{\sqrt{m(1+4\beta^2)}}\ g_{min} \left(\frac{M}{\sqrt{m(1+4\beta^2)}}\right) \tag{A2.30}$$

Since $g_{min}(y)$ approaches infinity as y tends to infinity, we can choose a positive number M_1 such that for $M > M_1$

$$q + r > 0 \ . \tag{A2.31}$$

This follows from the observation that the first and the second terms of (A2.22) and r are of order O(M), and that there exists a positive number β_0 such that

$$|\text{the first term + the second term} + r| \leq \beta_0 M \ . \tag{A2.32}$$

From (A2.25) and (A2.31) we conclude

$$x'Q(x) > 0 \tag{A2.33}$$

Thus (A2.11) holds.

Appendix 4. Proof of Lemma 3

From condition (I), it follows that

$$Y_{aa} \text{ is a positive-definitive matrix.} \tag{A3.1}$$

Hence,

$$|Y_{aa}| > 0 . \tag{A3.2}$$

Since $\Delta > 0$ for $D_F \to 0$ and $D_G \to \infty$, (14) implies that $\Delta > 0$ for all D_F and all D_G satisfying (15) or (16). (A3.3)

To prove Lemma 3, we need the following lemma:

Lemma A.1 Let $f(x) = f(x_1, x_2, \ldots, x_n)$ be a function of degree one in each variable x_μ ($\mu = 1, 2, \ldots, n$). Let S be a set of points such that $S = \{x | \alpha_\mu \leq x_\mu \leq \beta_\mu \ (\mu = 1, 2, \ldots, h);\ \alpha_\mu < x_\mu < \beta_\mu \ (\mu = h+1, \ldots, n)\}$. Here h may possibly be 0 or n. Then

$$f > 0 \text{ for all } x \in S \tag{A3.4}$$

if and only if the following three conditions are satisfied:

1) The function f evaluated at the "boundary" points where $x_\mu = \alpha_\mu$ or β_μ ($\mu = 1, 2, \ldots, n$) is nonnegative.

2) At least one of them is positive.

3) For any combination $x_\mu = \alpha_\mu$ or β_μ ($\mu = 1, 2, \ldots, h$) there exists a combination $x_\mu = \alpha_\mu$ or β_μ ($\mu = h+1, \ldots, n$) such that $f > 0$.

Proof of Lemma A.1: By the assumption of Lemma A.1, f can be written as

$$f = (x_1 - \alpha_1) f_0 + (\beta_1 - x_1) f_1 \tag{A3.5}$$

where

$$\left.\begin{aligned} f_0 &= f_0(x_2, x_3, \cdots x_n) = \frac{1}{\beta_1 - \alpha_1} f(\beta_1, x_2, x_3, \cdots x_n) \\ f_1 &= f_1(x_2, x_3, \cdots x_n) = \frac{1}{\beta_1 - \alpha_1} f(\alpha_1, x_2, x_3, \cdots x_n) \end{aligned}\right\} \tag{A3.6}$$

Similarly f_0 and f_1 can be written as

$$\left.\begin{aligned} f_0 &= (x_2-\alpha_2)f_{00} + (\beta_2-x_2)f_{01} \\ f_1 &= (x_2-\alpha_2)f_{10} + (\beta_2-x_2)f_{11} \end{aligned}\right\} \tag{A3.7}$$

where

$$\left.\begin{aligned} f_{00} &= f_{00}(x_3,x_4,\cdots x_n) = \frac{1}{\beta_2-\alpha_2} f_0(\beta_2,x_3,\cdots x_n) \\ f_{01} &= f_{01}(x_3,x_4,\cdots x_n) = \frac{1}{\beta_2-\alpha_2} f_0(\alpha_2,x_3,\cdots x_n) \\ f_{10} &= f_{10}(x_3,x_4,\cdots x_n) = \frac{}{\beta_2-\alpha_2} f_1(\beta_2,x_3,\cdots x_n) \\ f_{11} &= f_{11}(x_3,x_4,\cdots x_n) = \frac{1}{\beta_2-\alpha_2} f_1(\alpha_2,x_3,\cdots x_n) \end{aligned}\right\} \tag{A3.8}$$

Continuing this recursive procedure, we finally obtain

$$\begin{aligned} f = {} & (x_1-\alpha_1)(x_2-\alpha_2)\cdots(x_n-\alpha_n)f_{00\cdots0} \\ & + (x_1-\alpha_1)(x_2-\alpha_2)\cdots(x_{n-1}-\alpha_{n-1})(\beta_n-\alpha_n)f_{00\cdots01} \\ & + \cdots \\ & + (\beta_1-x_1)(\beta_2-x_2)\cdots(\beta_n-x_n)f_{11\cdots1} \end{aligned} \tag{A3.9}$$

where $f_{\varepsilon_1,\varepsilon_2\cdots\varepsilon_n}$ is a constant and is obtained by replacing x_μ by α_μ (when $\varepsilon_\mu = 1$) or β_μ (when $\varepsilon_\mu = 0$). It immediately follows from (A3.9) and (A3.4) that all of $f_{\varepsilon_1,\varepsilon_2\cdots\varepsilon_n}$ must be nonnegative and that at least one of them must be positive. In addition we can easily see that the condition 3) is necessary.

Conversely, if one of three conditions is not satisfied, then f can be made negative for some $x \in S$. This completes the proof of Lemma A.1.

Lemma A.1 holds even if some α_μ and β_μ are not finite, as demonstrated in the following:

Example A.1. Let

$$f(x_1,x_2) = \begin{vmatrix} a_{11}+b_{11}x_1 , & a_{12}+b_{12}x_2 \\ a_{21}+b_{21}x_1 , & a_{22}+b_{22}x_2 \end{vmatrix} . \tag{A3.10}$$

The function f satisfies the condition of Lemma A.1. Let S be a set such that $S = \{x|0 < x_\mu < \infty;\ \mu = 1,2\}$. Then $f > 0$ for all $x \in S$ if and only if

$$\left.\begin{array}{l} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \geqq 0 , \quad \begin{vmatrix} a_{11} & b_{12} \\ a_{11} & b_{22} \end{vmatrix} \geqq 0 \\ \\ \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} \geqq 0 , \quad \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \geqq 0 \end{array}\right\} . \tag{A3.11}$$

where at least one of the above equalities does not hold.

Example A.2. Consider (A3.10) again. Now let S be a set such that $S = \{x|0 \leqq x_1 \leqq \infty; 0 < x_2 < \infty\}$. In this case condition 1) implies (A3.11) and condition 3) implies that for $x_1 = 0$ either $\begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix}$ or $\begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix}$ must be positive, and that for $x_1 = \infty$ either $\begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix}$ or $\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$ must be positive.

Continuation of Proof of Lemma 3: Consider Δ in Eq. (14) as a function of α_μ ($\mu = 1,2,\cdots,k$) satisfying (15a) or (16a). Applying Lemma A.1 to Δ, we obtain Lemma 3.

Appendix 5. Proof of Lemma 4

Let us first derive the relationship between the matrix H in (29) and the admittance matrix $\tilde{Y}$ of the network in Fig. 24.

In order to calculate $\tilde{Y}$, we connect voltage sources U to each of the a-, b-, c-, and d-edges. Here, the elements of U are arranged in the order of a-, b-, c- and d-edges and

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{2k+m} \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} u_1 \\ u_2 \\ \vdots \\ u_{2k+m} \end{matrix}}\right\} 2k+m \\ \} \; m_0 \end{matrix} \tag{A4.1}$$

Let the current vector of the voltage source U be J. Then we have the standard loop equation

$$-HJ = U. \tag{A4.2}$$

(The minus sign in (A4.2) is due to the fact that the positive directions of the voltages are taken opposite to those of the current sources.) From (A4.2) it follows that $\tilde{Y}$ is given as the upper left $(2k+m) \times (2k+m)$ principal submatrix of H^{-1}.

Next consider Δ_∞ for K_1 and K_2 in (27). Clearly, Δ_∞ is equal to the determinant of the shaded submatrix of $\tilde{Y}$ in Fig. A.4. We can relate Δ_∞ with a minor of H by the following well-known lemma.

