THEORY OF ASYMMETRIC DOUBLE LAYERS

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ABSTRACT

We present *analytic* solutions for asymmetric double layers which satisfy the time stationary Vlasov-Poisson system and which require the double-valuedness of Sagdeev potential as a function of physical potential: it is pointed out that any distribution function having an *analytic* density representation as a polynomial power series of potential can never satisfy the asymmetric double layer boundary conditions. Considering K-dV like equation, it is found that there is some relationship between the speed of asymmetric double layer and the degree of asymmetry.

A *monotonic* double layer is simply an isolated pair of oppositely charged sheets which result in a narrow region of abrupt potential jump of some amplitude $\Delta\phi = \psi$; well outside of this localized jump, the potential is effectively uniform[1-20]. An *asymmetric* double layer refers actually to a localized region of *three* sheets of alternating charge sign, having an asymmetric potential jump profile and thus including sub regions of oppositely directed asymmetric electric fields. Asymmetric double layers were first reported in recent numerical simulations and in space observations on auroral field lines[9-11]; subsequent theoretical studies have numerically examined the time stationary Vlasov-Poisson system[7,8,12]. It has been suggested that small amplitude double layers may account for a large portion of the total potential along auroral field lines and may also explain the fine structure of auroral kilometric radiation: recent satellite measurements[11] are especially consistent with the asymmetric double layer, its potential depression (or the potential hump) at the low potential side (or high potential side). Recent studies of the thermal barrier in tandem mirror devices have also found that there exist states with abrupt potential depressions as a result of forced changes in the distribution functions[7]. The negative potential depression is thought to play a crucial role in the formation of double layers and to be responsible both for current disruptions (by reflecting the electrons) and also for high frequency noise excitation behind the double layer (by a two stream instability involving electrons that pass the negative potential peak[8-11]).

Although there have been many theoretical and experimental investigations of double layers, recent theoretical work has been limited to only numerical evaluations of the Vlasov-Poisson system (or of the fluid system) mainly because of the highly nonlinear properties of asymmetric double layers. One such recent numerical calculation[12] suggested that there may be a low amplitude limit for the monotonic double layer. In a previous paper[20], we gave analytic evidence for the existence of small amplitude monotonic double layers, which are the analytic extensions of the well known electron solitary hole and ion acoustic solitary hole[21,22]. In this report, we present the self-consistent *analytic* solution for *asymmetric* double layers, which satisfy the time stationary Vlasov-Poisson system. We present a derivation of a K-dV like equation describing a moving asymmetric double layer. It is shown that asymmetric double layer has a some relationship between the asymmetry parameter and the speed of double layer.

To describe propagation of an electrostatic double layer, we use a Vlasov-Poisson system that has been Galilean-transformed to the wave frame (where the wave is time stationary). In this frame, we can express the time stationary solution of Vlasov equation as any function of the constants of motion: (i) particle total energy and (ii) the sign of the velocity of the untrapped particles[23]. Besides the above usual constants of motion, it is important to realized that a third constant of motion exists for the reflected particles: $sgn\ (x - x_m)$ where x_m represents the position of potential minimum (or maximum) for the negatively charged particles (or the positively charged particles). It turns out that this final constant of motion plays an important role in order to construct the asymmetric double layers. With the above distribution function(any function of the three constantsof motion) , we can develop a general theory of double layers by expanding the corresponding density in half-integer powers of $\pm\phi$. However, in this paper we would like to give a physically more transparent description of our double layer problem, so we will proceed by using specific distribution functions.

In order to describe the asymmetric double layer accompanying a potential depression at the low potential side, we assume a Maxwell-Boltzmann ion distribution, $f_i = \frac{\tilde{n}_i \sqrt{\tau}}{\sqrt{2\pi}} e^{-\frac{\tau}{2}(v^2 + 2\phi)}$, and we consider the following electron distribution function[24]:

$$f_e = (2\pi)^{-1/2} \Big\{ \exp\{-1/2(sgn\ v\ \epsilon^{1/2} - v_d)^2\}\ \Theta(\epsilon) + (1 + f\ sgn(x - x_m))\exp\{-1/2(v_d^2 + \beta\ \epsilon)\}\ \Theta(-\epsilon) \Big\} \tag{1}$$

