LINEAR LONGITUDINAL OSCILLATIONS IN COLLISIONLESS PLASMA DIODES WITH THIN SHEATHS. PART I. METHOD

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LINEAR LONGITUDINAL OSCILLATIONS IN COLLISIONLESS PLASMA DIODES WITH THIN SHEATHS. PART I. METHOD

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A method is proposed for treating linear longitudinal perturbations in one-dimensional collisionless plasma diodes with a uniform plasma region and thin electrode sheaths. The method is comprehensive in that it allows for very general equilibrium, initial, boundary, and external- circuit conditions. Upon Laplace-transforming the Vlasov and Poisson equations in both space and time, appropriate evaluation of all pertinent relations leads to a set of $2+2n_\sigma$ (n_σ is the number of particle species) coupled integral equations in x and v for the following quantities (which are the time Laplace transforms of the respective physical perturbations): $\tilde{j}_e(\omega)$ (external-circuit current density), $\tilde{E}(x,\omega)$ (electrostatic field), $\tilde{f}_l^\sigma(v>0,\omega)$, and $\tilde{f}_r^\sigma(v<0,\omega)$ (velocity distribution functions of the plasma-bound particles at the left- and right-hand plasma boundaries, resp.), where σ is the species index. The formal solution of these integral equations and the inverse Laplace transformation ($\omega \rightarrow t$) are discussed in general terms. In particular, it is shown that the intrinsic eigenfrequencies are given by the zeros of the coefficient determinant of the integral equations. A comparison with previous treatments is given, and it is concluded that extensions of the method proposed to more

general systems should be feasible.

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I. INTRODUCTION AND SUMMARY

In the first part (§1) of his famous 1946 paper,[1] Landau derived the general solution to the initial-value problem for small-amplitude longitudinal electron oscillations in an infinite, uniform collisionless plasma. Any perturbation $\tilde{a}(x, v, t)$ was represented as a spatial Fourier integral, with every Fourier component

$$\tilde{a}(q, v, t)\, e^{-iqx} \qquad (q \text{ real})$$

evolving in time independently of the others. (Here and henceforth we use a notation conforming with the rest of this paper.) The key result was Landau's[1] Eq. (10), which provided an explicit expression for the time Laplace transform of the potential-perturbation amplitude associated with a given wavevector q.

In a straightforward extension to multi-species plasmas, Landau's[1] equation (10) can be re-written in the form

$$\tilde{V}(q, \omega) = \frac{4\pi i}{q^2 D(q, \omega)} \sum_{\sigma=1}^{n_\sigma} e^\sigma \int_{-\infty}^{\infty} \frac{dv}{qv + \omega}\, \tilde{f}_i^\sigma(q, v) \tag{1}$$

where q is real, ω is complex (cf. Subsec. II.B), $\tilde{V}(q, \omega)$ is the Fourier-Laplace transform of the electrostatic-potential perturbation $\tilde{V}(x, t)$, $\sigma = 1, ..., n_\sigma$ is the particle-species index, e^σ is the electric charge of a particle of species σ, $\tilde{f}_i^\sigma(q, v)$ is the Fourier transform of $\tilde{f}_i^\sigma(x, v)$ (the initial perturbation of the velocity distribution function of species σ), and

$$D(q, \omega) = 1 - \sum_{\sigma=1}^{n_\sigma} \frac{4\pi (e^\sigma)^2}{m^\sigma q} \int_{-\infty}^{\infty} \frac{dv}{qv + \omega} \frac{d\bar{f}^\sigma(v)}{dv} \tag{2}$$

is the well-known dielectric function for longitudinal plasma oscillations, with $\bar{f}^\sigma(v)$ representing the equilibrium velocity distribution function of species σ.

Here and henceforth, the bar and the tilde indicate equilibrium quantities and perturbations, respectively. For a given value of q, Eqs. (1) and (2) as they stand are valid only in the upper ω half-plane; if $\mathrm{Im}\,\omega \leq 0$, the path of integration must usually be distorted in the complex v-plane, so as to ensure proper analytic continuation with respect to ω.

Inverting $\tilde{V}(q,\omega)$ into $\tilde{V}(q,t)$ one finds that the latter contains, among other possible contributions, terms proportional to $\exp(-i\omega_\nu t)$, where the mode frequencies ω_ν $(\nu = \ldots, 1, 2, \ldots)$ are the zeros of the dielectric function,

$$D(q,\omega_\nu) = 0, \tag{3}$$

and hence also represent poles of $\tilde{V}(q,\omega)$ as given by (1).