Lemma A.2 [4]. Let A be a nonsingular matrix of order n. If $B = A^{-1}$, then for arbitrary

$$\left(1 \leq \begin{matrix} i_1 < i_2 < \cdots < i_p \\ h_1 < h_2 < \cdots < h_p \end{matrix} \leq n\right)$$

$$B\begin{pmatrix} i_1 i_2 \cdots i_p \\ h_1 h_2 \cdots h_p \end{pmatrix} = \frac{(-1)^{\sum_1^p i_\mu + \sum_1^p h_\mu} A\begin{pmatrix} h'_1 h'_2 \cdots h'_{n-p} \\ i'_1 i'_2 \cdots i'_{n-p} \end{pmatrix}}{A\begin{pmatrix} 1,2,\cdots,n \\ 1,2,\cdots,n \end{pmatrix}} \tag{A4.3}$$

where $i_1 < i_2 < \cdots < i_p$ and $i'_1 < i'_2 < \cdots < i'_{n-p}$ form a complete system of of indices $1,2,\cdots,n$, as do $h_1 < h_2 < \cdots < h_p$ and $h'_1 < h'_2 < \cdots < h'_{n-p}$.

Since Δ_∞ is a minor of H^{-1}, we can apply Lemma A.2 to Δ_∞. By setting

$$A = H$$

$$n = 2k + m + m_0$$

$$P = k + m$$

$$i_\mu = \mu \ (\mu = 1,2,\cdots,k)$$

$$i_{k+\mu} = 2k+\mu \ (\mu = 1,2,\cdots,m)$$

$$h_\mu = \mu \ (\mu = 1,2,\ldots,k_1)$$

$$h_{k_1+\mu} = k+k_1+\mu \ (\mu = 1,2,\cdots,k_2)$$

$$h_{k+\mu} = 2k+\mu \ (\mu = 1,2,\cdots,m)$$

we have

$$\Delta_\infty = (-1)^{kk_2}|H|^{-1}\delta_0 \ . \tag{A4.4}$$

Appendix 6. Proof of Lemma 5

Suppose that we calculate δ_0 in (31) by using the Binet-Cauchy's formula [4]. Since δ_1 and δ_2 depend on the choice of $Ⓗ_0$, we write them temporarily as $\delta_1(Ⓗ_0)$ and $\delta_2(Ⓗ_0)$. Let the principal minor of Ⓗ corresponding to $Ⓗ_0$ be $\eta(Ⓗ_0)$. Then Binet-Cauchy's formula says that

$$\delta_0 = \Sigma\delta_1(Ⓗ_0)\ \delta_2(Ⓗ_0)\ \eta(Ⓗ_0) \tag{A5.1}$$

where the summations are taken over all possible combinations of $Ⓗ_0$. Note that $\eta^{-1}(Ⓗ_0)$ is positive. If $\delta \geqq 0$ for each $Ⓗ_0$, then we have by (31), (30) and (A5.1) $\Delta_\infty \geqq 0$.

Conversely suppose that there exists a $Ⓗ_0$ such that $\delta < 0$. Then by (A5.1) we can make Δ_∞ negative by choosing the values of resistors included in $Ⓗ_0$ sufficiently small and those of all other resistors sufficiently large. This completes the proof.

References

[1] R. O. Nielsen and A. N. Willson, "A fundamental result concerning the topology of transistor circuits with multiple equilibria," Proc. IEEE, vol. 68, pp. 196-208, February 1980.

[2] L. O. Chua and N. N. Wang, "On the application of degree theory to the analysis of resistive nonlinear networks," Int. Journal of Circuit Theory and Applications, vol. 5, pp. 35-68, 1977.

[3] T. Nishi and L. O. Chua, "Topological Criteria for nonlinear resistive circuits containing controlled sources to have a unique solution," ERL Memorandum No. UCB/ERL M83/21, February 1983.

[4] F. R. Gantmacher, The Theory of Matrices, Chelsea Pub. Co., New York, NY, 1959.

Figure Captions

Fig. 1. Op amp, its model and its graph representation

Fig. 2. Characteristic of "Model-C" op amp

Fig. 3. Characteristic of "Model-D" op amp

Fig. 4. Simple op-amp circuit and its associated graph

Fig. 5. A graph contained in Fig. 4(b)

Fig. 6. A disallowed graph

Fig. 7. (a) Open-circuit operation $k \to \mathcal{O}(k)$

(b) Short-circuit operation $k \to \mathcal{S}(k)$

Fig. 8. Applying operation $\mathcal{O}/\mathcal{S}(\cdot)$ to an op amp $(k,\hat{k})$

Fig. 9. Applying operation $\mathcal{Z}(\cdot)$ to an op amp $(k,\hat{k})$

Fig. 10. Cactus graphs

Fig. 11. Graph with a complementary tree structure

Fig. 12. An op-amp circuit which does not have a unique solution

Fig. 13. Circuit for Example 2

Fig. 14. Circuit for Example 3

Fig. 15. Graphs obtained from Fig. 4(b) by applying operation (a)

Fig. 16. Circuit for Example 6

Fig. 17. Circuit for Example 7

Fig. 18. Circuit for Example 8

Fig. 19. Circuit (I) for Example 9

Fig. 20. Circuit (II) for Example 9

Fig. 21. Circuit for Example 10

Fig. 22. Two characteristics of nonlinear voltage-controlled voltage source

Fig. 23. Circuit containing "k" VCVS's and "m" nonlinear resistors

Fig. 24. Linear resistive (2k+m)-port corresponding to $\tilde{Y}$ in (24)

Fig. 25. Graph representation of each port

Fig. 26. Main part of the fundamental loop matrix of the graph $\tilde{G}$.

Fig. 27. The coefficient matrix H associated with the loop equation in (A4.2)

Fig. 28. Submatrix of B_T in Fig. 25. This is identified as the main part of the fundamental loop matrix of the graph $G^{(0)}$

Fig. 29. Matrix obtained from $B_T^{(0)}$

Fig. A.1. A complementary-tree-structure graph obtained from a circuit which satisfies Assumptions (1) and (2)

Fig. A.2. Loop maue exclusively of output edges of op amps and cutset made exclusively of input edges of op amps.

Fig. A.3. Illustration of (A2.4)

Fig. A.4. Admittance matrix $\tilde{Y}$ of the network in Fig. 24.

