NONLINEAR UNITY FEEDBACK SYSTEMS AND Q-PARAMETRIZATION

by

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Abstract

This paper concerns nonlinear systems, defines a new concept of stability, the $\mathcal{S}$-stability, and extends to unity-feedback systems the technique of Q-parametrization introduced for linear system by Zames and developed by Desoer, Chen and Gustafson. We specify 1) a global parametrization of all controllers that $\mathcal{S}$-stabilize a given $\mathcal{S}$-stable nonlinear plant; 2) a parametrization of a class of controllers that stabilize an unstable nonlinear plant, 3) necessary and sufficient conditions for a nonlinear controller to simultaneously stabilize two nonlinear plants.

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I. Introduction

The purpose of this paper is to obtain the broadest generalization within the context of nonlinear systems of a number of recent results pertaining to linear feedback systems.

In the context of linear systems, recent development in the field can be sketched as follows: Youla et al. studied the linear lumped case and obtained the first global parametrization of all compensators that stabilize a given plant [You. 1]. The same problem as well as the tracking problem was solved for linear distributed systems by Callier et al. [Cal. 1]. Then Desoer, Liu, Murray, and Saeks gave a general algebraic treatment of the global parametrization problem [Des. 1]. Later Vidyasagar, Schneider, and Francis introduced a slightly different factorization of transfer functions which leads them to a very natural topology of stable and unstable transfer functions [Vid. 1]. Zames studied, in particular, the case of stable plants and derived fundamental limitations on the performance of the feedback system [Zam. 1]. Pernebo obtained a parametrization of the I/O map and of the disturbance-to-output map for the two-input-one-output configuration [Per. 1]. The same configuration was treated in complete algebraic generality by Desoer and Gustafson [Des. 2].†

For the unity feedback configuration and for a stable plant, Zames proposed a parametrization of the controller in terms of a stable proper transfer function Q [Zam. 1]. This idea was further developed as a design procedure by Desoer and Chen and was used for computer aided design by Gustafson and Desoer [Des. 3, Gus. 1]. In this paper we use

†This review is admittedly very sktechy; for a more complete list of references see [[Des. 2], [Sae. 1], [Vid. 1], [Vid. 2] etc.].

also a Q-parametrization but in a nonlinear context. We first generalize the concept of finite-gain stability (incremental stability) to that of $\mathcal{S}$-stability (incremental $\mathcal{S}$-stability, resp.). In Thoerem 1, we establish for the nonlinear case, a global parametrization of all I/O maps and of all compensators that result in an $\mathcal{S}$-stable configuration. This theorem generalizes to the nonlinear case, the original results of Zames. As a consequence of the more general stability concept, Theorem 1 is a generalization of a previous result of Desoer and Liu [Des. 4].

In Section IV we consider the case where the plant is unstable. For the linear case, Zames established his "decomposition principle," i.e., stabilize the given linear plant P with a stable linear compensator F, and then proceed with the Q-parametrization as above. Anantharam et al. established a nonlinear version of this result [Ana. 1]. In Theorem 2 we establish a similar result in the more general concept of $\mathcal{S}$-stability and we weaken the requirement on the stabilizing feedback F: it need not be itself stable but need only lead to a stable feedback configuration of P and F. Note that Theorem 2 generalizes our previous work, first it uses the more general stability concept and, second, the method of proof is greatly improved [Des. 5].

The problem of simultaneous stabilization has been formulated and solved in the linear case by Saeks and Murray [Sae. 1]. Vidyasagar et al. also have interesting results along this line [Vid. 2]. In Section V we consider the nonlinear case: we are given two (possibly unstable) nonlinear plants $\overline{P}_1$ and $\overline{P}_2$ and we derive necessary and sufficient conditions for the existence of a fixed compensator that stabilizes both plants. Theorem 4 is a generalization for nonlinear plants and within the

$\mathcal{S}$-stability concept of the linear results of Vidyasagar et al., and of our previous work [Des. 6].

This paper starts by generalizing the concept of nonlinear I/O stability and then showing that a number of global parametrization results valid for linear feedback systems can be suitably generalized for nonlinear feedback systems. This paper is reasonably self-contained and the proofs are basically simple: the key tool being that the composition of $\mathcal{S}$-stable maps is $\mathcal{S}$-stable.

II. Definitions and Notations

Let $(\mathcal{L}, \|\cdot\|)$ be a normed space of "time functions": $\mathcal{J} \to \mathcal{V}$ where $\mathcal{J}$ is the time set (typically $\mathbb{R}_+$ or $\mathbb{N}$), $\mathcal{V}$ is a normed space (typically $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}^n$, ...) and $\|\cdot\|$ is the chosen norm in $\mathcal{L}$. Let $\mathcal{L}_e$ be the corresponding extended space [Wil. 1], [Des. 7], [Vid. 3].

A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class K iff ϕ is continuous and increasing. ϕ is said to belong to class K_0 iff $\phi \in K$ and $\phi(0) = 0$. If ϕ_1 and $\phi_2 \in K_0$, then $\phi_1 + \phi_2$ and $\alpha \mapsto \phi_1(\phi_2(\alpha)) \in K_0$. A nonlinear causal map $H : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$ is said to be $\mathcal{S}$-stable iff $\exists\ \phi \in K$ s.t. $\forall\ x \in \mathcal{L}_e^{n_i}$, $\forall\ T \in \mathcal{J}$,

$$\|Hx\|_T \leq \phi(\|x\|_T)$$

H is said to be incrementally $\mathcal{S}$-stable (incr. $\mathcal{S}$-stable) if (i) H is $\mathcal{S}$-stable, (ii) $\exists\ \tilde{\phi} \in K_0$ s.t. $\forall\ x, x' \in \mathcal{L}_e^{n_i}$, $\forall\ T \in \mathcal{J}$,

$$\|Hx-Hx'\|_T \leq \tilde{\phi}(\|x-x'\|_T)$$

It can be shown that if the nonlinear causal maps H_1 and H_2 are $\mathcal{S}$-stable, (incr. $\mathcal{S}$-stable), then $H_1 + H_2$ and $H_1 \circ H_2$ are $\mathcal{S}$-stable, (incr. $\mathcal{S}$-stable, resp.). (For simplicity, we drop in the following the symbol "$\circ$" denoting the composition of maps).

A feedback system is said to be well-posed iff the relation between the inputs of interest and the outputs of interest is a well-defined causal map between suitable extended spaces. More precisely, the system ${}^1S(P,C)$ of Fig. 1, where $P : \mathcal{L}_e^{n_i} \mapsto \mathcal{L}_e^{n_o}$, $C : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ are causal maps, is said to be well-posed iff $H : (u_1,u_2) \mapsto (e_1,e_2,y_1,y_2)$ is well-defined and causal. Note that ${}^1S(P,C)$ is well-posed implies that† $(I+PC)^{-1}$ and $(I+CP)^{-1}$ are well-defined and causal. We say that a well-posed nonlinear feedback system is $\mathcal{S}$-stable (incr. $\mathcal{S}$-stable) iff its I/O map is $\mathcal{S}$-stable (incr. $\mathcal{S}$-stable, resp.). For the system ${}^1S(P,C)$, since $e_1 = u_1 - y_2$, $e_2 = u_2 + y_1$, we see that $H_{yu} : (u_1,u_2) \mapsto (y_1,y_2)$ is $\mathcal{S}$-stable iff $H_{eu} : (u_1,u_2) \mapsto (e_1,e_2)$ is $\mathcal{S}$-stable iff ${}^1S(P,C)$ is $\mathcal{S}$-stable. The same equivalence holds for incr. $\mathcal{S}$-stability. These concepts of $\mathcal{S}$-stability and incr. $\mathcal{S}$-stability are generalizations of finite-gain stability and incremental stability [Des. 7]; they are in spirit closer to Safonov's work [Saf. 1]. We use "s.t." to abbreviate such that.