Ever since its appearance in the literature, the infinite-plasma dispersion relation (3) (or some analogous dispersion relation for more complex velocity structures) has been used by numerous authors for investigating the wave-propagation and stability properties of a great variety of plasma configurations.[2,3] As long as the wavelengths in question are much shorter than the relevant macroscopic dimensions, this approach is usually believed to be justified. However, due to its relative simplicity, and owing to the fact that a proper treatment of bounded plasma systems is usually much more difficult and, hence, not readily feasible (the eigenmodes are no longer identical with single Fourier modes), Landau's infinite-plasma result is often also applied to situations where a bounded-system analysis would be more appropriate.[4-6] (Rognlien et al.[7] found that the eigenmodes of systems described by fluid-type equations can under certain circumstances, but not generally, be constructed from the roots of the corresponding infinite-system dispersion relation. At the same time, however, they pointed out that their treatment did not in general apply to collisionless plasmas described by kinetic equations.)

Thus, the question we wish to answer in the present work is the following: What is the equivalent of Landau's fundamental result (1) for linear perturbations in collisionless bounded plasma systems, where not only equilibrium plasma properties and initial conditions, but also boundaries and external circuits have to be taken into account? May we expect explicit results similar to (1) (which would be the most favorable case), or will we end up with something considerably more complex? If the latter is true, how can we still extract information on dispersion and stability properties?

We decide to solve this problem for a system that is sufficiently simple to allow for a basically analytic treatment, but still complex enough to account for typical bounded-system effects.

The system we choose is the collisionless, one-dimensional plasma diode with "thin" sheaths (in a sense to be specified in Subsec. II.A) but otherwise allowing for very general equilibrium, boundary, and external-circuit conditions. The major geometrical assumptions, namely one-dimensionality and thin sheaths, are not believed to be crucial to the whole (integral-equation) approach as such (cf. Subsec. III.A) but are certainly helpful in this first stage of development of the theory.

The present work is divided into two parts. In Part I (this paper), the general formalism is developed (Sec. II) and conclusions of a general nature are drawn (Secs. III, IV). Part II[8] presents a detailed and quantitative application of the formalism to a very fundamental special case, namely to an extended Pierce-type problem involving a non-trivial external circuit. This is a striking example of how a plasma that is stable in the infinite-system limit may become violently unstable if part of a bounded system. The numerical results presented in Part II also demonstrate that the external circuit may have drastic effects on the stability and dispersion properties of the whole system, a fact which is frequently ignored.

The rest of this paper (Part I) is organized as follows. Subsection II.A describes the model considered and introduces the basic equations (Vlasov-Poisson) for the plasma region. In Subsec. II.B these equations are Laplace transformed with respect to t and x, the perturbations of the velocity distribution functions are eliminated, and the resulting form of Poisson's equation is Laplace inverted with respect to x. In Subsection II.C, the condition of total-current conservation is used to "link" the plasma to the external circuit at the left-hand electrode, and Poisson's equation is cast into a form symmetric with respect to both electrodes. Subsection II.D presents a very general class of boundary conditions, whence a set of integral equations is derived for the perturbations of the velocity distribution functions at the plasma-sheath boundaries. In Subsec. II.E we adopt a very general class of external-circuit conditions, thus closing the system of equations needed to determine the dynamics of our linearly perturbed diode system.

Subsection II.F reviews the structure of the problem, which in the most general case involves $2 + 2n_\sigma$ coupled integral equations in x and v for the following quantities, which are the time Laplace transforms of the corresponding physical perturbations: $\tilde{j}_e(\omega)$ (external-circuit current density), $\tilde{E}(x, \omega)$ (electrostatic field), $\tilde{f}_l^\sigma(v \geq 0, \omega)$ and $\tilde{f}_r^\sigma(v \leq 0, \omega)$ (velocity distribution functions for the plasma-bound particles at the left-hand and right-hand plasma boundaries, respectively). These integral equations basically represent the bounded-system analogue to Landau's infinite-plasma result (1) and can be easily shown to reduce to the latter in the appropriate limit. Explicit solutions for the time Laplace transforms $\tilde{j}_\sigma(\omega)$ etc. may be expected only for "sufficiently" simple systems. However, the intrinsic eigenfrequencies ω_ν (as opposed to frequencies generated externally) are shown to be the zeros of the coefficient determinant of the up to $2+2n_\sigma$ integral equations in a suitable basis-set representation. Thus, there is still a systematic way of analyzing stability and dispersion properties for any given application.

The discussion in Subsec. III.A leads to the main conclusion that the integral-equation approach adopted here is likely to be extendible to more general and complex problems. In Subsec. III.B we review some pertinent literature and hence conclude that the present method includes a significant class of previous treatments as special cases and, in addition, has potential for investigating problems that have hardly been touched in the previous literature. Finally, Sec. IV summarizes our main conclusions.

