NONLINEAR CIRCUITS

by

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ABSTRACT

This tutorial paper presents a sample of some interesting and practical results from three fundamental areas in nonlinear circuit theory; namely, <u>nonlinear circuit analysis</u>, <u>nonlinear circuit synthesis</u>, and <u>nonlinear device modelling</u>.

The first area covers both <u>resistive</u> and <u>dynamic</u> circuits. The former includes the following 5 topics: existence and uniqueness of solution, bounds on solutions, symmetry properties, stationary properties, and geometric properties. The latter includes the following topics: no finite forward-escape-times, existence of a unique periodic steady-state response, dc steady-state response of capacitor-diode circuits, nonlinear dynamics and energy-related concepts.

The second area covers driving-point and transfer characteristics, nonlinear n-port resistors, mutators, nonlinear algebraic elements, and the Hodge decomposition.

The third area covers nonlinear model validation and nonlinear model structure.

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I. INTRODUCTION

A selection of basic papers on various aspects of nonlinear circuits prior to 1975 has appeared in an IEEE Press book edited by A. N. Willson, Jr. [1]. Highlights of some of these papers can be found in a comprehensive survey by the editor [2]. In addition, the following special issues have served a very useful role in focusing relevant research topics and problems on nonlinear circuits:

Special Issue dedicated to Professor van der Pol
IRE Transactions on Circuit Theory, December, 1960.
Guest Editors: N. De Claris and L. A. Zadeh

Special Issue on Nonlinear Circuits
IEEE Transactions on Circuit Theory, November, 1972.
Guest Editor: C. A. Desoer

Special Issue on Nonlinear Circuits and Systems
IEEE Transactions on Circuits and Systems, November, 1980.
Guest Editor: R. W. Liu

Special Issue on Nonlinear Phenomena, Modelling, and Mathematics
IEEE Transactions on Circuits and Systems, Part I: Tutorial Papers (August 1983) and Part II: Contributed Papers (September 1983).
Guest Editor: L. O. Chua

In line with the expository nature of this centennial issue, this paper is intended for a broad (non-specialist) audience. Its objective is to sample a few interesting recent results on three fundamental aspects of nonlinear circuit theory; namely, nonlinear circuit analysis, nonlinear circuit synthesis, and nonlinear device modelling. Due to space limitation, only a small subset of recent advances in these areas can be described. We will focus on those results which can be easily understood and applied by the lay reader. In particular, we will present only those tools, methods, and algorithms which can be applied by inspection, by direct substitution, or by simple graphical construction.

Rather than overwhelming the reader with a deluge of lengthy theorems, mathematical conditions, and technicalities, we have opted to describe our results using an informal descriptive style with virtually no mathematical equations. Most results are not presented in their full generality in order to simplify their statements. Many examples are given for illustrative and motivational purposes. We hope this small dose of some interesting aspects of nonlinear circuit theory will whet the readers appetite for doing research in the many yet uncharted terrain of nonlinear circuits.

In keeping with the tutorial nature of this paper, only references closely related to the topic under discussion are included. Space limitation has precluded the inclusion of a more extensive bibliography. For example, all papers pertaining to numerical techniques and algorithms for solving nonlinear and piecewise-linear equations are excluded. Also papers concerned with the theoretical and mathematical aspects of nonlinear systems are excluded unless they contain some circuit examples and applications. We apologize for any inadvertent omission of any reference related to the topics covered in this paper as no slight is intended.

For convenience, the following is an outline of the topics presented:

PART I: ANALYSIS

A. RESISTIVE CIRCUITS

A nonlinear circuit is said to be resistive iff it is made of only 2-terminal, 3-terminal, and n-terminal resistors, in addition to independent voltage and current sources.

An n-terminal resistor is in general characterized by a relation involving only the terminal voltages and currents; namely,

$$f_j(v_1,v_2,\cdots,v_{n-1};i_1,i_2,\cdots,i_{n-1}) = 0, \quad j = 1,2,\cdots,n-1 \tag{1}$$

where v_j denotes the voltage from terminal j to an arbitrarily chosen datum node n, and i_j denotes the current entering terminal j.

For example, a transistor modelled by the Ebers-Moll equation

$$i_E = -I_{ES}\left(\exp\frac{-v_{EB}}{V_T} - 1\right) + \alpha_R I_{CS}\left(\exp\frac{-v_{CB}}{V_T} - 1\right) \tag{2a}$$

$$i_C = \alpha_F I_{ES}\left(\exp\frac{-v_{EB}}{V_T} - 1\right) - I_{CS}\left(\exp\frac{-v_{CB}}{V_T} - 1\right) \tag{2b}$$

(where I_{ES}, I_{CS}, α_R, and α_F are device parameters and V_T is the thermal voltage) is a 3-terminal resistor since (2a) and (2b) involve only the 4 terminal variables i_E, i_C, v_{EB} and v_{CB}.[1]

Likewise, ideal multi-terminal elements such as gyrators, ideal transformers, controlled sources and op amps (modelled by a voltage-controlled voltage source) are 4-terminal resistors.

There is a sound justification for calling all these elements resistors: arbitrary interconnection of such elements and independent sources always give rise to algebraic equations. Concepts and tools for analyzing this class of circuits are similar. They are distinct from those for analyzing dynamic circuits in Section B.

1. Existence and Uniqueness of Solution

Even simple nonlinear resistive circuits may have no solution, or multiple solutions. For example, the circuit in Fig. 1 has no solution for all $I > I_S$; that in Fig. 2 has multiple solutions for all $E_1 \le E \le E_2$. Since the pioneering work of Duffin in 1947 [3], a great deal of research has been done on the existence and uniqueness of solution of various classes of nonlinear resistive

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[1] It is interesting to note that the inventors of the transistor aptly recognized the "resistive" nature of this device at dc and decided to christen it a transfer resistor, or transistor for short.

circuits. The reader is referred to [1] for a collection of important papers on this subject over the last 25 years. Here, we will present a sample of some more recent results.

Theorem 1. Existence of Solution [4]

Let N be a network made of linear passive resistors, 2-terminal nonlinear resistors, and dc voltage and current sources. Then, except for some pathological cases, N has an odd number of solutions if the following 2 conditions are satisfied:

1. Interconnection condition: N contains no loop (resp., cut set) formed exclusively by nonlinear resistors and/or voltage sources (resp., current sources).
2. Element conditions:
   a. Each nonlinear resistor is characterized by either a continuous function $i = \hat{i}(v)$, $-\infty < v < \infty$, or a continuous function $v = \hat{v}(i)$, $-\infty < i < \infty$.
   b. Each nonlinear resistor is eventually passive in the sense that $vi \geq 0$ for all $|v| > k$, or $|i| > k$, for some finite k.

Remarks:

1. The v-i curve of an eventually-passive resistor need not be monotone increasing nor passive (restricted to the 1st and 3rd quadrants). However, it must lie in the 1st and 3rd quadrants beyond some finite distance (say k) from the origin. For example, the resistor described in Fig. 3(a) is not eventually passive. However, by reshaping the curve so that it eventually lies on the 1st and 3rd quadrants as in Fig. 3(b), the resistor becomes eventually passive.
2. Condition 2b can be generalized to include eventually-passive multi-terminal resistors, e.g., transistors modelled by the Ebers-Moll equation [4].
3. The circuit in Fig. 1 violates the interconnection condition: the diode and the current source formed a cut set. Recall that it has no solution if $I > I_S$.
4. A situation is said to be pathological in Theorem 1 if the "number" of solutions can be changed by an arbitrarily small perturbation of some circuit parameters. For example, the circuit in Fig. 2 is pathological when $E = E_1$ or $E = E_2$, because in either case, we can change from an "even" number (2) of solutions to an "odd" number (1 or 3) by a small perturbation in E_1 or E_2. Except for this easily avoided pathological situation, Theorem 1 asserts that N always has an odd number of solutions.
5. The interconnection condition is always satisfied if we insert a linear (passive) resistor in series (resp. parallel) with each voltage (resp., current) source.

6. The eventual-passivity condition can be satisfied by connecting a linear (passive) resistor in parallel (resp., series) with each voltage-controlled (resp., current-controlled) resistor which is not eventually passive, provided the v-i characteristics of all such resistors are bounded. The same holds, *mutatis mutandis*, for multi-terminal resistors.
7. Theorem 1 and Remarks 5 and 6 allow us to trivially modify a circuit model to ensure the existance of at least one solution.

Let us now turn to a most remarkable and useful existence and uniqueness theorem due to Nielsen and Willson [5] which is valid for any circuit made of linear passive resistors, npn and pnp transistors and dc voltage and current sources.

Definition 1. Feedback Structure

The above circuit is said to contain a feedback structure iff the set of connections involving two transistors shown in Fig. 4 is present when some combination of short and open circuits replaces all resistors, when each independent voltage source is replaced by a short circuit, when each independent current source is replaced by an open circuit, and when all but two of the transistors are replaced by one of the open and/or short circuit structures shown in Fig. 5. The transistor-like symbol used in Fig. 4 is meant to denote a transistor which can assume either of the two possible orientations with respect to its collector and emitter terminals.

Theorem 2. Existence and Uniqueness of Solution [5]

Let N_T denote any circuit made of linear passive resistors, npn and/or pnp transistors (modelled by the Ebers-Moll equation), and dc voltage and current sources. Then N_T has a unique solution for all values of the circuit's resistors and all values of the transistor parameters ($0 < \alpha_f, \alpha_r < 1$) if N_T possesses no feedback structure.

Example 1

Consider the Widlar-type [6] transistor voltage reference circuit shown in Fig. 6(a). This circuit is easily seen to contain a feedback structure formed by Q_3 and Q_4 in Fig. 6(b) upon performing the following operations: a) short circuit the battery E, b) short circuit resistor R_2, c) open circuit resistors R_1, R_3, and R_4 and d) replace transistors Q_1, Q_2, Q_5, and Q_6 by the structure in Fig. 5(a).

It follows from Theorem 2 that this circuit could have multiple solutions for some circuit and transistor parameters. Indeed, for the parameters $R_1 = 200\ \Omega$, $R_2 = 7\ \text{K}\Omega$, $R_3 = 500\ \Omega$, $R_4 = 4.5\ \text{K}\Omega$, and $E = 1.3$ V, two distinct solutions are found by computer simulation using the SPICE program, Version 2G-5A [7].[2]

Example 2. The transistor circuit shown in Fig. 7(a) consists of an emitter-coupled pair with an active load [6]. To search for the presence of a feedback structure, we short all batteries and open the current source I_{EE}, to obtain the simplified circuit shown in Fig. 7(b). We leave it to the reader to verify that it is impossible to obtain a feedback structure in Fig. 7(b) by replacing any two of the four transistors by any combination of the open and/or short circuit structures specified in Fig. 5. It follows therefore from Theorem 2 that this circuit has a unique solution for all possible values of V_1, V_2, V_{cc}, and I_{EE}, and all possible transistor parameters.

Remarks:

1. Theorem 2 gives a sufficient (but not necessary) condition for a unique solution. It is a Corollary of the Nielsen-Willson theorem which asserts that the non-existence of a feedback structure is both necessary and sufficient for the associated equation $\underset{\sim}{A}\underset{\sim}{F}(\underset{\sim}{x}) + \underset{\sim}{B}\underset{\sim}{x} = \underset{\sim}{c}$ to have a unique solution, provided the n-vector $\underset{\sim}{c}$ is allowed to take any values. Since $\underset{\sim}{c}$ represents the contribution due to the independent sources, it may not be possible to set $\underset{\sim}{c}$ to an arbitrary value by adjusting the values of the independent sources in a particular circuit. In other words, even if a circuit is found to contain a feedback structure, it may still possess a unique solution for all circuit and transistor parameters.
2. The feedback-structure condition in Theorem 2 becomes both necessary and sufficient if one is allowed to introduce additional independent sources (not present in the original circuit) at strategic locations.
3. Theorem 2 can be generalized to allow both bipolar transistors and FET [8].

Since most nonlinear multi-terminal devices (e.g., SCR, GaAs FET, op amp, etc.) can be realistically modelled by a circuit containing only 2-terminal nonlinear resistors and linear controlled sources [9], it is most desirable to derive general existence and uniqueness criteria which allow controlled sources. Unfortunately, nonlinear circuits containing controlled sources often exhibit multiple solutions over a wide range of controlling coefficients.

For example, consider the circuit shown in Fig. 8(a). Replacing the linear one-port to the right of the nonlinear resistor by its Thevenin equivalent resistance $R_{eq} = (1-\alpha)R$, we obtain the equivalent circuit shown in Fig. 8(b).

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[2] The transistor parameters in the program are chosen as follow: Q_1 and Q_2: $\beta_f = 20$, $I_S = 10^{-14}$; Q_4: $\beta_f = 50$, $I_S = 10^{-10}$, Q_3, Q_5, and Q_6: $\beta_f = 50$, $I_s = 10^{-14}$. All other parameters are assigned default values.

Since $R_{eq} < 0$ for all $\alpha > 1$, the load line in Fig. 8(c) has a positive slope. Note that there are 2 operating points even if the nonlinear resistor is characterized by a strictly-monotone increasing v_R-i_R curve.

The above example, as well as numerous others, tend to support the belief that it is impossible to derive general existence and uniqueness criteria which are independent of the circuit parameters for circuits containing controlled sources. A recent breakthrough on this problem, however, proves otherwise. We have succeeded in deriving a general topological criteria which give the necessary and sufficient conditions for a circuit containing strictly-monotone increasing 2-terminal nonlinear resistors and all 4 types of linear controlled sources (i.e., current-controlled voltage source, voltage-controlled current source, current-controlled current source and voltage-controlled voltage source) to have a unique solution for all circuit parameters and positive controlling coefficients [10].

To simplify the statement of our topological criteria, we will consider only the special case where the circuit contains only one type of controlled sources; namely, current-controlled current sources (CCCS). Each CCCS is represented in the directed graph G of the circuit by two disconnected branches, as shown in Fig. 9. The two branches are labelled k and $\hat{k}$, respectively, for the kth controlled source. Hence, the controlling branch k is called the input branch and the associated controlled branch $\hat{k}$ is called the output branch.

Like Theorem 2, our topological criteria depends on the absence of a particular topological structure defined below.

Definition 2. Cactus Graph

A connected graph G associated with a circuit made exclusively of CCCS is called a cactus graph iff every loop in G is made exactly of 2 branches labelled k and $\widehat{k+1}$, where $k = 1,2,\cdots,n$, and $\widehat{n+1} = \hat{1}$ (see Fig. 10(b)). Here k denotes the input (controlling) branch of the kth controlled source, $\hat{k}$ denotes the associated output (controlled) branch, and n denotes the number of controlled sources in G.

Remarks

1. We can visualize a cactus graph as the boundary of a cactus plant made of leaves (shaded area) hinged at the top and bottom only. For example, the cactus graph in Fig. 10(b) is associated with the cactus plant in Fig. 10(a).
2. Definition 2 requires that the branches in a cactus graph be labelled in the prescribed way. Note that the two branches k and $\hat{k}$ associated with the kth controlled source must appear in two different leaves (see Fig. 10(b)).

Violation of this labelling scheme disqualifies a candidate for a cactus graph.

Our topological criteria requires that we systematically simplify a given graph G using the following 4 graph operations to obtain a reduced graph G_0:

Operation 1. Replace each independent voltage source by a short circuit and each independent current source by an open circuit.

Operation 2. Replace each resistor by either a short circuit or an open circuit.

Operation 3. Replace the input (controlling) branch of some (possibly none) controlled sources by a short circuit and the associated output (controlled) branch by an open circuit.

Let G' denote any graph obtained after applying operations 1, 2, and 3 to G. Clearly, G' contains only branches associated with controlled sources. The simplified graph G' is said to have a complementary tree structure iff both the input branches (labelled $1,2,\cdots,n$) and their associated output branches (labelled $\hat{1},\hat{2},\cdots,\hat{n}$) form a tree of G'.

Operation 4. For each simplified graph G' having a complementary tree structure, replace the input branch of some (possibly none) controlled sources by a short circuit (or an open circuit) and the associated output branch by an open circuit (or a short circuit).

Theorem 3. Existence and Uniqueness of Solution [10].

Let N contain only 2-terminal linear passive resistors, 2-terminal nonlinear resistors (characterized by strictly-monotone-increasing v-i curves such that $|v| \to \infty$ as $|i| \to \infty$), independent voltage and current sources, and current-controlled current sources (having positive controlling coefficients[3]). Then N has a unique solution for all values of linear resistors and independent sources, and all (positive) controlling coefficients if and only if G can not be simplified by the above 4 graph operations into a cactus graph G_0 containing an even number (including zero) of similarly-directed loops.[4]

Example 1

Consider the circuit shown in Fig. 11(a) and its associated graph in Fig. 11(b). Note that we did not assign a direction in the branches corresponding

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[3] If a controlling coefficient is negative, we simply transpose the two terminals of the controlled source to obtain a positive coefficient.

