THE ROLE OF FUZZY LOGIC IN THE MANAGEMENT OF UNCERTAINTY IN EXPERT SYSTEMS

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ABSTRACT

Management of uncertainty is an intrinsically important issue in the design of expert systems because much of the information in the knowledge base of a typical expert system is imprecise, incomplete or not totally reliable.

In the existing expert systems, uncertainty is dealt with through a combination of predicate logic and probability-based methods. A serious shortcoming of these methods is that they are not capable of coming to grips with the pervasive fuzziness of information in the knowledge base, and, as a result, are mostly ad hoc in nature. An alternative approach to the management of uncertainty which is suggested in this paper is based on the use of fuzzy logic, which is the logic underlying approximate or, equivalently, fuzzy reasoning. A feature of fuzzy logic which is of particular importance to the management of uncertainty in expert systems is that it provides a systematic framework for dealing with fuzzy quantifiers, e.g., *most, many, few, not very many, almost all, infrequently, about 0.8*, etc. In this way, fuzzy logic subsumes both predicate logic and probability theory, and makes it possible to deal with different types of uncertainty within a single conceptual framework.

In fuzzy logic, the deduction of a conclusion from a set of premises is reduced, in general, to the solution of a nonlinear program through the application of projection and extension principles. This approach to deduction leads to various basic syllogisms

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which may be used as rules of combination of evidence in expert systems. Among syllogisms of this type which are discussed in this paper are the intersection/product syllogism, the generalized modus ponens, the consequent conjunction syllogism, and the major-premise reversibility rule.

Keywords: Expert systems, knowledge representation, fuzzy logic, fuzzy sets.

1. Introduction

An expert system, as its name implies, is an information system which provides the user with a facility for posing and obtaining answers to questions relating to the information stored in its knowledge base. Typically, such systems possess a nontrivial inferential capability and, in particular, have the capability to infer from premises which are imprecise, incomplete or not totally reliable.

Since the knowledge base of an expert system is a repository of human knowledge, and since much of human knowledge is imprecise in nature, it is usually the case that the knowledge base of an expert system is a collection of rules and facts which, for the most part, are neither totally certain nor totally consistent. Now, as a general principle, the uncertainty of information in the knowledge base of any question-answering system induces some uncertainty in the validity of its conclusions. Hence, to serve a useful purpose, the answer to a question must be associated -- explicitly or, at least, implicitly -- with an assessment of its reliability. For this reason, a basic issue in the design of expert systems is how to equip them with a computational capability to analyze the transmission of uncertainty from the premises to the conclusion and associate the conclusion with what is commonly called a *certainty factor*.

In the existing expert systems, the computation of certainty factors is carried out through a combination of methods which are based on or, at least, not far removed from, two-valued logic and probability theory. However, it is widely recognized at this juncture that such methods have serious shortcomings and, for the most part, are hard to rationalize. In particular, what is open to question is the universally made assumption that if each premise is associated with a numerical certainty factor then the certainty factor of the conclusion is a number which may be expressed as a function of the certainty factors of the premises. As will be seen in the sequel, this assumption is, in general, invalid. It regains its validity, however, if the certainty factors are represented as fuzzy rather than crisp numbers.

More generally, a point of view which is articulated in the present paper is that the conventional approaches to the management of uncertainty in expert systems are intrinsically inadequate because they fail to come to grips with the fact that much of the uncertainty in such systems is possibilistic rather than probabilistic in nature. As an alternative, it is suggested that a fuzzy-logic-based computational framework be employed to deal with both possibilistic and probabilistic uncertainty within a single conceptual system. In this system, test-score semantics -- which is the meaning-representational component of fuzzy logic -- forms the basis for the representation of knowledge, while the inferential component of fuzzy logic is employed to deduce answers to questions and, when necessary, associate each answer with a probability which is represented as a fuzzy quantifier.

The employment of fuzzy logic as a framework for the management of uncertainty in expert systems makes it possible to consider a number of issues which cannot be dealt with effectively or correctly by conventional techniques. The more important of these issues are the following.

1. The fuzziness of antecedents and/or consequents in rules of the form

   (a) If X is A then Y is B

   (b) If X is A then Y is B with $CF = \alpha$,

   where the antecedent, X is A, and the consequent, X is B, are fuzzy propositions, and α is a numerical value of the certainty factor, CF. For example,

   $$\textit{If } X \textit{ is small then } Y \textit{ is large with } CF = 0.8 \ ,$$

   in which the antecedent "*X is small*" and the consequent "*Y is large*" are fuzzy propositions because the denotations of the predicates *small* and *large* are fuzzy subsets *SMALL* and *LARGE* of the real line.

2. Partial match between the antecedent of a rule and a fact supplied by the user.

   Since the number of rules in an expert system is usually relatively small (i.e., of the order of two hundred), there are likely to be many cases in which a fact such as "X is A^*" may not match exactly the antecedent of any rule of the form "If X is A then Y is B with $CF = \alpha$". The conventional rule-based systems usually avoid this issue or treat it in an ad hoc manner because partial matching does not lend itself to analysis within the confines of two-valued logic. By contrast, the gradation of truth and membership in fuzzy logic provides a natural way of dealing with partial matching through the use of the compositional rule of inference and interpolation.

3. The presence of fuzzy quantifiers in the antecedent and/or the consequent of a rule.

   In many cases, the antecedent and/or the consequent of a rule contain implicit or explicit fuzzy quantifiers such as *most*, *many*, *few*, *many more*, *usually*, *much of*, etc. As an illustration, consider the disposition [1]

   $$d \triangleq \textit{Students are young} \;,$$

   which may be interpreted as the proposition

   $$p \triangleq \textit{Most students are young} \;,$$

   which in turn may be expressed as a rule, or, equivalently, as the conditional proposition

   $$r \triangleq \textit{If } x \textit{ is a student then it is likely that } x \textit{ is young} \;,$$

   in which the fuzzy probability *likely* has the same denotation, expressed as a fuzzy subset of the unit interval, as the fuzzy quantifier *most*.

In the following two sections, we shall consider first the problems which arise when the antecedent and consequent components of a rule are fuzzy propositions, but no certainty factors are involved. In Section 4, then, we shall consider the more general case of rules in which the certainty factor is represented as a fuzzy quantifier.

Our exposition of the role of fuzzy logic in the management of uncertainty in expert systems is not intended to be definitive and/or complete. Rather, its much more limited objective is to suggest that fuzzy logic provides a natural conceptual framework for the analysis and design of expert systems, and point to some of the basic problem-areas which are in need of exploration.

1 As defined in [89], a disposition is a proposition with implicit fuzzy quantifiers. Many of the rules in a typical expert system are, in effect, dispositions.

2. The Effect of Fuzziness in Facts and Rules

Typically, the knowledge base of an expert system consists of (a) a collection of propositions which represent the *facts*; and (b) a collection of conditional propositions which constitute the *rules*. For example, the facts may be: [2]

(a) Carol is a graduate student.

(b) Berkeley's population is *over 100,000.*

(c) San Francisco is a *foggy* city.

(d) John *has duodenal ulcer* (CF = 0.3).

In these examples, (a) is a nonfuzzy proposition since the class of graduate students is a crisp set; (b) is a fuzzy proposition because of an implicit understanding that *over 100,000* means *over 100,000 but not much over 100,000*; (c) is a fuzzy proposition because *foggy city* is a fuzzy predicate; and (d) is a fuzzy proposition because the predicate *has a duodenal ulcer* is a fuzzy predicate in the sense that having a duodenal ulcer is a matter of degree. Furthermore, although the certainty factor is stated to be equal to 0.3, it should be interpreted as a fuzzy number which is *approximately equal* to 0.3.

A typical rule in MYCIN [61] is exemplified by

Rule 0-47

If: 1) The site of the culture is blood, and

2) The identity of the organism *is not known with certainty*, and

3) The stain of the organism is gramneg, and

4) The morphology of the organism is rod, and

5) The patient has been *seriously burned*

then: There is *weakly suggestive* evidence that the identity of the organism is pseudomonas.

Another typical rule in MYCIN reads:

2 In these and the following examples, the italicized constituents are fuzzy predicates. A proposition is defined to be fuzzy if it contains fuzzy predicates.

If: 1) The route of the administration of the penicillin is oral, and

2) There is a gastrointestinal factor which *may interfere* with the *absorption* of the penicillin

then: There is *suggestive evidence* (0.6) that the route of administration of the penicillin is not *adequate*.