II. METHOD

A. Model and basic equations

We consider a one-dimensional diode as shown in Fig. 1. The surfaces of the (ideally conducting) electrodes are located at $x = 0$ ("left-hand electrode") and $x = L$ ("right-hand electrode"). Let the intervening space be filled with a collisionless plasma consisting of n_σ particle species, and let the far ends of the electrodes be connected through an external circuit with specified properties (Subsec. II.E).

For the d.c. state to be perturbed (henceforth referred to as "equilibrium") we assume a uniform plasma with a constant plasma potential $\bar{V}_p$, extending over the region $x_l \leq x \leq x_r$. In the regions $0 < x < x_l$ and $x_r < x < L$ we allow for space-charge sheaths, which are crucial in shaping the equilibrium velocity distribution functions.[9-12] With respect to the perturbations, these sheaths are assumed to be "thin" ($x_l \to 0, x_r \to L$).[13] This approximation is justified if (i) the typical times of particle transit through the sheath regions are short compared with the characteristic time scales of the perturbations (cf. Subsec. II.D), and (ii) the sheath widths are small with respect to the typical scale lengths of the perturbations (cf. Subsec. II.E). Globally speaking, these requirements restrict the validity of our analysis to perturbations which do not crucially depend on the fine structures of the sheaths, and for which the sheath impedances can be neglected.

The thin-sheath approximation is of relevance, e.g., for longitudinal modes in a single-ended Q machine[14,15] at "moderate" interelectrode biases, whereas for "very high" biases the sheath widths may no longer be negligible.[16,17] As an example, Fig. 1 shows a "one-emitter" potential distribution, which represents one of the self-consistent equilibrium configurations of the ideal one-emitter plasma diode or single-ended Q machine.[9-11]

In the plasma region, the small-amplitude longitudinal oscillations we wish to study are described by the linearized Vlasov and Poisson equations:

$$\frac{\partial \tilde{f}^{\sigma}}{\partial t} + v\frac{\partial \tilde{f}^{\sigma}}{\partial x} = -\frac{e^{\sigma}}{m^{\sigma}}\bar{f}^{\sigma}_{v}\tilde{E} \tag{4}$$

$$\frac{\partial \tilde{E}}{\partial x} = 4\pi \sum_{\sigma=1}^{n_{\sigma}} e^{\sigma} \int_{-\infty}^{\infty} dv\, \tilde{f}^{\sigma}, \tag{5}$$

where $E(x,t) = \tilde{E}(x,t)$ is the electrostatic field, m^{σ} and $f^{\sigma}(x,v,t) = \bar{f}^{\sigma}(v) + \tilde{f}^{\sigma}(x,v,t)$ are the particle mass and velocity distribution function of species σ, and $\bar{f}^{\sigma}_{v} = d\bar{f}^{\sigma}(v)/dv$.

Equation (4) has the following formal solution, which will be needed later on:

$$\tilde{f}^{\sigma}(x,v,t) = \tilde{f}^{\sigma}(\hat{x}_o, v, \hat{t}_o) - \frac{e^{\sigma}}{m^{\sigma}}\frac{\bar{f}^{\sigma}_{v}(v)}{v}\int_{\hat{x}_o}^{x} d\hat{x}\, \tilde{E}(\hat{x},\hat{t}) \tag{6}$$

where $\{\hat{x} = x + v(\hat{t} - t), \hat{v} = v, \hat{t}\}$, with $\hat{t}$ as the basic parameter, is any point along the equilibrium particle trajectory passing through the given point (x,v,t), and $\{\hat{x}_o = x + v(\hat{t}_o - t), \hat{v}_o = v, \hat{t}_o\}$ is the starting point for integration along the trajectory.

The problem to be solved consists in predicting the linear evolution in time of the perturbed diode system for given equilibrium, initial, boundary, and external-circuit conditions. In the rest of this section we present a very general method for tackling this problem, which implies introducing several additional relations and unknowns. The following subsections describe the single steps which will eventually (Subsec. II.F) lead to a set of coupled integral equations for the time Laplace transforms of the dynamic variables chosen.

B. Laplace transformations and x-inversion

In order to solve the linearized basic equations in the plasma region, we choose to make use of Laplace transformations in both space and time. Let these be defined by

$$S(q) = \int_0^\infty dx\, e^{iqx}\, S(x), \qquad S(x) = \int_{-\infty+i(\mathrm{Im}\,q)_1}^{\infty+i(\mathrm{Im}\,q)_1} \frac{dq}{2\pi}\, e^{-iqx}\, S(q) \qquad \text{(7a,b)}$$

and

$$T(\omega) = \int_0^\infty dt\, e^{i\omega t}\, T(t), \qquad T(t) = \int_{-\infty+i(\mathrm{Im}\,\omega)_1}^{\infty+i(\mathrm{Im}\,\omega)_1} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, T(\omega), \qquad \text{(8a,b)}$$

where, as usual, $\mathrm{Im}\,q$, $(\mathrm{Im}\,q)_1$, $\mathrm{Im}\,\omega$, and $(\mathrm{Im}\,\omega)_1$ must be "sufficiently" positive. A major advantage of using these transformations lies in the fact that they permit one to handle boundary and initial conditions in a very natural and convenient manner. Note that, although the spatial transformation is formally applied to the whole half-space ($0 < x < \infty$), only the plasma region ($x_l \leq x \leq x_r$) will ultimately be of physical interest.