where $\tau = T_e/T_i$ and $\epsilon = v^2 - 2\phi$ for $0 \leqslant \phi \leqslant \psi$. Here the electron velocity, the wave potential and the spatial coordinates are normalized to the electron thermal velocity $(T_e/m_e)^{1/2}$, the electron temperature T_e/e and the electron Debye length $\lambda_e = (T_e/4\pi n_0 e^2)^{1/2}$, respectively; v_d represents the electrons drift velocity. The ion velocity has been normalized by the ion acoustic velocity. Here $\tilde{n}_i$ and f are constants and β represents an inverse temperature. Here $-1 \leqslant f \leqslant 1$ in order to obtain *positive* trapped particle distribution function.

With these distribution functions f_i and f_e, the Poisson equation may be written in the small amplitude limit by introducing a Sagdeev potential $V(\phi)$:

$$\phi_{xx} = -\frac{dV(\phi)}{d\phi} \tag{2}$$

$$\begin{aligned}\phi_{xx} = {} & \{1 - \tilde{n}_i\} + \frac{2\,e^{-E_d}}{\pi^{1/2}} f\ sgn(x - x_m)\,\phi^{1/2} + \{\tilde{n}_i\tau - 1/2 Z'_r(\sqrt{E_d})\}\phi \\ & + \frac{4e^{-E_d}}{3\pi^{1/2}}\{2E_d + (1 + f sgn(x - x_m))\beta - 1\}\phi^{3/2} \\ & + \{1/2(1 - \tilde{n}_i \tau^2 e^{E_d}) + \frac{3e^{E_d}}{8E_d{}^2}\}e^{-E_d}\phi^2 \\ & + \{\text{higher order terms}\}\end{aligned} \tag{3}$$

where E_d is the electron drift energy and $Z'_r(x)$ represents the real part of the derivative of the complex plasma dispersion function(see Fig. 1. (a)). The double layer solution may be found by considering the following nonlinear eigenvalue conditions:

- Charge neutrality at $x = \pm\infty$ requires that the *rhs* of Eq.(3) should vanish at the boundaries $\phi = \psi_1, \psi_2$.
- Existence of the asymmetric double layer requires that the Sagdeev potential be identically zero at $\phi = 0, \psi_1, \psi_2$, so that the electric field equals zero at those values of potential.
- An additional condition on the Sagdeev potential{see Fig. 2. (a)} requires that $V(\phi) < 0$ for $0 < \phi < \psi_1$, $0 < \phi < \psi_2$.

Here it is important to note that the Sagdeev potential should be a *double* valued function of ϕ for $0 < \phi \leqslant \min(\psi_1, \psi_2)$ in order to yield the asymmetric double layer solution, while the existence of the monotonic double layer and the solitary hole does not require the double valuedness of Sagdeev potential. Furthermore, it should be noted that from the double valuedness of Sagdeev potential(as a function of ϕ) for the existence of the asymmetric double layer, we can make a following statement: any distribution function having analytic density representation as any polynomial power series of potential ($n = \int_{-\infty}^{+\infty} f\, dv = \sum_n C_n\, (\pm\phi)^n$ with $C_n = constant$) can never satisfy the asymmetric double layer boundary conditions. In our case, double-valuedness of Sagdeev potential is guaranted by the use of the constant of motion for reflected particles.

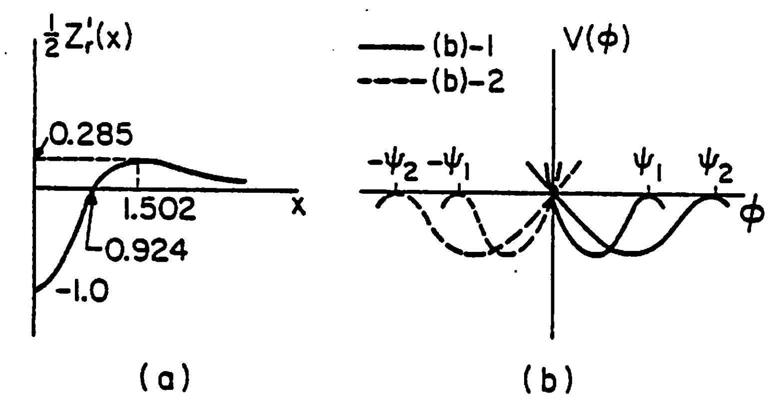

FIG. 1. (a) The real part of derivative of the complex plasma dispersion function: $\frac{1}{2}Z'_r(x)$. (b) The Sagdeev potential for the asymmetric double layer.