[4] A loop in a cactus graph is said to be similarly-directed iff both branches are directed in the same direction. For example, only loops $\{1,\hat{2}\}$, $\{3,\hat{4}\}$, and $\{4,\hat{5}\}$ in Fig. 10(b) are similarly directed.

to the battery and resistor because they are irrelevant in the criteria. Note also that the direction of branch $\hat{1}$ is opposite to the arrow head in Fig. 11(a). By applying only operations 1 and 2 (short-circuiting the battery E and resistor R_1, and open circuiting the resistor R_2) we obtain the cactus graph shown in Fig. 11(c). Since this cactus graph has "zero" (hence even) similarly-directed loops, it follows from Theorem 3 that this circuit has multiple solutions for some values of E, $R_2 > 0$, and $\alpha > 0$. Indeed, simply choose $E = 0$ and $\alpha > 1$ and the circuit reduces to that shown in Fig. 8(a).

Example 2

Consider the circuit shown in Fig. 11(a) but with the controlled source terminals interchanged. In this case, operations 1-4 will reduce the associated graph G to a cactus graph containing an odd number (one) of similarly-directed loops. Since it is impossible to obtain a cactus graph with an even number of similarly-directed loops in this case, it follows from Theorem 3 that this modified circuit has a unique solution for all values of E, $R_2 > 0$ and $\alpha > 0$. This conclusion can be independently verified by noting the Thevenin equivalent resistance $R_{eq} = (1+\alpha)R_2 > 0$ for all $\alpha > 0$. Hence the load line in Fig. 8(c) must be drawn with a negative slope, thereby intersecting the strictly-monotone-increasing v_R-i_R curve at only one point.

Example 3

Consider the circuit shown in Fig. 12(a) containing 3 CCCS's. Applying operations 1, 2, and 3, we obtain the cactus graphs in Figs. 12(b) and (c) which contain an odd number (one) of similarly-directed loops. However, unlike Example 2 which contains only one controlled source, we can not rush to conclude that this circuit has a unique solution because other combinations of operations 1-4 may give rise to a cactus graph with an even number of similarly-directed loops.

Applying operations 1 and 2 we obtain the graph G' shown in Fig. 12(d), which has a complementary tree structure (since {1,2,3} and $\{\hat{1},\hat{2},\hat{3}\}$ both form a tree). Applying operation 4 to G', we obtain the two distinct cactus graphs shown in Figs. 12(e) and 12 (f) respectively. Note that the cactus graphs in Figs. 12(e) and (f) have an even (zero) number of similarly-directed loops. It follows from Theorem 3 that this circuit has multiple solutions for some circuit paremeters.

Example 4

Consider the circuit shown in Fig. 12(a) but with the terminals of each controlled source interchanged. The reader can easily verify that no combination of operations 1-4 can reduce G to a cactus graph with an even number of similarly-directed loops. It follows from Theorem 3 that this modified circuit has a unique solution for all values of E, I, $\alpha_1 > 0$, $\alpha_2 > 0$, and $\alpha_3 > 0$.

Remarks

1. Theorem 3 can be generalized to allow all 4 types of linear controlled sources [10].
2. Existence and uniqueness criteria analogous to Theorem 3 can be derived for circuits containing operational amplifiers modelled by a nonlinear voltage-controlled voltage source [11].
3. The criteria in Theorems 1, 2, 3 are strictly topological in the sense that they do not involve any numerical calculations. For simple circuits, these criteria can often be applied by inspection.

2. Bounds on Solutions

Since equations associated with nonlinear circuits can not be obtained in closed form in general, the next best thing that can be done is to derive bounds on the solutions. This information is particularly useful in computer simulation [9] where a good initial guess of the solution can greatly reduce the computation time.

One might conjecture that the voltage solution in a resistive circuit can not exceed the sum of the power supply voltages, a situation commonly observed in electronic circuits. Unfortunately, this conjecture is false even for linear resistive circuits: simply apply a voltage source E across port 1 of an ideal transformer and observe that the voltage across port 2 is greater than E if the turns ratio $n_2/n_1 > 1$. On the other hand, many circuits do satisfy the above property and it is convenient to give it a name.

Definition 3. No-gain Property [12-15]

A circuit N made of 2-terminal, 3-terminal, ⋯, n-terminal resistors and independent voltage and current sources is said to have the no-gain property iff each solution of N satisfies the following 2 properties:

1. the magnitude of the voltage between any pair of nodes in the network is less than or equal to the sum of the magnitudes of the voltages appearing across the independent voltage and current sources.
2. the magnitude of the current flowing into each terminal of every element is less than or equal to the sum of the magnitudes of the currents flowing through the independent voltage and current sources.

Theorem 4. No-gain Criterion [15]

A resistive circuit N has the no-gain property if and only if each n-terminal resistor ($n = 2,3,\cdots$) in N satisfies the following criterion:

For each dc operating point

$$(\underset{\sim}{V}_Q,\underset{\sim}{I}_Q) \triangleq [V_1,V_2,\cdots,V_{n-1};\ I_1,I_2,\cdots,I_{n-1}] \tag{3}$$

satisfying the resistor characteristic (1), there exists a connected n-terminal circuit N_Q containing n-1 linear positive resistors which has the same operating point.

An n-terminal resistor satisfying the above criterion is called a no-gain element.

For example, consider a strictly passive resistor characterized by $v_i > 0$, except at origin, as shown in Fig. 13(a). Note that for each operating point Q, the nonlinear resistor in Fig. 13(b) and its associated linear resistor R_Q in Fig. 13(c) have the same operating point. It follows that every 2-terminal resistor is a no-gain element if and only if it is strictly passive [12].

Remarks

1. A strictly passive n-terminal resistor may not satisfy the no-gain criterion if n > 2. Consequently, the elegant proof of the no-gain criterion given by Wolaver [12] can not be generalized for n > 2.
2. The first rigorous proof of the no-gain criterion for n = 3 is given by Willson [14]. Because it depends on geometrical arguments, Willson's proof can not be generalized for n > 3. The lengthy proof for the n > 3 case is given in [15].
3. Equivalent analytical conditions for testing the no-gain criterion are given in [14] for n = 3 and [15] for $n \geq 3$.
4. An n-teriminal circuit made of arbitrary interconnection of no-gain elements is also a no-gain element.
5. Transistors, FET, op amps, and most electronic devices modelled by a resistive circuit model are no-gain elements. Hence, most electronic circuits satisfy the no-gain property.

Corollary 1. Maximum and Minimum Voltage-node Property [15]

Let N be a connected resistive circuit made of 2-terminal, 3-terminal, ··· n-teriminal no-gain elements and independent dc voltage and current sources. Then the highest potential node (resp. lowest potential node) must necessarily be connected to either a voltage, or a current source.

Corollary 2. Bounding Region for Voltage Transfer Characteristic [15]

Let N be a connected resistive circuit made of 2-terminal, 3-terminal, ··· n-terminal no-gain elements and independent dc voltage sources as shown in

Fig. 14(a), where nodes b and d are not necessarily connected to each other. Then:

a. The v_0 versus v_{in} transfer characteristic must lie within the shaded region shown in Fig. 14(b), where E_0 denotes the sum of the voltage magnitude of all internal voltage sources.

b. If all internal sources of N, as well as terminals b and d, are grounded in Fig. 14(a), then the transfer characteristic shrinks to within the shaded region in Fig. 14(c), where E_{max} and $-E_{min}$ denote, respectively, the maximum and minimum node-to-ground voltage of the internal voltage sources.

3. Symmetry Properties

By exploiting various forms of symmetry in nonlinear elements and circuits, one often achieves drastic simplifications in the ensuing analysis. For example, one-port and 2-port resistive circuits made of complementary symmetric [16] elements (e.g., linear multi-terminal elements, npn and pnp transistors, n-channel and p-channel FET, op amps, etc.) exhibit an odd symmetric driving-point and transfer characteristic, respectively. Moreover, many practical circuits are deliberately made symmetrical (e.g., push-pull and complementary symmetric amplifiers) to achieve some desired properties, such as 2nd-harmonic cancellation. Using simple group-theoretic techniques a unified theory of symmetry in resistive nonlinear circuits has been developed in [17-18].

4. Stationary Properties

Maxwell was the first person to recognize that the solution of a circuit made of linear 2-terminal resistors and dc voltage sources coincides with the stationary point of an associated scalar potential function [19]. This property has been generalized by Millar [20] and Duinker [21] for circuits containing 2-terminal nonlinear resistors. A further generalization to allow reciprocal n-terminal nonlinear resistors ($n \geq 2$) is given in [22]. Some applications of these properties are given in [23].

The stationary property of nonlinear resistive circuits has been exploited by Dennis [24] and Stern [25] to solve linear programming and quadratic programming problems by simulating the inequality constraints with ideal diodes. A generalization of this approach for solving any nonlinear programming problem has been achieved recently [26]. This generalization allows an engineer with virtually no nonlinear programming background to solve nonlinear optimization problems with nonlinear inequality constraints with the help of a circuit simulation program [27], such as SPICE [7].

5. Geometric Properties

A deeper understanding of resistive nonlinear circuits requires a careful study of the geometrical properties of the characteristics of the nonlinear resistors and the circuit in which they are imbedded. For example, the driving-point and transfer characteristics of even simple nonlinear resistive circuits can be extremely complicated: they are generally multivalued as in Fig. 15(a) (corresponding to a voltage-controlled resistor and a current-controlled resistor in series) and may contain several branches as in Fig. 15(b) (corresponding to two voltage-controlled resistors in series [28]. How do we interpret and classify these bizarre characteristics? How do singular elements such as nullator and norator fit into a unified theory of resistive nonlinear circuits? Under what conditions are the solutions structurally stable in the sense that their geometrical properties are preserved under small perturbations of circuit parameters? All of these fundamental questions can only be resolved with the help of differential geometry and topology. Readers interested in this subject are referred to [29-31].

B. DYNAMIC CIRCUITS

A circuit N is said to be dynamic iff it contains at least one resistor and a capacitor, or inductor, and is nonlinear iff N contains at least one nonlinear element. The elements may be 2-terminal, 3-terminal, or multi-terminal. An n-terminal resistor is described by (1), i.e., by a relation involving only the terminal voltages and currents. An n-terminal capacitor is described by a relation involving the terminal voltages and charges; namely,

$$f_j(q_1,q_2,\cdots,q_{n-1};\ v_1,v_2,\cdots,v_{n-1}) = 0, \quad j = 1,2,\cdots,n-1 \tag{4}$$

where $i_j \triangleq \dot{q}_j$. An n-terminal inductor is described by a "dual" relation involving the terminal current and fluxes; namely,

$$f_j(\phi_1,\phi_2,\cdots,\phi_{n-1};\ i_1,i_2,\cdots,i_{n-1}) = 0, \quad j = 1,2,\cdots,n-1 \tag{5}$$

where $v_j \triangleq \dot{\phi}_j$.

Dynamic nonlinear circuits are much more untractable and difficult to analyze than resistive nonlinear circuits. Research on dynamic nonlinear circuits began in earnest in the early sixties with the main attention focused almost entirely on the formulation of a system of nonlinear state equations for various types of nonlinearities [28, 32-37] and their geometrical interpretations [38-42]. As a result, a solid foundation on this basic subject is now firmly in place. It is but natural that recent research on dynamic nonlinear circuits has focused on the extremely rich qualitative properties and nonlinear dynamics exhibited by nonlinear state equations. A survey of the main results of this research activity prior to 1980 is given in [41]. Readers interested in more details are referred to [43-49]. In the following sections, we will sample a few recent results.

1. No Finite Forward-Escape-Times

Consider a 1F linear capacitor in parallel with a nonlinear resistor described by a square-law characteristic $i_R = -v_R^2$. It has a well-defined state equation

$$\dot{v}_c = v_c^2 \tag{6}$$

The solution of (6) with an initial capacitor voltage $v_c(0) = 1V$ is given by

$$v_c(t) = \frac{1}{1-t}\,, \quad t \geq 0 \tag{7}$$

Since this solution blows up at a finite time (called a finite forward-escape-time in the literature) $t = 1$, the above circuit model is clearly defective and must be remodelled in order to obtain a more realistic answer.

Rather than wasting time blindly simulating such non-physical circuits in a computer, it is important to derive simple criteria which rules out such non-physical situations. The following result is a case in point.

Theorem 5. No Finite Forward-escape Time Criteria [44]

Let N be a circuit made of 2-terminal resistors, inductors, capacitors, and independent voltage and current sources. Then N has no finite forward-escape-time if the following conditions are satisfied:

a. There are no loop and no cut set made exclusively of inductors and/or capacitors.
b. Every voltage source (resp., current source) is in series (resp. in parallel) with a linear positive resistor.
c. Each inductor (resp., capacitor) is characterized by a differentiable ϕ-i (resp., q-v) curve with positive slopes.
d. Each resistor is eventually passive.

Remarks

1. Theorem 5 is a Corollary of a much more general theorem proved in [44].
2. The square-law resistor in the above example is not eventually passive and therefore violates condition d.
3. The four conditions in the above criteria are satisfied by most realistic circuit models.

2. Existence of a Unique Periodic Steady-State Response [45]

Consider the RLC circuit shown in Fig. 16(a), driven by a periodic voltage source $v_s(t) = \frac{1}{3}\cos 3t$. Its state equation is given by:

$$\dot{v}_C = i_L \tag{8a}$$

$$\dot{i}_L = i_L - \frac{4}{3} i_L^3 - v_C + \cos 3t \tag{8b}$$

Depending on the initial condition, (8) has at least 3 distinct periodic solutions at $\frac{1}{3}$ of the input frequency; namely,

$$v_C(t) = \sin(t + \tfrac{2}{3}k\pi)\ , \quad k = 0,1,2 \tag{9a}$$

$$i_L(t) = \cos(t + \tfrac{2}{3}k\pi)\ , \quad k = 0,1,2 \tag{9b}$$

Since the "output" frequency is smaller than the "input" frequency, we say the circuit has at least 3 distinct subharmonic solutions.

In many applications (e.g., circuits in a power substation, amplifiers, transducers, etc.) it is important that the circuit have a unique periodic steady-state solution having the same frequency as the input. The following criteria can be used for this purpose.

Theorem 6. Unique Periodic Steady-state Response Criteria [45]

Let N be a circuit made of 2-terminal resistors, inductors and capacitors, and let N be driven by a periodic voltage or current excitation of frequency ω. Then N has a unique periodic steady-state response (independent of initial conditions) with the same frequency ω if the following conditions are satisfied:

a. There is no loop (resp. no cut set) formed exclusively by capacitors, inductors, and/or voltage sources (resp., current sources).

b. All inductors (resp., capacitors) are linear with a positive inductance (resp., capacitance)

c. Each resistor is characterized by a differentiable v-i curve with positive slopes.

Remarks

1. Theorem 6 is a Corollary of a more general theorem proved in [45].
2. The non-monotonic resistor in Fig. 16(b) violates condition c.
3. If the input is a quasi-periodic signal made of several incommensurable frequency components, then Theorem 6 still holds with the frequency spectrum of the unique steady-state response made up of only harmonics and intermodulation components of the input frequencies.

4. Theorem 6 is valid even if $\omega = 0$; i.e., when the circuit is autonomous. In this case, the criteria guarantee that the circuit has a unique and hence globally-asymptotically-stable equilibrium point.

3. dc Steady-State Response of Capacitor-Diode Circuits

Consider next the circuit shown in Fig. 17(a) where the diode v_R-i_R curve $i_R = g(v_R)$ is shown in Fig. 17(b). Its state equation is given by

$$\dot{v}_C = g(E \sin t - v_C) \tag{10}$$

Since $g(v_R) \geq 0$, the current charging the capacitor can never be negative. Hence the capacitor voltage either remains constant (if $i_C = 0$) or increases, monotonically. Now observe that if $v_C(0) > E$, then the capacitor voltage will remain constant at E volts for all times $t > 0$. Hence, this circuit has infinitely many distinct dc steady-state responses to the same periodic excitation; namely, one for each initial condition $v_C(0) > E$.

In applications involving capacitors and diode circuits (e.g., voltage multipliers) the capacitors are usually at zero initial condition before the input is applied. The following theorem illustrates how the solution of certain classes of dynamic nonlinear circuits can be calculated by inspection.

Theorem 7. dc Steady-State Response [46]

Let N be a circuit containing linear positive capacitors and diodes modelled by the v_R-i_R curve shown in Fig. 17(b). Let N be driven by a sinusoidal voltage source $E\sin \omega t$ and assume the following topological conditions are satisfied:

a. The voltage source and the capacitors form a tree.

b. The voltage source and the diodes form a tree

c. The voltage source and the diodes form a cut set.

d. The diodes, possibly with the voltage source, form a similarly-directed path.

Assuming the capacitors are initially uncharged, N has a unique dc steady-state solution. Each steady-state capacitor voltage is given explicitly by

$$v_{C_j} = k_j E \tag{11}$$

where k_j is the number of diodes contained in the fundamental loop defined by capacitor C_j with respect to the voltage-source-diode tree, provided the reference polarity of v_{C_j} is aligned with the diode's forward direction.

Example

Consider the circuits shown in Figs. 18(a) and (b). Note that conditions a-d are satisfied in each case and hence the dc steady-state voltage across each capacitor can be determined by inspection, as indicated in Figs. 18(a) and (b).

4. The Hopf Bifurcation Theorem

Electronic sinusoidal oscillators are usually designed as a feedback circuit made of locally active saturation type nonlinear devices (e.g., transistors, op amps, etc.) and phase shifting elements. The circuit parameters are so chosen that the linearized circuit has a pair of complex conjugate poles which is slightly to the right of the imaginary axis. It is argued that thermal noise will start an oscillation whose amplitude will grow until limited by the nonlinear devices. The oscillation frequency is taken to be the frequency ω_0 at which the pole crosses the imaginary axis when some parameter (such as the loop gain) varies [57]. This approach, called the Barkhausen Criterion [58], usually works but is obviously not rigorous. Indeed, counterexamples exist where the circuit will not oscillate.