To our knowledge, the only previous solution of the linearized Vlasov equation by means of the double Laplace transform method has been given by Evans.[18] In Refs. 19 and 20 this method was also applied in the context of cold-fluid type systems. However, all of these treatments were concerned with semi-infinite systems, which may be viewed as special limiting cases of the diode configuration considered here. A preliminary version of the present method was given in Ref. 21.

On application of (8a) (first) and (7a) (second), Eqs. (4) and (5) turn into

$$\tilde{f}^{\sigma}(q,v,\omega)=\frac{i}{qv+\omega}\left[\tilde{f}^{\sigma}_{i}(q,v)+v\tilde{f}^{\sigma}_{l}(v,\omega)-\frac{e^{\sigma}}{m^{\sigma}}\bar{f}^{\sigma}_{v}(v)\tilde{E}(q,\omega)\right] \tag{9}$$

and

$$\tilde{E}_{l}(\omega)+iq\tilde{E}(q,\omega)=-4\pi\sum_{\sigma=1}^{n_{\sigma}}e^{\sigma}\int_{-\infty}^{\infty}dv\,\tilde{f}^{\sigma}(q,v,\omega), \tag{10}$$

where the subscripts i and l indicate values at $t=0$ (initial values) and $x=x_{l}$, respectively. (Analogously, values at $x=x_{r}$ will be denoted by subscript r.) Let us insert the perturbations $\tilde{f}^{\sigma}(q,v,\omega)$ as given by (9) into the Laplace-transformed Poisson equation (10), which then becomes

$$\begin{aligned}\tilde{E}(q,\omega)=&\frac{1}{qD(q,\omega)}\left\{i\tilde{E}_{l}(\omega)\right.\\&\left.-4\pi\sum_{\sigma=1}^{n_{\sigma}}e^{\sigma}\int_{-\infty}^{\infty}\frac{dv}{qv+\omega}\left[\tilde{f}^{\sigma}_{i}(q,v)+v\,\tilde{f}^{\sigma}_{l}(v,\omega)\right]\right\}.\end{aligned} \tag{11}$$

At this point we may most easily recover the infinite-plasma result (1) in the limit of an infinitely long diode. We first note that the double Laplace transform of the definition equation

$$\tilde{E}(x,t)=-\frac{\partial\tilde{V}(x,t)}{\partial x} \tag{12}$$

is given by

$$\tilde{E}(q,\omega)=\tilde{V}_{l}(\omega)+iq\tilde{V}(q,\omega), \tag{13}$$

so that (11) can be easily re-written in terms of $\tilde{V}(q,\omega)$. Let us, just for the present purpose and without lack of generality, transform the interval $0\leq x\leq L$ into the new one $-L/2\leq x\leq L/2$, with $L\to\infty$. Since Landau[1] used Fourier integrals to represent his perturbations, the latter were tacitly assumed to be localized in space. In particular, this is the case for $\tilde{f}^{\sigma}(x,v,t)$, $\tilde{E}(x,t)$, and

$\tilde{V}(x,t)$, so that these quantities all become zero at $x = \pm L/2$ as L tends to infinity. This means that the quantities $\tilde{E}_i(\omega)$ and $\tilde{f}_i^{\sigma}(v,\omega)$ in Eq. (11) must vanish. Finally, letting $\operatorname{Im} q$ become real leads us, via (13), to Eq. (1).

By inspection of Eq. (11) we see that the dependence of $\tilde{E}(q,\omega)$ on q is explicit once the equilibrium distribution functions $\bar{f}^{\sigma}(v)$ and the initial perturbations $\tilde{f}_i^{\sigma}(x,v)$ have been specified. Hence, we can now apply to (11) the inverse spatial transformation (7b), which leads to

$$\begin{aligned}\tilde{E}(x,\omega) = -4\pi \sum_{\sigma=1}^{n_\sigma} e^{\sigma} \int_{-\infty}^{\infty} \frac{dv}{v}\, \tilde{k}_1^{\sigma}(x,v,\omega) \quad &+ \quad i\, k_2(x,\omega)\, \tilde{E}_i(\omega) \\ -4\pi \sum_{\sigma=1}^{n_\sigma} e^{\sigma} \int_{-\infty}^{\infty} dv\, k_3(x,v,\omega)\, \tilde{f}_i^{\sigma}(v,\omega)&. \end{aligned} \tag{14}$$

The known functions $\tilde{k}_1^{\sigma}$, k_2 and k_3 introduced here are defined in Eqs. (A1) through (A3) of Appendix A. There we have also listed a number of other functions which will be encountered in the course of this study and may be considered as being known, in the sense that they can, in principle, be evaluated once the equilibrium, initial, boundary, and external-circuit conditions have been specified.