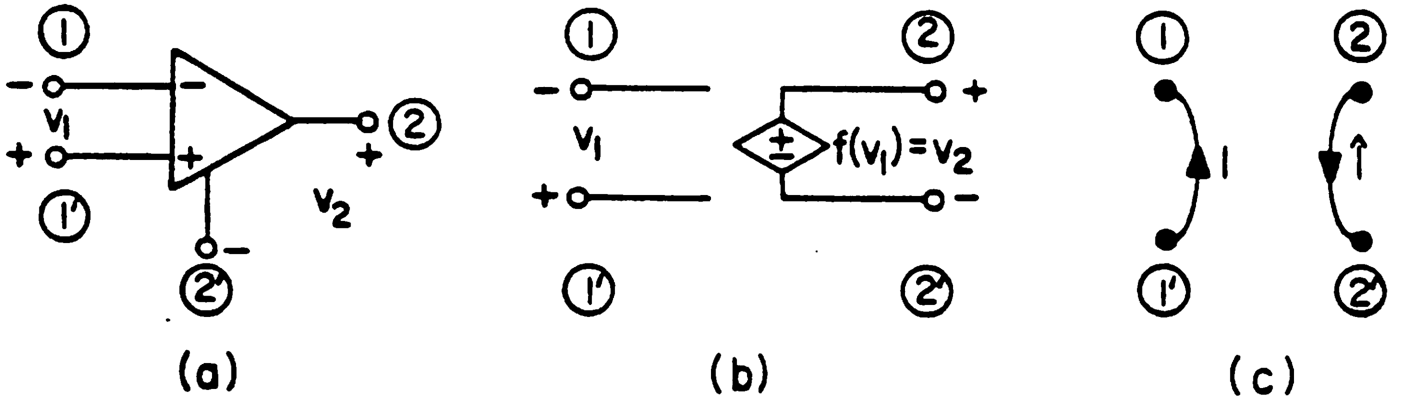


Fig. 1

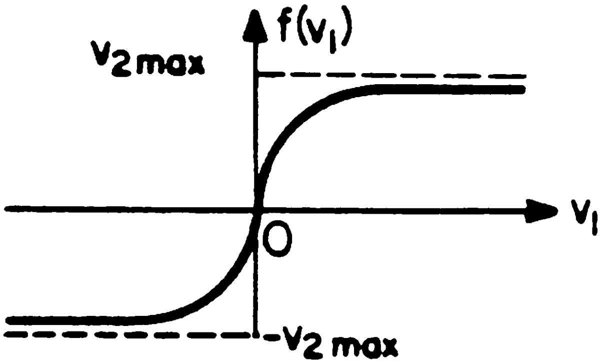


Fig. 2

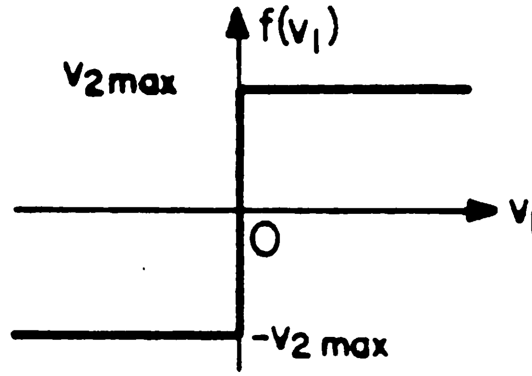


Fig. 3

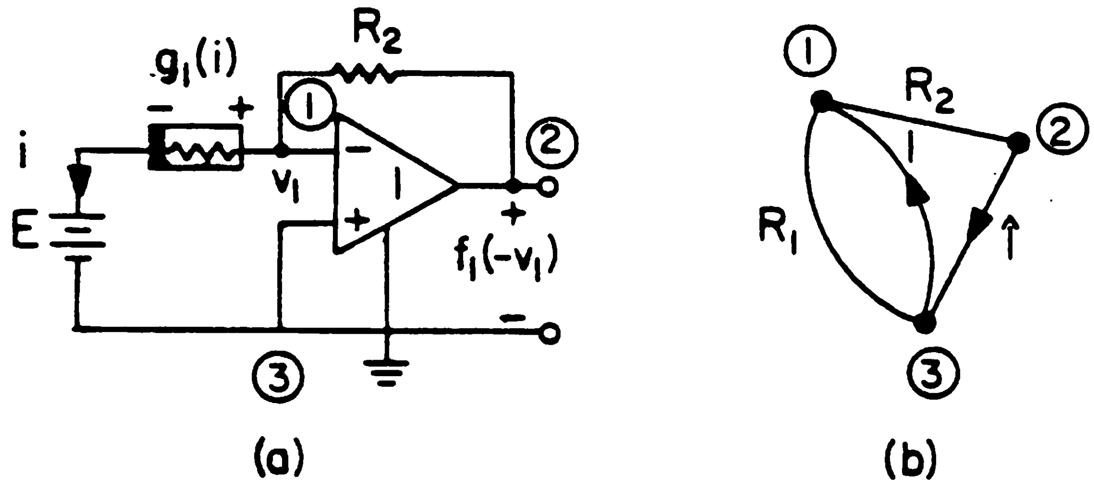


Fig. 4

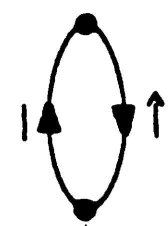


Fig. 5

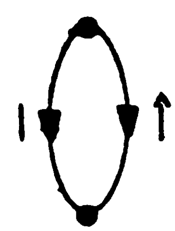