Typical rules in PROSPECTOR [20] are exemplified by:

a) **If:** *Abundant* quartz sulfide *veinlets* with *no apparent alteration halos*

b) **then:** (*LS*, *LN*) *alteration favorable* for the *potassic stage*.

a) **If:** Volcanic rocks in the region are *contemporaneous* with the *intrusive system* (*coeval volcanic rocks*)

b) **then:** (*LS*, *LN*) the level of erosion is *favorable* for a *prophyry copper deposit*.

In these rules, LS and LN are real numbers representing likelihood ratios. Informally, if the ratio LS which is associated with a hypothesis H and evidence E is greater than unity, then the odds on H are increased by the presence of E. On the other hand, if LN is greater than unity, then the odds on H are increased by the absence of E. In consequence of the definitions of LS and LN, they cannot be simultaneously greater than unity.

The point we wish to make through these examples is that most of the facts and rules in expert systems contain fuzzy predicates and thus are fuzzy propositions. This is particularly true of the heuristic rules which are encoded as production rules in what are, in effect, fuzzy algorithms [72], [57], [74]. An example of such heuristic rules in the case of a rule-based system for playing poker is provided by the following excerpt [61], [66]:

"If your hand is *excellent* then bet *low* if the opponent *tends* to be a *conservative* player and has just bet *low*. Bet *high* if the opponent is not *conservative*, is not *easily bluffed*, and has just made a *sizable* bet. Call if the pot is *extremely large*, and the opponent has just made a *sizable bet*.

In the existing expert systems, the fuzziness of the knowledge base is ignored because neither predicate logic nor probability-based methods provide a systematic basis for dealing with it. As a consequence, fuzzy facts and rules are generally manipulated as if they were nonfuzzy, leading to conclusions whose validity is open to question.

As a simple illustration of this point, consider the fact

John has duodenal ulcer (CF = 0.3) .

Since *has duodenal ulcer* is a fuzzy predicate, so that John may have it to a degree, the meaning of the certainty factor becomes ambiguous. More specifically, does CF = 0.3 mean that (a) John has duodenal ulcer to the degree 0.3; or (b) that the probability of the fuzzy event "John has duodenal ulcer" is 0.3? Note that in order to make the latter interpretation meaningful, it is necessary to be able to define the probability of a fuzzy event. This can be done in fuzzy logic [71], [30] but not in classical probability theory.

As another illustration, consider a rule of the general form:

If X is A then Y is B with probability β ,

where X and Y are variables, A and B are fuzzy predicates and β is fuzzy probability expressed as a fuzzy number, e.g., *about 0.8*, or as a linguistic probability, e.g., *very likely*. For example,

If Hans has a new red Porsche then it is likely that his wife is young,

in which case $X \triangleq$ make of Hans' car; $A \triangleq$ fuzzy set of new red Porsches; $Y \triangleq$ age of Hans' wife; $B \triangleq$ fuzzy subset *YOUNG* of the Age scale; and $\beta \triangleq$ *LIKELY*, which is a fuzzy subset of the unit interval.

Expressed as a conditional probability, the rule in question may be written as

$$Pr\ \{Y \text{ is } B \mid X \text{ is } A\} \text{ is } \beta \ . \tag{2.1}$$

In the existing expert systems, such a rule would be treated as an ordinary conditional probability, from which it would follow that

$$Pr\ \{Y \text{ is not } B \mid X \text{ is } A\} \text{ is } 1-\beta \ . \tag{2.2}$$

However, as shown in [88], this conclusion is, in general, incorrect if A is a fuzzy set. The correct conclusion is weaker than (2.2), namely,

$$Pr\ \{Y\ is\ not\ B \mid X\ is\ A\} + Pr\ \{Y\ is\ B \mid X\ is\ A\} \geq 1\ , \tag{2.3}$$

with the understanding that the probabilities in question may be fuzzy numbers.

In short, an assumption which is treated as a truism in expert systems is that

$$P(H \mid E) = 1 - P(\neg H \mid E)\ , \tag{2.4}$$

where $P(H \mid E)$ is the conditional probability of a hypothesis H given an evidence E, and $\neg H$ is the negation of H. Our discussion shows that, in general, this assumption is not valid when E is a fuzzy proposition, as is frequently the case in most knowledge bases.

As was alluded to earlier, perhaps the most serious deficiency of the existing expert systems relates to the ways in which (a) the traditional rules of inference are applied to fuzzy rules; and (b) the computation of certainty factors is carried out when two or more rules are combined through conjunction, disjunction or chaining. In the case of chaining, in particular, the standard inference rule -- *modus ponens* -- loses much of its validity and must be replaced by the more general *compositional rule of inference* [75]. Furthermore, as shown in [88], the transitivity of implication, which forms the basis for both forward and backward chaining [7] in most expert systems, is a *brittle* property which must be applied with great caution. We shall discuss these issues in greater detail in the following sections.

3. Inference in Fuzzy Logic

Fuzzy logic provides a natural framework for the management of uncertainty in expert systems because -- in contrast to traditional logical systems -- its main purpose is to provide a systematic basis for representing and inferring from imprecise rather than precise knowledge. In effect, in fuzzy logic everything is allowed to be -- but need not be -- a matter of degree.

The greater expressive power of fuzzy logic derives from the fact that it contains as special cases the traditional two-valued as well as multi-valued logics. The main features of fuzzy logic which are of relevance to the management of uncertainty in expert systems are the following.

(a) In two-valued logical systems a proposition, p, is either true or false. In multi-valued logical systems, a proposition may be true, false, or have an intermediate truth-value which may be an element of a finite or infinite truth-value set T. In fuzzy logic, the truth-values are allowed to range over the fuzzy subsets of T. For example, if T is the unit interval, then a truth-value in fuzzy logic, e.g., *very true*, may be interpreted as a fuzzy subset of the unit interval which defines the possibility distribution associated with the truth-value in question. In this sense, a fuzzy truth-value may be viewed as an imprecise characterization of an intermediate truth-value.

(b) The predicates in two-valued logic are constrained to be crisp in the sense that the denotation of a predicate must be a nonfuzzy subset of the universe of discourse. In fuzzy logic, the predicates may be crisp, e.g., *mortal*, *even*, *father of*, etc. or, more generally, fuzzy, e.g., *ill*, *tired*, *large*, *tall*, *much heavier*, *friend of*, etc.

(c) Two-valued as well as multi-valued logics allow only two quantifiers: *all* and *some*. By contrast, fuzzy logic allows, in addition, the use of fuzzy quantifiers exemplified by *most*, *many*, *several*, *few*, *much of*, *frequently*, *occasionally*, *about ten*, etc. Such quantifiers may be interpreted as fuzzy numbers which provide an imprecise characterization of the cardinality of one or more fuzzy or nonfuzzy sets. In this perspective, a fuzzy quantifier may be viewed as a second order fuzzy predicate. Based on this view, fuzzy quantifiers may be used to represent the meaning of propositions containing fuzzy probabilities and thereby make it possible to manipulate probabilities within fuzzy logic.

(d) Fuzzy logic provides a method for representing the meaning of both nonfuzzy and fuzzy predicate-modifiers (or, simply, modifiers) exemplified by *not*, *very*, *more or less*, *extremely*, *slightly*, *much*, *a little*, etc. This, in turn, leads to a system for computing with *linguistic variables* [48], [77], that is, variables whose values are words or sentences in a natural or synthetic language. For example, *Age* is a linguistic variable when its values are assumed to be: *young*, *old*, *very young*, *not very old*, etc., with each value interpreted as a possibility distribution over the real line.

(e) In two-valued logical systems, a proposition, p, may be qualified, principally by associating with p a truth-value, *true* or *false*; a modal operator such as *possible* or *necessary*; and an intensional operator such as *know*, *believe*, etc. In fuzzy logic, the three principal modes of qualification are: (a) *truth qualification*, expressed as p is τ, in which τ is a fuzzy truth-value; *probability qualification*, expressed as p is λ, in which λ is a fuzzy probability; and *possibility qualification*, expressed as p is π, in which π is a fuzzy possibility, e.g., *quite possible*, *almost impossible*, etc. Furthermore, *knowing* and *believing* are assumed to be binary, second-order, fuzzy predicates.