III. Global Parametrization of Nonlinear $\mathcal{S}$-stable I/O Maps

Consider the well-posed nonlinear unity feedback system ${}^1S(P,C)$ shown in Fig. 1, where $P : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$, $C : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ are nonlinear causal maps, and (u_1,u_2), (y_1,y_2) and (e_1,e_2) are the "input", "output", and "error" respectively. Theorem 1 is a generalization of a result of Desoer and Liu [Des. 4], it gives a global parametrization of all achievable input-output maps, and of all stabilizing compensators, under the

†The meaning of $(I+PC)^{-1}$ deserves clarification: the map C is composed with P then the identity is added, and the resulting map is inverted. Although this formula has the same form as the linear case, it has a completely different interpretation.

assumption that P is incr. $\mathcal{S}$-stable. This theorem is an extension to the nonlinear case, the well-known linear Q-parametrization result, proved by Zames in a very general algebraic context [Zam. 1].

Theorem 1. (Global parametrization of stable ${}^{1}S(P,C)$).

Let $P : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$, $C : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ be nonlinear causal maps. Assume that (i) the system ${}^{1}S(P,C)$ is well-posed, and

(ii) P is incr. $\mathcal{S}$-stable.

Under these conditions (U.t.c.),

(a) H_{yu} is $\mathcal{S}$-stable $\Leftrightarrow \exists\, Q : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ $\mathcal{S}$-stable s.t.

$$C = Q(I-PQ)^{-1} \tag{3.1}$$

(b) $$C = Q(I-PQ)^{-1} \Leftrightarrow Q = C(I+PC)^{-1} \tag{3.2}$$

(c) With $u_2 = 0$ and with $C = Q(I-PQ)^{-1}$,

$$H_{y_2u_1} = PQ \tag{3.3}$$

Comments

(i) Equivalence (b) above requires only that ${}^{1}S(P,C)$ be well-posed.

(ii) Equivalence (a) gives a global parametrization of $\mathcal{C}(P)$, the family of all compensators that result in an $\mathcal{S}$-stable system ${}^{1}S(P,C)$; more precisely:

$$\mathcal{C}(P) = \{C \mid C = Q(I-PQ)^{-1},\ Q \text{ is } \mathcal{S}\text{-stable}\}.$$

(iii) From (c), $\mathcal{H}_{y_2u_1}$, the class of achievalbe $\mathcal{S}$-stable I/O maps is given by

$$\mathcal{H}_{y_2u_1}(P) = \{PQ|Q \text{ is } \mathcal{S}\text{-stable}\}.$$

(iv) Practical design considerations such as robustness of stability, disturbance rejection, plant saturation, etc. impose additional restrictions on Q. (See e.g., [Des. 3], [Gus. 1]).

(v) The equation (3.3), $H_{y_2u_1} = PQ$, raises a number of new problems: given a nonlinear map P, how can one describe the constraints imposed by P on the achievable I/O map $H_{y_2u_1}$? If we have a desired I/O map $H_{y_2u_1}$ and a given P, how does one find a Q_a such that in some appropriate sense, $PQ_a \simeq H_{y_2u_1}$? Then having such a Q_a how does one synthesize C?

<u>Proof</u>:

<u>(I) Proof of (b)</u>.

($\Rightarrow$) By assumption,

$$C = Q(I-PQ)^{-1}.$$

Composing with P and adding identity we obtain successively,

$$I + PC = I + PQ(I-PQ)^{-1} = (I-PQ)^{-1}$$

By taking the inverse, and composing with C, we obtain

$$C(I+PC)^{-1} = Q(I-PQ)^{-1}(I-PQ) = Q$$

Hence, $\quad Q = C(I+PC)^{-1}.$

($\Leftarrow$) By assumption,

$$Q = C(I+PC)^{-1}.$$

Composing with P and adding identity we obtain successively,

$$I - PQ = I - PC(I+PC)^{-1} = (I+PC)^{-1}$$

By taking the inverse, and composing with Q, we obtain

$$Q(I-PQ)^{-1} = C(I+PC)^{-1}(I+PC) = C$$

Hence, $\quad C = Q(I-PQ)^{-1}$.

<u>(II) Proof of (a)</u>.

($\Rightarrow$) Set $u_2 = 0$, the map $H_{y_1u_1} : u_1 \mapsto y_1$ is given by $H_{y_1u_1} = C(I+PC)^{-1}$ which by assumption is $\mathcal{S}$-stable. Let $Q := C(I+PC)^{-1}$, then Q is $\mathcal{S}$-stable and from (b), we have $C = Q(I-PQ)^{-1}$.

($\Leftarrow$) Refer to Fig. 1, write the summing node equations

$$e_1 = u_1 - Pe_2 \tag{3.4}$$

$$e_2 = u_2 + Ce_1 \tag{3.5}$$

Define

$$\tilde{u}_1 := PC\, e_1 - P(u_2+Ce_1) \tag{3.6}$$

Using (3.5) and (3.6), rewrite (3.4) as

$$e_1 = u_1 + \tilde{u}_1 - PC\, e_1 \tag{3.7}$$

From equation (3.7)

$$e_1 = (I+PC)^{-1}(u_1+\tilde{u}_1) \tag{3.8}$$

$$y_1 = Ce_1 = C(I+PC)^{-1}(u_1+\tilde{u}_1) = Q(u_1+\tilde{u}_1) \tag{3.9}$$

Now, since P is incr. $\mathcal{S}$-stable, $\exists\ \tilde{\phi}_p \in K_o$ s.t. $\forall\ (u_1,u_2) \in \mathcal{L}_e^{n_o} \times \mathcal{L}_e^{n_i}$, $\forall\ T \in \mathcal{T}$,

$$\|\tilde{u}_1\|_T = \|P(Ce_1)-P(u_2+Ce_1)\|_T \leq \tilde{\phi}_P(\|u_2\|_T) \leq \tilde{\phi}_P(\|u_1\|_T+\|u_2\|_T) \tag{3.10}$$

Hence the map $\tilde{\pi} : (u_1,u_2) \mapsto \tilde{u}_1$ is $\mathcal{S}$-stable. Define the projection map $\pi_i : (u_1,u_2) \to u_i$, $i = 1,2$. From (3.9), the map $H_{y_1u} : (u_1,u_2) \mapsto y_1$ is given by

$$H_{y_1u} = Q(\pi_1+\tilde{\pi}) \tag{3.11}$$

Since π_1 and $\tilde{\pi}$ are $\mathcal{S}$-stable, and by assumption Q is $\mathcal{S}$-stable, the map H_{y_1u} is $\mathcal{S}$-stable.

From Fig. 1, we have

$$y_2 = P(u_2+y_1) \tag{3.12}$$

Hence the map $H_{y_2u} : (u_1,u_2) \mapsto y_2$ is given by

$$H_{y_2u} = P(\pi_2+H_{y_1u}) \tag{3.13}$$

Now π_2 and H_{y_1u} are $\mathcal{S}$-stable, and by assumption P is $\mathcal{S}$-stable, it follows that H_{y_2u} is $\mathcal{S}$-stable. Therefore H_{yu} is $\mathcal{S}$-stable.