In deriving Eq. (14) we have interchanged the order of performing the inverse spatial Laplace transformation (7b) and the velocity integration. It is not difficult to justify this procedure for the case of a finite number of cold beams. Moreover, since any plasma may be approximated by a sufficiently large number of cold beams, we conclude that it is also admissible for plasmas with continuous velocity distribution functions.

If the perturbations $\tilde{E}_i(\omega)$ and $\tilde{f}_i^{\sigma}(v,\omega)$ were explicitly known as functions of ω, Eq. (14) could now be Laplace inverted for $\tilde{E}(x,t)$, so that our problem would be paractically solved. This is, in fact, possible at least for Landau's infinite-plasma problem (cf. Sec. I), but in general the above functions must be eliminated or determined via suitable additional relations. In the following subsections, these relations are introduced and incorporated into our formalism.

C. Total-current conservation and symmetric form of Poisson's equation

In a one-dimensional system of the form considered here, the total (i.e., convection plus displacement) current density depends on time only.[22] At the left-hand electrode this condition assumes the form

$$\tilde{j}_e(t) = \frac{1}{4\pi}\frac{d\tilde{E}_l(t)}{dt} + \tilde{j}_l(t), \tag{15}$$

where $\tilde{j}_e(t)$ is the perturbation of the external-current density, and $\tilde{j}_l(t)$ is the perturbation of the convection or particle current density at $x = x_l$:

$$\tilde{j}_l(t) = \sum_{\sigma=1}^{n_\sigma} e^\sigma \int_{-\infty}^{\infty} dv\, v\, \tilde{f}_l^\sigma(v,t). \tag{16}$$

Inserting (16) into (15) and applying the time Laplace transformation (8a) we find

$$\tilde{E}_l(\omega) = \frac{i}{\omega}\left[\tilde{E}_{li} + 4\pi\tilde{j}_e(\omega) - 4\pi\sum_{\sigma=1}^{n_\sigma} e^\sigma \int_{-\infty}^{\infty} dv\, v\, \tilde{f}_l^\sigma(v,\omega)\right]. \tag{17}$$

The constant $\tilde{E}_{li} \equiv \tilde{E}(x = x_l,\, t = 0)$ can be calculated in terms of other initial values by integrating (5) twice with respect to x; at $t = 0$:

$$\tilde{E}_{li} = -\frac{\Delta\tilde{V}_{pi}}{L} - \frac{4\pi}{L}\sum_{\sigma=1}^{n_\sigma} e^\sigma \int_0^L dx \int_0^x dx' \int_{-\infty}^{\infty} dv\, \tilde{f}_i^\sigma(x',v) \tag{18}$$

where $\Delta\tilde{V}_{pi} \equiv \Delta\tilde{V}_p(t = 0)$, $\Delta\tilde{V}_p(t)$ being the perturbation of the potential drop across the plasma region (cf. Fig. 1 and Eq. (12)):

$$\Delta\tilde{V}_p(t) = -\int_{x_l}^{x_r} dx\,\tilde{E}(x,t). \tag{19}$$

According to Sec. II.E below, the initial potential-difference perturbation $\Delta\tilde{V}_{pi}$ is essentially determined by the initial state of the external circuit.

We now insert $\tilde{E}_l(\omega)$ as given by Eq. (17) into Poisson's equation (14) and, after rearranging terms, obtain

$$\begin{aligned}\tilde{E}(x,\omega) = {} & \tilde{k}_4(x,\omega) + k_5(x,\omega)\tilde{j}_e(\omega) \\ & + 4\pi\sum_{\sigma=1}^{n_\sigma} e^\sigma \int_{-\infty}^{\infty} dv\, k_6(x,v,\omega)\,\tilde{f}_l^\sigma(v,\omega),\end{aligned} \tag{20}$$

where the (known) functions $\tilde{k}_4$, k_5, and k_6 are defined by Eqs. (A4) through (A6). Although Eq. (20) contains the same number of unknowns as Eq. (14), progress has been made in that $\tilde{E}_l(\omega)$ has been replaced with the more useful quantity $\tilde{j}_e(\omega)$, which establishes a link between the plasma and the external circuit.