Types of propositions

Since the inference processes in the existing expert systems are based on two-valued logic and/or probability theory, the principal tools for inference are the modus ponens and/or its probabilistic analog - the Bayes rule and its variations. As was alluded to earlier, the validity of these inference processes is open to question since most of the information in the knowledge base of a typical expert system consists of a collection of fuzzy rather than nonfuzzy propositions.

To deal with such propositions, fuzzy logic draws, first, on test-score semantics [86] to represent their meaning; and, second, on a combination of the entailment and extension principles [83] to deduce the answer to a given question. In this way, the problem of inference from fuzzy propositions is reduced, in general, to the solution of a nonlinear program.

The basic ideas which underlie the deduction processes in fuzzy logic may be summarized as follows.

Assume that the knowledge base, KB, consists of a collection of propositions $\{p_1, \ldots, p_N\}$, some or all of which may be fuzzy in nature.

The propositions in a typical knowledge base may be divided into four principal categories.

(1) An unconditional, unqualified proposition.
Example: *Carol has a young daughter.*
Canonical form: [3] X is F, where X is a variable and F is a fuzzy predicate (i.e., a fuzzy subset of the domain of X).

(2) An unconditional, qualified proposition.
Example 1: *It is very likely that Carol has a young daughter*, or, equivalently, *Carol has a young daughter is very likely*, which exhibits more directly the proposition *Carol has a young daughter* and the qualifying fuzzy probability *very likely*.
Example 2: *Carol is very vivacious most of the time*. In this case, the qualifier *most of the time* plays the role of a fuzzy quantifier.

3 Canonical forms are discussed in greater detail at a later point in this section.

Canonical form 1: X is F is λ, where X is a variable, F is a fuzzy predicate and λ is a fuzzy probability.

Canonical form 2: Q U's are F's, where Q is a fuzzy quantifier and F is a fuzzy subset of the universe of discourse, U.

(3) A conditional, unqualified proposition.

Example: *If X is a man then X is mortal.*

Canonical form: *If X is F then Y is G*, where X and Y are variables and F and G are fuzzy predicates.

(4) A conditional, qualified proposition.

Example 1: *If a car is old then it probably is not very reliable* or, more precisely, *If X is an old car then (X is not very reliable is probable).*

Example 2: *If X is a young man then (Y is a young woman is likely)*, where Y denotes the girlfriend of X and *likely* is the qualifying fuzzy probability.

Example 3: *Most Swedes are blond.* This proposition may be expressed in the equivalent form *If X is a Swede* then (*X is blond is likely*) in which the fuzzy probability *likely* is equal, as a fuzzy number, to the fuzzy quantifier *most.*

Canonical form 1: *If X is F then Y is G is* λ , where X and Y are variables, F and G are fuzzy predicates and λ is a fuzzy probability.

Canonical form 2: Q F's are G's, where Q is a fuzzy quantifier, and F and G are fuzzy predicates.

Most of the rules in the knowledge base of a typical expert system are of Types 3 and 4. In particular, many of the rules of Type 4 are *dispositions*, that is, propositions with implicit fuzzy quantifiers, e.g., *Snow is white*, *Small cars are unsafe*, *Young men like young women*, etc. Dispositions play an especially important role in the representation of – and inference from – commonsense knowledge [89].

Canonical forms

As a preliminary to applying the rules of inference of fuzzy logic to propositions in *KB*, it is necessary to represent their meaning in a canonical form which places in evidence the constraints which are induced by each proposition. In fuzzy logic, this is accomplished through the use of test-score semantics [80], [86].

More specifically, the basic idea underlying test-score semantics is that a proposition, p, in a natural language may be viewed as a collection of *focal* *variables* $X_1, \ldots, X_n$ taking values in $U_1, \ldots, U_n$, respectively, which are constrained by a system of elastic (or fuzzy) constraints $F_1, \ldots, F_n$. In general, the variables $X_1, \ldots, X_n$ and the associated constraints $F_1, \ldots, F_n$ are implicit rather than explicit in p.

In more concrete terms, any unconditional proposition, p, may be represented in a *canonical form*, $cf(p)$,

$$p \rightarrow cf(p) \triangleq X \text{ is } F \ , \tag{3.1}$$

in which $X \triangleq (X_1, \ldots, X_n)$ is an n-ary focal variable whose components are $X_1, \ldots, X_n$; and F is a fuzzy relation in $U_1 \times \cdots \times U_n$ which represents an elastic constraint on X. Informally, what this means is that a proposition, p, may be viewed as a system of elastic constraints, and that the representation of the meaning of p is the process by which the implicit constraints in p are made explicit by expressing p in a form which places in evidence the constrained variable X and the elastic (or fuzzy) constraint F which is induced by p.

Let Π_X denote the possibility distribution of X, that is, the fuzzy set of possible values which X can take in $U \triangleq U_1 \times \cdots \times U_n$ [84]. Then, as shown in [86], the canonical proposition X *is* F may be interpreted as the *possibility assignment equation*

$$\Pi_X = F \tag{3.2}$$

or, more explicitly, as

$$\begin{aligned} \pi_X(u) &\triangleq Poss\{X = u\} \\ &= \mu_F(u) \ , \end{aligned} \tag{3.3}$$

where $u \triangleq (u_1, \ldots, u_n)$ is a generic point in $U = U_1 \times \cdots \times U_n$; $\pi_X : U \rightarrow [0,1]$ is the *possibility distribution function* associated with Π_X; $\mu_F : U \rightarrow [0,1]$ is the membership function of the fuzzy relation F; and $Poss\{X = u\}$ should be read as "the possibility that X may take u as its value." In this way, p may be *translated into* its canonical form (3.1) or, equivalently, its possibility assignment equation

$$p \rightarrow \Pi_X = F \tag{3.4}$$

in which $\Pi_{(X_1, \ldots, X_n)} = F$ is the possibility distribution induced by p.

If p is a conditional proposition, e.g., *If Roberta works near Washington then Roberta lives near Washington*, the canonical form of p may be expressed as

$$cf(p) \triangleq \textit{If } X \textit{ is } F \textit{ then } Y \textit{ is } G \ , \tag{3.5}$$

where the focal variables X and Y take values in U and V, respectively; F and G are fuzzy subsets of U and V; and X *is* F and Y *is* G are the canonical forms of the antecedent and consequent components of p. Correspondingly, the possibility assignment equation associated with p becomes [83]

$$p \rightarrow \Pi_{(Y|X)} \textit{ is } H \ , \tag{3.6}$$

in which $\Pi_{(Y|X)}$ denotes the conditional possibility distribution of Y given X and H is defined in terms of F and G by [4]

$$\mu_H(u,v) = 1 \wedge (1 - \mu_F(u) + \mu_G(v)) \ , \tag{3.7}$$

where u and v are generic values of X and Y, respectively; $\mu_H: U \times V \rightarrow [0,1]$ is the membership function of H; $\mu_F: U \rightarrow [0,1]$ and $\mu_G: V \rightarrow [0,1]$ are the membership functions of F and G; and $\wedge$ is the min operator. Thus, expressed in terms of the possibility distributuion functions, (3.7) implies that

$$\begin{aligned} \pi_{(Y|X)}(u,v) &\triangleq Poss\{Y = v \mid X = u\} \\ &= 1 \wedge (1 - \mu_F(u) + \mu_G(v)) \ . \end{aligned} \tag{3.8}$$

In general, the translation of p into its canonical form requires the construction of (a) an *explanatory database*; (b) a test procedure which tests and scores the constratins induced by p; and (c) an aggregation function which combines the partial test scores into a single (or, more generally, a vector) test score τ which represents the compatibility of p with the explanatory database [86]. In the case of expert systems, however, the propositions in *KB* are usually simple enough to be amenable to translation by inspection. For example:

(a) *Carol has dark hair* $\rightarrow$ X *is* F,

where

4 Equation (3.7) expresses a particular definition of the conditional possibility distribution which is consistent with the definition of implication in Lukasiewicz's $L_{Aleph\,1}$ logic. A more detailed discussion of various forms of $\Pi_{(Y|X)}$ may be found in [5], [18] and [43].

$X \triangleq$ Color (Hair (Carol))

$F \triangleq$ DARK

and *DARK* is a fuzzy subset of the set of colors of human hair.

(b) *John lives about two miles from Henry* $\rightarrow$ *X is F*,

where

$X \triangleq$ Distance (Location (Residence (John))),
Location (Residence (Henry))

$F \triangleq$ ABOUT 2

(c) *Henry is much younger than George* $\rightarrow$ (X_1, X_2) *is* F,

where

$X_1 \triangleq$ Age (Henry)

$X_2 \triangleq$ Age (George)

$F \triangleq$ MUCH.YOUNGER

and the fuzzy relation MUCH.YOUNGER is a fuzzy subset of $[0,120] \times [0, 120]$.