<u>(III) Proof of (c)</u>.

Since $C = Q(I-PQ)^{-1}$, from (b) we have $Q = C(I+PC)^{-1}$. With $u_2 = 0$,

$H_{y_1u_1} = H_{e_2u_1} = C(I+PC)^{-1} = Q$.

Hence, $\quad H_{y_2u_1} = PQ$.

IV. Two-Step Stabilization of Nonlinear Plants

The equivalence (a) of Theorem 1 above requires that the plant be incr. $\mathcal{S}$-stable. In practice, unstable plants do occur (e.g., chemical reactors, high performance airplanes, etc.), it is important to extend this method to include unstable plants.

Theorem 2. (Two-step stabilization of nonlinear plants).

Let $P : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$, $F : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ be nonlinear causal maps such that

(A.1) the system ${}^1S(P,F)$ of Fig. 2 is well-posed,

(A.2) ${}^1S(P,F)$ is incr. $\mathcal{S}$-stable, and

(A,3) for all the C's under consideration, the system ${}^3S(P,F,C-F)$ of Fig. 3 is well-posed.

Let $P_1 := P[I-F(-P)]^{-1}$.

U.t.c., if

$$C := F + Q(I-P_1Q)^{-1} \text{ for some } \mathcal{S}\text{-stable } Q : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}, \tag{4.1}$$

then

(a) ${}^1S(P,C)$ is $\mathcal{S}$-stable, and

(b) ${}^3S(P,F,C-F)$ is $\mathcal{S}$-stable.

Comments.

(i) None of the maps P, C. F, C-F are required to be stable.

(ii) The key assumptions are (a) well-posedness, (b) ${}^1S(P,F)$ is incr. $\mathcal{S}$-stable, (c) $C = F + Q(I-P_1Q)^{-1}$ where $P_1 = P[I-F(-P)]^{-1}$ and Q is $\mathcal{S}$-stable.

(iii) By definition, ${}^3S(P,F,C-F)$ is $\mathcal{S}$-stable iff the map $(u_1,u_2,u_3) \mapsto (y_1,y_2,y_3)$ is $\mathcal{S}$-stable.

(iv) If P is incr. $\mathcal{S}$-stable, then by choosing F the zero map, we have $P_1 = P$, $C = Q(I-PQ)^{-1}$, and Theorem 2 reduces to Theorem 1.

(v) In the proof we show that (b) implies (a), a simple example shows that (a) does not imply (b). However, if F is incr. $\mathcal{S}$-stable, then (a) and (b) are equivalent [Ana. 1, Thm. 3].

Proof:

(I) Proof of (b): ${}^3S(P,F,C-F)$ is $\mathcal{S}$-stable

Consider the system ${}^1S(P,F)$ if Fig. 2, let $\psi = (\psi_2,\psi_3) : (e_2'',u_3) \mapsto (y_2,y_3)$ be the I/O map. Note that $P_1(\cdot) := P[I-F(-P)]^{-1}(\cdot) = \psi_2(\cdot,0)$. By (A.2), ψ is incr. $\mathcal{S}$-stable, hence P_1 is incr. $\mathcal{S}$-stable; further from assumption (4.1), Q is $\mathcal{S}$-stable and $C-F = Q(I-P_1Q)^{-1}$; hence, by Theorem 1, these three conclusions imply that the system ${}^1S(P_1,C-F)$ shown in Fig. 4 is $\mathcal{S}$-stable.

Next consider Fig. 3 which shows the system ${}^3S(P,F,C-F)$ with input (u_1,u_2,u_3) and output (y_1,y_2,e_2'',y_3). We claim that the map ${}^3H : (u_1,u_2,u_3) \mapsto (y_1,y_2,e_2'',y_3)$ is $\mathcal{S}$-stable. Let

$$\Delta\tilde{y}_2 := \psi_2(e_2'',u_3) - \psi_2(e_2'',0). \tag{4.2}$$

Drive the system ${}^3S(P,F,C-F)$ with input $(u_1-\Delta\tilde{y}_2,u_2,0)$, call the corresponding output $(\tilde{y}_1,\tilde{y}_2,\tilde{e}_2'',\tilde{y}_3)$, and note that $\tilde{y}_2 = P[I-F(-P)]^{-1}\tilde{e}_2'' = P_1\tilde{e}_2''$; thus if we ignore $\tilde{y}_3$, the system reduces to ${}^1S(P_1,C-F)$, (which has just been shown to be $\mathcal{S}$-stable), with input $(u_1-\Delta\tilde{y}_2,u_2)$ and output $(\tilde{y}_1,\tilde{y}_2,\tilde{e}_2'')$. Hence, for ${}^3S(P,F,C-F)$, the partial map (with respect to

${}^3H$), ${}^2\tilde{H} : (u_1-\Delta\tilde{y}_2,u_2,0) \mapsto (\tilde{y}_1,\tilde{y}_2,\tilde{e}_2'')$ is $\mathcal{S}$-stable. Since ψ_2 is incr. $\mathcal{S}$-stable, $\exists\ \tilde{\phi}_2 \in K_o$ s.t. $\forall\ e_2''$, $\forall\ u_3$, $\forall\ T$,

$$\|\Delta\tilde{y}_2\|_T = \|\psi_2(e_2'',u_3)-\psi_2(e_2'',0)\|_T \le \tilde{\phi}_2(\|u_3\|_T) \le \tilde{\phi}_2(\|u_1\|_T+\|u_2\|_T+\|u_3\|_T) \tag{4.3}$$

Hence the map $(u_1,u_2,u_3) \mapsto \Delta\tilde{y}_2$ is $\mathcal{S}$-stable. Therefore, the map ${}^2\pi : (u_1,u_2,u_3) \mapsto (u_1-\Delta\tilde{y}_2,u_2,0)$ is $\mathcal{S}$-stable. Considering the composition ${}^2\tilde{H}\ {}^2\pi$ we see that, for ${}^3S(P,F,C-F)$, the map $(u_1,u_2,u_3) \mapsto (\tilde{y}_1,\tilde{y}_2,\tilde{e}_2'')$ is $\mathcal{S}$-stable.

Now, we claim that $y_1 = \tilde{y}_1$, $y_2 = \tilde{y}_2 + \Delta\tilde{y}_2$, $e_2'' = \tilde{e}_2''$, and hence the map $(u_1,u_2,u_3) \mapsto (y_1,y_2,e_2'')$ is $\mathcal{S}$-stable. To prove this, write the equations for ${}^3S(P,F,C-F)$ with input (u_1,u_2,u_3) and with input $(u_1-\Delta\tilde{y}_2,u_2,0)$, respectively:

$y_1 = (C-F)(u_1-y_2)$	(4.4a)	$\tilde{y}_1 = (C-F)(u_1-\Delta\tilde{y}_2-\tilde{y}_2)$	(4.5a)
$y_2 = \psi_2(e_2'',u_3)$	(4.4b)	$\tilde{y}_2 = \psi_2(\tilde{e}_2'',0)$	(4.5b)
$e_2'' = y_1 + u_2$	(4.4c)	$\tilde{e}_2'' = \tilde{y}_1 + u_2$	(4.5c)