By solving Eqs.(2) and (3) subject to these conditions, we get the following double layer solution:

$$\phi = \bar{\psi}\{\, a_s + \tanh \pm|\kappa_i|\, x\,\}^2 \tag{4}$$

where for convenience consider only $\psi_2 > \psi_1$ and we have defined

$$\bar{\psi} = \frac{(\sqrt{\psi_1}+\sqrt{\psi_2})^2}{4}\,, \quad a_s = \frac{\sqrt{\psi_2}-\sqrt{\psi_1}}{\sqrt{\psi_1}+\sqrt{\psi_2}}\,, \quad \kappa_i^{\,2} = \frac{\nu\bar{\psi}}{6} > 0\,. \tag{5}$$

Here $\bar{\psi}$, a_s, κ_i are related to our system parameters as follows:

$$1-\bar{n}_i = \frac{1}{3}\psi_1\psi_2\,\nu\,, \tag{6}$$

$$\frac{2}{\sqrt{\pi}} f\, e^{-E_d} = \pm\sqrt{\psi_1}\sqrt{\psi_2}(\sqrt{\psi_1}-\sqrt{\psi_2})\,\nu\,, \tag{7}$$

$$\{\, \bar{n}_i\tau - \frac{1}{2}Z'_r(\sqrt{E_d})\,\} = \frac{2}{3}(\psi_1+\psi_2-4\sqrt{\psi_1}\sqrt{\psi_2})\,\nu\,, \tag{8}$$

$$\frac{4e^{-E_d}}{3\sqrt{\pi}} f\,\beta = \mp\frac{1}{6}(\sqrt{\psi_1}-\sqrt{\psi_2})\,\nu\,, \tag{9}$$

$$\nu = \{\, \frac{1}{2}(\,1-\bar{n}_i\,\tau^2\,e^{E^d}\,) + \frac{3e^{E_d}}{8E_d^{\,2}}\,\}\, e^{-E_d} \tag{10}$$

For example if we set $\psi_1 = 0$ ($a_s = 0$) in Eq.(4), then we would obtain our previous monotonic *electron* double layer solution[20], which is the nonlinear version of slow electron acoustic wave in the limit $\tau \rightarrow 0$, $\psi_2 \rightarrow 0$. Similarly, the condition $\psi_1 = \psi_2$ ($a_s = 0$) corresponds to the solitary ion hole solution. More generally the condition $|a|<1$, the above solution Eq.(4) describes an *asymmetric* double layer, with a potential depression at the low potential side. Therefore, we may consider a_s to be the asymmetry parameter. From Eq.(6), we see that the existence of the asymmetric double layer, with a potential depression at the low potential side, requires $\tilde{n}_i < 1$ in order to have a positive curvature at $\phi = 0$.

Having considered the asymmetric double layer with a potential depression at the low potential side, we now turn to the problem of an asymmetric double layer having a potential hump at the high potential side. In order to describe this class of double layer, we assume a Maxwell-Boltzmann electron distribution $f_e = \tilde{n}_e (2\pi)^{-1/2} e^{-\frac{1}{2\tau}(2\tau v^2 - 2\phi)}$ and we consider the following ion distribution[24]:

$$f_i = (2\pi)^{-1/2}\Big[\exp\{-1/2(sgn\ v\ \epsilon_i^{1/2} + v_0)^2\}\ \Theta(\epsilon_i) + (1 + h\ sgn\ (x - x_m)) \exp\{-1/2\alpha\epsilon_i\}\ \Theta(-\epsilon_i) \Big] \tag{11}$$

where $\epsilon_i = v^2 + 2\tau\phi$ with $-\psi \leqslant \phi \leqslant 0$, $\tau = T_e/T_i$ and ion velocity has been normalized to the ion thermal velocity $(T_i/m_i)^{1/2}$. Here v_0 and E_0 represent the ion drift velocity and the ion drift energy, respectively. Here $\tilde{n}_e$ and h are constants and α represents an inverse temperature. Here $-1 \leqslant h \leqslant 1$ in order to have *positive* trapped particle distribution function. With the above distribution functions, the small amplitude limit Poisson equation becomes