(d) *If Tong is blond then he is not Chinese* $\rightarrow$ *If X is F then Y is G*,

where

$X \triangleq$ Color (Hair (Tong))

$X \triangleq$ Nationality (Tong)

$F \triangleq$ BLOND

$G \triangleq$ CHINESE

(e) *John has three sons* $\rightarrow$ *X is F*,

where

$X \triangleq$ Count (Sons (John))

$F \triangleq 3$

and Count (Sons (John)) is the count of the number of elements in the set Sons (John).

(f) *John has three young sons* $\rightarrow$ (X_1, X_2) *is* F,

where

$X_1 \triangleq$ Count (Sons (John))

$X_2 \triangleq$ Age (Oldest.Son (John))

$F \triangleq (F_1, F_2)$

$F_1 \triangleq 3$

$F_2 \triangleq$ YOUNG

and YOUNG is a fuzzy subset of the interval [0, 120].

A basic point which these simple examples are intended to illustrate is that either by inspection or, more generally, through the application of test-score semantics, a constituent proposition, p, in *KB* may be expressed in a canonical form which places in evidence the variables constrained by p and the elastic or, equivalently, fuzzy constraints to which they are subjected.

By interpreting the canonical form of p as the possibility assignment equation, we are led to the possibility distribution, $\Pi^p_{(X_1, \ldots, X_n)}$, which is induced by p. In this way, each constituent proposition in *KB* is converted into a possibility distribution which constrains the variables in *KB*. Then, through conjunction, we can construct the global possibility distribution, $\Pi_{(X_1, \ldots, X_n)}$, which is induced by the totality of propositions in *KB*. As we shall see in the sequel, this is the point of departure for deduction in fuzzy logic.

Deduction

Assuming, as we have done already, that the knowledge base, *KB*, consists of a finite collection of propositions $\{p_1, \ldots, p_N\}$, let $\Pi^{p_j}_{(X_1, \ldots, X_n)}$ or, simply, Π^j, denote the possibility distribution induced by p_j, $j = 1, \ldots, N$. [5] Then, under the assumption that the p_i are non-interactive [84], the possibility distribution function of the global possibility distribution may be expressed as [6]

$$\pi_{(X_1, \ldots, X_n)} = \pi^1_{(X_1, \ldots, X_n)} \wedge \cdots \wedge \pi^N_{(X_1, \ldots, X_n)} \tag{3.9}$$

where $\wedge \triangleq \min$, and

$$\pi_{(X_1, \ldots, X_n)}(u_1, \ldots, u_n) \triangleq Poss\{X_1 = u_1, \ldots, X_n = u_n\}, \; u_i \in U_i, \tag{3.10}$$

5 Note that there is no loss of generality in assuming that the constituent propositions in *KB* have the same set of focal variables since the set $\{X_1, \ldots, X_n\}$ may be taken to be the union of the focal variables associated with each proposition.

6 In the present paper, it will suffice for our purposes to use the standard connectives $\wedge \triangleq \min$ (conjunction) and $\vee \triangleq \max$ (disjunction). A more general treatment of connectives in fuzzy logic may be found in [18] and [32].

$$i = 1, \ldots, n.$$

Now suppose that we are interested in the possible values of a subset, $\{X_{i1}, \ldots, X_{ik}\}$, of the *KB* variables $\{X_1, \ldots, X_n\}$. In other words, we are interested in determining the *marginal possibility distribution* $\Pi_{(X_{i1}, \ldots, X_{ik})}$ from the knowledge of the global possibility distribution $\Pi_{(X_1, \ldots, X_n)}$. As we shall see presently, the desired possibility distribution is given by the *projection* of $\Pi_{(X_1, \ldots, X_n)}$ on $U_{i1} \times \cdots \times U_{ik}$, which is written for simplicity as [80]

$$x_{i1} \times \cdots \times x_{ik} \, \Pi_{(X_1, \ldots, X_n)} \triangleq Proj \; on \; U_{i1} \times \cdots \times U_{ik} \; of \; \Pi_{(X_1, \ldots, X_n)}. \tag{3.11}$$

For convenience, let $X_{(s)}$ denote the subvariable of the variable $X \triangleq (X_1, \ldots, X_n)$ which is the focus of our interest, i.e.,

$$X_{(s)} = (X_{i1}, \ldots, X_{ik}) \; ,$$

where the index sequence $s \triangleq (i1, \ldots, ik)$ is a subsequence of the sequence $(1, \ldots, n)$. Using the same notational device, any k-tuple of the form $(A_{i1}, \ldots, A_{ik})$ may be expressed more compactly as $A_{(s)}$.

Let $\Pi_{X(s)}$ be the *projection* of the global possibility distribution Π_X on $U_{(s)} \triangleq U_{i1} \times \cdots \times U_{ik}$. Then, by definition [83],

$$\pi_{X(s)}(u_{(s)}) \triangleq sup_{u_{(s')}} \pi_X(u) \; , \tag{3.12}$$

where $s' \triangleq (j1, \ldots, jm)$ is the index subsequence which is complementary to s. (E.g., if $n = 5$ and $s \triangleq (2,3,5)$ then $s' = (1,4)$.)

Now the *entailment principle* [83] of fuzzy logic asserts that from any fuzzy proposition p we can infer a fuzzy proposition q if the possibility distribution induced by p is contained in that induced by q. This may be represented in the schematic form

$$\begin{array}{l} p \rightarrow \Pi^p{}_X = F \\ \quad \downarrow \\ q \leftarrow \Pi^q{}_X = G \supset F \; , \end{array} \tag{3.13}$$

where the reverse arrow $\leftarrow$ signifies that q is a *retranslation* of the possibility assignment equation $\Pi^q{}_X = G$ if the latter is a translation of q. For simplicity, we shall say that $\Pi^q{}_X$ may be inferred from $\Pi^p{}_X$ if $\Pi^q{}_X \supset \Pi^p{}_X$, that is,

$$\pi^q{}_X(u) \geq \pi^p{}_X(u) \quad \text{for } all \ u \in U \ . \tag{3.14}$$

From the definition of $\pi_{X_{(s)}}(u_{(s)})$ as expressed by (3.12), it follows at once that

$$\pi_{X_{(s)}}(u_{(s)}) \geq \pi_X(u) \quad \text{for } u \in U \ , \tag{3.15}$$

and hence that $\Pi_{X_{(s)}}$ may be inferred from Π_X. As a consequence, $\Pi_{X_{(s)}}$ as defined by (3.12) represents the desired possibility distribution of the variable of interest, namely, $X_{(s)} \triangleq (X_{i1}, \ldots, X_{ik})$. Since $X_{(s)}$ is given by the projection of Π_X on $U_{(s)}$, we shall refer to the inference rule which yields $\Pi_{X_{(s)}}$ as the *P-rule*, with P standing for *projection*.

As a simple illustration, assume that the knowledge base contains three propositions p_1, p_2 and p_3 which induce, respectively, the possibility distributions $\Pi^1{}_{(X_2,X_3)}$,$\Pi^2{}_{(X_1)}$, and $\Pi^3{}_{(X_3,X_4)}$. Suppose that the variables of interest are X_2 and X_4. Then, the possibility distribution function of X_2 and X_4 is given by

$$\pi_{(X_2,X_4)}(u_2,u_4) = sup_{u_1,u_3} \left[\pi^1{}_{(X_2,X_3)}(u_2,u_3) \wedge \pi^2{}_{(X_1)}(u_1) \wedge \pi^3{}_{(X_2,X_4)}(u_2,u_4)\right\} \tag{3.16}$$

which reduces to

$$\pi_{(X_2,X_3)}(u_2,u_3) = sup_{u_3} \left[\pi^1{}_{(X_2,X_3)}(u_2,u_3) \wedge \pi^3{}_{(X_3,X_4)}(u_3,u_4)\right] \tag{3.17}$$

if $\Pi^1{}_{X_1}$ is a *normal* possibility distribution, i.e.,

$$sup_{u_1} \pi_{X_1}(u_1) = 1 \ . \tag{3.18}$$

The right-hand member of (3.17) constitutes the *composition* of $\Pi^1{}_{(X_2,X_3)}$ and $\Pi^3{}_{(X_3,X_4)}$ with respect to X_3 [75].