Fig. 6

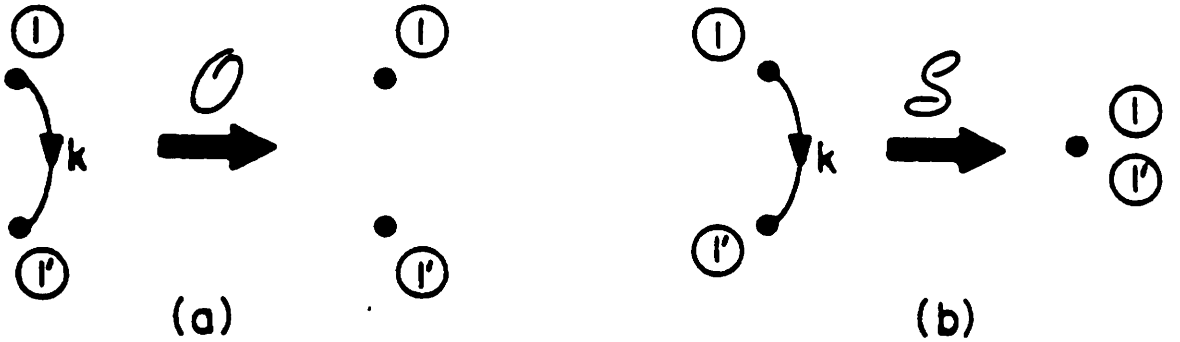


Fig. 7

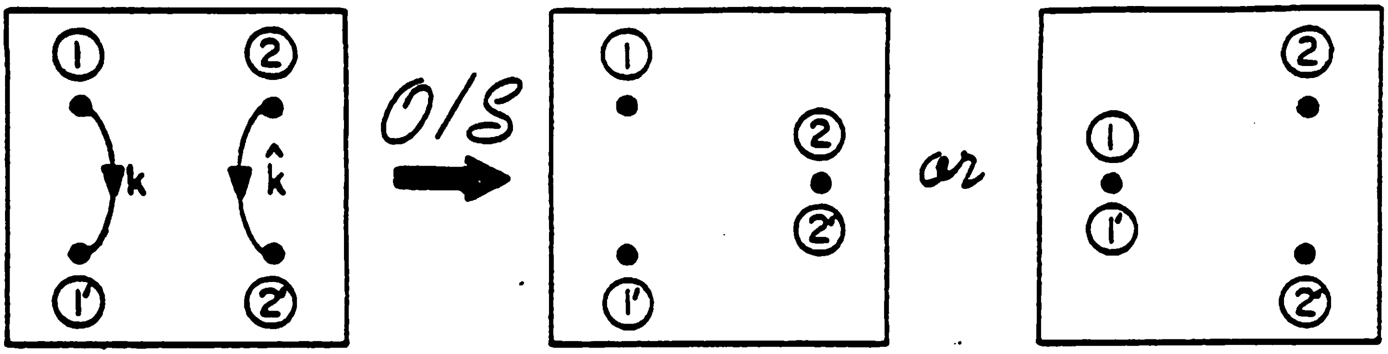


Fig. 8

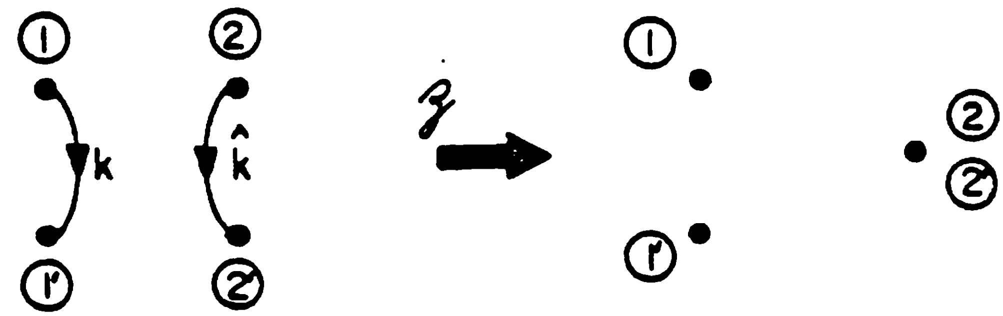


Fig. 9

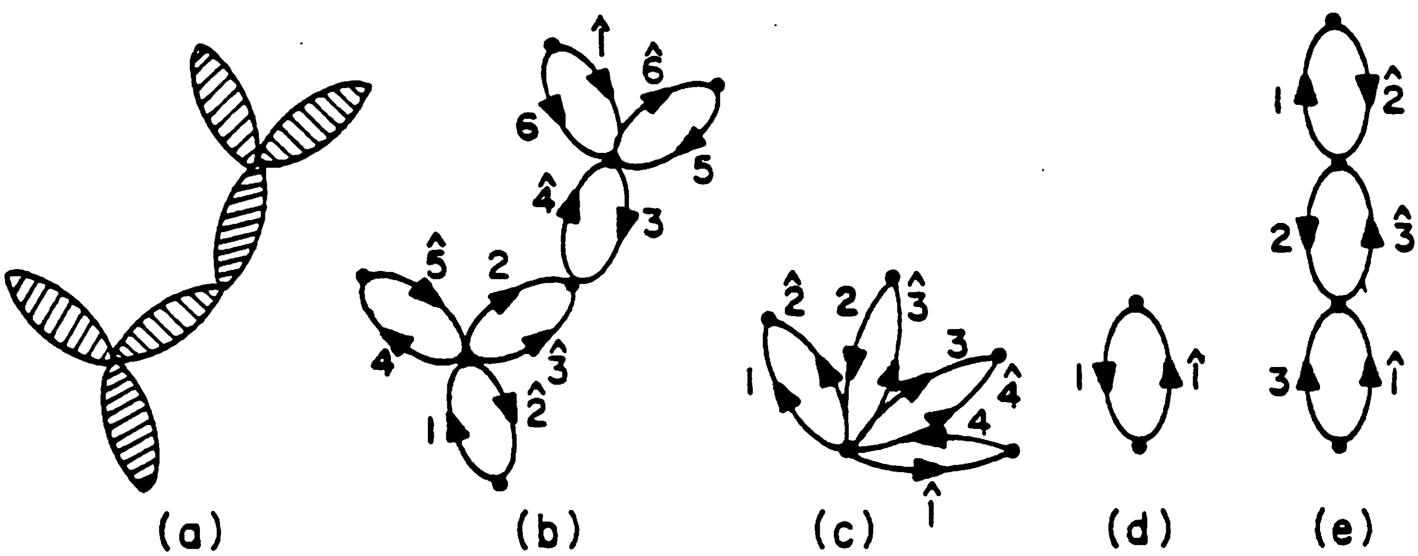


Fig. 10

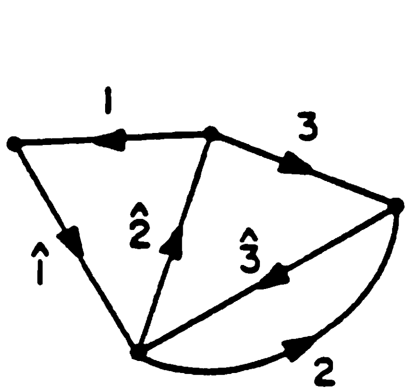


Fig. 11