The famous Hopf bifurcation theorem [59] gives rigorous conditions for oscillation that are almost the same as the intuitive design method we have just described. The theorem deals with the appearance and growth of a limit cycle as a parameter is varied in a nonlinear system. (The parameter may be, but does not have to be, a natural system parameter: one can always introduce it artificially if necessary).

The original Hopf bifurcation theorem as stated in the time domain (in terms of a system of nonlinear differential equations) requires an excessive amount of numerical calculations and is impractical for circuit applications. The following is a reformulation of this theorem in the frequency domain in a form reminiscent of the familiar Nyquist criterion.

Theorem 8. Frequency Domain Hopf Bifurcation Theorem [47,53]

Consider the single loop feedback system shown in Fig. 19(a), where $G(j\omega)$ denotes a scalar transfer function and $f(\cdot)$ denotes a continuously differentiable (at least 4 times) nonlinear function which may depend on some parameter μ. Consider the following graphical algorithm:

Step 1. Solve the nonlinear equation

$$G(0)\ f(y) + y = 0 \tag{12}$$

for the equilibrium point (i.e., dc operating point) y_Q.

Step 2. Define the linearized open-loop transfer function

$$\lambda(j\omega) \triangleq G(j\omega)\ f'(y_Q) \tag{13}$$

where $f'(y_Q)$ denotes the first derivative at $y = y_Q$.

Step 3. Sketch the "Nyquist Loci" Γ of $\lambda(j\omega)$ in the Im $\lambda(j\omega)$ versus Re $\lambda(j\omega)$ plane, as shown in Fig. 19(b). Identify the "resonant frequency" ω_R where Γ intersects the real axis; i.e., Im $\lambda(j\omega_R) = 0$.

Step 4. Calculate the closed-loop transfer function

$$H(s) \triangleq \frac{G(s)}{G(s)f'(y_Q)+1} \tag{14}$$

at $s = 0$ and $s = j2\omega_R$ to obtain $H(0)$ and $H(j2\omega_R)$.

Step 5. Calculate the number

$$\boxed{\zeta(\omega_R) \triangleq G(j\omega_R)\left\{f''(y_Q)^2\left[\frac{1}{4}H(0) + \frac{1}{8}H(j2\omega_R)\right] - \frac{1}{8}f'''(y_Q)\right\}} \tag{15}$$

where f'' and f''' denote the 2nd and 3rd derivative of f, respectively.

Step 6. Draw a vector through the point $-1+j0$ in the direction specified by the (generally complex) number $\zeta(\omega_R)$.

Conclusion. If the above vector intersects Γ at a point (with frequency $\omega = \omega_0$) sufficiently close to $-1+j0$ and points outwards as shown in Fig. 19(b), then the feedback system has a stable almost sinusoidal oscillation at a frequency nearly equal to ω_0, and an amplitude A nearly equal to

$$A = \sqrt{B/|\zeta(\omega_R)|} \tag{16}$$

Example

Consider the tunnel diode circuit shown in Fig. 20(a) and the $V_D - I_D$ curve shown in Fig. 20(b). The state equations are given by

$$\dot{i}_L = (1/L)v_C\ , \quad \dot{v}_C = (1/C)\,[g(V_B - v_C) - i_L] \tag{17}$$

The feedback system in Fig. 19(a) is described by the same state equations provided we choose[5]

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[5] A simple procedure is given in [47] for transforming any state equation into an equivalent feedback system.

$$G(s) \triangleq \frac{s}{s^2+s+1} \quad \text{and} \quad f(y) \triangleq -g(\mu-y) - y \tag{18}$$

Here, $\mu \triangleq V_B$ is a parameter to be adjusted in this example, and $L = 1$, $C = 1$.

To apply the above graphical algorithm, we first solve $G(0)\, f(y) + y = 0$ to obtain the equilibrium point $y_Q = 0$. The Nyquist loci Γ defined by

$$\lambda(j\omega) = [g'(\mu) - 1] \left[\frac{j\omega}{(1-\omega^2)+j\omega}\right] = \frac{-\omega^2-j\omega(1-\omega^2)}{(1-\omega^2)^2+\omega^2} \tag{19}$$

is sketched in Fig. 20(c) for the parameter $\mu = \mu_0$ (corresponding to the maximum point in Fig. 20(b). Note that Γ intersects the real axis at $\omega_R = 1$ in this case.

Next we calculate

$$H(0) = [G(0)f'(\mu_0) + 1]^{-1}G(0) = 0, \quad H(j2) = [G(j2)f'(\mu_0) + 1]^{-1}G(j2) = -j\,\frac{2}{3} \tag{20}$$

and

$$f''(0) = -g''(\mu_0), \quad f'''(0) = g'''(\mu_0) \tag{21}$$

Substituting (19)-(20) into (15), we obtain the <u>complex</u> number

$$\begin{aligned}\zeta(\omega_R) &= \zeta(1) = G(j1)\left\{g''(\mu_0)^2\left[\tfrac{1}{4}\,(0) + \tfrac{1}{8}\,(-j\,\tfrac{2}{3})\right] - \tfrac{1}{8}\, g'''(\mu_0)\right\} \\ &= -\frac{1}{8}\, g'''(\mu_0) - \frac{j}{12}\, g''(\mu_0)^2\end{aligned} \tag{22}$$

This is a vector pointed towards the 3rd quadrant. Translating it to the point $-1+j0$, we obtain the picture shown in Fig. 20(c) with $B = 0$. It follows from Theorem 8 that this circuit will <u>begin</u> to oscillate at a frequency $\omega_0 = 1$ when $V_B = \mu_0$ (with a "zero" amplitude).

Repeating the above graphical algorithm for several other values of μ, we obtain the corresponding pictures shown in Figs. 20(d), (f), (g) and (h). It follows from these pictures that the tunnel diode circuit has a stable nearly sinusoidal oscillation with a frequency ω_0 (where the vector intersects Γ) whenever $\mu_0 < V_B < \mu_0'$.

<u>Remarks</u>

1. Theorem 8 is a corollary of a much more general theorem which is applicable for any transfer function matrix $\underset{\sim}{G}(j\omega)$ and any nonlinear <u>vector</u> function $\underset{\sim}{f}$ [47,53].
2. The Nyquist locus in Fig. 20(b) can be measured experimentally, as is common in the design of microwave oscillators.

3. Theorem 8 can be interpreted as a "nonlinear" Nyquist criterion.
4. Using degree theory [60], it is possible to derive an explicit formula for calculating the oscillation frequency ω_0 and amplitude A [51].
5. Nonlinear Dynamics

Circuits containing modern microwave devices, such as Gunn diode, IMPATT, and Josephson junction, are often imbued with extremely complex and exotic dynamics. Indeed, many experimentally observed phenomena are so bizarre that no satisfactory explanations--let alone a rigourous analysis--are yet in sight. Remarkable progress on nonlinear dynamics, however, has been achieved during the last decade and a powerful toolkit [61] for nonlinear analysis is now available.

Using these modern tools, the Josephson constant-voltage-step phenomenon which has baffled engineers and physicists alike since its discovery has finally yielded to a rigorous analysis [48, 54].

Likewise, the elusive synchronization phenomenon, which has not been well understood previously, has been proved to be chaotic in a typical threshold triggered oscillator [56].

Roughly speaking, a circuit is chaotic iff its solution is neither a periodic (possibly constant) nor an almost periodic function. Such solution waveforms are characterized by a broad frequency spectrum [62,66]. Many chaotic circuits and systems have been reported during the last 5 years [52, 56, 66-70] and the list is certain to grow at a rapid rate in the coming years. It is now widely believed that a large class of practical nonlinear circuits can become chaotic if the circuit parameters are allowed to drift beyond some "failure" boundary. The goal of research in chaotic circuits is to determine such failure boundaries.

6. Energy-Related Concepts

Linear n-ports such as ideal transformers, gyrators, circulators, hybrid coils, etc., are said to be non-energic because the instantaneous power entering the n-port is zero at all times and hence they are incapable of storing energy. Are there any non-energic nonlinear n-ports? If so, what are their characteristic properties? Linear time-invariant n-port capacitor and inductors can not be non-energic. Are there any non-energic nonlinear capacitors or inductors? The answers to all these foundational questions are given in an exhaustive study on non-energic n-ports [71].

The concepts of passivity and losslessness are the corner stones of linear passive circuit theory. Indeed, the discipline of circuit theory fluorished only after Brune's classic research on the synthesis of linear passive one-ports. The generalization of Brune's work, and others, on linear one-ports to the linear n-port case was accomplished by the early sixties. What are the analogs of these results for dynamic nonlinear n-ports?

Research on this fundamental question began in earnest in the early seventies. Due to the uncovering of many disturbing anomalies in the original definitions of passivity and losslessness (when applied to nonlinear n-ports), we went through a lengthy period of intense debate and soul searching, hoping to patch up a few holes in these defective definitions. It took 5 years before all parties involved in the deliberations were convinced that the old definitions had to be junked in favor of a totally new definition before any self-consistent criteria can be derived. The results of this concerted joint research project are well documented in a series of provocative and exhaustive papers [72-76].

PART II: SYNTHESIS

Research on nonlinear circuit synthesis is motivated by two areas of applications in practice. One area calls for the design of nonlinear circuits with an optimum performance (e.g., minimization of switching times, nonlinear distortion, switching speed, slew-rate, etc.). Since the finished product in this case is an operational piece of hardware, the synthesis techniques are acceptable only if the circuit elements used in the synthesis are either available commercially, or can be built with available components or state-of-the-art integration technology (for applications involving large volume productions).

The second area calls for the synthesis of nonlinear circuit models (of devices, machinery, or systems) to be used in a (digital) computer simulation [9]. Since the finished product in this case is a circuit diagram on paper, any circuit element that is included in the repertoire of elements in the computer simulation program is acceptable. For example, a negative capacitance would be quite acceptable in modelling applications but would be rejected as impractical in the first area.

In Part II, we sample some basic results on nonlinear synthesis from both areas of applications.

1. Driving-Point and Transfer Characteristics

The most basic problem in nonlinear synthesis is the practical realization of a one-port having a prescribed v versus i driving-point characteristic, or a 2-port having a prescribed v_0 versus v_{in} voltage transfer characteristic. It is not surprising therefore that the early research on nonlinear synthesis had focused on this problem [16]. The following is a sample of some of these results.

a. Monotone-increasing characteristics: they can be realized using only passive components (e.g., positive resistors, pn junction diodes and zener diodes) [16]. If a tight tolerance is called for, they can be realized with great precision using a combination of concave resistors (defined in Fig. 21(a)) and convex resistors (defined in Fig. 21(b)) [16]. These two fundamental building blocks can be accurately built using simple op-amp feedback circuits [77].

b. Non-monotone characteristics: Using the Nielsen-Willson theorem [5,8], it is possible to systematically generate a large number of passive one-ports having a type N (Fig. 22(a)) or type S (Fig. 22(b)) v-i characteristics using only positive resistors and two transistors (bipolar and/or FET) [78-79]. Combinations of these passive one-ports would generate more complicated non-monotonic characteristics.

Other specialized techniques using rotators [80-82], reflectors [81-83], scalars [83], linear transformation converters [84-86] and related techniques [87-90] are useful in many occasions. An ingenious approach using analog multipliers which make use of linear circuit synthesis techniques is described in [91]. However, by far, the most practical and accurate technique is the piecewise-linear approach described in [77].

Not included in the above survey are the important research programs spearheaded by Professor Eugen Philippow and his students [92] at the Technical University at Ilmenau of the German Democratic Republic, and by Professors J. Cajka, F. Kouril, and K. Vrba [93] and their students at the Technical University of BRNO, Czechoslovakia. An impressive list of publications on nonlinear synthesis covering more than 10 years of their research have appeared in Ilmenau and BRNO. Unfortunately, they are not available in English.

2. Nonlinear N-Port Resistors

The second most basic problem in nonlinear synthesis is the realization of an n-port resistor having a prescribed voltage-controlled representation $\underset{\sim}{i} = \hat{\underset{\sim}{i}}(\underset{\sim}{v})$, current-controlled representation $\underset{\sim}{v} = \hat{\underset{\sim}{v}}(\underset{\sim}{i})$, or hybrid representation $\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x})$, where $x_j = v_j$ (or i_j) and $y_j = i_j$ (or v_j), $j = 1,2,\cdots,n$. Two basic approaches are available:

a. Canonical Piecewise-Linear Approach: This approach assumes the n-port is specified in terms of a multi-dimensional canonical piecewise-linear representation [94-95].

$$\boxed{\underset{\sim}{y} = \underset{\sim}{a} + \underset{\sim}{B}\,\underset{\sim}{x} + \sum_{i=1}^{p} \underset{\sim}{c}_i \left| \langle \underset{\sim}{\alpha}_i , \underset{\sim}{x} \rangle - \beta_i \right|} \tag{23}$$

where $\underset{\sim}{a}$, $\underset{\sim}{c}_i$, $\underset{\sim}{\alpha}_i$ are n-vectors, $\underset{\sim}{B}$ is an nxn matrix, β_i is a scalar, and $\langle\cdot,\cdot\rangle$ denotes the vector dot product between $\underset{\sim}{\alpha}_i$ and $\underset{\sim}{x}$.[6]

The most remarkable property of this representation is that it involves only one nonlinearity; namely, the absolute -value function. In the 1-dimensional case, any continuous piecewise-linear function can be represented by (23), namely [96],

$$y = a + bx + \sum_{i=1}^{p} c_i |x - \beta_i| \tag{24}$$

While not all n-dimensional piecewise-linear functions can be represented by (23) if $n > 1$, it is shown in [95] that all n-ports made of 2-terminal piecewise-linear functions can be described by (23). This observation makes it possible to develop a unified theory of nonlinear circuits in terms of the canonical piecewise-linear representation. It also allows a much more practical computer implementation of several fundamental piecewise-linear algorithms [97-98].

Any n-port described by (23) can be synthesized by the technique described in [77].

If the n-port is reciprocal (i.e., when the Jacobian matrix $\partial\hat{\underset{\sim}{i}}(\underset{\sim}{v})/\partial\underset{\sim}{v}$ or $\partial\hat{\underset{\sim}{v}}(\underset{\sim}{i})/\partial\underset{\sim}{i}$ is symmetric), it can be synthesized by the method in [99]. This approach uses only 2-terminal piecewise-linear resistors, independent sources, and a (p+q)-port transformer [100].

b. Sectionwise-Piecewise-Linear Approach: This approach assumes the n-port is specified in terms of a multi-dimensional Sectionwise piecewise-linear representation [96]

$$\boxed{f(x_1,x_2,x_3,\cdots,x_n) = \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \cdots \sum_{k_n=1}^{N} \alpha(k_1,k_2,k_3,\cdots,k_n) \cdot \prod_{j=1}^{n} \phi_{k_j}(x_j)} \tag{25}$$

[6]This global analytical representation completely overcomes the main objection of multi-dimensional piecewise-linear functions; namely, bookkeeping the matrices and vectors for each polyhedral region and their equally cumbersome boundary specifications.

where

$$\left.\begin{aligned}
\phi_1(x_j) &= 1 \\
\phi_2(x_j) &= x_j \\
\phi_3(x_j) &= |x_j - x_{j1}| \\
&\vdots \\
\phi_N(x_j) &= |x_j - x_{jN-2}| \ .
\end{aligned}\right\} \tag{26}$$

and $\alpha(k_1,k_2,\cdots,k_a)$ denotes a constant coefficient. Any continuous nonlinear function $g(x_1,x_2,\cdots,x_n)$ can be approximated arbitrarily closely by (25). Again, the only nonlinearity in (26) is the absolute-value function. Unlike (23), however, the argument of each absolute-value function in (26) consists of only one variable. In other words, (25) consists of sums of products of functions of only one variable. This remarkable property is reminiscent of the well-known Kolmogoroff's representation [101]. Unlike Kolmogoroff's representation, however, which is computationally impractical, the coefficients $\alpha(k_1,k_2,\cdots,k_n)$ in (25) can be efficiently estimated by multiple regression or other optimization technique.

Any n-port described by (25) can be synthesized by the techniques described in [102]. Another method can be found in [103].

3. Mutators

Let X and Y denote two distinct types of 2-terminal circuit elements. For example, X can be an inductor and Y a capacitor. A linear time-invariant 2-port is called an X-Y mutator iff element X can be realized by connecting an element Y across port 2, as shown in Fig. 23(a).

Mutators can be built using only op amps and linear passive resistors and capacitors [83]. The unique transformation property of an L-R mutator has been measured and the relevant oscilloscope tracings are shown in Figs. 23(b) and (c), respectively. Note that Fig. 23(b) "mutates" a nonlinear resistor (R) with a v-i curve into a nonlinear inductor (L) with an identical characteristic in the ϕ_1-i_1 plane. Likewise, Fig. 23(c) "mutates" a nonlinear inductor described by a hysteresis loop into a nonlinear resistor described by the same hysteresis loop in the v_2-i_2 plane. Note also that an L-C mutator is just a gyrator.

Mutators play a fundamental role in nonlinear circuit synthesis: it reduces the general nonlinear element realization problem to the synthesis of 2-terminal nonlinear resistors [104].