Using (4.2), rewrite the equations (4.4) as

$$y_1 = (C-F)[u_1-\Delta\tilde{y}_2-(y_2-\Delta\tilde{y}_2)] \tag{4.6a}$$

$$y_2 - \Delta\tilde{y}_2 = \psi_2(e_2'',0) \tag{4.6b}$$

$$e_2'' = y_1 + u_2 \tag{4.6c}$$

From Eqs. (4.5) and (4.6), we see that $(y_1,y_2-\Delta\tilde{y}_2,e_2'')$ and $(\tilde{y}_1,\tilde{y}_2,\tilde{e}_2'')$ satisfy the same equations, by the well-posedness assumption (A.3), Eqs. (4.5) and (4.6) both have a unique solution, hence $y_1 = \tilde{y}_1$, $y_2 = \tilde{y}_2 + \Delta\tilde{y}_2$, $e_2'' = \tilde{e}_2''$. Since $y_3 = \psi_3(e_2'',u_3)$ and ψ_3 is $\mathcal{S}$-stable, the map $(u_1,u_2,u_3) \mapsto y_3$ is $\mathcal{S}$-stable. Consequently, the map ${}^3H: (u_1,u_2,u_3) \mapsto (y_1,y_2,e_2'',y_3)$ is $\mathcal{S}$-stable and (b) is established.

<u>(II) Proof of (a): ${}^1S(P,C)$ is $\mathcal{S}$-stable</u>.

Write the summing node equations for ${}^3S(P,F,C-F)$ in terms of e_1, e_2, e_3, and e_2'': (see Fig. 3),

$$e_1 = u_1 - Pe_2 \tag{4.7a}$$

$$e_2'' = u_2 + (C-F)e_1 \tag{4.7b}$$

$$e_2 = e_2'' + Fe_3 \tag{4.7c}$$

$$e_3 = u_3 - Pe_2 \tag{4.7d}$$

Let $u_1 = u_3$, then, by (4.7a) and (4.7d), $e_1 = e_3$; thus by adding (4.7c) and (4.7b) we have

$$e_1 = u_1 - Pe_2 \tag{4.8a}$$

$$e_2 = u_2 + Ce_1 . \tag{4.8b}$$

The equations (4.8) describe ${}^1S(P,C)$. Since ${}^3S(P,F,C-F)$ is $\mathcal{S}$-stable, the map $(u_1,u_2,u_1) \mapsto (e_1,e_2)$ defined by (4.8) is $\mathcal{S}$-stable. Hence ${}^1S(P,C)$ is $\mathcal{S}$-stable.

V. <u>Simultaneous Stabilization of Nonlinear Plants</u>

In this section we study the problem of simultaneous stabilization of nonlinear plants. The main result is Theorem 4: a necessary and sufficient condition for given two nonlinear plants be simultaneously stabilized by a compensator. We assume, throughout this section, that <u>all</u> the systems under consideration are <u>well-posed</u>.

<u>Theorem 3.</u>

Let $\overline{P} : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$ and $C, F : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ be nonlinear causal maps.
Let $P := \overline{P}[I - F(-\overline{P})]^{-1}$.
U.t.c., if F is incr. <u>$\mathcal{S}$-stable</u>, then
${}^1S(\overline{P}, C+F)$ is $\mathcal{S}$-stable $\Leftrightarrow$ ${}^1S(P,C)$ is $\mathcal{S}$-stable.

<u>Comments</u>.

(i) None of the maps $\overline{P}$, P, and C are required to be stable.

(ii) The Theorem is false if F is not incr. $\mathcal{S}$-stable. Consider the following example: let $\overline{P} = (s-1)/(s+3) =: \overline{n}/\overline{d}$, $F = 3/(s-1)$, and $C = 3/1 =: n_c/d_c$. By calculation, $C + F = 3s/(s-1) =: n_{c+f}/d_{c+f}$, and $P = \overline{P}[I-F(-\overline{P})]^{-1} = (s-1)/(s+6) =: n/d$. The system ${}^1S(P,C)$ is $\mathcal{S}$-stable, since its characteristic polynomial is $nn_c + dd_c = 4s + 3$. However, the system ${}^1S(\overline{P}, C+F)$ is <u>unstable</u>, since its characteristic polynomial is $\overline{n}n_{c+f} + \overline{d}d_{c+f} = (s-1)(4s+3)$.

(iii) Traditionally the loop transformation theorem [Des. 7] requires that F be linear, so Theorem 3 is a generalization of the usual stability results obtainable from the loop transformation theorem.

<u>Lemma</u>:

Let $\overline{P} : \mathcal{L}_e^{n_i} \mapsto \mathcal{L}_e^{n_o}$ and $F : \mathcal{L}_e^{n_o} \mapsto \mathcal{L}_e^{n_i}$. If $P := \overline{P}[I-F(-\overline{P})]^{-1}$, then $\overline{P} = P[I+F(-P)]^{-1}$.

<u>Proof</u>:

By assumption,

$$P = \overline{P}[I-F(-\overline{P})]^{-1}.$$

Composing $-P$ with F and then adding identity, we have

$$I + F(-P) = I + F(-\overline{P})[I-F(-\overline{P})]^{-1} = [I-F(-\overline{P})]^{-1}$$

By taking the inverse, and composing with P, we obtain

$$P[I+F(-P)]^{-1} = \overline{P}[I-F(-\overline{P})]^{-1}[I-F(-\overline{P})] = \overline{P}$$

Hence, $\overline{P} = P[I+F(-P)]^{-1}$.

Comments

(i) By using the relation $P = \overline{P}[I-F(-\overline{P})]^{-1}$, ($\overline{P}=P[I+F(-P)]^{-1}$), the system ${}^{1}S(P,C)$ of Fig. 1 (${}^{1}S(\overline{P},C+F)$ of Fig. 5, resp.) can be redrawn as the system of Fig. 6 (Fig 7, resp.).

(ii) Note that the system in Fig. 6 (Fig. 7) and the system ${}^{1}S(P,C)$ (${}^{1}S(\overline{P},C+F)$, resp.) have the same I/O map $\psi_C : (u_1,u_2) \mapsto (e_1,e_2)$ ($\psi''_{C+F} : (u_1,u_2) \mapsto (e_1,e''_2)$, resp.).

Proof:

($\Rightarrow$) We show that for the system ${}^{1}S(P,C)$ map $\psi_C : (u_1,u_2) \to (e_1,e_2)$ is $\mathcal{S}$-stable. For the system shown in Fig. 6, write the equations defining e_1 and e''_2:

$$e_1 = u_1 - \overline{P}e''_2 \tag{5.1a}$$

$$\begin{aligned} e''_2 &= u_2 + Ce_1 + F(-\overline{P}e''_2) \\ &= u_2 + Ce_1 + F(e_1-u_1) \end{aligned} \tag{5.1b}$$

Rewrite (5.1b) as

$$e_2'' = u_2 + (C+F)e_1 + [F(e_1-u_1)-Fe_1] \tag{5.2}$$

Let

$$\tilde{u}_1 := u_1 \tag{5.3a}$$

$$\tilde{u}_2 := u_2 + [F(e_1-u_1)-Fe_1] \tag{5.3b}$$

Then, Eqs. (5.1) read

$$e_1 = \tilde{u}_1 - \overline{P}e_2'' \tag{5.4a}$$

$$e_2'' = \tilde{u}_2 + (C+F)e_1 \tag{5.4b}$$

Note that equations (5.4) describe ${}^1S(\overline{P},C+F)$ with input $(\tilde{u}_1,\tilde{u}_2)$; by assumption ${}^1S(\overline{P},C+F)$ is $\mathcal{S}$-stable. Hence the map $\tilde{\psi}_{C+F} : (u_1,u_2) \mapsto (e_1,e_2'')$, specified by (5.4), is $\mathcal{S}$-stable.