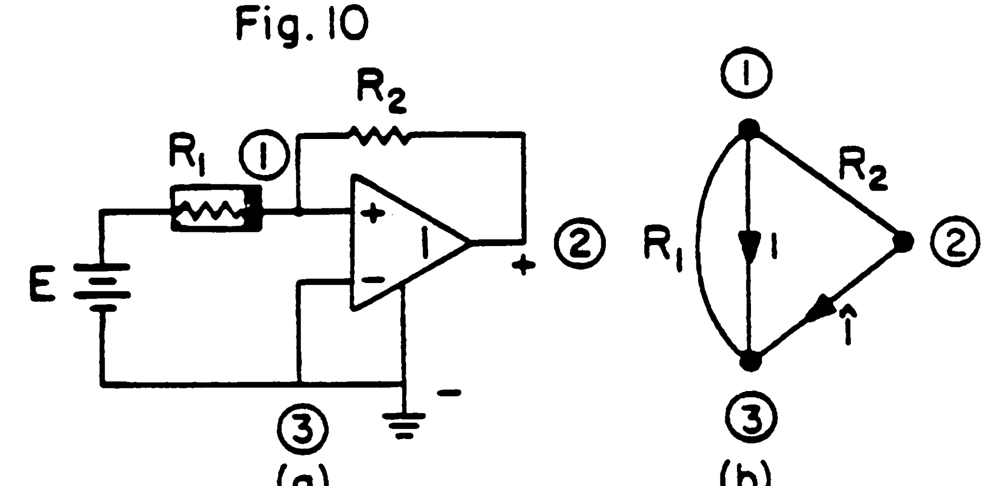


Fig. 12

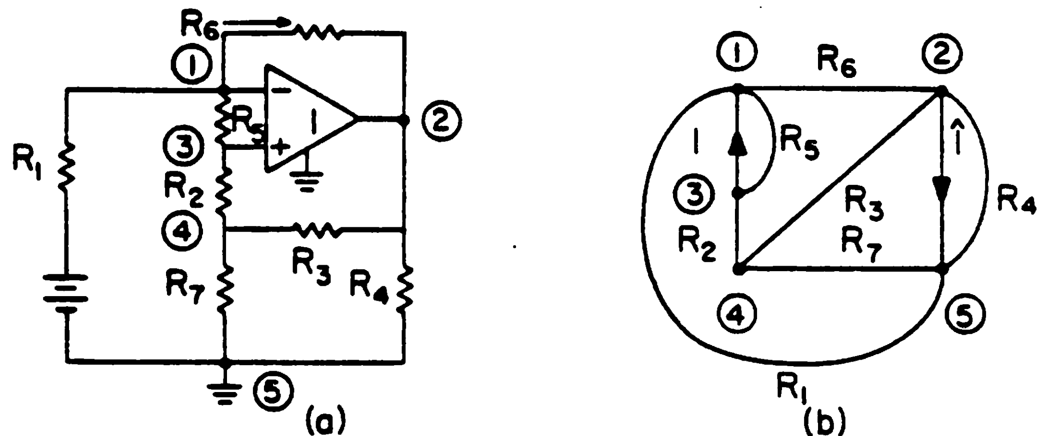


Fig. 13

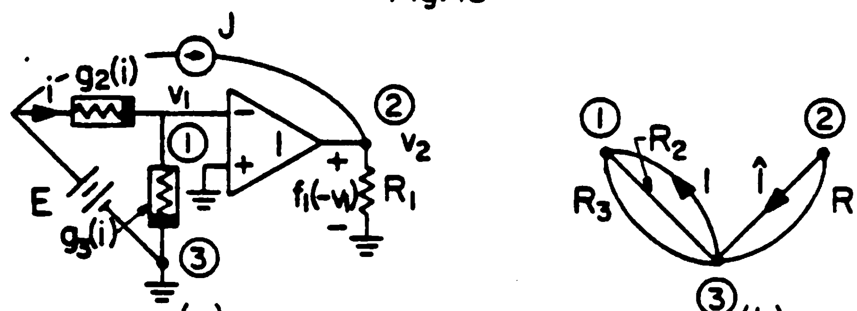


Fig. 14

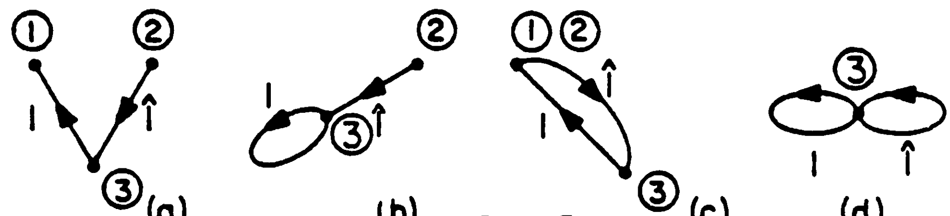


Fig. 15

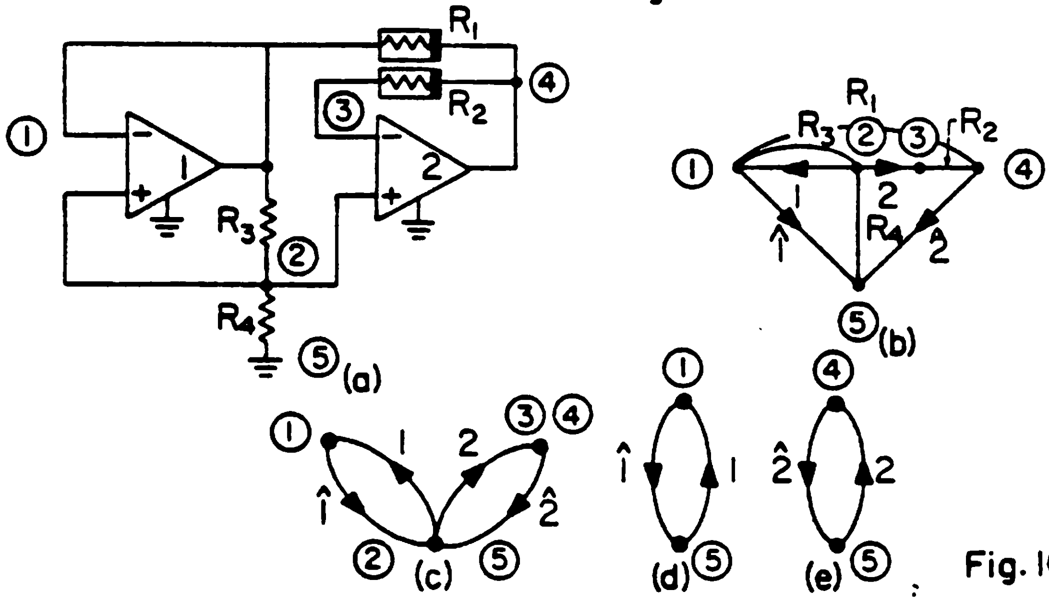


Fig. 16

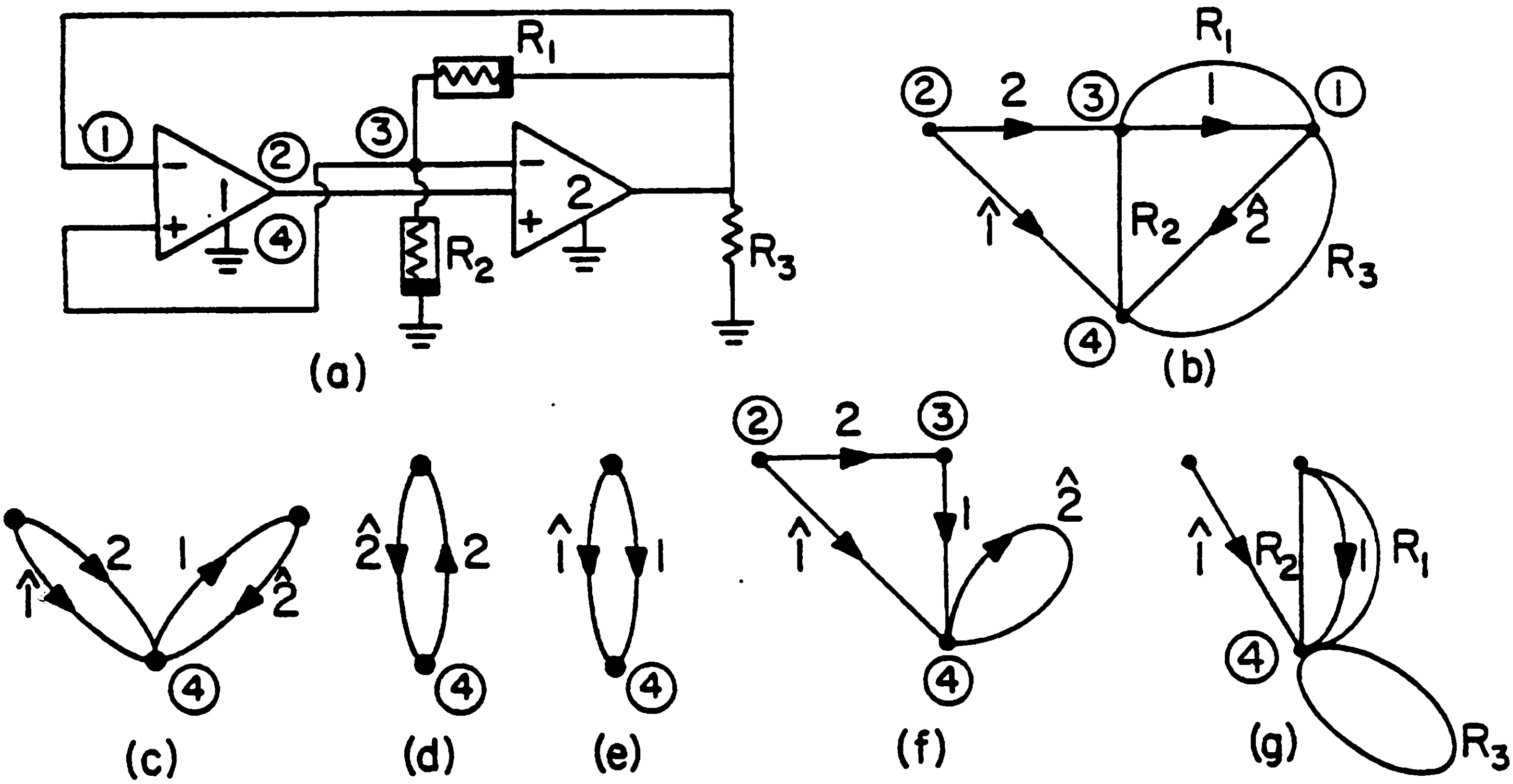