In summary, given a knowledge base, *KB*, the possibility distribution, $\Pi_{X_{(s)}}$, of a specified subvariable, $X_{(s)}$, may be obtained by projecting the global possibility distribution Π_X on $U_{(s)}$. The resulting possibility distribution, $\Pi_{X_{(s)}}$, may be expressed as the composition with respect to $X_{(s')}$ of those constituent possibility distributions which contain variables which are linked directly or transitively to the variables in $X_{(s)}$. [7] For this reason, the P-rule may be described, more

7 X_l and X_m, are linked directly if they appear in the same possibility distribution. X_l and X_m, are linked transitively if there exists a chain of variables $X_{c_1}, \ldots, X_{c_r}$, such that $(X_l, X_{c_1}), \ldots, (X_{c_r}, X_m)$ are linked directly.

suggestively, as the *compositional rule of inference* [75], [83]. Expressed as a sequence of operations, the application of this rule to *KB* may be represented as the chain:

$$\{p_1, \ldots, p_N\} \ \underrightarrow{\text{translation}} \ \{\Pi^1, \ldots, \Pi_N\} \ \underrightarrow{\text{conjunction}} \tag{3.19}$$
$$\{\pi_1 \wedge \cdots \wedge \pi_N\} \ \underrightarrow{\text{projection}} \ \{\Pi_{X_{(s)}}\} \ \underrightarrow{\text{retranslation}} \ q \ ,$$

where the *inferent* proposition q is a retranslation of $\Pi_{X_{(s)}}$, which is the marginal possibility distribution of the subvariable of $(X_1, \ldots, X_n)$, $X_{(s)}$, which is the target of the inference process.

Among the traditional rules of inference -- which may be viewed as special cases of the compositional rule of inference -- is the *modus ponens*. To establish this fact, we shall first derive from the compositional rule of inference a more general version of the modus ponens which in fuzzy logic is referred to as the *generalized modus ponens* [83], [35].

Consider a pair of propositions $\{p_1, p_2\}$ of the form

$$p_1 \triangleq \textit{If } X \textit{ is } F \textit{ then } Y \textit{ is } G \tag{3.20}$$
$$p_2 \triangleq X \textit{ is } F^* \ ,$$

in which F, F^* and G are fuzzy sets (or, equivalently, fuzzy predicates).

Applying (3.4), (3.6) and (3.7), the translations of p_1 and p_2 may be expressed as

$$p_1 \rightarrow \Pi_{(Y|X)} = H \tag{3.21}$$

$$p_2 \rightarrow \Pi_X = F^* \ , \tag{3.22}$$

where $\mu_H(u,v)$ is given by (3.7).

By applying the compositional rule of inference to (3.21) and (3.22), the possibility distribution of Y is found to be given by

$$\Pi_Y = H \circ F^* \ , \tag{3.23}$$

where the right-hand member of (3.23) represents the composition of H and F^* with respect to X. More concretely,

$$\mu_Y(v) = sup_u \left[\mu_H(u,v) \wedge \mu_{F^*}(u)\right] \tag{3.24}$$
$$= sup_u \left[\mu_{F^*}(u) \wedge (1 - \mu_F(u) + \mu_G(v))\right] \ .$$

This conclusion may be stated in the form of the syllogism

$$
\begin{array}{l}
\textit{If } X \textit{ is } F \textit{ then } Y \textit{ is } G \\
\underline{X \textit{ is } F^{*} \qquad\qquad\qquad} \\
Y \textit{ is } H \circ F^{*}
\end{array}
\tag{3.25}
$$

where $H \circ F^{*}$ is defined by (3.24). This syllogism expresses the generalized modus ponens.

The generalized modus ponens differs from its classical version in two respects. First, F^{*} is not required to be identical with F, as it is in the classical case. And second, the predicates F, G and F^{*} are not required to be crisp. It can readily be verified that when $F = F^{*}$ and the predicates are crisp, $H \circ F^{*}$ reduces to G and (3.25) becomes [8]

$$
\begin{array}{l}
\textit{If } X \textit{ is } F \textit{ then } Y \textit{ is } G \\
\underline{X \textit{ is } F \qquad\qquad\qquad} \\
Y \textit{ is } G
\end{array}
\tag{3.26}
$$

The PE-Rule

The compositional rule of inference makes it possible to deduce from KB the possibility distribution of a specified subvariable, $X_{(s)}$, of the KB variable $X \triangleq (X_1, \ldots, X_n)$. More generally, however, the target of the inference process is not $X_{(s)}$ but a specified function of $X_{(s)}$, say $f(X_{(s)})$. This may be viewed as a general formulation of the problem of finding an answer to a question which relates to the information resident in KB.

In fuzzy logic, the compositional rule of inference plays an essential role in the formulation and solution of this problem by making it possible to decompose the problem into two subproblems: (1) determination of the possibility distribution of $X_{(s)}$; and (2) determination of the possibility distribution of $f(X_{(s)})$ from the knowledge of that of $X_{(s)}$.

More specifically, assume that a question relates to the possibility

8 Depending on the way in which $\Pi_{(Y|X)}$ is defined in terms of μ_F and μ_G, (3.26) may or may hold when $F = F^{*}$ but F and G are not crisp. A more detailed discussion of this issue may be found in [22] and [43].

distribution of a function $f(X_{(s)})$ whose argument is a subvariable of the *KB* variable $X \triangleq (X_1, \ldots, X_n)$, and that through the use of the compositional rule of inference we have determined the possibility distribution of $X_{(s)}$. Then, by the extension principle [83], the determination of the possibility distribution of $f(X_{(s)})$ is reduced to the solution of the following nonlinear program:

$$\pi_f(v) = sup_{u_{(s)}}(\pi_{X_{(s)}}(u_{(s)})) \tag{3.27}$$

subject to

$$f(u_{(s)}) = v \quad , \tag{3.28}$$

where v is a generic value of f, and

$$\pi_f(v) \triangleq Poss\{f = v\} \tag{3.29}$$

Deduction in fuzzy logic is based, in the main, on the solution of the nonlinear program expressed by (3.27). Summarizing what we have said so far, assume that we are interested in determining the value of an unknown variable q which may be expressed as a function of a set of variables $X_{(s)}$ which are constrained by a collection of propositions in a knowledge base *KB*. If $X_{(s)}$ is a proper subset of the variables $X \triangleq (X_1, \ldots, X_n)$ which are constrained by *KB* -- as would usually be the case -- then we first find the possibility distribution of $X_{(s)}$ by projecting the global possibility distribution Π_X on $U_{(s)}$. Then, we apply the extension principle -- as in (3.27) -- to reduce the determination of the possibility distribution of q to the solution of a constrained maximization problem in which the function in question is treated as a constraint and the objective function is the possibility distribution function of $X_{(s)}$. For convenience, we shall refer to this deduction process as the *PE-rule*, with P and E standing for ***projection*** and ***extension***, respectively.

A simple illustration of the PE-rule is provided by the following problem. Suppose that the knowledge base contains the following propositions

$p_1 \triangleq$ *John lives about two miles from Henry*

$p_2 \triangleq$ *Henry lives about three miles from Ed*

$q \triangleq$ *How far away is John from Ed?*

The *KB* variables in this case are the coordinates of the residences of John, Henry and Ed, i.e. (X_J, Y_J), (X_H, Y_H) and (X_E, Y_E). Upon translation of p_1 and p_2, the possibility distribution functions which constrain these variables are found to be expressed by

$$\pi^1(X_J, Y_J, X_H, Y_H) = \mu_{ABOUT.2}\left[((X_J - X_H)^2 + (Y_J - Y_H)^2)^{\frac{1}{2}}\right] \qquad (3.30)$$

$$\pi^2(X_H, Y_H, X_E, Y_E) = \mu_{ABOUT.3}\left[((X_H - X_E)^2 + (Y_H - Y_E)^2)^{\frac{1}{2}}\right] , \qquad (3.31)$$

where $\mu_{ABOUT.2}$ and $\mu_{ABOUT.3}$ are the membership functions of the fuzzy sets *ABOUT*. 2 and *ABOUT*. 3, respectively.