$$\phi_{xx} = (\tilde{n}_e - 1) + \frac{2\sqrt{\tau}\, e^{-E_0}}{\sqrt{\pi}}\ h\ sgn\,(x - x_m\,)(-\phi)^{1/2} + \{\ \tilde{n}_e - \frac{\tau}{2}\ Z'_r(\sqrt{E_0}\,)\ \}\phi$$
$$- \frac{4\tau}{3\sqrt{\pi}} e^{-E_0}\{2E_0 + (\ 1 + h\ sgn\,(x - x_m)\ )\ \alpha - 1\}(-\phi)^{3/2}$$
$$+ \tau^2\{\ \frac{\tilde{n}_e}{2\tau^2} - e^{-E_0}(1/2 + \frac{3e^{E_0}}{8E_0^2})\ \}\phi^2 + \{\text{ higher order terms }\} \tag{12}$$

Charge neutrality at $x = \pm\infty$ requires that $V'(\phi) = 0$ at $\phi = -\psi_1\,, -\psi_2$. The conditions for zero electric field lead to $V(\phi) = 0$ at $\phi = 0\,, -\psi_1\,, -\psi_2$. Again solving Eq.(12) and the above nonlinear dispersion relations together with the requirement $V(\phi) < 0$ {see Fig. 2. (b)} for $-\psi_1 < \phi < 0\,, -\psi_2 < \phi < 0$, we obtain the following asymmetric double layer solution with a potential hump at the high potential side($|a_s| < 1$):

$$\phi = -\tilde{\psi}\{\ a_s + \tanh \pm |\kappa_e|\ x\ \}^2 \tag{13}$$

where for convenience we have defined

$$\tilde{\psi} = \frac{(\sqrt{\psi_1} + \sqrt{\psi_2})^2}{4}\ ,\ a_s = \frac{\sqrt{\psi_2} - \sqrt{\psi_1}}{\sqrt{\psi_1} + \sqrt{\psi_2}}\ ,\ \kappa_e^2 = -\frac{\tilde{\psi}\mu}{6}\ . \tag{14}$$

Here ψ_1, ψ_2 and κ_e are related to our system parameters as follows:

$$\tilde{n}_e - 1 = \frac{1}{3}\psi_1\psi_2\mu\ , \tag{15}$$

$$\frac{2\sqrt{\tau}h}{\sqrt{\pi}} e^{-E_0} = \pm\ \sqrt{\psi_1}\sqrt{\psi_2}(\sqrt{\psi_1} - \sqrt{\psi_2})\mu\ , \tag{16}$$

$$\{\ \tilde{n}_e - \frac{1}{2}\tau Z'_r(\sqrt{E_0})\ \} = -\frac{2}{3}(\psi_1 + \psi_2 - 4\sqrt{\psi_1}\sqrt{\psi_2})\mu\ , \tag{17}$$

$$-\frac{4\tau^{3/2}}{3\sqrt{\pi}} e^{-E_0} h\alpha = \mp\ \frac{1}{6}(\sqrt{\psi_1} - \sqrt{\psi_2})\mu\ , \tag{18}$$

$$\mu = \{\ \frac{\tilde{n}_e}{2} - \tau^2 e^{-E_0}\ (\ \frac{1}{2} + \frac{3e^{E_0}}{8E_0^2})\ \}\ . \tag{19}$$

For example if we set $\psi_1 = 0$ ($a_s = 1$) in Eq.(10), then we would recover our previous *ion* acoustic monotonic double layer solution[20], which is the nonlinear version of the slow ion acoustic wave. Similarly the condition $\psi_1 = \psi_2$ ($a_s = 0$) would return the electron solitary hole solution. From Eq.(10), we note that the existence of the asymmetric double layer with a potential hump at the high potential side, requires $\bar{n}_e < 1$ in order to have a negative curvature at $\phi = 0$.

Having obtained the analytic solutions for the time stationary double layers, we can now present a derivation of the K-dV like equation, which describes the one dimensional asymptotic behavior of the asymmetric double layers, having a potential depression at the low potential side, of small but finite amplitude.