Since F is incr. $\mathcal{S}$-stable, $\exists\ \tilde{\phi}_F \in K_o$, s.t. $\forall\ u_1$, $\forall\ e_1$, $\forall$ T,

$$\|F(e_1-u_1)-Fe_1\|_T \le \tilde{\phi}_F(\|u_1\|_T) \le \tilde{\phi}_F(\|u_1\|_T+\|u_2\|_T) \tag{5.5}$$

Hence the map $\tilde{\pi} : (u_1,u_2) \to (\tilde{u}_1,\tilde{u}_2)$ defined by (5.1) and (5.3) is $\mathcal{S}$-stable. Define $\tilde{\psi}_C'' = \tilde{\psi}_{C+F}\tilde{\pi}$, since both $\tilde{\psi}_{C+F}$ and $\tilde{\pi}$ are $\mathcal{S}$-stable, so is $\psi_C'' : (u_1,u_2) \mapsto (e_1,e_2'')$; hence for the system of Fig. 6 the map $(u_1,u_2) \mapsto (e_1,e_2'')$ is $\mathcal{S}$-stable.

Now from Fig. 6,

$$e_2 = e_2'' - F(e_1-u_1) \tag{5.6}$$

Since ψ_C'' and F are $\mathcal{S}$-stable, the map $(u_1,u_2) \mapsto e_2$ is $\mathcal{S}$-stable. It then follows that, for the system ${}^1S(P,C)$, $\psi_C : (u_1,u_2) \mapsto (e_1,e_2)$ is $\mathcal{S}$-stable.

($\Leftarrow$) We show that, for the system ${}^1S(\overline{P},C+F)$, the map $\psi''_{C+F} : (u_1,u_2) \mapsto (e_1,e''_2)$ is $\mathcal{S}$-stable.

Using the Lemma $\overline{P} = P[I+F(-P)]^{-1}$ and redraw ${}^1S(\overline{P},C+F)$ as in Fig. 7. Write the equations defining (e_1,e_2) in Fig. 7.

$$e_1 = u_1 - Pe_2 \tag{5.7a}$$

$$\begin{aligned} e_2 &= u_2 + (C+F)e_1 - F(-Pe_2) \\ &= u_2 + Fe_1 - F(e_1-u_1) + Ce_1 \end{aligned} \tag{5.7b}$$

Let

$$\overline{u}_1 := u_1 \tag{5.8a}$$

$$\overline{u}_2 := u_2 + Fe_1 - F(e_1-u_1) \tag{5.8b}$$

Since F is incr. $\mathcal{S}$-stable, the map $\overline{\pi} : (u_1u_2) \mapsto (\overline{u}_1,\overline{u}_2)$ defined by (5.7) and (5.8) is $\mathcal{S}$-stable. Now, with (5.8), equations (5.7) read

$$e_1 = \overline{u}_1 - Pe_2 \tag{5.9a}$$

$$e_2 = \overline{u}_2 + Ce_1 \tag{5.9b}$$

Note that equations (5.9) describe ${}^1S(P,C)$ with input $(\overline{u}_1,\overline{u}_2)$. By assumption ${}^1S(P,C)$ is $\mathcal{S}$-stable, hence the map $\overline{\psi}_C : (\overline{u}_1,\overline{u}_2) \rightarrow (e_1,e_2)$, specified by (5.9), is $\mathcal{S}$-stable. Define $\psi_{C+F} = \overline{\psi}_C\overline{\pi}$, since both $\overline{\psi}_C$ and $\overline{\pi}$ are $\mathcal{S}$-stable, so is $\psi_{C+F} : (u_1,u_2) \mapsto (e_1,e_2)$; hence for the system of Fig. 7, the map $(u_1,u_2) \mapsto (e_1,e_2)$ is $\mathcal{S}$-stable.

Now from Fig. 7,

$$e''_2 = e_2 + F(e_1-u_1) \tag{5.10}$$

Since ψ_{C+F} and F are $\mathcal{S}$-stable, equation (5.10) implies that the map $(u_1,u_2) \mapsto e_2''$ is $\mathcal{S}$-stable. Consequently, we have shown that for the system ${}^1S(\overline{P},C+F)$, $\psi''_{C+F} : (u_1,u_2) \mapsto (e_1,e_2'')$ is $\mathcal{S}$-stable.

Theorem 4. (Simultaneous Stabilization)

Given two nonlinear causal plants $\overline{P}_1, \overline{P}_2 : \mathcal{L}_e^{n_i} \mapsto \mathcal{L}_e^{n_o}$. Suppose $\exists$ incr. $\mathcal{S}$-stable $F : \mathcal{L}_e^{n_o} \mapsto \mathcal{L}_e^{n_i}$ s.t.
$P_1 := \overline{P}_1[I-F(-\overline{P}_1)]^{-1}$ is incr. $\mathcal{S}$-stable. Let $P_2 := \overline{P}_2[I-F(-\overline{P}_2)]^{-1}$.
For any $C : \mathcal{L}_e^{n_o} \mapsto \mathcal{L}_e^{n_i}$, let

$$Q := C(I+P_1C)^{-1} \tag{5.11}$$

U.t.c.

${}^1S(\overline{P}_1,C+F)$ and ${}^1S(\overline{P}_2,C+F)$ are $\mathcal{S}$-stable

$\Leftrightarrow$

Q is $\mathcal{S}$-stable and ${}^1S(P_2-P_1,Q)$ is $\mathcal{S}$-stable (see Fig. 8).

Comments

(i) By Theorem 1, Eq. (5.11) is equivalent to that $C = Q(I-P_1Q)^{-1}$.

(ii) None of the maps $\overline{P}_1$, $\overline{P}_2$, P_2, and C are required to be stable.

(iii) The meaning of the theorem is the following: given two nonlinear, not necessarily stable, plants $\overline{P}_1$ and $\overline{P}_2$, if by applying an incr. $\mathcal{S}$-stable feedback F around $\overline{P}_1$ (see Fig. 6), the resulting closed-loop I/O map $P_1 := \overline{P}_1[I-F(-\overline{P}_1)]^{-1}$ is incr. $\mathcal{S}$-stable, then any compensator of the form $Q(I-P_1Q)^{-1}+F$, for some $\mathcal{S}$-stable Q such that ${}^1S(P_2-P_1,Q)$ is $\mathcal{S}$-stable, will stabilize both $\overline{P}_1$ and $\overline{P}_2$.

(iv) If $\overline{P}_1$ is incr. $\mathcal{S}$-stable, take $F = \Theta$, the zero map from

$\mathcal{L}_e^{n_o} \mapsto \mathcal{L}_e^{n_i}$, then $P_1 = \overline{P}_1$ and $P_2 = \overline{P}_2$. The theorem shows, for this special case, that given two <u>nonlinear</u> plants $\overline{P}_1$ and $\overline{P}_2$, with $\overline{P}_1$ incr. $\mathcal{S}$-stable, then the problem of finding a compensator to stabilize both $\overline{P}_1$ and $\overline{P}_2$ is equivalent to that of finding an <u>$\mathcal{S}$-stable</u> compensator to stabilize $\overline{P}_2 - \overline{P}_1$. This result was proven for the linear case in [Vid. 2, Corollary 3.1.1.].