The function which characterizes the question q in this case is

$$f(X_J, Y_J, X_E, Y_E) = \left[((X_J - X_E)^2 + (Y_J - Y_E)^2)^{\frac{1}{2}}\right] , \qquad (3.32)$$

and hence the nonlinear program to which the solution of the problem reduces may be expressed as

$$\mu_f(d) = sup_{X_J, Y_J, X_H, Y_H, X_E, Y_E}\left[\mu_{ABOUT.2}(((X_J - X_H)^2 + (Y_J - Y_H)^2)^{\frac{1}{2}})\right] \wedge \qquad (3.33)$$

$$\left[\mu_{ABOUT.3}(((X_H - X_E)^2 + (Y_H - Y_E)^2)^{\frac{1}{2}})\right] ,$$

subject to

$$d = \left[(X_J - X_E)^2 + (Y_J - Y_E)^2)\right]^{\frac{1}{2}} ,$$

where d denotes the distance of John from Ed.

This problem can readily be solved by employing the level-set technique in fuzzy mathematical programming [14], [46]. The solution yields d as a fuzzy interval (or, equivalently, a fuzzy number) which may be represented as

$$ABOUT.3 \ominus ABOUT.2 \le d \le ABOUT.3 \oplus ABOUT.2 , \qquad (3.34)$$

where $\oplus$ and $\ominus$ denote fuzzy addition and fuzzy subtraction, respectively [18].

A few observations concerning the solution are in order. First, the answer is similar in form to what it would be if the distances in p_1 and p_2 were specified as 2 miles and 3 miles instead of *about 2* miles and *about 3* miles, i.e.,

$$3 - 2 \le d \le 3 + 2 . \qquad (3.35)$$

However, whereas in the case of (3.34) we start with fuzzy distances, expressed as fuzzy numbers, and arrive at a fuzzy answer, likewise expressed as a fuzzy number, in the case of (3.35) we start with real numbers and end up with an interval-valued answer. Obviously this is so because the information in the knowledge base is incomplete in relation to the posed question. What is important to recognize is that this is a pervasive phenomenon in the case of expert systems, and is the reason -- as was alluded to earlier -- why the certainty factor of a conclusion should in general be an interval-valued or fuzzy number, rather than a real number, as it is usually assumed to be.

Second, if in the statement of the problem the distances were specified precisely, the upper and lower bounds in (3.35) would be given by the solutions of the following nonlinear programs

$$d_{max} = sup_{X_J, Y_J, X_H, Y_H, X_E, Y_E}\left[(X_J - X_E)^2 + (Y_J - Y_E)^2\right]^{\frac{1}{2}} , \tag{3.36}$$

$$d_{min} = inf_{X_J, Y_J, X_H, Y_H, X_E, Y_E}\left[(X_J - X_E)^2 + (Y_J - Y_E)^2\right]^{\frac{1}{2}} , \tag{3.37}$$

subject to

$$\left[(X_J - X_H)^2 + (Y_J - Y_H)^2\right]^{\frac{1}{2}} = 2 \tag{3.38}$$

$$\left[(X_H - X_E)^2 + (Y_H - Y_E)^2\right]^{\frac{1}{2}} = 3 \ .$$

What is of interest to observe is that the roles of the constraints and objective functions in (3.36) and (3.37) are interchanged in relation to those in (3.33). In effect, the more general formulation expressed by (3.33) subsumes (3.36) and (3.37) and is dual to them.

Interpolation

As was pointed out earlier, an important problem which arises in the operation of any rule-based system is the following. Suppose that the user supplies a fact which in its canonical form may be expressed as X *is* F^*, where F is a fuzzy or nonfuzzy predicate. Furthermore, suppose that there is no conditional rule in *KB* whose antecedent matches F exactly. The question which arises is: Which rules should be executed and how should their results be combined?

The approach suggested by fuzzy logic involves the use of an interpolation technique which is based on the P-rule and is in the spirit of the generalized modus ponens [75].

Specifically, suppose that upon translation a group of propositions in KB may be expressed as a fuzzy relation of the form

(3.39)

R	X_1	X_2	$\cdot$	X_n	X_{n+1}
	R_{11}	R_{12}	$\cdot$	R_{1n}	Z_1
	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
	R_{m1}	R_{m2}	$\cdot$	R_{mn}	Z_m

in which the entries are fuzzy sets; the input variables are $X_1, \ldots, X_n$, with domains $U_1, \ldots, U_n$; and the output variable is X_{n+1}, with domain U_{n+1}. The problem is: Given an input n-tuple $(R^*_1, \ldots, R^*_n)$, in which R^*_j, $j = 1, \ldots, n$, is a fuzzy subset of U_j, what is the value of X_{n+1} expressed as a fuzzy subset of U_{n+1}?

A fuzzy relation which is represented by a tableau of the form (3.39) may be defined in different ways. The definition which will be used here is that given in [78], which is in the spirit of the standard definition of a relation as a collection of tuples. Specifically,

$$R = R_{11} \times \cdots \times R_{1n} \times Z_1 + \cdots + R_{m1} \times \cdots \times R_{mn} \times Z_m \ , \qquad (3.40)$$

where $\times$ and $+$ denote the cartesian product and union, respectively.

Based on this interpretation of R, the desired value of X_{n+1} may be obtained as follows.

First, we compute for each pair (R_{ij}, R^*_j) the degree of consistency of the input R^*_j with the R_{ij} element of R, $i = 1, \ldots, m$, $j = 1, \ldots, n$. The degree of consistency, γ_{ij}, is defined as

$$\gamma_{ij} \triangleq sup\,(R_{ij} \cap R^*_j) \qquad (3.41)$$

$$= sup_{u_j}(\mu_{R_{ij}}(u_j) \wedge \mu_{R^*_j}(u_j)) \ ,$$

in which $\mu_{R_{ij}}$ and $\mu_{R^*_j}$ are the membership functions of R_{ij} and R^*_j, respectively, and u_j is a generic element of U_j.

Next, we compute the overall degree of consistency, γ_i, of the input n-tuple $(R^*_1, \ldots, R^*_n)$ with the ith row of R, $i = 1, \ldots, n$, by employing $\wedge$ (min) as the aggregation operator. Thus

$$\gamma_i = \gamma_{i1} \wedge \gamma_{i2} \wedge \cdots \wedge \gamma_{in} \ . \tag{3.42}$$

As expressed by (3.42), γ_i may be interpreted as a conservative measure of agreement between the input n-tuple $(R^*_1, \ldots, R^*_n)$ and the ith row n-tuple $(R_{i1}, \ldots, R_{in})$. Then, employing γ_i as a weighting coefficient, the desired expression for X_{n+1} may be written as a "linear" combination

$$X_{n+1} = \gamma_1 \wedge Z_1 + \ldots + \gamma_m \wedge Z_m \tag{3.43}$$

in which + denotes the union, and $\gamma_i \wedge Z_i$ is a fuzzy set defined by

$$\mu_{\gamma_i \wedge Z_i}(u_{i+1}) = \gamma_i \wedge \mu_{Z_i}(u_{i+1}) \quad , i = 1, \ldots, n \ . \tag{3.44}$$

It should be observed that if no row of R has a high degree of consistency with the input n-tuple, the value of the output variable X_{n+1} will be a subnormal fuzzy set, that is, its maximal grade of membership will be smaller than unity. Furthermore, the lower the degree of consistency, the higher the degree of subnormality. Thus, to achieve a high degree of normality of the output, it is necessary that at least one of the rows of R have a high degree of consistency with the input n-tuple.

4. Inference from Quantified Propositions

In the preceding section, we have restricted our discussion of inference in fuzzy logic to unqualified propositions. In the case of qualified propositions, the problem of inference becomes considerably more complex, and our discussion of it will touch upon only a few of the many issues which arise when some of the propositions in the knowledge base are associated with certainty factors. [9]

For simplicity, we shall restrict our attention to quantified propositions whose canonical form may be expressed as

$$q \triangleq Q \ A\text{'s are } B\text{'s} \ , \tag{4.1}$$

9 Some of the definitions and examples in this section are drawn from [88] and [89].

e.g., *Most small cars are unsafe*, where A and B are fuzzy predicates and Q is a fuzzy quantifier such as *most*, *many*, *not very many*, *approximately 0.8*, *much more than a half*, etc. For convenience, A and B will be referred to as the *antecedent* and *consequent* of q. This is motivated by the observation that q may be interpreted as the conditional probability assignment

$$Prob\ \{B\,|\,A\}\ is\ Q\ , \tag{4.2}$$

in which $Prob\ \{B\,|\,A\}$ denotes the conditional probability of the fuzzy event $\{X\ is\ B\}$ given the fuzzy event $\{X\ is\ A\}$ [71]. In both (4.1) and (4.2), Q plays the role of a fuzzy number which is a fuzzy subset of the unit interval.