Fig. 17

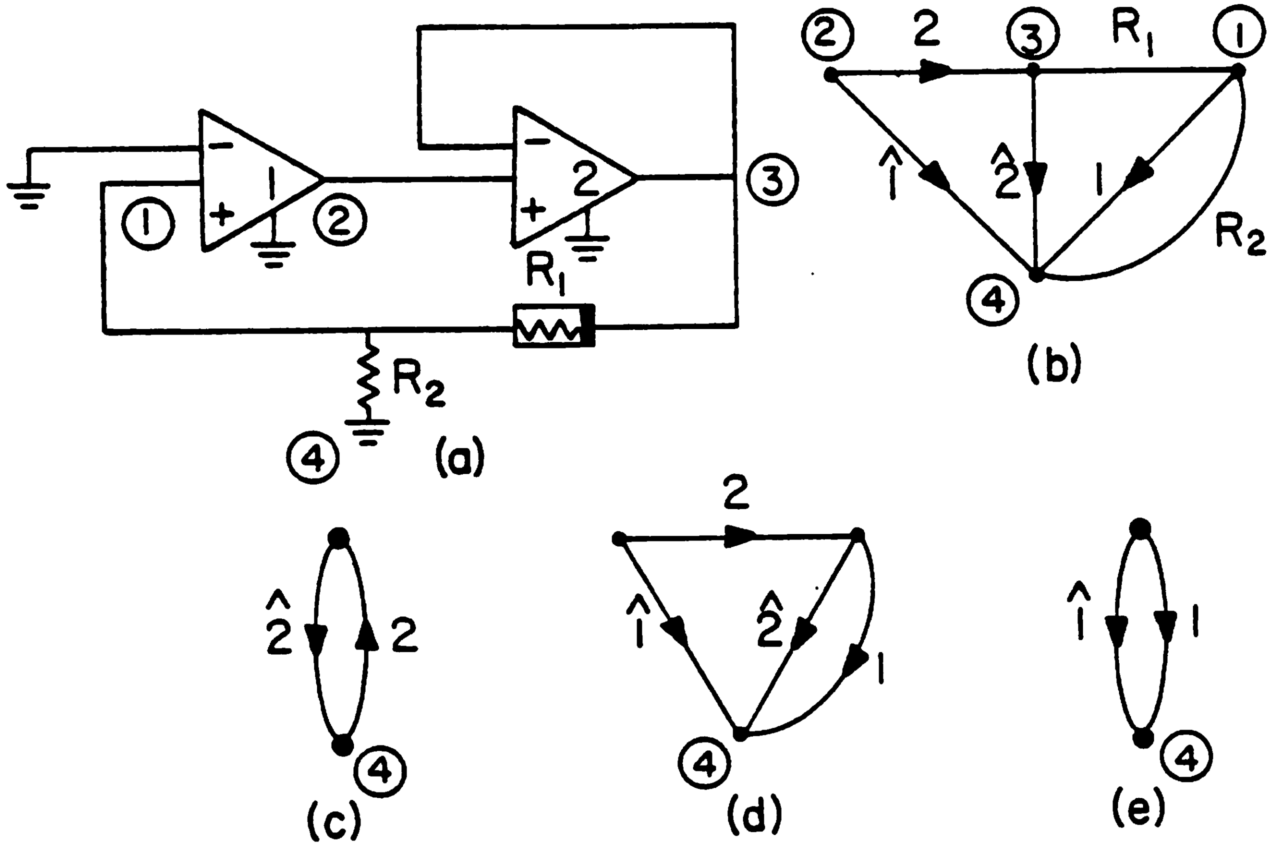


Fig. 18

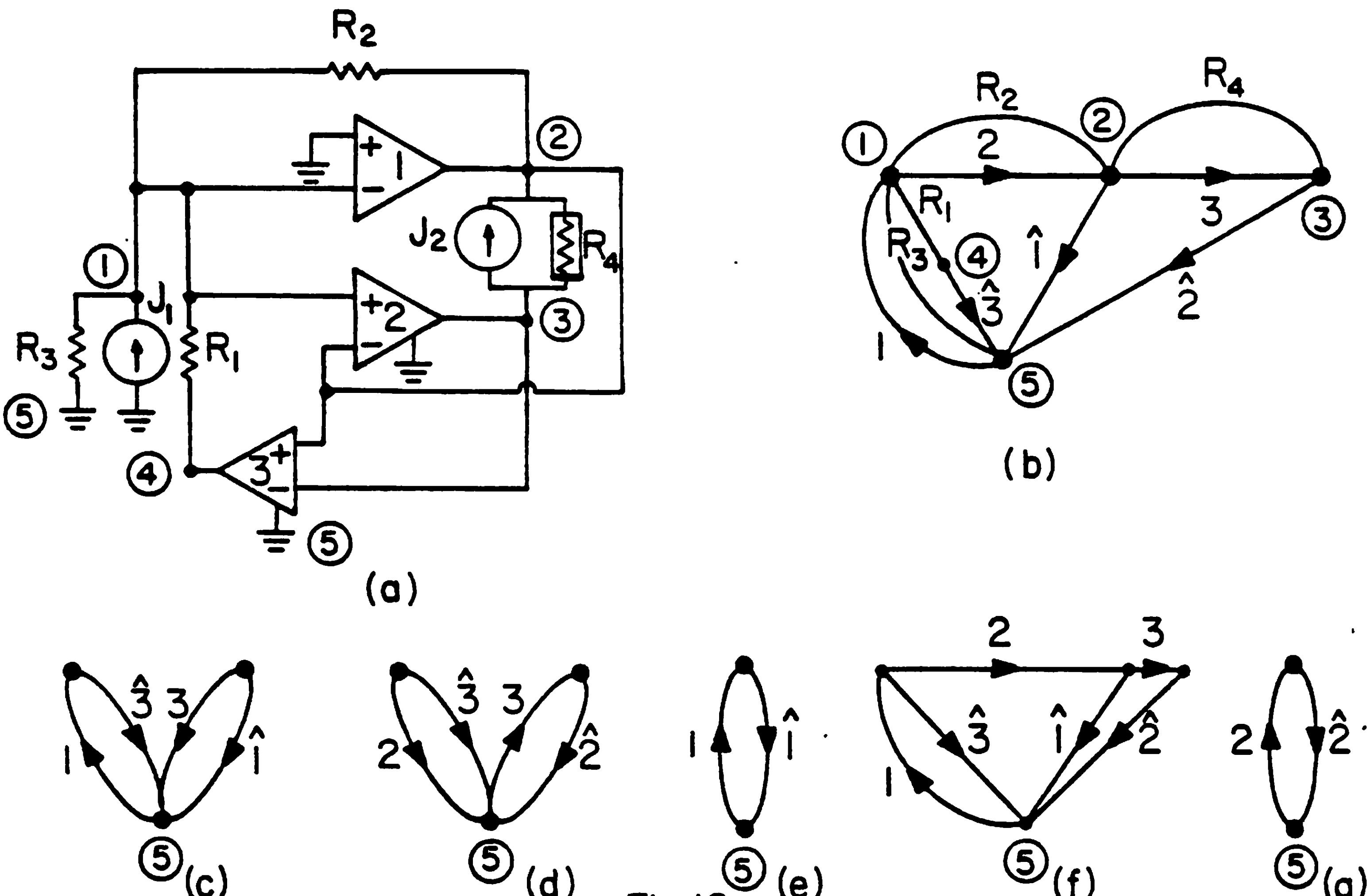


Fig.19

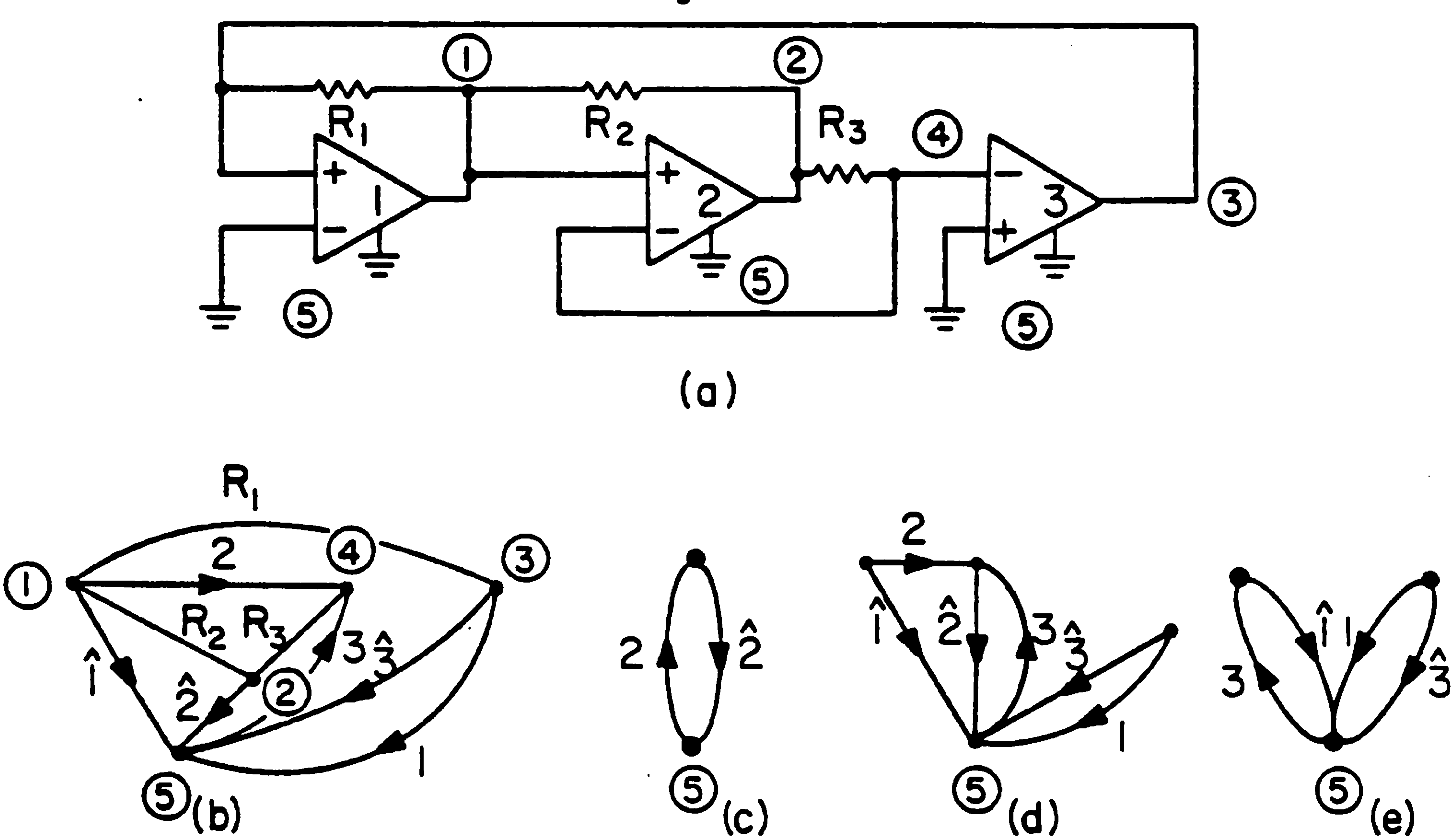
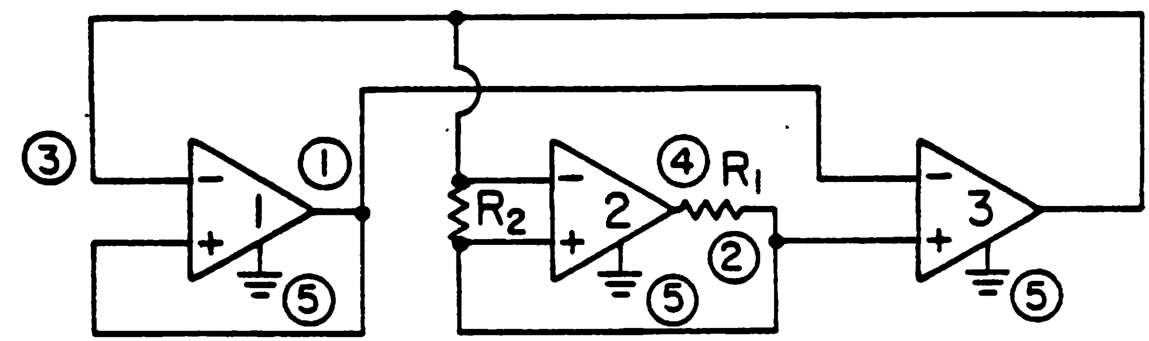
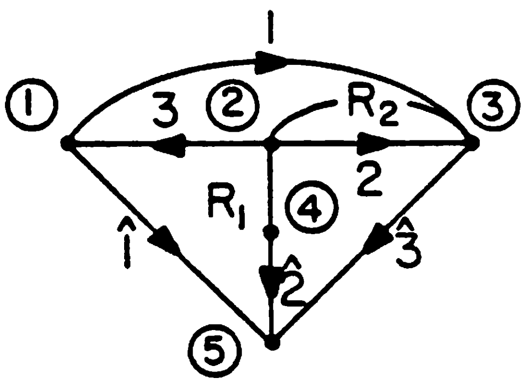


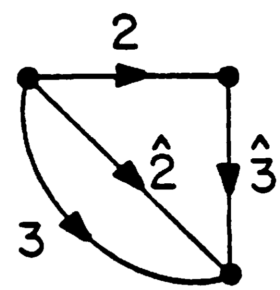
Fig. 20



(a)