(v) Suppose that we have n nonlinear plants $\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_n$, then we may apply successively the theorem to the pairs $(\overline{P}_i, \overline{P}_1)$, $i = 2, 3, \ldots n$, thus ${}^1S(\overline{P}_i, C+F)$ is $\mathcal{S}$-stable for $i = 1, 2, \ldots, n$ iff $Q := C(I+P_1C)^{-1}$ is $\mathcal{S}$-stable, and ${}^1S(P_i - P_1, Q)$ is $\mathcal{S}$-stable for $i = 2, 3, \ldots, n$.

(iv) To the best of the authors' knowledge, there are no known general conditions under which a general nonlinear plant is stabilizable by a compensator, incr. $\mathcal{S}$-stable or not.

<u>Proof</u>:

(I) We first show that the system ${}^1S(\overline{P}_1, C+F)$ is $\mathcal{S}$-stable. By assumption, P_1 is incr. $\mathcal{S}$-stable and Q is $\mathcal{S}$-stable, hence, with $C = Q(I-P_1Q)^{-1}$, the system ${}^1S(P_1, C)$ is $\mathcal{S}$-stable. Now F is incr. $\mathcal{S}$-stable, so by Theorem 3, the system ${}^1S(\overline{P}_1, C+F)$ is $\mathcal{S}$-stable.

(II) Consider the system ${}^2S(P_2 - P_1, C, P_1)$ shown in Fig. 9, with input (u_1, u_2, u_3) and output (y_1, y_2, e_1', y_3). We claim that the map ${}^3H' : (u_1, u_2, u_3) \mapsto (y_1, y_2, e_1', y_3)$ is $\mathcal{S}$-stable.

Let

$$\Delta\tilde{y}_3 := P_1(u_3 + y_1) - P_1(y_1) \tag{5.12}$$

Drive ${}^2S(P_2 - P_1, C, P_1)$ with input $(u_1 - \Delta\tilde{y}_3, u_2, 0)$, and call the corresponding output $(\tilde{y}_1,\tilde{y}_2,\tilde{e}_1',\tilde{y}_3)$. Note that $\tilde{y}_1 = C(I+P_1C)^{-1}\tilde{e}_1' = Q\tilde{e}_1'$, thus if we ignore $\tilde{y}_3$, the system reduces to ${}^1S(P_2-P_1,Q)$ with input $(u_1-\Delta\tilde{y}_3,u_2)$ and output $(\tilde{y}_1,\tilde{y}_2,\tilde{e}_1')$. By assumption, ${}^1S(P_2-P_1,Q)$ is $\mathcal{S}$-stable, hence, for the system ${}^2S(P_2-P_1,C,P_1)$, the map ${}^2H:(u_1-\Delta\tilde{y}_3,u_2,0)\mapsto(\tilde{y}_1,\tilde{y}_2,\tilde{e}_1')$ is $\mathcal{S}$-stable. Since P_1 is incr. $\mathcal{S}$-stable, $\exists\ \tilde{\phi}_1 \in K_o$ s.t. $\forall\ u_3$, $\forall\ y_1$, $\forall\ T$,

$$\|\Delta\tilde{y}_3\|_T = \|P_1(u_3+y_1)-P_1y_1\|_T \leq \tilde{\phi}_1(\|u_3\|_T) \leq \tilde{\phi}_1(\|u_1\|_T+\|u_2\|_T+\|u_3\|_T) \quad (5.13)$$

From inequality (5.13), it follows that, for the system ${}^2S(P_2-P_1,C,P_1)$, the map ${}^2\tilde{\pi}: (u_1,u_2,u_3)\mapsto(u_1-\Delta\tilde{y}_3,u_2,0)$ is $\mathcal{S}$-stable. Define ${}^2H' := {}^2H\,{}^2\tilde{\pi}$, since both ${}^2H$ and ${}^2\tilde{\pi}$ are $\mathcal{S}$-stable, so is ${}^2H' : (u_1,u_2,u_3)\mapsto(\tilde{y}_1,\tilde{y}_2,\tilde{e}_1')$, hence for the system ${}^2S(P_2-P_1,C,P_1)$, the map $(u_1,u_2,u_3)\mapsto(\tilde{y}_1,\tilde{y}_2,\tilde{e}_1')$ is $\mathcal{S}$-stable.

Now, we claim that $y_1 = \tilde{y}_1$, $y_2 = \tilde{y}_2$, $e_1' = \tilde{e}_1' + \Delta\tilde{y}_3$, and hence the map $(u_1,u_2,u_3)\mapsto(y_1,y_2,e_1')$ is $\mathcal{S}$-stable. To prove this, write the equations for ${}^2S(P_2-P_1,C,P_1)$ with input (u_1,u_2,u_3) and with input $(u_1-\Delta\tilde{y}_3,u_2,0)$, respectively:

$$e'_1 = u_1 - y_2 \quad (5.14a) \qquad \tilde{e}_1' = u_1 - \Delta\tilde{y}_3 - \tilde{y}_2 \quad (5.15a)$$

$$y_1 = C(e_1'-P_1(u_3+y_1)) \quad (5.14b) \qquad \tilde{y}_1 = C(\tilde{e}_1-P_1\tilde{y}_1) \quad (5.15b)$$

$$y_2 = (P_2-P_1)(u_2+y_1) \quad (5.14c) \qquad \tilde{y}_2 = (P_2-P_1)(u_2+\tilde{y}_1) \quad (5.15c)$$

Using (5.12), rewrite the equations (5.14) as

$$e_1' - \Delta\tilde{y}_3 = u_1 - \Delta\tilde{y}_3 - y_2 \quad (5.16a)$$

$$y_1 = C(e_1'-\Delta\tilde{y}_3-P_1y_1) \quad (5.16b)$$

$$y_2 = (P_2 - P_1)(u_2 + y_1) \tag{5.16c}$$

From Eqs. (5.15) and (5.16), we see that $(y_1, y_2, e_1' - \Delta\tilde{y}_3)$ and $(\tilde{y}_1, \tilde{y}_2, \tilde{e}_1')$ satisfy the same equations. By the well-posedness assumption, Eqs. (5.15) and (5.16) both have a unique solution, hence $y_1 = \tilde{y}_1$, $y_2 = \tilde{y}_2$, $e_1' = \tilde{e}_1' + \Delta\tilde{y}_3$. Thus, we have showed that, for the system ${}^2S(P_2-P_1,C,P_1)$, the map $(u_1,u_2,u_3) \mapsto (y_1,y_2,e_1')$ is $\mathcal{S}$-stable. Since P_1 is $\mathcal{S}$-stable and $y_3 = P_1(y_1+u_3)$, hence for ${}^2S(P_2-P_1,C,P_1)$, the map $(u_1,u_2,u_3) \mapsto y_3$ is $\mathcal{S}$-stable. Consequently, for the system ${}^2S(P_2-P_1,C,P_1)$, the map ${}^3H' : (u_1,u_2,u_3) \mapsto (y_1,y_2,e_1',y_3)$ is $\mathcal{S}$-stable.

(III) We prove that ${}^1S(\overline{P}_2,C+F)$ is $\mathcal{S}$-stable.