Cardinality of fuzzy sets

To make the concept of a fuzzy quantifier meaningful, it is necessary to define a way of counting the number of elements in a fuzzy set or, equivalently, to determine its cardinality.

There are several ways in which this can be done [88]. For our purposes, it will suffice to employ the concept of a *sigma-count*, which is defined as follows.

Let F be a fuzzy subset of $U = \{u_1, \ldots, u_n\}$

expressed symbolically as

$$F = \mu_1/u_1 + \ldots + \mu_n/u_n = \Sigma_i \mu_i/u_i \tag{4.3}$$

or, more simply, as

$$F = \mu_1 u_1 + \ldots + \mu_n u_n\ , \tag{4.4}$$

in which the term μ_i/u_i, $i = 1, \ldots, n$, signifies that μ_i is the grade of membership of u_i in F, and the plus sign represents the union. [10]

The sigma-count of F is defined as the arithmetic sum of the μ_i, i.e.,

$$\Sigma Count(F) \triangleq \Sigma_i \mu_i\ ,\ i = 1, \ldots, n\ , \tag{4.5}$$

with the understanding that the sum may be rounded, if need be, to the nearest integer. Furthermore, one may stipulate that the terms whose grade of membership falls below a specified threshold be excluded from the summation.

10 In most cases, the context is sufficient to resolve the question of whether a plus sign should be interpreted as the union or the arithmetic sum.

The purpose of such an exclusion is to avoid a situation in which a large number of terms with low grades of membership become count-equivalent to a small number of terms with high membership.

The *relative sigma-count*, denoted by $\Sigma Count(F/G)$, may be interpreted as the proportion of elements of F which are in G. More explicitly,

$$\Sigma Count(F/G) = \frac{\Sigma Count(F \cap G)}{\Sigma Count(G)}, \tag{4.6}$$

where $F \cap G$, the intersection of F and G, is defined by

$$F \cap G = \Sigma_i (\mu_B(u_i) \wedge \mu_G(u_i)) / u_i \ , \ i = 1, \ldots, n \ . \tag{4.7}$$

Thus, in terms of the membership functions of F and G, the relative sigma-count of F in G is given by

$$\Sigma Count(F/G) = \frac{\Sigma_i \mu_F(u_i) \wedge \mu_G(u_i)}{\Sigma_i \mu_G(u_i)} \ . \tag{4.8}$$

The concept of a relative sigma-count provides a basis for interpreting the meaning of propositions of the form Q *A's are B's* , e.g., *Most young men are healthy.* More specifically, if the focal variable in the proposition in question is taken to be the proportion of B's in A's, then the corresponding translation rule may be expressed as

$$Q \ A\text{'s are } B\text{'s} \rightarrow \Sigma Count(B/A) \text{ is } Q \tag{4.8}$$

or, equivalently, as

$$Q \ A\text{'s are } B\text{'s} \rightarrow \Pi_X = Q \tag{4.9}$$

where

$$X = \frac{\Sigma_i \mu_A(u_i) \wedge \mu_B(u_i)}{\Sigma_i \mu_A(u_i)} , \tag{4.10}$$

The intersection/product syllogism

In fuzzy logic, a basic rule of inference for quantified propositions is the *intersection/product syllogism* which may be expressed as the schema [88]

Q_1 *A's are B's* (4.11)

$\underline{Q_2 \ (A \text{ and } B)\text{'s are } C\text{'s}}$

$(Q_1 \otimes Q_2)$ *A's are* $(B$ and $C)$*'s* ,

in which A, B and C are fuzzy predicates and $Q_1 \otimes Q_2$ is a fuzzy number which is the fuzzy product of the fuzzy numbers Q_1 and Q_2 [18]. For example, as a special case of (4.11), we may write

Most students are single (4.12)

A little more than a half of single students are male

(*Most* $\otimes$ *A little more than a half*) *of students are single and male* .

Since the intersection of B and C is contained in C, the following corollary of (4.11) is its immediate consequence

Q_1 *A's are B's* (4.13)

Q_2 $(A$ and $B)$*'s are C's*

$\geq (Q_1 \otimes Q_2)$ *A's are C's* ,

where the fuzzy number $\geq (Q_1 \otimes Q_2)$ should be read as *at least* $(Q_1 \otimes Q_2)$, with the understanding that $\geq (Q_1 \otimes Q_2)$ represents the composition of the binary non-fuzzy relation $\geq$ with the unary fuzzy relation $(Q_1 \otimes Q_2)$. In particular, if the fuzzy quantifiers Q_1 and Q_2 are monotone increasing (e.g., when $Q_1 = Q_2 \triangleq$ *most*), then as is stated in [88],

$$\geq (Q_1 \otimes Q_2) = Q_1 \otimes Q_2 \ , \tag{4.14}$$

and (4.13) becomes

Q_1 *A's are B's* (4.15)

Q_2 $(A$ and $B)$*'s are C's*

$(Q_1 \otimes Q_2)$ *A's are C's* .

The consequent conjunction syllogism

Another basic syllogism in fuzzy logic is the *consequent conjunction syllogism* [88] which may be expressed as the schema

Q_1 *A's are B's* (4.16)

Q_2 *A's are C's*

Q A's are (*B* and *C*)*'s* ,

where

$$0 \circledvee (Q_1 \oplus Q_2 \ominus 1) \leq Q \leq Q_1 \circledwedge Q_2 \ .$$

in which the operators $\circledwedge$, $\circledvee$, $\oplus$ and $\ominus$, and the inequality $\leq$ are the extensions of $\wedge$, $\vee$, $+$, $-$ and $\leq$, respectively, to fuzzy numbers [18].

The consequent conjunction syllogism plays the same role in fuzzy logic as the rule of combination of evidence for conjunctive hypotheses does in MYCIN [60] and PROSPECTOR [20]. However, the latter rules are ad hoc in nature whereas the consequent conjunction syllogism is not. Furthermore, the consequent conjunctive syllogism shows that the certainty-factor values used in the case of conjunctive hypotheses in MYCIN and PROSPECTOR correspond to the upper bound in (4.16).

As a simple illustration of the consequent conjunction syllogism, assume that

$$p_1 \triangleq \textit{Most Frenchmen are not tall} \tag{4.17}$$

$$p_2 \triangleq \textit{Most Frenchmen are not short}$$

In this case, by the application of (4.16), we can infer that

Q Frenchmen are not tall and not short

where

$$0 \circledvee (2\, most \ominus 1) \leq Q \leq most \ . \tag{4.18}$$

In the above example, the variable of interest is the conjunction of the consequents *not tall* and *not short*. In a more general setting, the variable of interest may be a specified function of the variables constrained by the knowledge base. The following variation on (4.16) is intended to give an idea of how the value of the variable of interest may be inferred by an application of the PE-rule.

Infer from the propositions

$$p_1 \triangleq \textit{Most Frenchmen are not tall} \tag{4.19}$$

$$p_2 \triangleq \textit{Most Frenchmen are not short}$$

the answer to the question

$$q \triangleq \textit{What is the average height of a Frenchman?} \tag{4.20}$$

Assume that p_1 and p_2 refer to a population of Frenchmen, $\{Frenchman_1, \ldots, Frenchman_n\}$ with respective heights $X_1, \ldots, X_n$ which play the role of *KB* variables. On denoting the membership functions of the fuzzy predicates ***tall*** and ***short*** by μ_{TALL} and μ_{SHORT}, and that of the fuzzy quantifier ***most*** by μ_{MOST}, the possibility distributions induced by p_1 and p_2 may be expressed as

$$\pi^1_{(X_1,\ldots,X_n)}(h_1, \ldots, h_n) = \mu_{ANTMOST}\left[\frac{1}{n}\Sigma_i \mu_{TALL}(h_i)\right] \tag{4.21}$$

$$\pi^2_{(X_1,\ldots,X_n)}(h_1, \ldots, h_n) = \mu_{ANTMOST}\left[\frac{1}{n}\Sigma_i \mu_{SHORT}(h_i)\right] \tag{4.22}$$

where $h_1, \ldots, h_n$ are generic values of $X_1, \ldots, X_n$, and ***ant*** is an abbreviation for ***antonym*** [80], i.e.,

$$\mu_{ANTMOST}(u) \triangleq \mu_{MOST}(1-u) \quad ,u \in [0,1] \ . \tag{4.23}$$

Now the variable of interest is the average height, Y, which, as a function of the *KB* variables $X_1, \ldots, X_n$ may be expressed as

$$Y = \frac{1}{n}(X_1 + \ldots + X_n) \ . \tag{4.24}$$

Consequently, by applying the PE-rule (3.27), the determination of the possibility distribution of Y is reduced to the solution of the nonlinear program

$$\mu_Y(h) = sup_{h_1,\ldots,h_n} \left(\mu_{ANTMOST}\left(\frac{1}{n}\Sigma_i \mu_{TALL}(h_i)\right) \wedge \right. \tag{4.25}$$

$$\left. \mu_{ANTMOST}\left(\frac{1}{n}\Sigma_i \mu_{SHORT}(h_i)\right)\right)$$

subject to

$$h = \frac{1}{n}\Sigma_i h_i \ ,$$

where h is the generic value of Y.