4. Nonlinear Algebraic Elements

Resistors, inductors, and capacitors are only three among many other distinct types of ideal circuit elements which are essential to the development of a unified theory of nonlinear circuits. Some of the less familiar circuit elements which have appeared in the literature include the traditors [105], the FDNR [106], the charge store element [107], etc. Do these elements differ from each other, or are they just fancy examples of nonlinear resistors, inductors, or capacitors? From the modelling and circuit-theoretic points of view, it is essential that each circuit element be classified in a consistent and scientific manner.

Just as circuits are logically classified into two classes--resistive or dynamic--circuit elements can also be logically classified into algebraic or dynamic elements. Since a dynamic element is any circuit element which is not algebraic, it suffices for us to define an algebraic element.

Definition. Algebraic Element

An n-port N is said to be algebraic iff it is described by an algebraic relation involving at most 2 port variables $v_j^{(\alpha)}$ and $i_j^{(\beta)}$ for each port j, $j = 1,2,\cdots,n$, where $v_j^{(\alpha)}$ denotes the αth time derivative of v_j if $\alpha > 0$, or the αth time integral of v_j if $\alpha < 0$. Likewise, $i_j^{(\beta)}$ denotes the βth time derivative of i_j if $\beta > 0$, or the βth time integral of i_j if $\beta < 0$.

Remarks

1. The 2 variables associated with each port must be of a different type in the sense that one is associated with the port voltage, the other with the port current.
2. The integer "α" (as well as β) associated with two distinct ports may be distinct.
3. The relation defining a resistor, inductor, or capacitor involve only the pairs $(v^{(0)},i^{(0)}),(v)^{(-1)},i^{(0)})$, and $(v^{(0)},i^{(-1)})$, respectively.

Example 1

A 2-terminal element characterized by a relation $f(\phi,q) = 0$ between the two variables $v^{(-1)} \triangleq \phi$ and $i^{(-1)} \triangleq q$ is an algebraic one-port. This element is called a memristor (contraction of memory resistor) because it behaves like a resistor with memory [108].

Example 2

A frequency-dependent negative resistance (FDNR) [106] defined by $i = md^2v/dt^2$ is an algebraic 1-port involving only the variables $(v^{(2)},i^{(0)})$.

Example 3

A 2-terminal element described by the relation $i = q^3$ is not an algebraic element because the two variables i and q are not derived from v and i, respectively (Remark 1). It is therefore a dynamic element. Indeed, it is described by a differential equation in q; namely, $\dot{q} = q^3$.

Example 4

The L-R mutator described by $\phi_1 = v_2$ and $i_1 = -i_2$ [83] is an algebraic 2-port because only the variables $(v_1^{(-1)}, i_1^{(0)})$ and $(v_2^{(0)}, i_2^{(0)})$ are involved.

Example 5

The unconventional element called a charge store widely used in the modelling of diodes and transistors [107] is actually a 2-port defined by $v_1 = 0$ and $i_2 = kq_1$, where q_1 is the charge in the "short-circuited" port 1; namely, a charge-controlled current source. Since only the variables $(v_1^{(0)}, q_1^{(0)})$ and $i_2^{(0)}$ are involved, this element is also an algebraic 2-port.

Example 6

Consider a type I traditor defined by [105] $v_1 = -Aq_2 i_3$, $v_2 = -Aq_1 i_3$, and $\phi_3 = Aq_1 q_2$. Since only the variables $(v_1^{(0)}, i_1^{(-1)})$, $(v_2^{(0)}, i_2^{(-1)})$, and $(v_3^{(-1)}, i_3^{(0)})$ are involved, this element is an algebraic 3-port.

It follows from the above examples that the class of algebraic elements includes not only resistors, inductors, and capacitors, but many unconventional elements which in the past have baffled and confused many circuit engineers. The readers are referred to a more detailed study of the interconnection properties of these elements in [104]. Although these elements are presently more relevant to the study of nonlinear device modelling, they can be synthesized, if necessary, with the help of appropriate types of mutators [109].

5. The Hodge Decomposition

Every linear n-port N characterized by an admittance matrix $\underset{\sim}{Y}$ can be decomposed into a parallel connection between a reciprocal n-port N, and an anti-reciprocal n-port N_2, as shown in Fig. 24 for the 2-port case. What is the analog of this fundamental result for the nonlinear case?

It was shown in [110] that only a very small subclass of nonlinear n-ports can be similarly decomposed into a reciprocal and anti-reciprocal nonlinear n-ports. Hence, the next best thing that can be done is to relax the requirement of N_2 while still requiring N_1 to be reciprocal. It is desirable to keep N_1 reciprocal because its synthesis would require only reciprocal elements [99]. The basic problem is therfore to identify the structure of N_2 so that every voltage-controlled n-port can be decomposed as shown in Fig. 24.

This problem was solved in 1975 for the n = 2 case where an explicit decomposition formula was given in [110]. Solution for the n > 2 case turns out to be much harder and the problem was not solved until 3 years later. The answer is that N_2 must be solenoidal in the sense that if N_2 is described by $\underset{\sim}{i} = \underset{\sim}{g}(\underset{\sim}{v})$, then

$$\operatorname{div} \underset{\sim}{g}(\underset{\sim}{v}) \triangleq \frac{\partial g_1(\underset{\sim}{v})}{\partial v_1} + \frac{\partial g_2(\underset{\sim}{v})}{\partial v_2} + \cdots + \frac{\partial g_n(\underset{\sim}{v})}{\partial v_n} = 0$$

We summarize this basic result as follow:

Theorem 9. Hodge Decomposition [111]

Every voltage-controlled nonlinear n-port described by a smooth vector function i = i(v) can be decomposed into a parallel connection between a reciprocal nonlinear n-port resistor and a solenoidal nonlinear n-port resistor.

Remarks

1. Theorem 9 is a corollary of a more general theorem proved in [111].
2. Theorem 9 is related to a deep result from Algebraic geometry due to Hodge [112], hence the name Hodge decomposition.

PART III: MODELLING

Research in nonlinear circuits would become academic without a parallel development of nonlinear device modelling. Since modern electronic devices are generally nonlinear and rich in high-frequency dynamics, they must be replaced by realistic nonlinear circuit models before any meaningful analysis is possible.

The goal of research on nonlinear modelling is to develop a systematic methodology for synthesizing and validating nonlinear circuit models for modern devices.

Two basic approaches to device modelling have been pursued in the past: the physical approach and the black box approach. Examples of several electronic devices modelled via a physical approach include the pn junction diode [113], the IMPATT [114] , the Gunn diode [115], the SCR [116] and the GaAs FET [117]. Examples modelled via a black box approach include the hysteresis phenomenon in an iron-core inductor [118-119], memristive devices [108,120], and the analog multiplier [121]. Space limitation precludes even a cursory description of these models, each one interesting and illuminating in a different way. A summary of some of these models is given in a recent state-of-the-art survey article [104]. In this final section, we will describe the highlights on two important aspects of nonlinear modelling.

1. Nonlinear Model Validation

Every model is an approximation. A useful circuit model, however, must have some predictive ability independent of the circuit to which it is connected and the excitation applied. Although the solution waveforms obtained by computer simulation will seldom if ever match those actually measured in the laboratory, they must be "close" quantitatively and "similar" qualitatively. Otherwise, engineers will have no confidence in any design based on such models. In other words, regardless of the approach used in deriving the model, it must be tested for its predictive ability before being declared valid.

Unlike linear systems, the predictive ability of a nonlinear model can not be ascertained from its response to a simple set of testing signals, say steps of different amplitudes. Indeed, it is easy to contrive several distinct nonlinear circuit models which have identical responses to this set of testing signals but yield different responses to other signals.

For example, consider the following 2 discrete-time nonlinear systems with input u and output y:

System N_1: $y(n) = u^2(n) + u(n)u(n-1) - u^2(n-1)$

System N_2: $y(n) = u^2(n)$

Note that the output y of both N_1 and N_2 to step inputs of level α is just a step of level α^2! However, the responses of N_1 and N_2 to the signal $u(k) = 0$, $k < 0$, $u(1) = 1$, $u(2) = -1$, $u(k) = 0$, $k > 2$ are different. This example shows that any model which merely matches the response of a device to steps of all levels has no predictive ability--here in fact it confuses a memoryless system (N_2) with a dynamic one (N_1)!

On the other hand, every nonlinear device that has a Volterra series expansion[7]

$$
\begin{aligned}
y(t) = &\int_{-\infty}^{\infty} h_1(\tau_1)\, u(t-\tau_1)d\tau_1 \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)\, u(t-\tau_1)\, u(t-\tau_2)d\tau_1 d\tau_2 \\
&+ \cdots \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\tau_2,\cdots\tau_n)u(t-\tau_1)u(t-\tau_2)\cdots u(t-\tau_n)d\tau_1 d\tau_2\cdots d\tau_n \qquad (27)\\
&+ \cdots
\end{aligned}
$$

is associated with a unique set of (symmetric) Volterra kernels $h_1(t_1)$, $h_2(t_1,t_2)$, $\cdots$, $h_n(t_1,t_2,\cdots,t_n)$, $\cdots$ about a dc operating point.

The n-dimensional Laplace transform

$$
H_n(s_1,s_2,\cdots,s_n) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(t_1,t_2,\cdots,t_n)e^{-s_1t_1-s_2t_2\cdots-s_nt_n}dt_1dt_2\cdots dt_n \qquad (28)
$$

of $h_n(t_1,t_2,\cdots,t_n)$ is called the nth order transfer function because it has a natural circuit interpretation in the frequency domain [128-130].

For example, consider the nonlinear system shown in Fig. 25(a) where the output of the two linear filters $H_1(s)$ and $H_2(s)$ are multiplied to obtain y. The 2nd-order transfer function for this system is simply equal to the symmetrical product of $H_1(s_1)$ and $H_2(s_2)$ [129-130]. Hence, if we choose $H_1(s)$ to be a

[7]The literature on Volterra series is immense and we refer to three recent textbooks [122-124] for a good introduction. Readers interested on the mathematical foundation of Volterra series are referred to a recent series of fundamental research papers by Sandberg [125], de Figueiredo [126] and Fliess [127].

1st-order low-pass filter and $H_2(s)$ to be a 1st-order high-pass filter, then

$$H_2(s_1,s_2) = \text{SYM} \frac{\frac{s_2}{\omega_2}}{(1+\frac{s_1}{\omega_1})(1+\frac{s_2}{\omega_2})} \tag{29}$$

where $\text{SYM}\ f(s_1,s_2) \triangleq \frac{1}{2} f(s_1,s_2) + \frac{1}{2} f(s_2,s_1)$.

Similarly, if we choose $H_1(s)$ to be a 2nd-order bandpass filter and $H_2(s)$ to be a 1st-order high-pass filter, then

$$H_2(s_1,s_2) = \text{SYM} \frac{(\frac{s_1}{\omega_1})(\frac{s_2}{\omega_2})}{(1+\frac{s_1}{\omega_1}+\frac{s_1^2}{\omega_1^2})(1+\frac{s_2}{\omega_2})} \tag{30}$$

Just like the frequency response of a linear system, much physical insight is gained by plotting the magnitude of $H_2(s_1,s_2)$ in the f_1-f_2 plane, where $s_1 = j2\pi f_1$ and $s_2 = j2\pi f_2$. The 2-dimensional surface $|H_2(f_1,f_2)|$ corresponding to (29) is plotted in Fig. 25(b) with $\omega_1 = \omega_2 = 1$. That corresponding to (30) is plotted in Fig. 25(c) with $\omega_1 = \omega_2 = 1$.

Since every nonlinear device having a Volterra series expansion has a unique set of (symmetric) nth order transfer functions $H_1(s_1)$, $H_2(s_1,s_2)$, $\cdots$, $H_n(s_1,s_2,\cdots,s_n)$ about a dc operating point, it follows that a nonlinear model has predictive ability if and only if its associated nth order transfer functions are "close" to those measured from the device.

Unfortunately, up until only recently, no practical and reliable method exists for measuring the higher-order transfer functions. However, a recent breakthrough for measuring 2nd-order transfer functions has been achieved [131]. This practical and robust measurement scheme consists basically of generating a multi-tone probing signal made of a sum of many sinusoids (typically more than 15) whose frequencies are generated by a simple algorithm which guarantees that no two pairs of input frequencies give rise to the same intermodulation frequency. Since this signal must be of extremely high quality, it can not be generated by an electronic function generator. Rather, it is generated by a microcomputer and a D/A converter. Since high precision D/A's and crystal clocks are readily and cheaply available, generating these intricate signals is no problem.

To measure the 2nd-order transfer function of a nonlinear device or system, the above probing signal is applied to the device at several amplitude levels and its output sampled and collected. The resulting data is then processed in accordance with a numerically robust algorithm described in [131].

The above measurement approach has been calibrated against several nonlinear circuits (with a 2nd-order part like that shown in Figure 25(a)) whose 2nd-order transfer function can be calculated exactly. In each case, the measured transfer functions are virtually indistinguishable from those calculated by exact formulas, such as (29) and (30).

The 2nd-order transfer functions of several nonlinear devices and systems have been successfully measured by the above scheme. Figure 26(a) shows the spectrum of the probing signal applied to a high-quality pressure transducer used in our bioengineering laboratory. The spectrum of the response is shown in Fig. 26(b). The associated 2nd-order transfer function calculated automatically by a dedicated micro-computer is shown in Fig. 26(c).

2. Nonlinear Model Structure

For nonlinear devices and systems whose internal operation, mechanism, and composition are not well understood (e.g., amorphous devices [132] and biological systems [133]), it is essential to resort to a black box modelling approach. In this situation, one of the most difficult and fundamental problems is to determine what cirucit topology or system structure best describes the real device or system. In other words, from input/output measurements alone, is it possible to identify a unique model topology or structure?

The answer to the above question is clearly "no" for a linear device or system. Indeed, given any linear one-port N, there exist many equivalent one-ports having distinct circuit topologies which are not distinguishable from input/output measurements alone. For example, the Thevenin and the Norton equivalent one-ports are indistinguishable in this sense.

However, this lack of uniqueness does not carry over to nonlinear devices and systems. As it turns out, the same complexity which makes nonlinear systems difficult to represent, analyze, and design (e.g., noncommutativity, non-distributivity,···) also allows much more information about the internal structure to be extracted from input/output measurements.

This rather surprising observation is the culmination of many years of futile effort at deriving equivalent nonlinear circuit structures analogous to the many well-known results in Linear Circuit Theory. For example, much effort has gone into deriving an analogous Δ-Y transformation theorem for nonlinear circuits. The only results that have been obtained so far are applicable only to resistive nonlinear circuits [32,134]. The extremely stringent conditions imposed in [32, 134] already portend even more stringent conditions for the dynamic case. What is ironic, however, is that it took nearly 20 years for the following definitive theorem to see the light of day.

Theorem 10. Uniqueness of Model Structures [135-136]

Consider the three common system structures shown in Figs. 27(a), (b), and (c), where $F(\cdot)$ denotes a memoryless nonlinearity and the H's denote linear time-invariant systems. Suppose $F(\cdot)$ and $H_{fb}(s)$ are not constant, $H_{pre}(s)$ and $H_{post}(s)$ are not identically zero, and $H_{fb}(\infty) \triangleq \lim_{s\to\infty} H_{fb}(s) = 0$. Then two such systems each with structure (a), (b), or (c) are equivalent (i.e., have identical input/output measurements) if and only if:

1. they have the same structure.
2. the corresponding $F(\cdot)$ and $H'(s)$ are related by scaling, and possibly shuttling some delay between $H_{pre}(s)$ and $H_{post}(s)$.

Remarks

1. Although conjectured nearly 3 years ago, we did not succeed in proving this theorem until only very recently. The complete proof is given in [136].
2. In its present form only one scalar nonlinearity is allowed although we strongly believe similar results hold also for the multiple nonlinearity case. Even then, this theorem already has profound consequences, some of which are stated in the following corollaries.

Corollary 1 [136]

If two dynamic one-ports containing a 2-terminal nonlinear resistor are equivalent, then their respective resistor v-i characteristics, as well as their associated linear 2-ports, are "similar" in the sense that they can differ by at most a scaling and possibly a delay transformation.[8]

Corollary 2 [135]

Every algebraic 2-terminal element defined by a nonlinear relation

$$f(v^{(\alpha)}, i^{(\beta)}) = 0 \tag{31}$$

between $v^{(\alpha)}$ and $i^{(\beta)}$ (as defined earlier in Section 4, Part II) is unique in the sense that its order (α,β) and its characteristic curve $f(x,y) = 0$ are unique, that is, such elements have only one description as algebraic elements.[9]

[8] See [136] for the explicit transformation formulas.

[9] Corollary 2 implies that whereas only 3 basic linear 2-terminal circuit elements (R,L,C) are adequate in linear circuit theory, an infinite number of distinct 2-terminal nonlinear algebraic elements [104] must be defined in a general theory of nonlinear circuits.

Corollary 3 [135]

Given any system N having at least two non-zero Volterra kernels, the only linear time-invariant systems which commute with N are delays (or delays and negation if N has only odd-order Volterra kernels).

Corollary 4 [135]

The two common model structures shown in Figs. 29(a) and (b) can never be equivalent.

We close this paper on a lighter note by giving a mundane application of theorem 10.