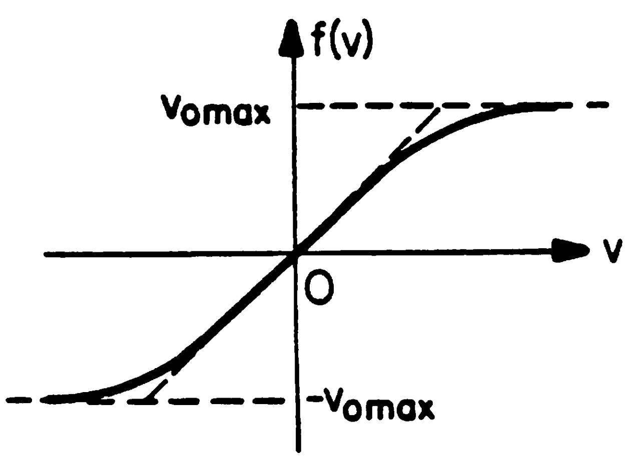


(b)

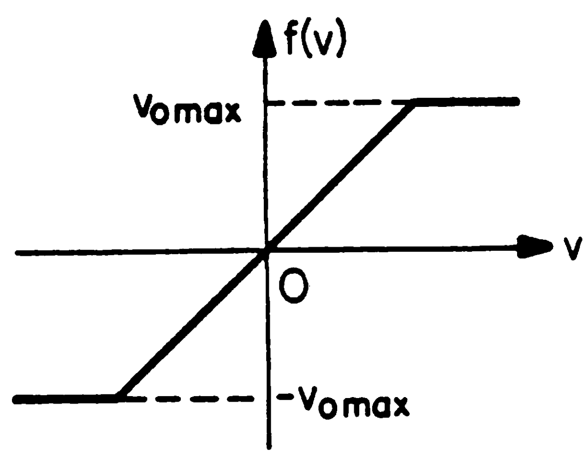


(c)

Fig. 21



(a)



(b)