Equation (20) contains a velocity integral to be evaluated at $x = x_l$, but no such integral for $x = x_r$. It is thus asymmetric with respect to the two electrodes, none of which is, however, *a priori* assigned a preferred role. We choose to remove this asymmetry by expressing $\tilde{f}_l^\sigma(v<0,\omega)$ in terms of $\tilde{f}_r^\sigma(v<0,\omega)$, to which end we first evaluate (6) at $x = x_l \to 0$:

$$\begin{aligned}\tilde{f}_l^\sigma(v<0,t) = {} & U(t+\frac{L}{v})\left\{\tilde{f}_r^\sigma(v,t+\frac{L}{v}) - \frac{e^\sigma}{m^\sigma}\frac{\bar{f}_v^\sigma(v)}{v}\int_L^0 d\hat{x}\,\tilde{E}(\hat{x},t+\frac{\hat{x}}{v})\right\} \\ & + U(-t-\frac{L}{v})\left\{\tilde{f}_i^\sigma(-vt,v) - \frac{e^\sigma}{m^\sigma}\frac{\bar{f}_v^\sigma(v)}{v}\int_{-vt}^0 d\hat{x}\,\tilde{E}(\hat{x},t+\frac{\hat{x}}{v})\right\},\end{aligned} \tag{21}$$

where U is the Heaviside unit step function. Obviously, the term with $U(t+L/v)$ represents the contribution of those particles which took off at the right-hand electrode at a time $\hat{t}_0 > 0$. The second term describes the influence of those (slower) particles which at time $\hat{t}_0 = 0$ were already on their way towards the left-hand electrode and thus are the immediate carriers of the information contained in the initial conditions. The time Laplace transform (8a) of Eq. (21) is

$$\tilde{f}_l^\sigma(v<0,\omega) = -\frac{1}{v}\tilde{k}_7^\sigma(v,\omega) + \frac{e^\sigma}{m^\sigma}\frac{\bar{f}_v^\sigma(v)}{v}\int_0^L dx'\,\exp(-i\omega\frac{x'}{v})\tilde{E}(x',\omega) + \exp(-i\omega\frac{L}{v})\,\tilde{f}_r^\sigma(v,\omega). \tag{22}$$

For later use we also write down the analogous relations for the right-hand electrode, namely

$$\tilde{f}_r^\sigma(v>0,t) = U(\frac{L}{v}-t)\left\{\tilde{f}_i^\sigma(L-vt,v) - \frac{e^\sigma}{m^\sigma}\frac{\bar{f}_v^\sigma(v)}{v}\int_{L-vt}^L d\hat{x}\,\tilde{E}(\hat{x},\hat{t})\right\} + U(t-\frac{L}{v})\left\{\tilde{f}_l^\sigma(v,t-\frac{L}{v}) - \frac{e^\sigma}{m^\sigma}\frac{\bar{f}_v^\sigma(v)}{v}\int_0^L d\hat{x}\,\tilde{E}(\hat{x},\hat{t})\right\} \tag{23}$$

and its time Laplace transform

$$\tilde{f}_r^\sigma(v>0,\omega) = \frac{1}{v}\exp(i\omega\frac{L}{v})\,\tilde{k}_7^\sigma(v,\omega) - \frac{e^\sigma}{m^\sigma}\frac{\bar{f}_v^\sigma(v)}{v}\exp(i\omega\frac{L}{v})\int_0^L dx'\,\exp(-i\omega\frac{x'}{v})\tilde{E}(x',\omega) + \exp(i\omega\frac{L}{v})\,\tilde{f}_l^\sigma(v,\omega). \tag{24}$$

Inserting (22) into (20) we obtain

$$\tilde{E}(x,\omega) = \tilde{k}_8(x,\omega) + k_5(x,\omega)\tilde{j}_e(\omega) + \int_0^L dx'\,k_9(x,x',\omega)\tilde{E}(x',\omega) + 4\pi\sum_{\sigma=1}^{n_\sigma} e^\sigma\left\{\int_0^\infty dv\,k_6(x,v,\omega)\tilde{f}_l^\sigma(v,\omega) + \int_{-\infty}^0 dv\,k_6(x,v,\omega)\exp(-i\omega\frac{L}{v})\tilde{f}_r^\sigma(v,\omega)\right\}, \tag{25}$$

with the newly introduced known functions $\tilde{k}_8$ and k_9 defined in Appendix A.

Equation (25) is the more symmetric version of Poisson's equation that we have been looking for. It contains two source terms corresponding to the particles entering the plasma region at the plasma-sheath boundaries. If the boundary perturbations $\tilde{f}_l^\sigma(v > 0, \omega)$ and $\tilde{f}_r^\sigma(v < 0, \omega)$ were explicitly known, (25) would represent a Fredholm-type integral equation for $\tilde{E}(x, \omega)$ only, with ω and $\tilde{j}_e(\omega)$ as parameters. However, in many cases of interest these functions will not be known *a priori* but have to be determined with the help of the relevant boundary conditions. Hence, the latter must now be specified. In the following subsection we introduce a very general class of boundary conditions and therewith derive additional equations for our unknowns.