Consider a communications system consisting of N cable-repeater sections (Fig. 30(a)), each with frequency response R(s) (Fig. 30(b)). Suppose the output stage of the kth repeater drifts off bias and starts distorting slightly.

To locate the faulty repeater we could tap into the system at various points, possibly at great trouble. But a much simpler mehtod is to recognize the system in this case can be modelled by the structure shown in Fig. 27(a). If we let $f(\cdot)$ denote the nonlinear output stage, then the faulted system can be described by $R^{N-k}f(\cdot)R^k$. It follows from Theorem 10 that from input/output measurements alone (of the whole system), we can locate the faulty repeater. It is remarkable to note that this method would not work for a linear fault: suppose an element in the kth repeater amplifier drifts in such a way as to say, halve the bandwidth of the repeater. The kth repeater in this case is still linear but with frequency response $\tilde{R}(s)$. Since the system's linear (and only) frequency response is $R(s)^{N-1}\tilde{R}(s)$, no matter where the fault is, no input/output measurements alone can locate this fault.

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Figure Captions

Fig. 1. This pn-junction diode circuit has no solution if $I > I_S$.

Fig. 2. This tunnel diode circuit has 3 solutions if $E_1 < E < E_2$.

Fig. 3. Of the 2 v-i curves, only (b) is eventually passive.

Fig. 4. A feedback structure.

Fig. 5. Open and/or short-circuit structure used to replace transistors.

Fig. 6. A transistor circuit containing a feedback structure.

Fig. 7. A transistor circuit which does not contain a feedback structure.

Fig. 8. (a) Circuit containing a CCCS (b) Equivalent circuit (c) Load Line method yields two operating points.

Fig. 9. Directed graph associated with a CCCS. Note the output branch is directed opposite to the arrowhead inside the diamond-shape symbol.

Fig. 10. Example of a cactus graph.

Fig. 11. A circuit having an even (zero in this case) number of similarly-directed loops in the reduced cactus graph.

Fig. 12. A circuit containing a reduced cactus graph (e) containing an even (zero) number of similarly-directed loops.

Fig. 13. Illustration of the no-gain criterion for a 2-terminal resistor.

Fig. 14. Bounding region for v_0 versus v_{in} transfer characteristic.

Fig. 15. (a) A voltage-controlled v_1-i_1 curve 1 in series with a current-controlled v_2-i_2 curve 2 gives rise to a multivalued v-i curve. (b) A voltage-controlled v_1-i_1 curve 1 in series with another voltage-controlled v_2-i_2 curve 2 gives rise to 2 disconnected branches in the resulting v-i curve.

Fig. 16. A nonlinear RLC circuit having 3 distinct (subharmonic) steady state responses.

Fig. 17. A circuit having infinitely many distinct dc steady state solutions.

Fig. 18. Two capacitor-diode circuits satisfying the 4 topological conditions in Theorem 7.

Fig. 19. (a) Single loop feedback system (b) The intersection of the Nyquist loci Γ with the vector drawn through -1+j0 having the direction $\xi(\omega_R)$ gives the oscillation frequency.

Fig. 20. A tunnel diode circuit and its associated Nyquist loci for various values of $\mu \triangleq V_B$ drawn with $L = 1$ and $C = 1$.

Fig. 21. (a) Symbol and v-i curve defining a concave resistor: they are uniquely specified by two parameters, the slope G (in V) and the voltage intercept E. (b) Symbol and v-i curve defining a convex resistor: they are uniquely specified by two parameters, the reciprocal slope R (in Ω) and the current intercept I.

Fig. 22. (a) A type-N voltage-controlled characteristic. (b) A type-S current-controlled characteristic.

Fig. 23. (a) An X-Y Mutator transforms a type Y 2-terminal element into a type X 2-terminal element. (b) Oscilloscope tracings of the v-i characteristic of a nonlinear resistor on the right and an identical ϕ-i characteristic on the left. (c) Oscilloscope tracings of the "hysteretic" ϕ-i characteristic on the left and an identical v-i characteristic on the right.

Fig. 24. Decomposition of a 2-port N into two 2-ports N_1 and N_2 in parallel.

Fig. 25. (a) Simple nonlinear system made of two linear filters and a multiplier. (b) Surface representing $|H_2(f_1,f_2)|$ when $H_1(s)$ is a 1st-order low pass filter and $H_2(s)$ is a 1st-order high-pass filter. (c) Surface representing $|H_2(f_1,f_2)|$ when $H_1(s)$ is a 2nd-order band pass filter and $H_2(s)$ is a 1st-order high-pass filter.

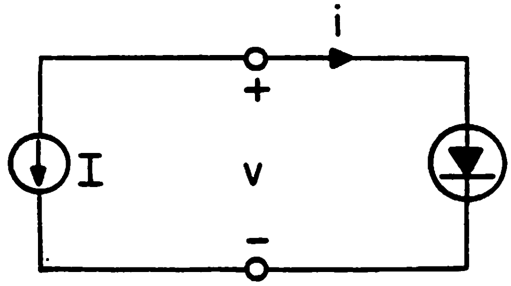
Fig. 26. (a) Spectrum of probing signal to pressure transducer. (b) Spectrum of output signal from pressure transducer. (c) 2nd-order transfer function of pressure transducer.

Fig. 27. Three common model structures: $F(\cdot)$ denotes a scalar nonlinear function, $H_{pre}(s)$, $H_{post}(s)$ and $H_{fb}(s)$ are linear time-invariant transfer functions (usually referred to as the pre-filter, post-filter, and feedback filter respectively).

Fig. 28. Every dynamic one-port whose only nonlinear element is a 2-terminal nonlinear resistor R is equivalent to a linear 2-port terminated in a nonlinear resistor R_{eq}, where R and R_{eq} are related by a scaling transformation

Fig. 29. These two model structures are completely disjoint.

Fig. 30. (a) Section consisting of a cable and a repeater (b) A long communication chain of N cable-repeaters, each section modelled by a linear transfer function R(s). (c) A nonlinear fault at the nth repeater reduces the linear system to a nonlinear one having the structure shown in Fig. 27(a). (d) Corresponding situation but with a linear fault.



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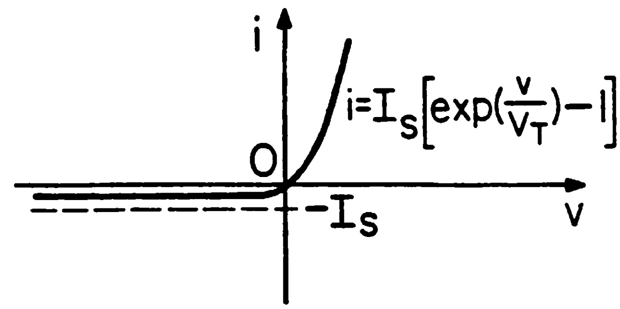
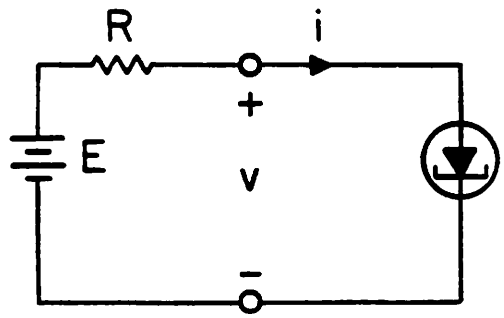


(b)

Fig. 1



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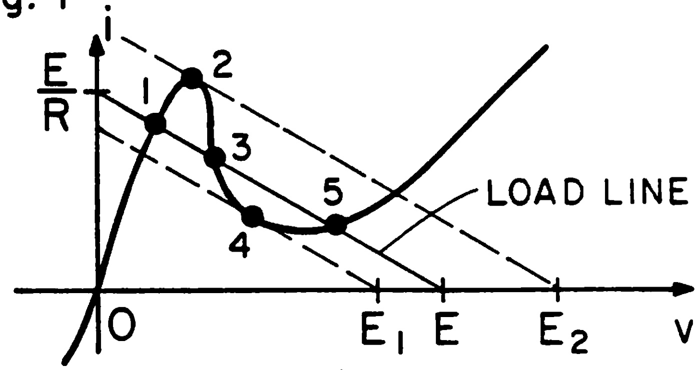
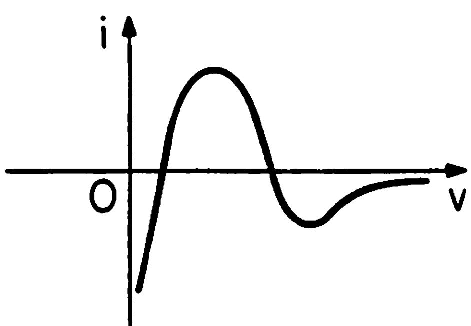


(b)

Fig. 2



(a)

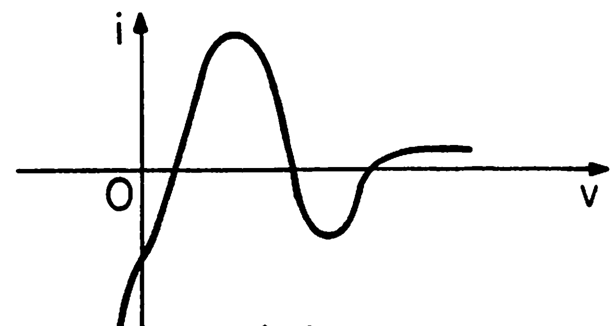


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Fig. 3

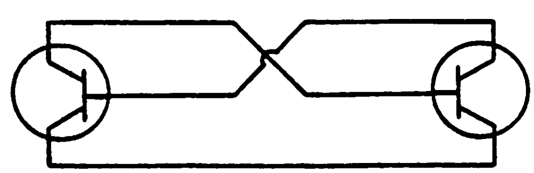
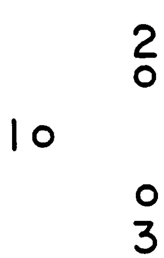
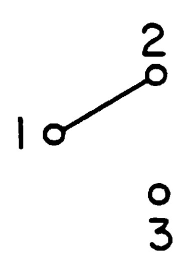


Fig. 4



(a)



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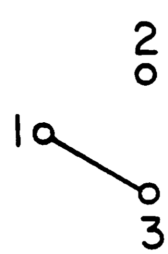
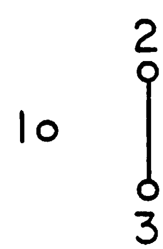


(c)



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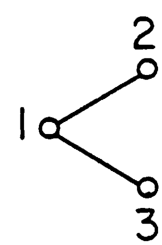
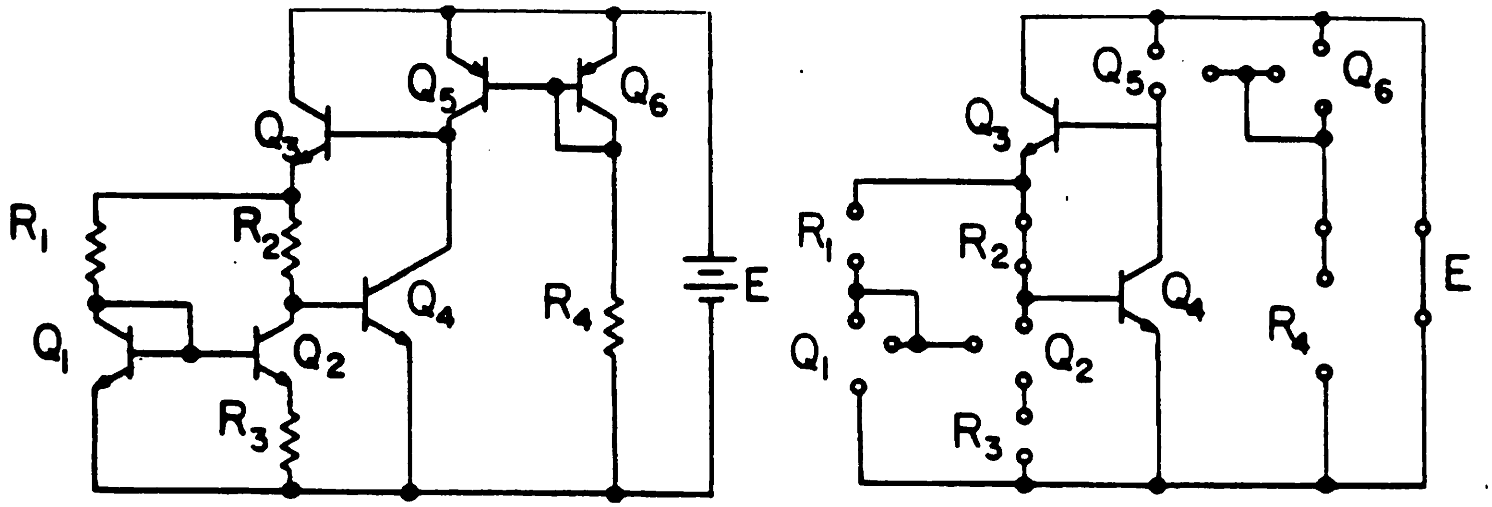


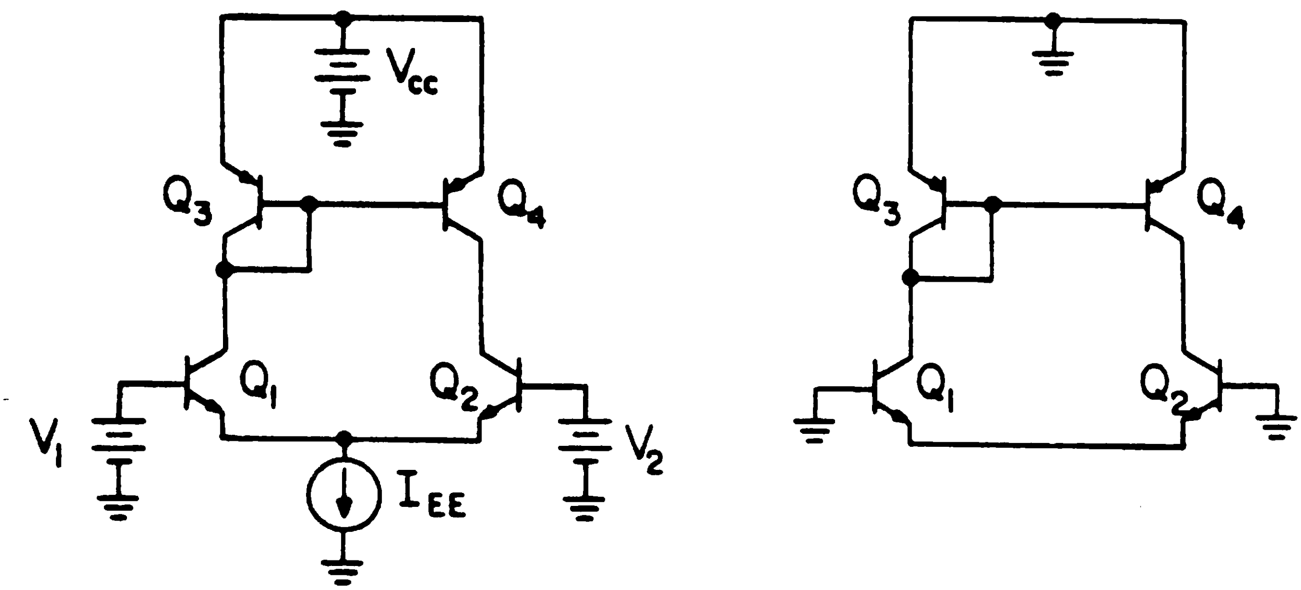
(e)

Fig. 5



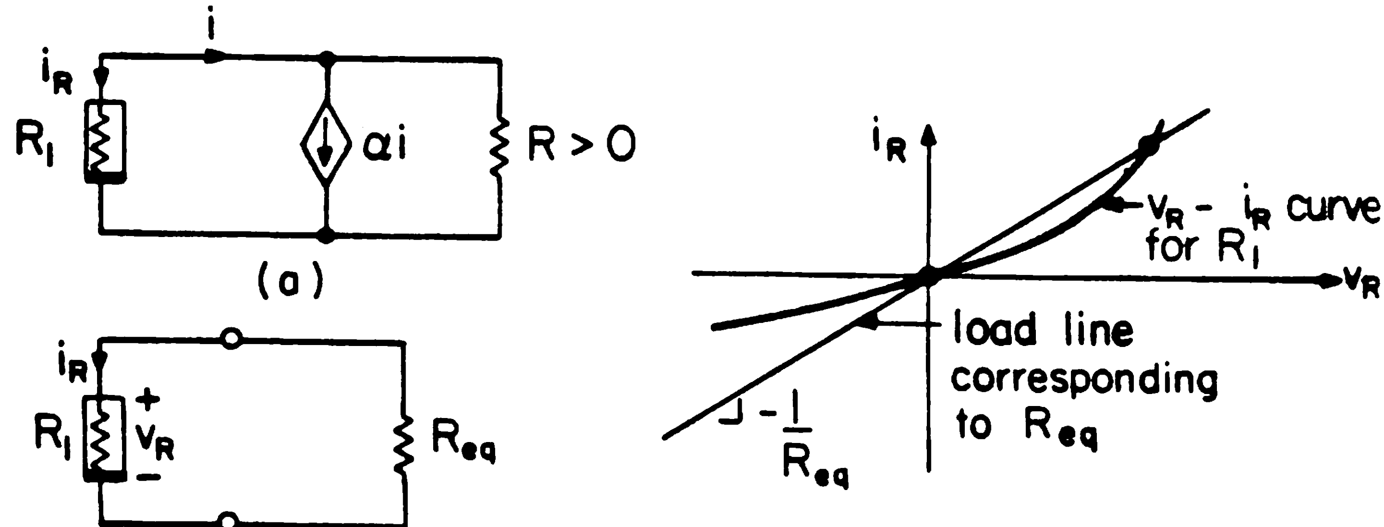
(a) (b)