Fig. 22

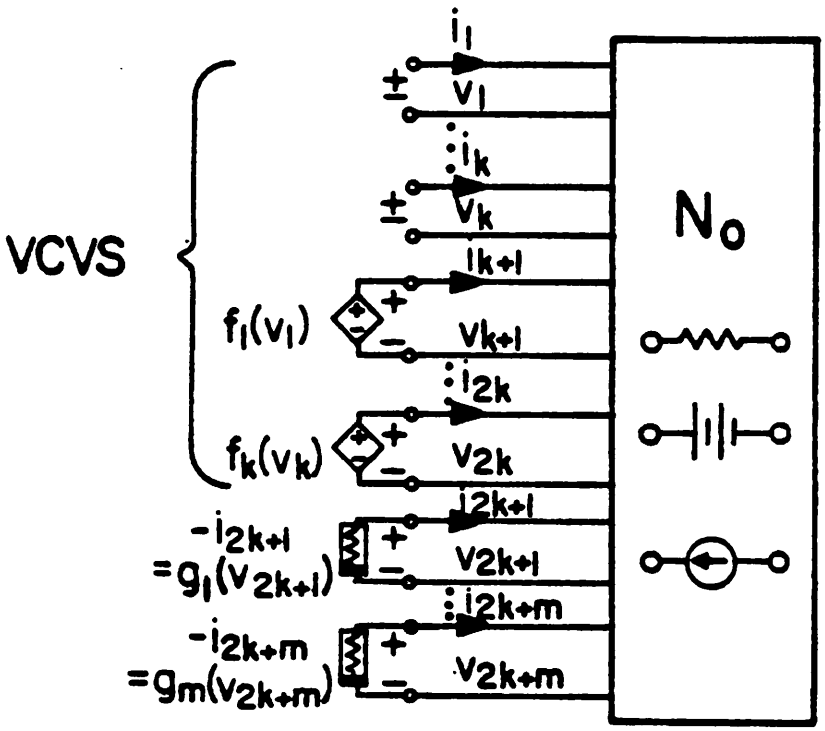


Fig. 23

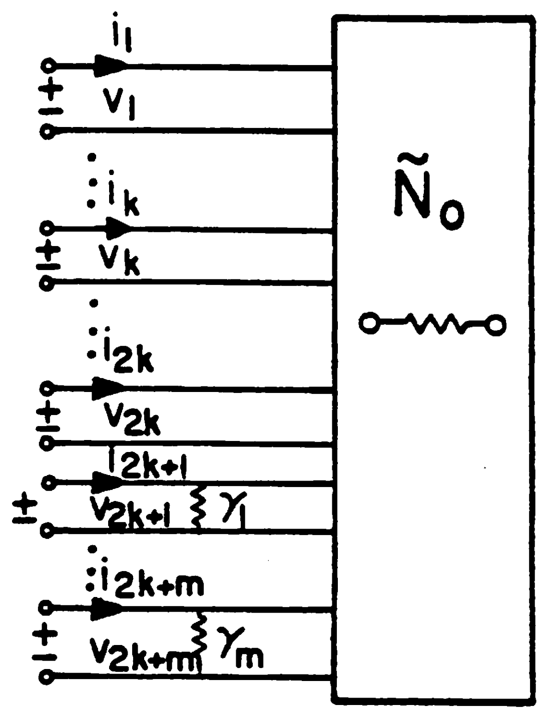


Fig. 24

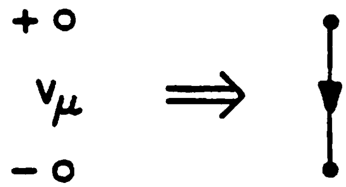


Fig. 25

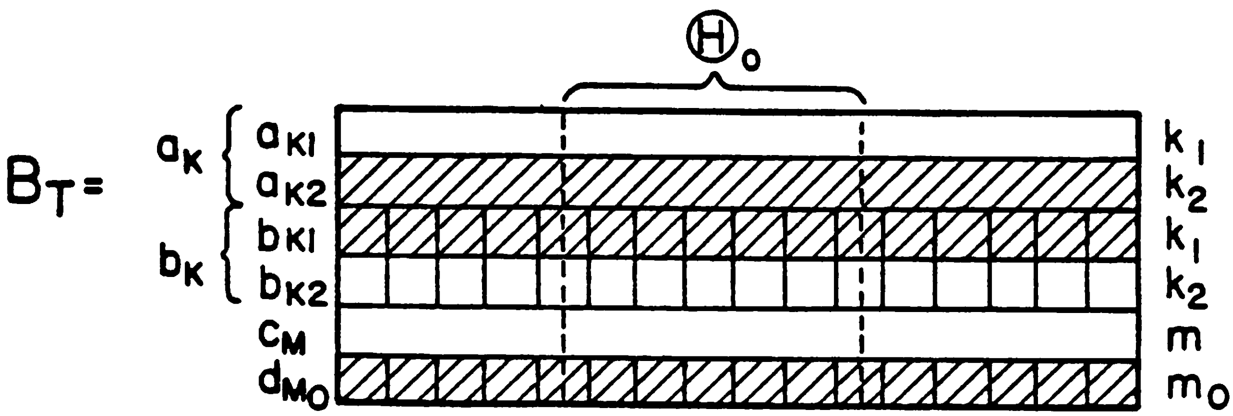


Fig. 26

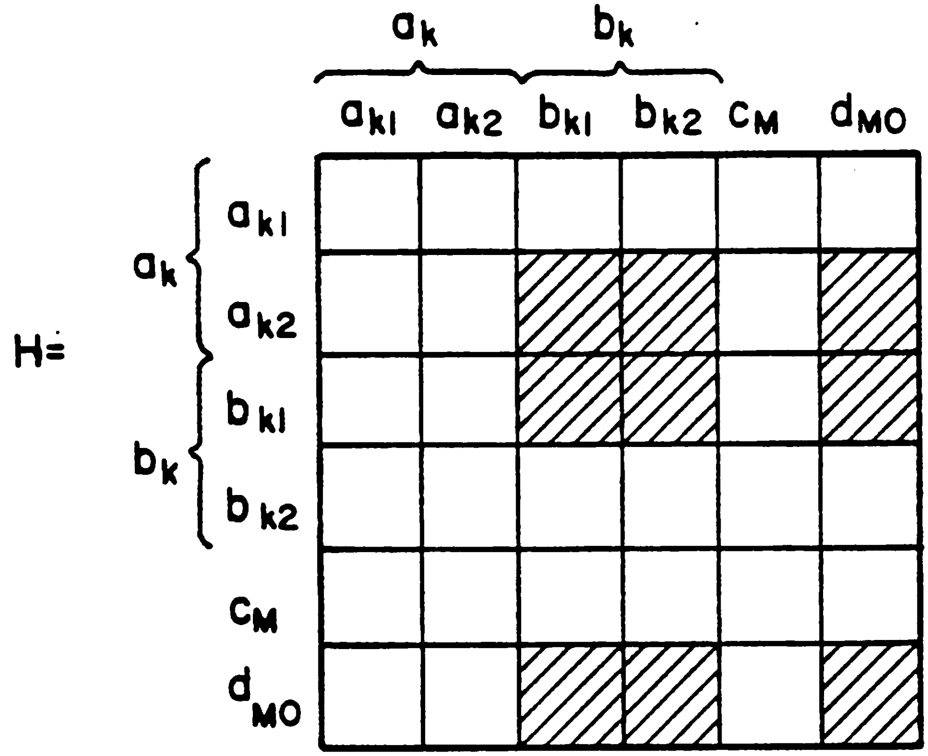


Fig. 27

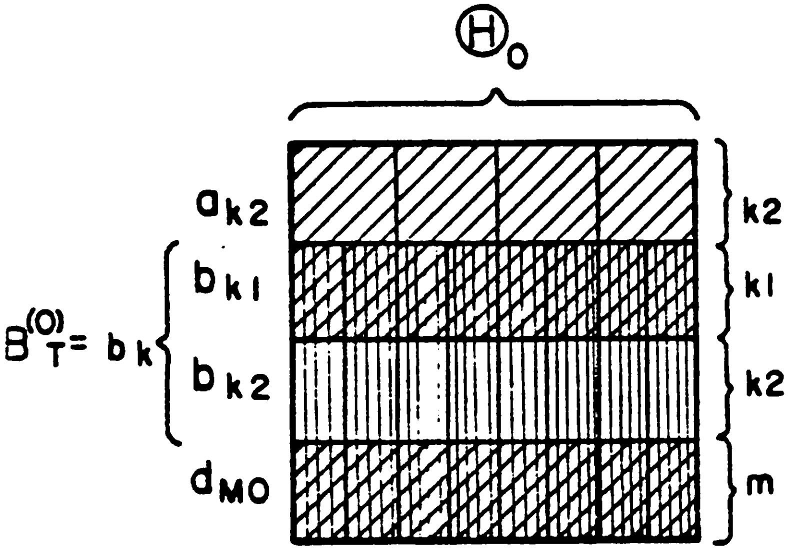


Fig. 28

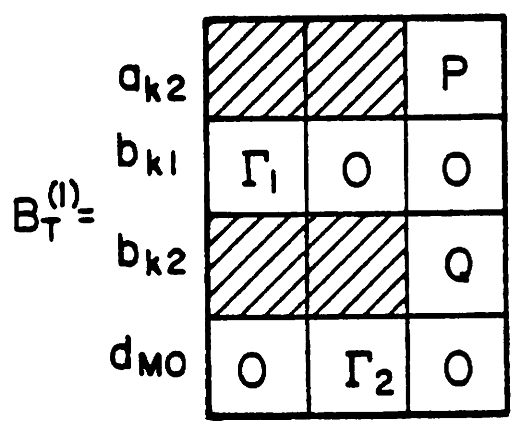


Fig. 29

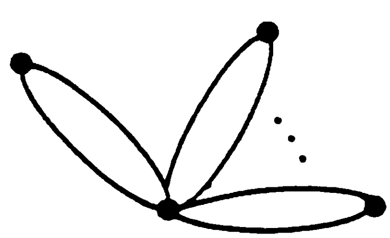
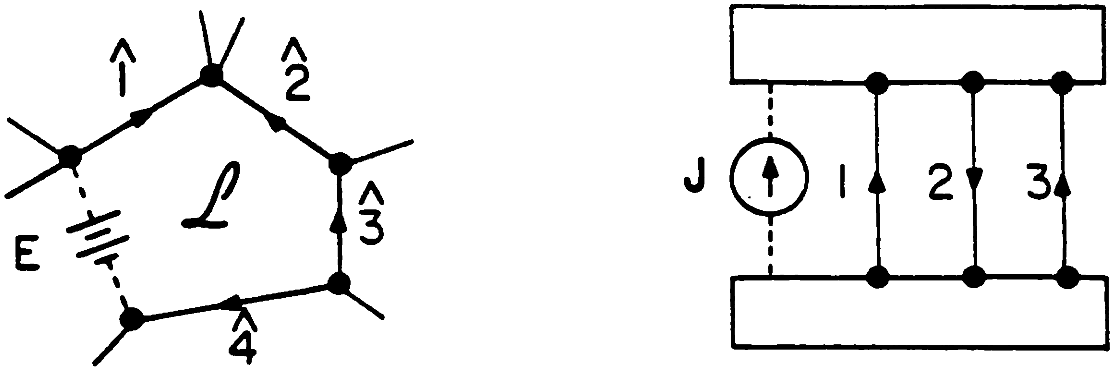
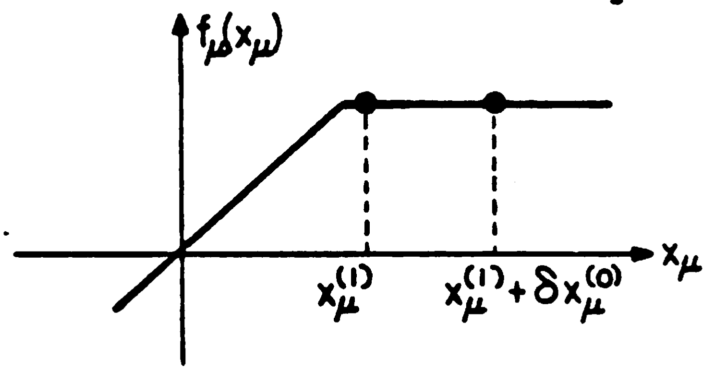


Fig. A.1

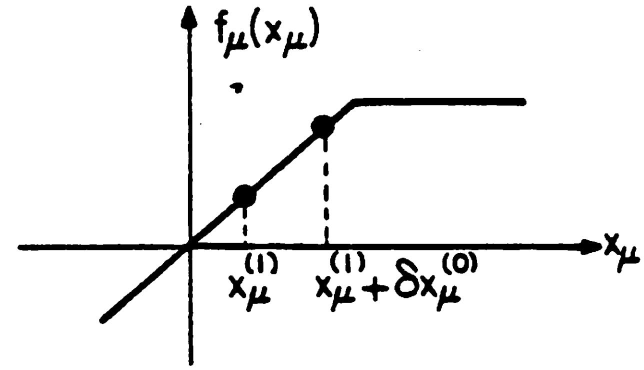


(a) (b)

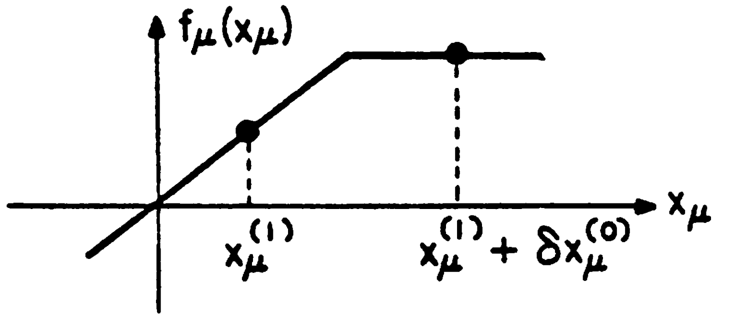
Fig. A.2



(a) The case $d_{f\mu} = 0$



(b) The case $d_{f\mu} = \alpha_\mu$



(c) The case $0 < d_{f\mu} < \alpha_\mu$

Fig. A.3

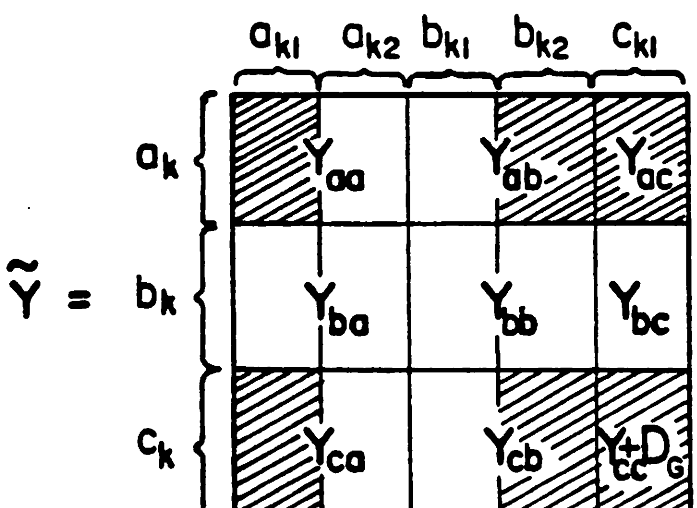


Fig. A.4