Write the equations of ${}^2S(P_2-P_1,C,P_1)$ in terms of e_1', e_1, e_2, e_3:

$$e_1' = u_1 - (P_2-P_1)e_2 \tag{5.17a}$$

$$e_1 = e_1' - P_1e_3 \tag{5.17b}$$

$$e_2 = u_2 + Ce_1 \tag{5.17c}$$

$$e_3 = u_3 + Ce_1 \tag{5.17d}$$

If $u_2 = u_3$, then $e_2 = e_3$, thus Eqs. (5.17) reduce to

$$e_1 = u_1 - P_2e_2 \tag{5.18a}$$

$$e_2 = u_2 + Ce_1 \tag{5.18b}$$

$$e_3 = e_2$$

Equations (5.18) describe ${}^1S(P_2,C)$. Since the system ${}^2S(P_2-P_1,C,P_1)$ is $\mathcal{S}$-stable, so is the system ${}^1S(P_2,C)$. By Theorem 3, the ${}^1S(\overline{P}_2,C+F)$ is

$\mathcal{S}$-stable. This together with (I) completes the proof.

$(\Rightarrow)$

By assumption the systems ${}^1S(\overline{P}_1,C+F)$ and ${}^2S(\overline{P}_2,C+F)$ are $\mathcal{S}$-stable, and F is incr. $\mathcal{S}$-stable. Hence, by Theorem 3, the systems ${}^1S(P_1,C)$ and ${}^1S(P_2,C)$ are $\mathcal{S}$-stable. Thus, $Q = C(I+P_1C)^{-1}$ is $\mathcal{S}$-stable.

Consider the system ${}^2S(P_2,Q,-P_1)$ of Fig. 10, with input (u_1,u_2,u_3), and output (y_1,y_2,e_1,y_3). Let $\Delta\overline{y}_3 := P_1(y_1)-P_1(y_1+u_3)$. Drive the system with input $(u_1-\Delta\overline{y}_3,u_2,0)$, call the corresponding output $(\overline{y}_1,\overline{y}_2,\overline{e}_1,\overline{y}_3)$. Note that $\overline{y}_1 = Q(I-P_1Q)^{-1}e_1 = Ce_1$, thus if we ignore $\overline{y}_3$, the system reduces to ${}^1S(P_2,C)$ with input $(u_1-\Delta\overline{y}_3,u_2)$ and output $(\overline{y}_1,\overline{y}_2,\overline{e}_1)$. Since ${}^1S(P_2,C)$ is $\mathcal{S}$-stable, it follows then, by similar arguments as those in the proof of the other implication, that the system ${}^1S(P_2-P_1,Q)$ is $\mathcal{S}$-stable.

This completes the proof.

VI. Summary

In this paper, we introduce a generalized concept of stability: $\mathcal{S}$-stability and incremental $\mathcal{S}$-stability, both applicable to nonlinear systems. Theorem 1 generalizes to the nonlinear case the Q-parametrization results established by Zames [Zam. 1]. Theorem 2 extends Theorem 1 to include unstable plants. Finally, in Theorem 4, we give a necessary and sufficient condition for the existence of a fixed compensator that stabilizes two given nonlinear plants. It is surprising that these three theorems generalize the linear theory to the nonlinear case and the general formulas of the theory are almost unchanged in form.

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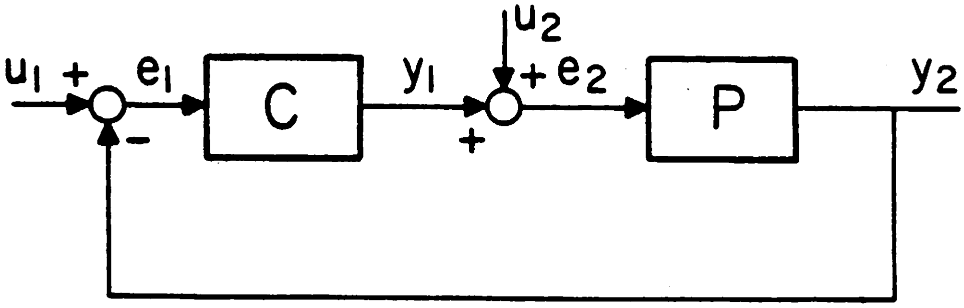

Fig. 1. Shows the system ${}^1S(P,C)$.

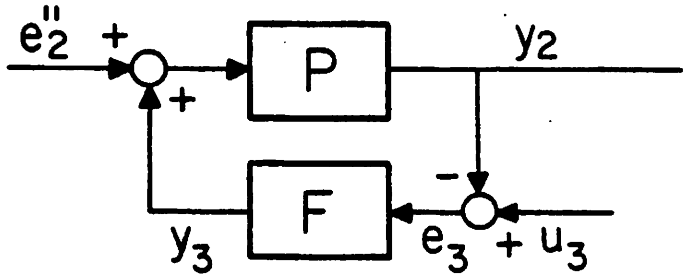

Fig. 2. Shows the system ${}^1S(P,F)$ in which F stabilizes P.

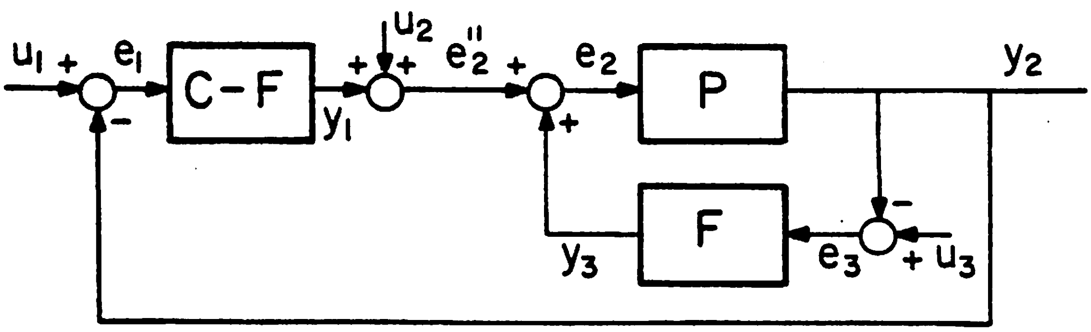

Fig. 3. Shows the system ${}^3S(P,F,C\text{-}F)$.

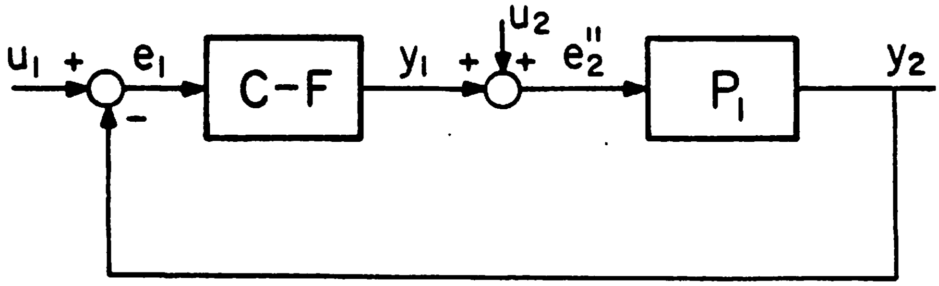

Fig. 4. Shows the system ${}^1S(P_1,C\text{-}F)$.