Fig. 6



(a) (b)

Fig. 7



(b) Fig. 8 (c)

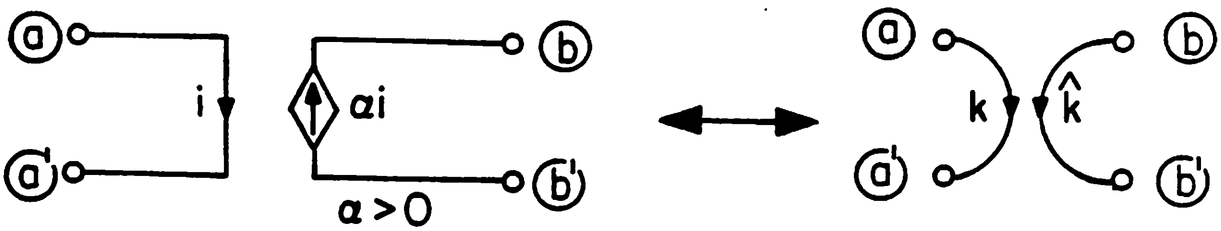


Fig. 9

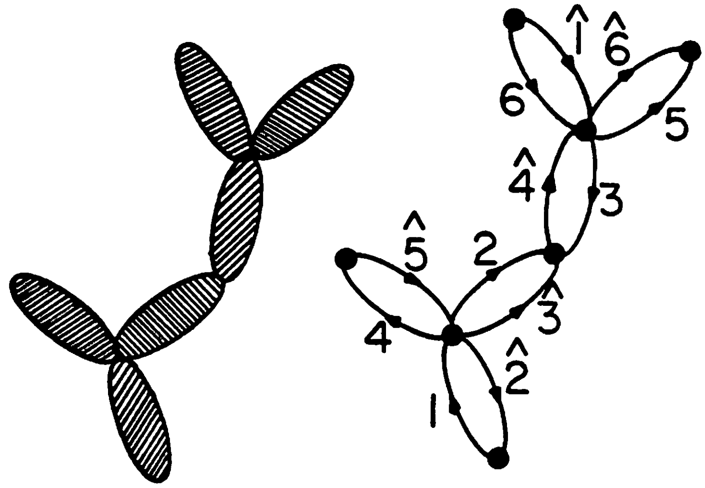


Fig. 10

(a) (b)

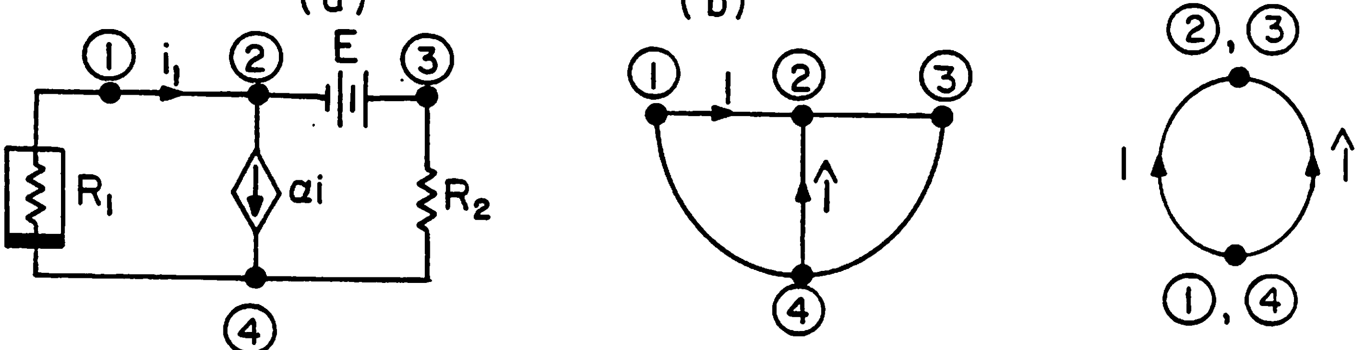


Fig. 11

(a) (b) (c)

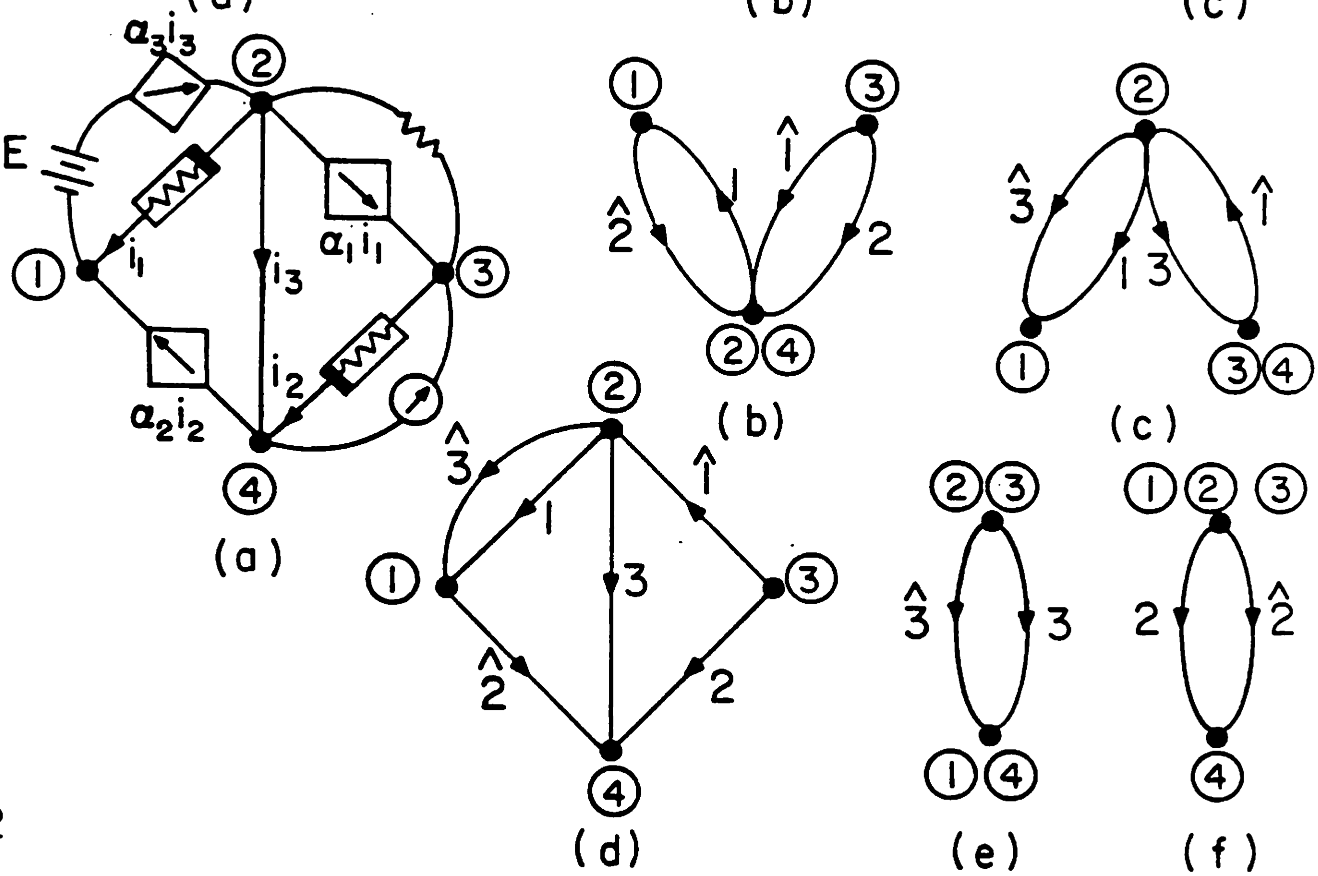


Fig. 12

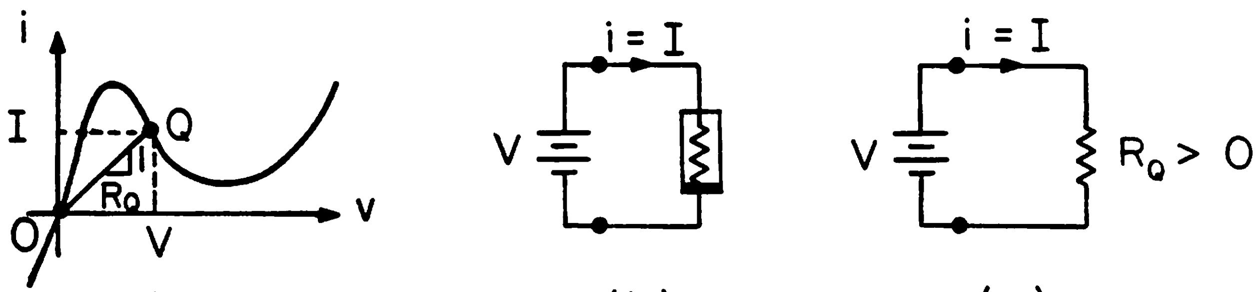


Fig.13 (a) (b) (c)

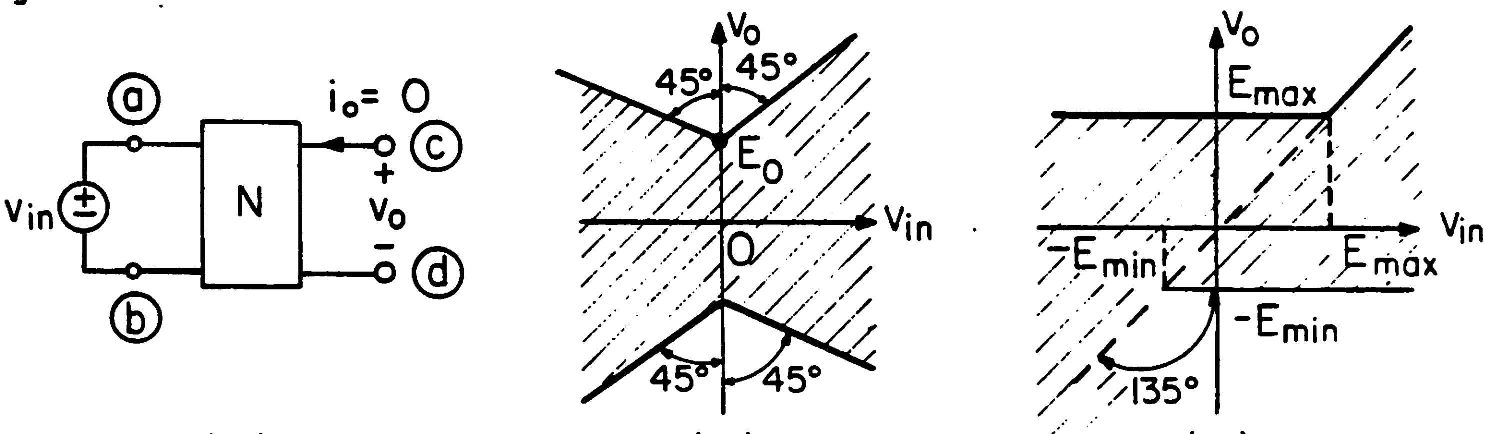


Fig.14 (a) (b) (c)

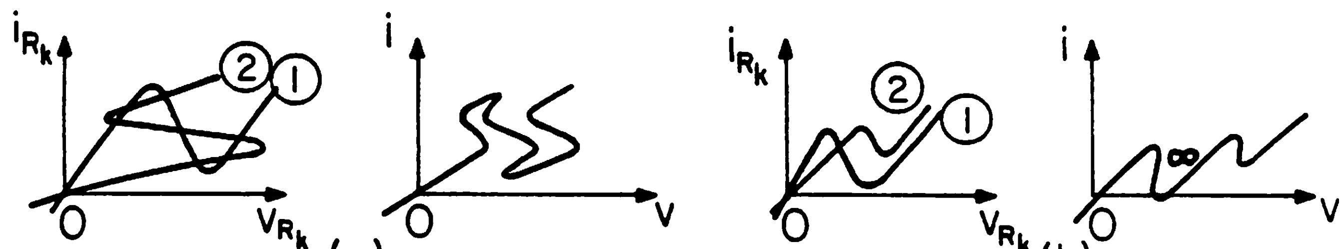


Fig.15 (a) (b)

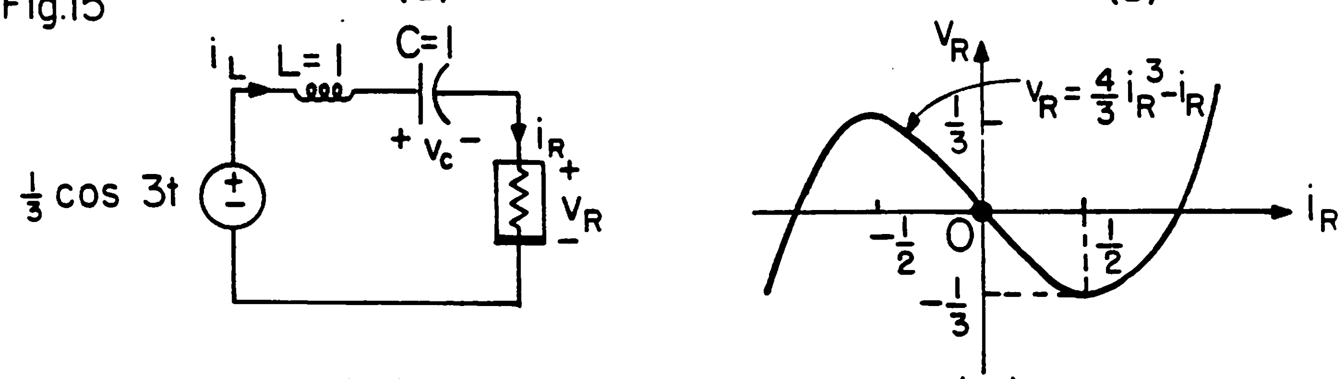


Fig.16 (a) (b)

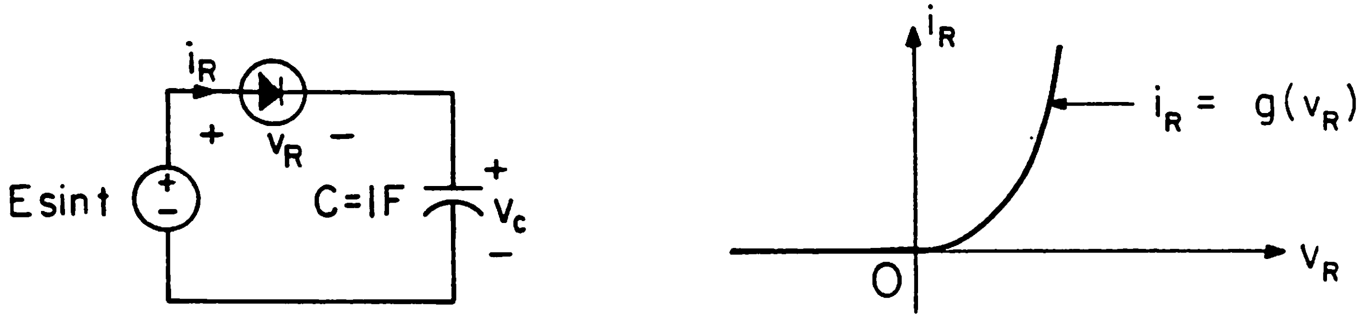


Fig.17 (a) (b)