D. Boundary conditions on particles and resulting integral equations

Let us assume that, at the plasma-sheath boundaries $x = x_l$ and $x = x_r$, the perturbations of the velocity distribution functions of the plasma-bound particles are related to those of the sheath-bound particles as follows:

$$\tilde{f}_l^\sigma(v > 0, t) = \tilde{f}_{lg}^\sigma(v, t) + \sum_{\sigma'=1}^{n_\sigma} \int_{-\infty}^{0} dv'\, b_l^{\sigma\sigma'}(v, v')\, \tilde{f}_l^{\sigma'}(v', t) \tag{26a}$$

$$\tilde{f}_r^\sigma(v < 0, t) = \tilde{f}_{rg}^\sigma(v, t) + \sum_{\sigma'=1}^{n_\sigma} \int_{0}^{\infty} dv'\, b_r^{\sigma\sigma'}(v, v')\, \tilde{f}_r^{\sigma'}(v', t), \tag{26b}$$

where $\tilde{f}_{lg}^\sigma$ and $\tilde{f}_{rg}^\sigma$ are externally generated (and, hence, explicitly given) perturbations, and the functions $\tilde{b}_l^{\sigma\sigma'}(v > 0, v' < 0)$ and $\tilde{b}_r^{\sigma\sigma'}(v < 0, v' > 0)$ essentially represent the probabilities for a sheath-bound particle of species σ' with velocity v' to "produce" a plasma-bound particle of species σ with velocity v. The time Laplace transforms of (26a,b) are

$$\tilde{f}_l^{\sigma}(v > 0, \omega) = \tilde{f}_{lg}^{\sigma}(v, \omega) + \sum_{\sigma'=1}^{n_\sigma} \int_{-\infty}^{0} dv' \, b_l^{\sigma\sigma'}(v, v') \, \tilde{f}_l^{\sigma'}(v', \omega) \tag{27a}$$

$$\tilde{f}_r^{\sigma}(v < 0, \omega) = \tilde{f}_{rg}^{\sigma}(v, \omega) + \sum_{\sigma'=1}^{n_\sigma} \int_{0}^{\infty} dv' \, b_r^{\sigma\sigma'}(v, v') \, \tilde{f}_r^{\sigma'}(v', \omega). \tag{27b}$$

As an example of an external perturbation mechanism let us mention the fluctuations in the flux of neutral alkali atoms incident on a hot metal electrode in a Q machine, which may cause variations in the flux of the plasma-bound ions produced by contact ionization.

The homogeneous terms in (26a,b) are capable of describing a wide class of generalized reflection processes that a sheath-bound particle of species σ' incident with velocity v' can undergo. Among them are some simple ones most commonly used (cf. Subsec. III.B). If, e.g., the sheath-bound particle is always absorbed without releasing any other particle, we have

$$b_l^{\sigma\sigma'}(v > 0, v' < 0) = 0 \tag{28a}$$

$$b_r^{\sigma\sigma'}(v < 0, v' > 0) = 0 \tag{28b}$$

for all permitted values of σ and v. For specular reflection, the b's become

$$b_l^{\sigma\sigma'}(v > 0, v' < 0) = \delta_{\sigma\sigma'} \, \delta(v + v') \tag{29a}$$

$$b_r^{\sigma\sigma'}(v < 0, v' > 0) = \delta_{\sigma\sigma'} \, \delta(v + v'), \tag{29b}$$

where δ_{ij} is the Kronecker delta symbol and $\delta(x)$ is the Dirac delta function. Clearly, the boundary conditions (26) can also describe processes like inelastic reflection or secondary-particle emission, but these should be discussed in specific future applications rather than in the present general outline of the theory.

In (26), the perturbations of the velocity distribution functions for the plasma-bound particles are evaluated at the same time t as those for the sheath-bound particles. This involves two major physical assumptions, namely that both (i) the time needed for a particle to traverse the sheath regions, and (ii) the time scales of the generalized reflection processes taking place at the electrode surfaces proper, are negligibly small in comparison with the time scales of the dynamic phenomena of interest. However, these "instant-reflection" conditions are likely to be satisfied in many cases of practical interest.