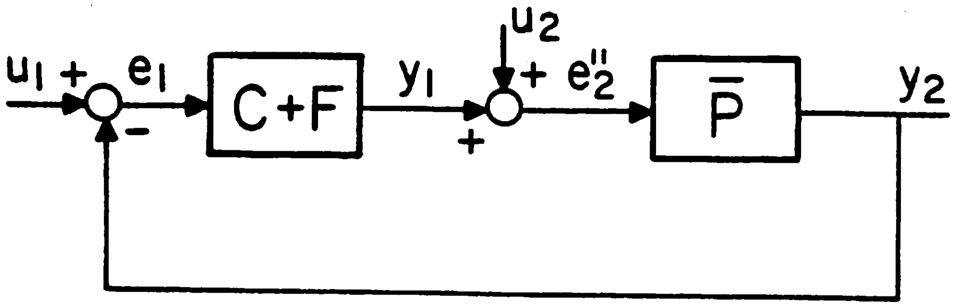

Fig. 5. Shows the system $^1S(\overline{P},C+F)$.

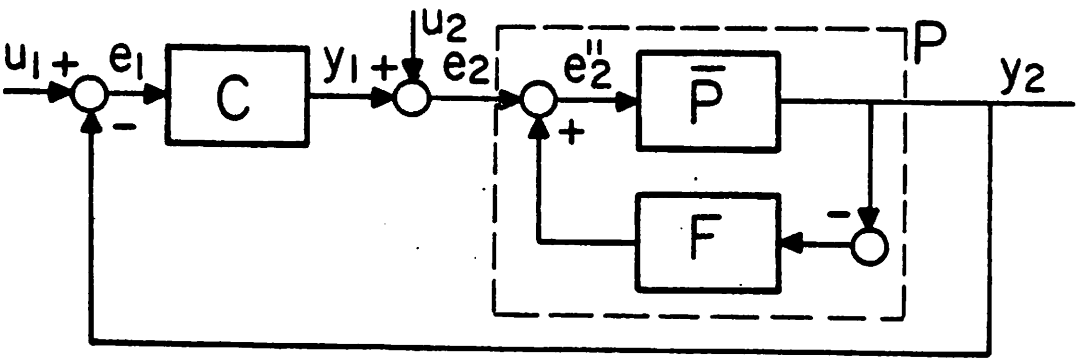

Fig. 6. Shows the system $^3S(\overline{P},F,C)$ with $u_3 \equiv 0$.

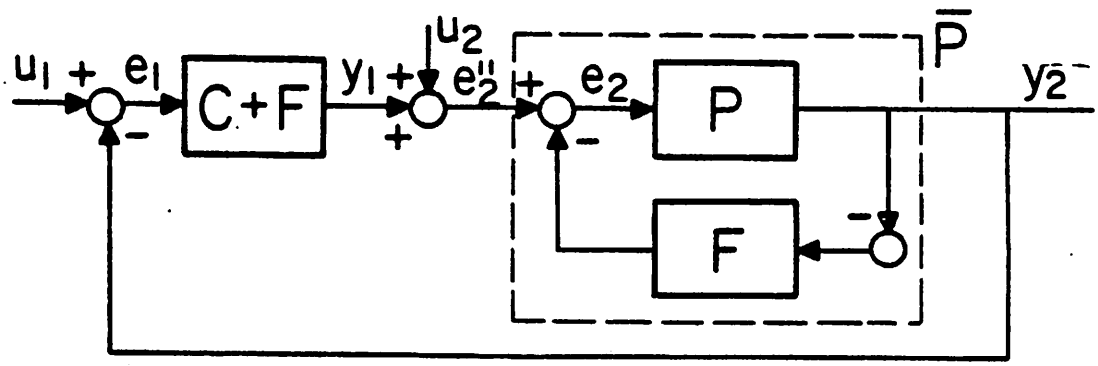

Fig. 7. Shows the system $^3S(P,-F,C+F)$ with $u_3 \equiv 0$.

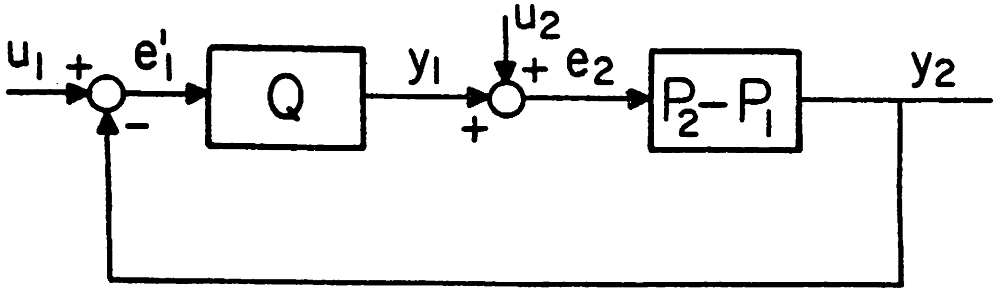

Fig. 8. Shows the system $^1S(P_2-P_1,Q)$.

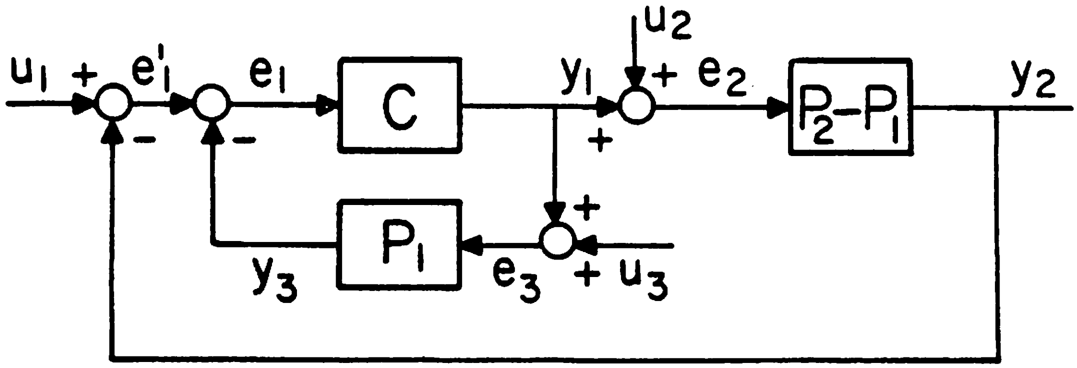

Fig. 9. Shows the system ${}^2S(P_2-P_1,C,P_1)$.

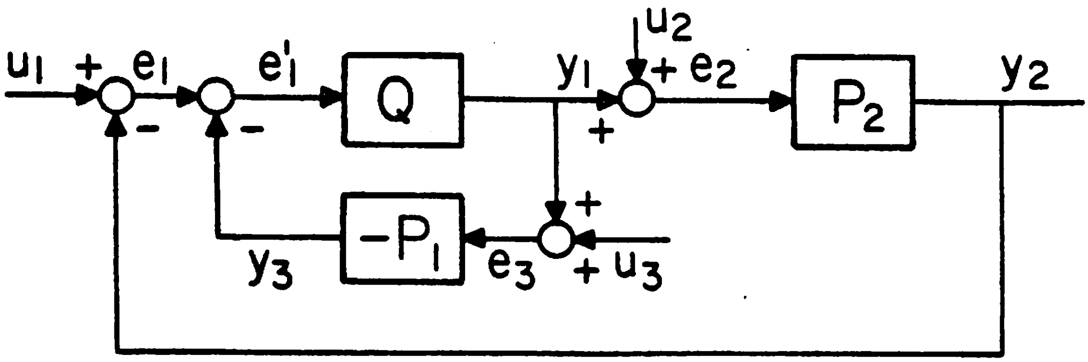

Fig. 10. Shows the system ${}^2S(P_2,Q,-P_1)$.