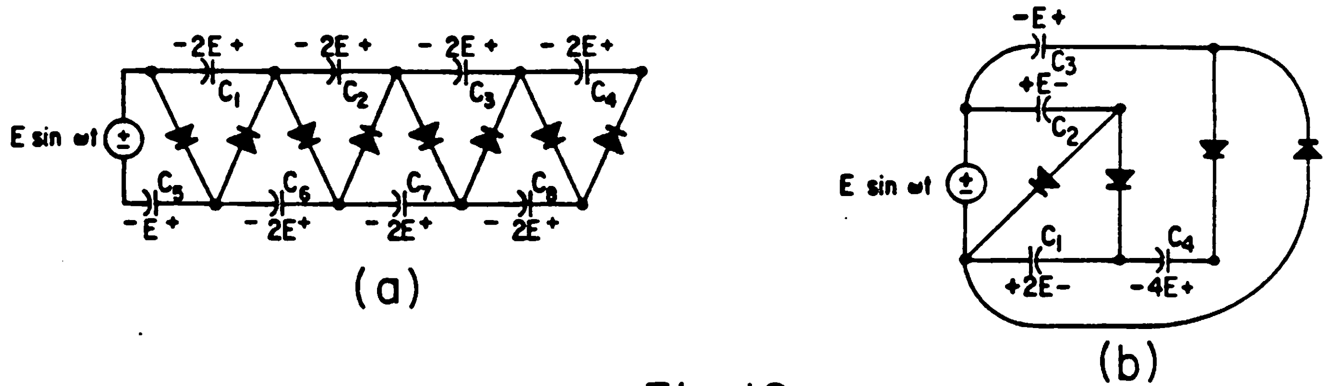
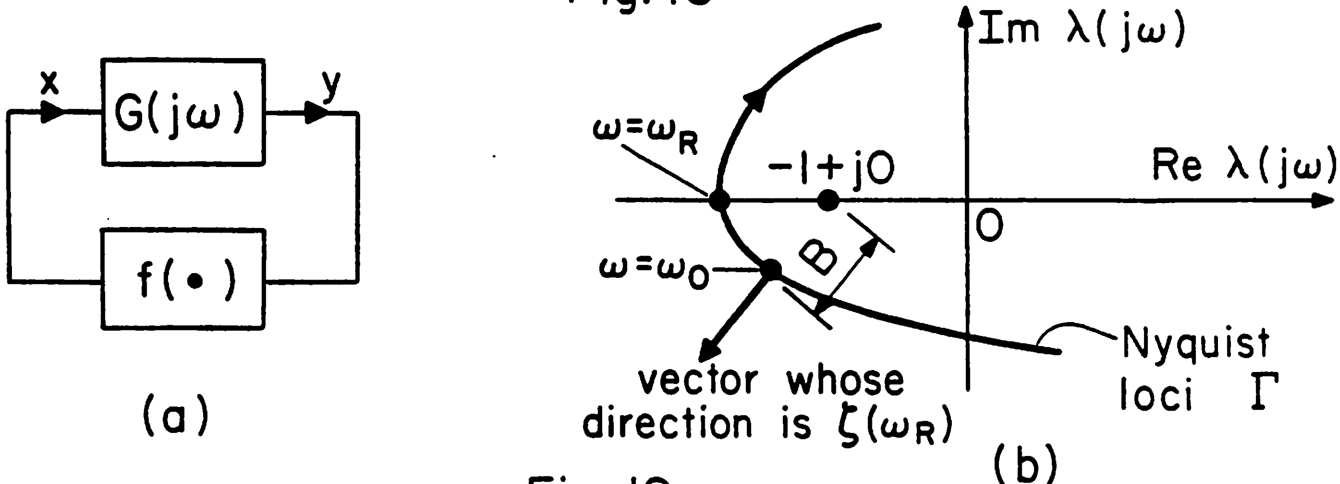


Fig. 18



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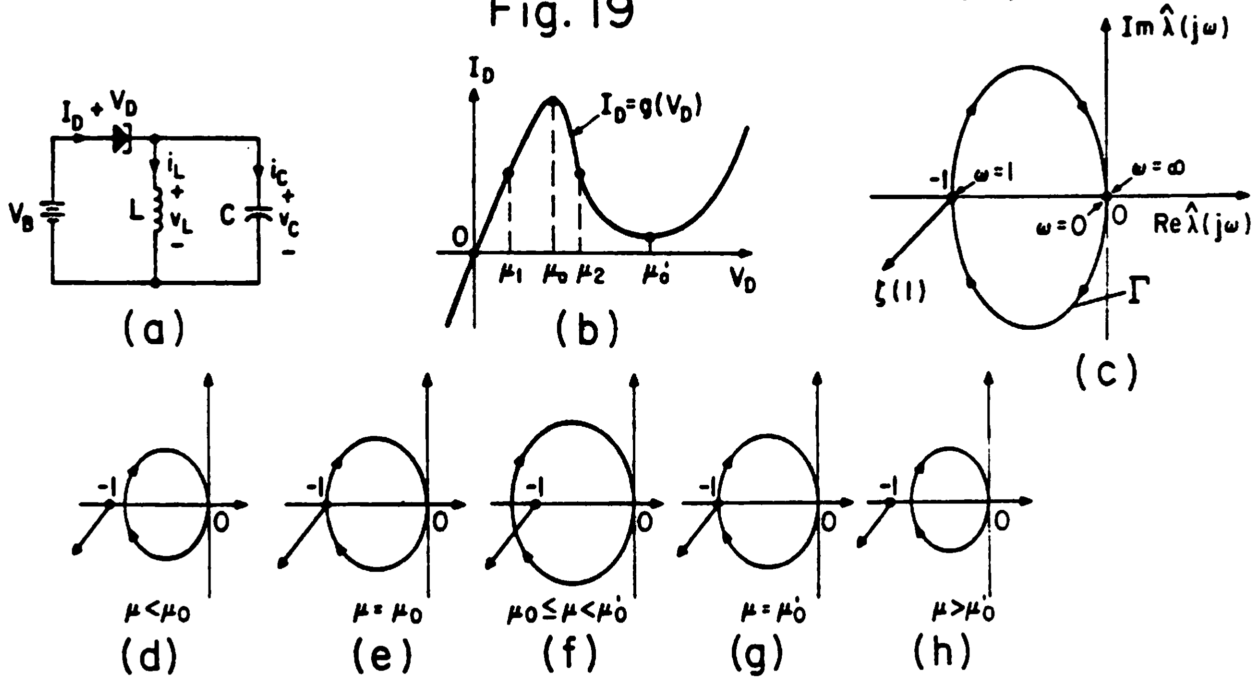


Fig. 20

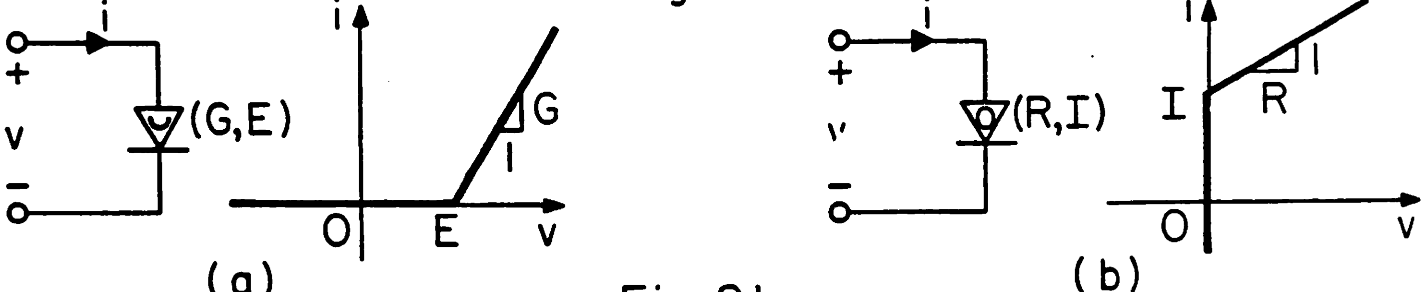


Fig. 21

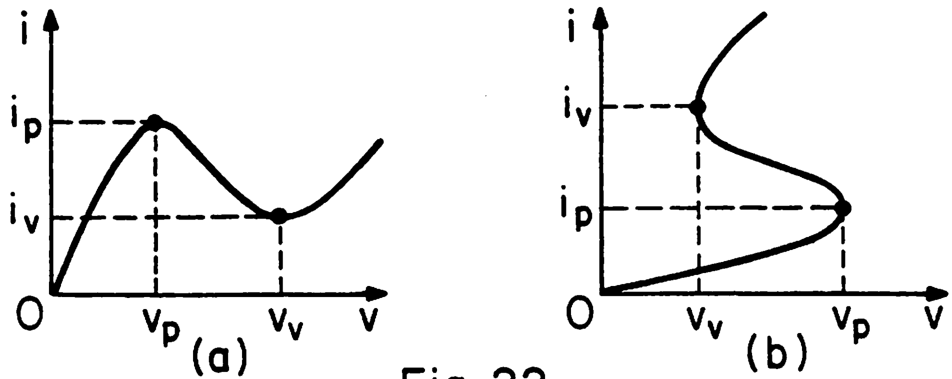


Fig. 22

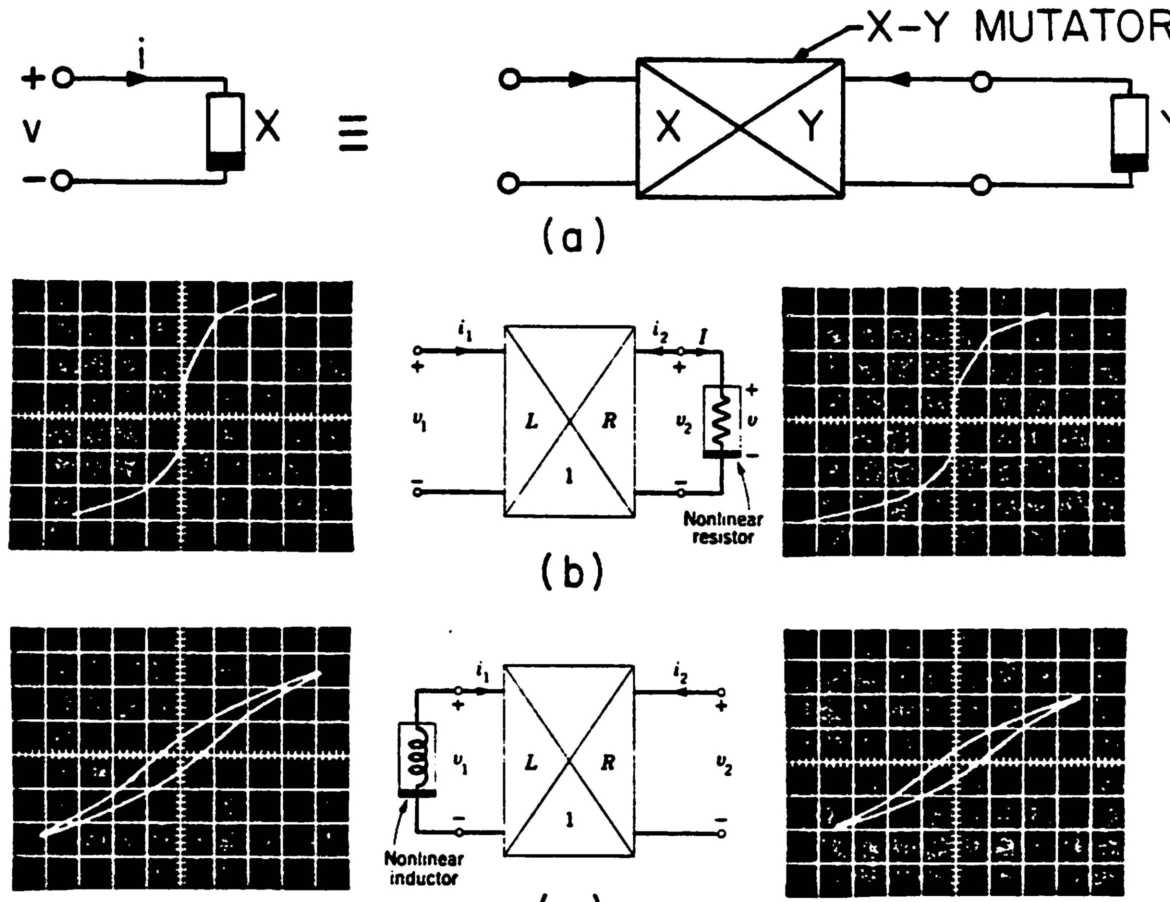


Fig. 23

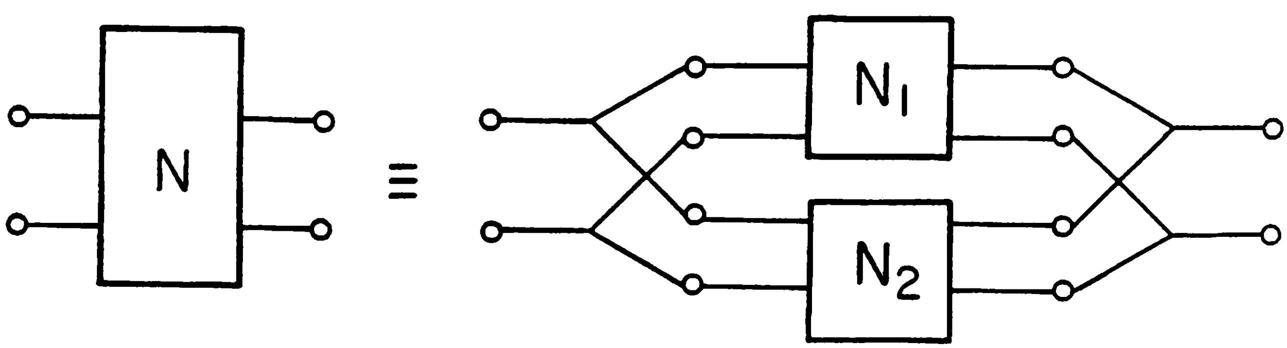
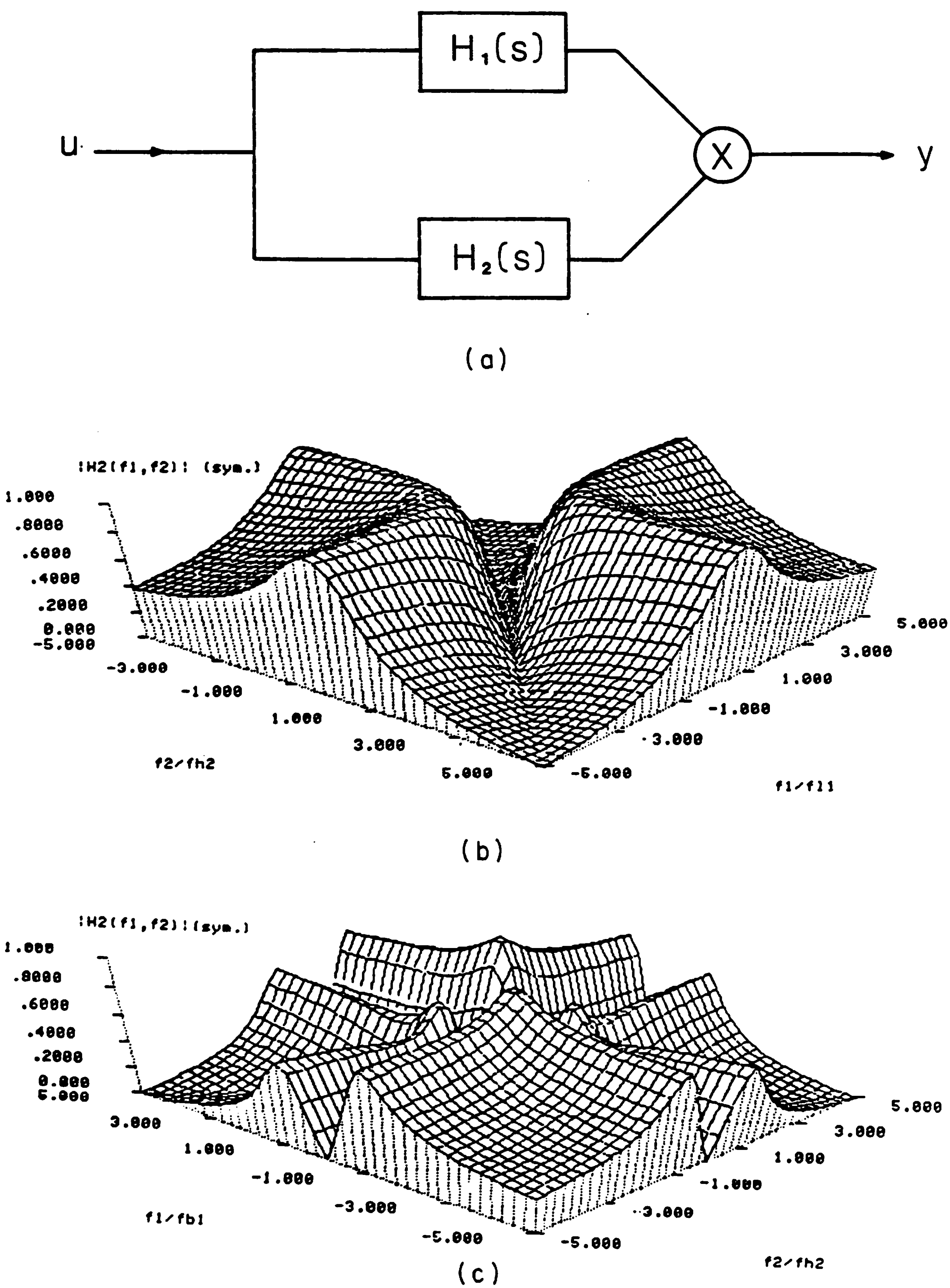


Fig. 24



H1(s)
H2(s)
u
X
y
(a)
|H2(f1,f2)| (sym.)
1.000
.8000
.6000
.4000
.2000
0.000
-5.000
-3.000
-1.000
1.000
3.000
5.000
f2/fh2
-5.000
-3.000
-1.000
1.000
3.000
5.000
f1/fl1
(b)
|H2(f1,f2)|(sym.)
1.000
.8000
.6000
.4000
.2000
0.000
5.000
3.000
1.000
-1.000
-3.000
-5.000
f1/fb1
-5.000
-3.000
-1.000
1.000
3.000
5.000
f2/fh2
(c)