Alternatively, a simpler but less informative answer may be formulated by forming the intersection of the possibility distributions of Y which are induced separately by p_1 and p_2. More specifically, let $\Pi_{Y|p_1}$, $\Pi_{Y|p_2}$, $\Pi_{Y|p_1 \wedge p_2}$ be the possibility distributions of Y which are induced by p_1, p_2, and the conjunction of p_1 and p_2, respectively. Then, by using the minimax inequality [73], it can readily be shown that

$$\Pi_{Y|p_1} \cap \Pi_{Y|p_2} \supset \Pi_{Y|p_1 \wedge p_2} , \tag{4.26}$$

and hence we can invoke the entailment principle [83] to validate the intersection in question as the possibility distribution of Y. For the example under consideration, the desired possibility distribution is readily found to be given by

$$Poss\{Y = h\} = \mu_{ANTMOST}(\mu_{TALL}(h)) \wedge \mu_{ANTMOST}(\mu_{SHORT}(h)) \quad . \tag{4.27}$$

Chaining of propositions

An ordered pair, (p_1, p_2), of quantified propositions of the form

$$p_1 \triangleq Q_1 \, A\text{'s } \mathit{are} \; B\text{'s} \tag{4.28}$$

$$p_2 \triangleq Q_2 \, B\text{'s } \mathit{are} \; C\text{'s} \quad .$$

are said to form a *chain*. More generally, an *n-ary chain* may be represented as an ordered n-tuple

$$(Q_1 A_1\text{'s } \mathit{are} \; B_1\text{'s} , Q_2 A_2\text{'s } \mathit{are} \; B_2\text{'s} , \ldots , Q_n A_n\text{'s } \mathit{are} \; B_n\text{'s}) , \tag{4.29}$$

in which $B_1 = A_2$, $B_2 = A_3 , \ldots , B_{n-1} = A_n$.

Now assume that p_1 and p_2 appear as premises in the inference schema

$$\begin{array}{ll} Q_1 \, A\text{'s } \mathit{are} \; B\text{'s} & (\textit{major premise}) \\ \underline{Q_2 \, B\text{'s } \mathit{are} \; C\text{'s}} & (\textit{minor premise}) \\ ?Q \, A\text{'s } \mathit{are} \; C\text{'s} & (\textit{conclusion}) \end{array} \tag{4.30}$$

in which $?Q$ is a fuzzy quantifier which is to be determined. This schema corresponds to the combining rules in MYCIN and PROSPECTOR in which uncertain evidence is combined with an uncertain rule.

When $Q_1 = Q_2 = \mathit{all}$, the transitivity of fuzzy set containment or, equivalently, the rule of property inheritance in AI, implies that $Q = \mathit{all}$, i.e.,

$$\mathit{all} \; A\text{'s } \mathit{are} \; B\text{'s} \tag{4.31}$$

$$\frac{\textit{all } B\text{'s are } C\text{'s}}{\textit{all } A\text{'s are } C\text{'s}} \quad .$$

However, as shown in [88], property inheritance is a *brittle* property in the sense that even when Q_1 and Q_2 are arbitrarily close to *all*, there is nothing that can be said about Q, which means that, as a fuzzy number, $Q = [\,0,1]$. Thus, to be able to constrain Q it is necessary to make restrictive assumptions about Q_1, Q_2, A and B.

As an illustration, assume that $B \subset A$ and hence that $A \cap B = B$. In this case, the intersection/product syllogism (4.11) yields

$$\begin{array}{l} Q_1 \; A\text{'s are } B\text{'s} \\ \underline{Q_2 \; B\text{'s are } C\text{'s} \qquad} \\ \geq (Q_1 \otimes Q_2) \; A\text{'s are } C\text{'s} \end{array} \tag{4.32}$$

which implies that $Q = \geq (Q_1 \otimes Q_2)$. If, in addition, it is assumed that Q_1 and Q_2 are monotone increasing, so that $\geq (Q_1 \otimes Q_2) = Q_1 \otimes Q_2$, we obtain the *product chain rule* [89],

$$\begin{array}{l} Q_1 \; A\text{'s are } B\text{'s} \\ \underline{Q_2 \; B\text{'s are } C\text{'s} \qquad} \\ (Q_1 \otimes Q_2) \; A\text{'s are } C\text{'s} \quad . \end{array} \tag{4.33}$$

In this case, the chain (Q_1 A's are B's , Q_2 B's are C's) will be said to be *product transitive*. [11]

As an illustration of (4.32), we can assert that

most students are undergraduates

most undergraduates are young

most^2 *students are young* ,

where *most*^{2} represents the product of the fuzzy number *most* with itself.

11 More generally, an n-ary chain (Q_1 A_1's *are* B_1's , . . . , Q_n A_n's *are* B_n's) will be said to be *product transitive* if from the premises which constitute the chain it may be inferred that $\geq (Q_1 \otimes \cdots \otimes Q_n)$ A_1's *are* B_n's.

Chaining under reversibility

An important chaining rule which is approximate in nature relates to the case where the major premise in the inference chain

$$\begin{array}{l} Q_1\ A\text{'s are } B\text{'s} \\ \underline{Q_2\ B\text{'s are } C\text{'s}} \\ Q\ A\text{'s are } C\text{'s} \end{array} \tag{4.34}$$

is *reversible* in the sense that

$$Q_1\ A\text{'s are } B\text{'s} \leftrightarrow Q_1\ B\text{'s are } A\text{'s}\ , \tag{4.35}$$

where $\leftrightarrow$ denotes approximate semantic equivalence [80]. For example,

$$\textit{Most American cars are big} \leftrightarrow \textit{Most big cars are American}\ . \tag{4.36}$$

Under the assumption of reversibility, it is shown in [89] that the following syllogism holds in an approximate sense

$$\begin{array}{l} Q_1\ A\text{'s are } B\text{'s} \\ \underline{Q_2\ B\text{'s are } C\text{'s}} \\ \geq(0 \otimes (Q_1 \oplus Q_2 \ominus 1))\ A\text{'s are } C\text{'s}\ . \end{array} \tag{4.37}$$

We shall refer to this syllogism as the *MPR-rule*, with MPR standing for *major premise reversibility.* The transitivity of fuzzy set containment which is implied by the MPR-rule will be referred to as the *BR-transitivity*, with B and R standing for Bezdek and Ruspini, respectively, who have employed this type of transitivity of fuzzy relations in their work on fuzzy clustering [10], [55].

Concluding remark

In the foregoing analysis, we have considered some of the representative problems which arise in the management of uncertainty in expert systems. Our analysis is intended to suggest that fuzzy logic provides a natural conceptual framework for knowledge representation and inference from knowledge bases which are imprecise, incomplete, or not totally reliable. Generally, the use of fuzzy logic reduces the problem of inference to that of solving a nonlinear program and leads to conclusions whose uncertainty is a cumulation of the uncertainties in the premises from which the conclusions are derived. As a consequence, the conclusions as well as the certainty factor (or their equivalents) are fuzzy sets which are characterized by their possibility distributions.

In our analysis, we have considered some -- but by no means all -- of the inference rules in fuzzy logic which are needed for the combination of evidence in expert systems. In particular, we have not considered the ***antecedent conjunction syllogism***, which is needed when the antecedents of fuzzy rules are combined conjunctively. In devising such syllogisms, it may be necessary to employ the theory of dispositions [89] to make use of the ultrafuzzy information about the antecedent and consequent predicates. Otherwise, little can be said, in general, about the value of the certainty factor of the conclusion.

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