To describe a collisionless plasma of cold ions and warm electrons, we consider the following set of equations:

$$n_t + (nv)_x = 0\ , \quad v_t + vv_x + \phi_x = 0\ , \quad \phi_{xx} = n_e - n\ , \tag{20}$$

where the density, velocity, potential and spatial coordinate are normalized to the unperturbed density n_0, ion acoustic velocity $(T_e/m_i)^{1/2}$, the electron temperature T_e/e and the electron Debye length, respectively. Assuming the electrons to be in a quasi-equilibrium with the low-frequency ion acoustic wave, we may expand the electron density as before $n_e = 1 + c_0\, sgn\ (x - x_m)\ (\phi)^{1/2} + c_1\phi + c_2\, sgn\ (x - x_m)\ (\phi)^{3/2} + c_3\phi^2 + \ldots$. By introducing the Gardner-Morikawa coordinate transformation $\xi = \delta^{1/2}\ (x-t)$ and $\tau = \delta^{3/2}t$, then expanding n, v, ϕ in powers of small parameter[25] δ, and then assuming c_0 and c_2 to be order of $\delta^{3/2}$ and $\delta^{1/2}$, respectively, we obtain the following K-dV like equation by using the boundary conditions for an asymmetric double layer with a potential depression at the low potential side as before:

$$\begin{aligned} 0 = {} & \frac{1}{\bar{\psi}\kappa^2}\phi_t + \frac{1}{2\bar{\psi}\kappa^2}\phi_{xxx} \\ & - \Bigg[\pm\, 6a_s(1-a_s^2)\ sgn(x - x_m)\left(\frac{\phi}{\bar{\psi}}\right)^{1/2} \\ & \qquad + \{4(3a_s^2-1) - \frac{M}{\kappa^2}\}\frac{\phi}{\bar{\psi}} \mp 10a_s\ sgn(x - x_m)\left(\frac{\phi}{\bar{\psi}}\right)^{3/2} + 3\left(\frac{\phi}{\bar{\psi}}\right)^2 \Bigg]_x \end{aligned} \tag{21}$$

where M represents the velocity of the double layer, $\kappa^2 = \dfrac{(M-1)}{4(3a_s^2 - 1)}$ and $\bar{\psi} > 0$. It should be noted that we have used our boundary condition for a moving asymmetric double layer so that we can extract some useful physics. Here we have the same definition of $\bar{\psi}$ and a_s as in Eq.(5). The evolution equation for the asymmetric moving double layer with a potential hump at the high potential side can also be obtained similarly by using the corresponding asymmetric double layer boundary conditions; it is given by simply letting $\bar{\psi} \rightarrow -\bar{\psi} < 0$ in Eq.(21), using the definition of $\bar{\psi}$ as in Eq.(14). The corresponding moving asymmetric double layer solution, with a potential depression at the low potential side, of Eq.(21) is given by

$$\phi(x,t) = \bar{\psi}\{\ a_s + \tanh \pm\sqrt{\frac{M-1}{4(3a_s^2 - 1)}}(x-Mt)\ \}^2 \tag{22}$$

where x_m is given by the equation $\phi\ (\ x_m = x - Mt\ ,\ 0\ ) = 0$. Here it should be noted that $M > 1$ for $3a_s^2 - 1 > 0$ and that $M < 1$ for $3a_s^2 - 1 < 0$.

In conclusion, we have obtained two different asymmetric double layer analytic solutions: one has a potential hump at the high potential side, the other has a potential depression at the low potential side. By considering the double-valuedness properties of the Sagdeev potential (required for the existence of an asymmetric double layer), we have proven the following fact: any distribution function having analytic density representation as any polynomial power series of potential (for example $n = \int_{-\infty}^{+\infty} f\ dv = \sum_n C_n\ (\ \pm\phi\ )^n$ with $C_n = constant$) can never satisfy the asymmetric double layer boundary conditions. This shows the importance of using the third constant of motion for reflected particles, in order to satisfy the double-valuedness of Sagdeev potential for the asymmetric double layers. We have also given a derivation of the K-dV like equation, which describes the asymmetric moving double layer with a potential depression at the low potential side[26]. Thus we have found that there is some relationship between the speed of asymmetric double layer and the degree of asymmetry(a_s).

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26 We have also extended our theory to the case of obliquely propagating double layers in magnetoplasmas by considering fluid-Vlasov-Poisson system.