Fig. 25

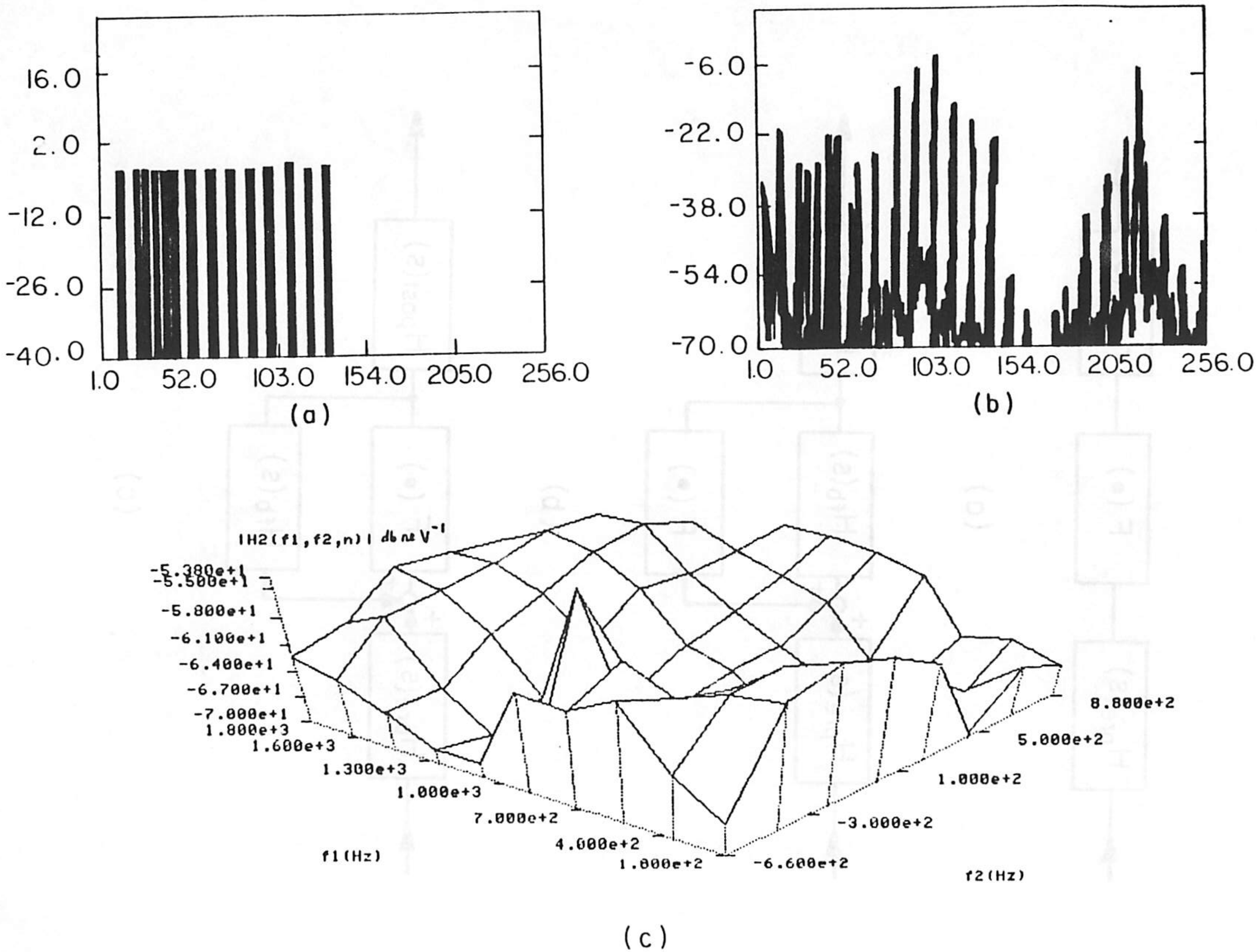
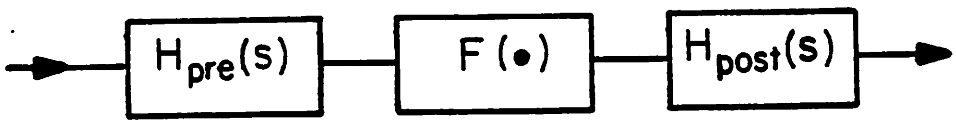
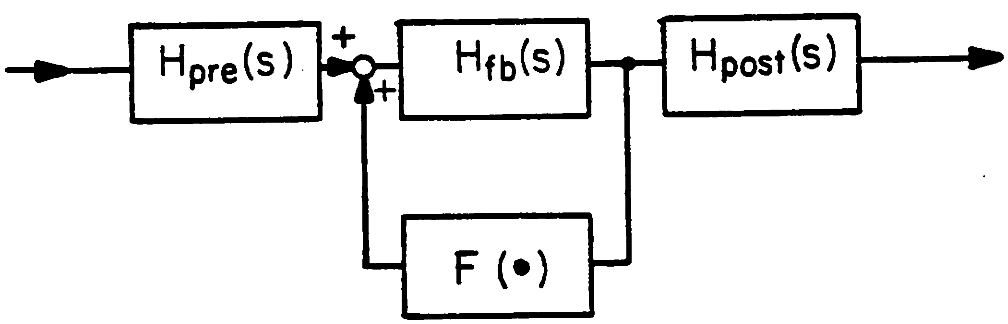

16.0
2.0
-12.0
-26.0
-40.0
1.0
52.0
103.0
154.0
205.0
256.0
(a)
-6.0
-22.0
-38.0
-54.0
-70.0
1.0
52.0
103.0
154.0
205.0
256.0
(b)
|H2(f1,f2,n)| db nt V^-1
-5.380e+1
-5.500e+1
-5.800e+1
-6.100e+1
-6.400e+1
-6.700e+1
-7.000e+1
1.800e+3
1.600e+3
1.300e+3
1.000e+3
7.000e+2
4.000e+2
1.000e+2
f1(Hz)
-6.600e+2
-3.000e+2
1.000e+2
5.000e+2
8.800e+2
f2(Hz)
(c)

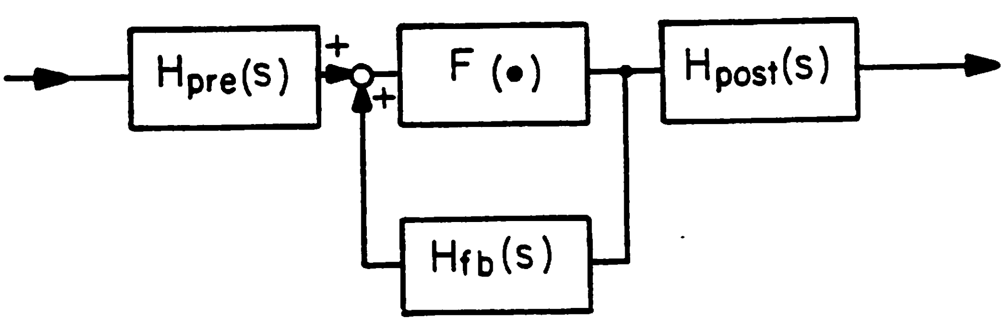
Fig. 26


$H_{pre}(s)$
$F(\bullet)$
$H_{post}(s)$

(a)

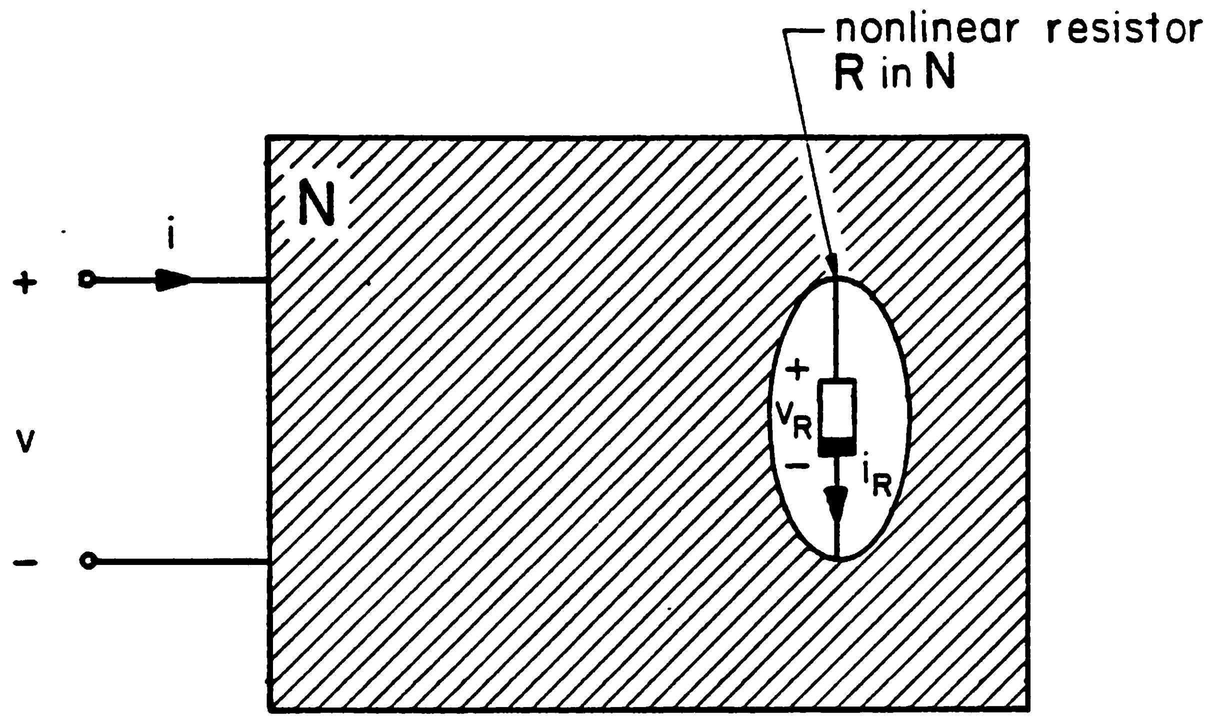

$H_{pre}(s)$
+
+
$H_{fb}(s)$
$H_{post}(s)$
$F(\bullet)$

(b)

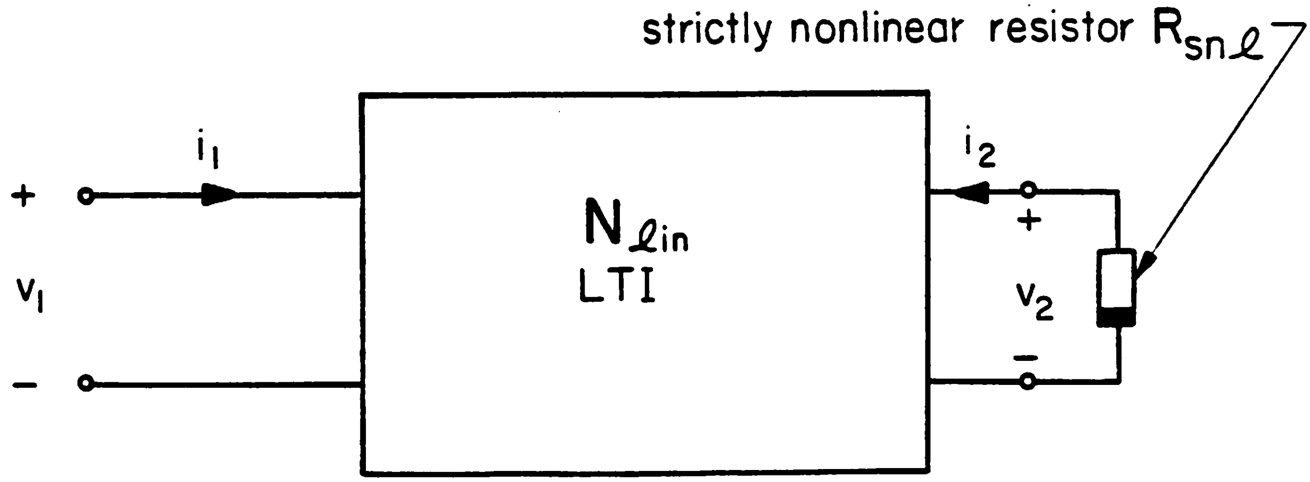

$H_{pre}(s)$
+
+
$F(\bullet)$
$H_{post}(s)$
$H_{fb}(s)$

(c)

Fig.27


nonlinear resistor
R in N
N
i
+
v
−
+
v_R
−
i_R

(a)


strictly nonlinear resistor $R_{sn\ell}$
i_1
+
v_1
−
$N_{\ell in}$
LTI
i_2
+
v_2
−

(b)

Fig. 28

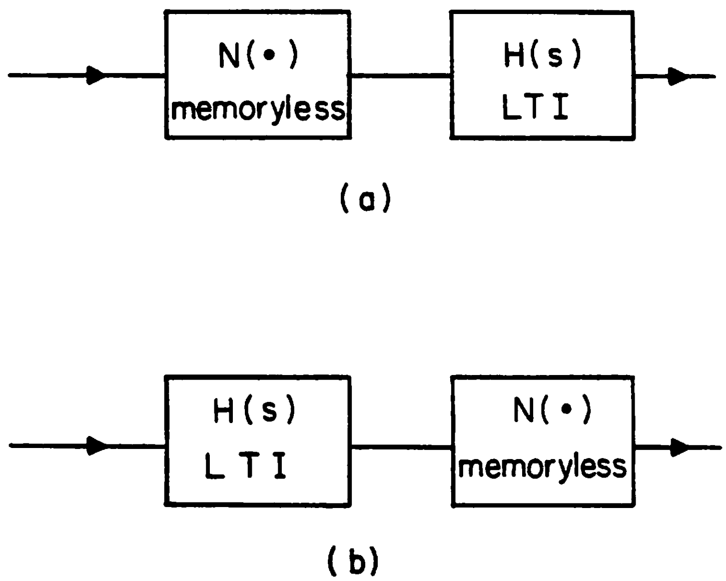

N(•)
memoryless
H(s)
LTI
(a)
H(s)
LTI
N(•)
memoryless
(b)

Fig. 29

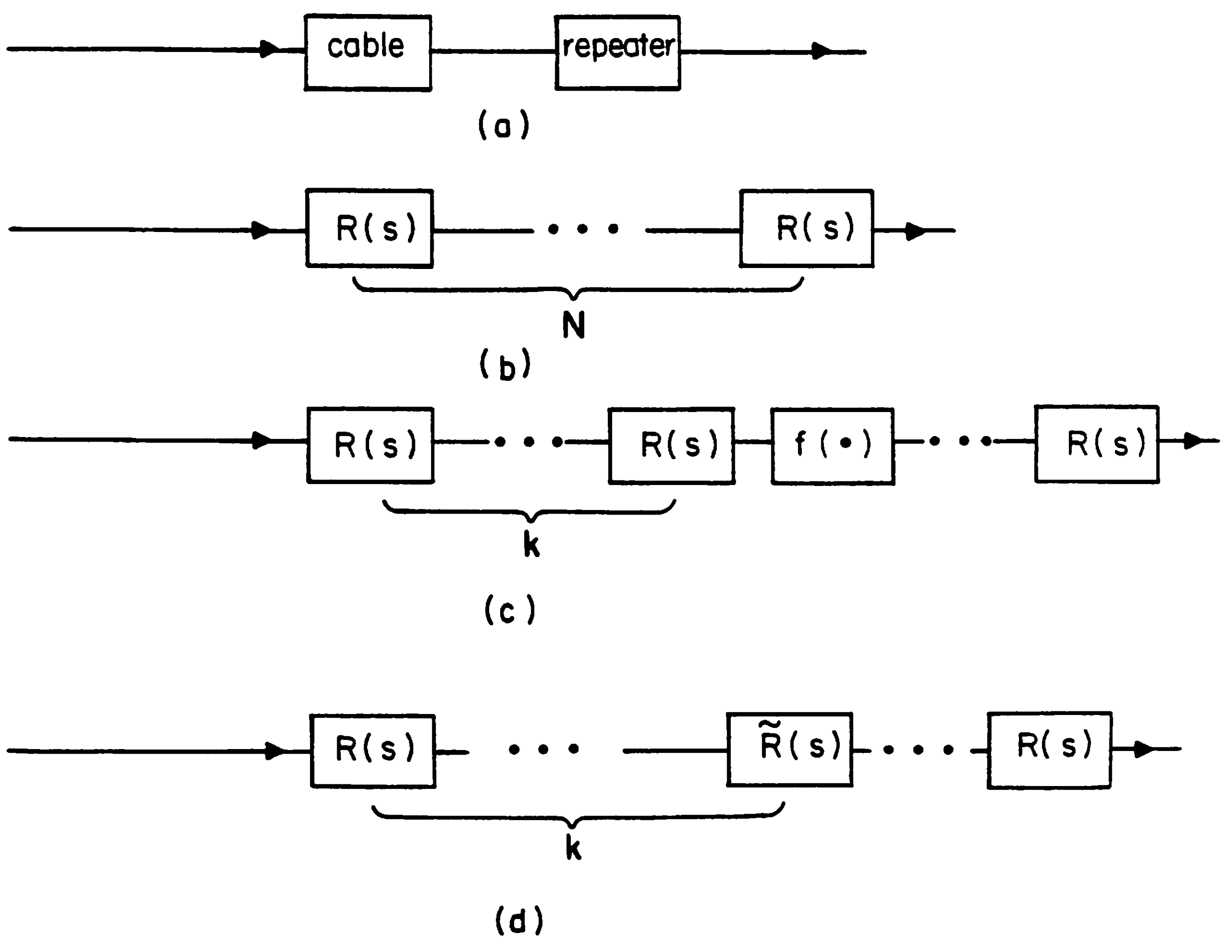

cable
repeater
(a)
R(s)
R(s)
N
(b)
R(s)
R(s)
f(•)
R(s)
k
(c)
R(s)
R̃(s)
R(s)
k
(d)

Fig. 30