It should be borne in mind that the "boundary" conditions (26) are not only determined by the electrode surface properties but also account, in a global manner, for the sheath regions. (This global treatment is made possible by the assumption in Subsec. II.A of "thin" sheaths, whereas a finite-sheath theory would require a more refined approximation to the solution of the Vlasov-Poisson system in the sheath regions.) Consider, e.g., a singly charged positive ion of mass m moving to the right in the configuration shown in Fig. 1. If $v < [2e(\bar{V}_L - \bar{V}_p)/m]^{1/2}$, the ion is reflected somewhere in the right-hand sheath region ($x_l < x < L$), which in terms of (26) counts as a specular reflection, cf. (29b). In the opposite case the ion hits the electrode surface proper and there may be neutralized, undergo inelastic reflection, or even release secondary particles. Clearly, the description of these latter processes will require a form of $b_r^{\sigma\sigma'}$ considerably more complex than (29b).

We are now in a position to derive the additional relations required for determining the functions $\tilde{f}_l^{\sigma}(v > 0, \omega)$ and $\tilde{f}_r^{\sigma}(v < 0, \omega)$, which occur in Poisson's equation (25). Let us start with $\tilde{f}_l^{\sigma}(v > 0, \omega)$ as given by (27a), and express $\tilde{f}_l^{\sigma}(v < 0, \omega)$ on the r.h.s. step by step in terms of $\tilde{f}_r^{\sigma}(v < 0, \omega)$,

$\tilde{f}_r^\sigma(v > 0, \omega)$, and $\tilde{f}_l^\sigma(v > 0, \omega)$ by successive application of Eqs. (22), (27b), and (24). What comes out is the integral equation

$$\tilde{f}_l^\sigma(v > 0, \omega) = \tilde{k}_{10}^\sigma(v, \omega) + \int_0^L dx'\, k_{11}^\sigma(x', v, \omega)\, \tilde{E}(x', \omega) + \sum_{\sigma'=1}^{n_\sigma} \int_0^\infty dv'\, k_{12}^{\sigma\sigma'}(v, v', \omega)\, \tilde{f}_l^{\sigma'}(v', \omega). \tag{30}$$

In an analogous manner we can express $\tilde{f}_r^\sigma(v < 0, \omega)$ consecutively in terms of $\tilde{f}_r^\sigma(v > 0, \omega)$, $\tilde{f}_l^\sigma(v > 0, \omega)$, $\tilde{f}_l^\sigma(v < 0, \omega)$, and $\tilde{f}_r^\sigma(v < 0, \omega)$ by using Eqs. (27b), (24), (27a), and (22). The result is

$$\tilde{f}_r^\sigma(v < 0, \omega) = \tilde{k}_{13}^\sigma(v, \omega) + \int_0^L dx'\, k_{14}^\sigma(x', v, \omega)\, \tilde{E}(x', \omega) + \sum_{\sigma'=1}^{n_\sigma} \int_{-\infty}^0 dv'\, k_{15}^{\sigma\sigma'}(v, v', \omega)\, \tilde{f}_r^{\sigma'}(v', \omega). \tag{31}$$

Again, the newly introduced known functions are defined in Appendix A.

Equations (25), (30), and (31) constitute a system of $1+2n_\sigma$ coupled integral equations in x and v for the $1 + 2n_\sigma$ functions $\tilde{E}(x, \omega)$, $\tilde{f}_l^\sigma(v > 0, \omega)$, and $\tilde{f}_r^\sigma(v < 0, \omega)$. Since both ω and $\tilde{j}_e(\omega)$ enter as parameters, yet another relation is required in order to specify the ω-dependence of the above functions and to thus make them invertible into $\tilde{E}(x, t)$ etc. In the following subsection we show how this relation can be obtained by combining the overall potential balance with the external-circuit condition.

E. Overall potential balance and external-circuit condition

So far we have essentially concentrated on the plasma region, with $\tilde{j}_e(\omega)$ entering as a free parameter, cf. Eq. (25). Thus, in order to close the problem we need yet another suitable equation that will completely specify the ω-dependence of the quantities involved. In the present subsection we derive this equation by combining (i) the overall potential balance in the diode/external-circuit system, (ii) the potential balance in the external circuit, and (iii) the condition of zero sheath impedance, which is consistent with the thin-sheath approximation.

The overall balance for the potential perturbations may be written

$$\Delta\tilde{V}_{e}(t) = \Delta\tilde{V}_{sl}(t) + \Delta\tilde{V}_{p}(t) + \Delta\tilde{V}_{sr}(t), \tag{32}$$

where $\Delta\tilde{V}_{sl}(t) = \tilde{V}(x_l,t) - \tilde{V}(0,t)$ and $\Delta V_{sr}(t) = \tilde{V}(L,t) - \tilde{V}(x_r,t)$ are the perturbations of the potential drops across the left-hand and the right-hand sheath regions, respectively. The potential balance for the external circuit ("external-circuit condition") we choose to write in the form

$$\Delta\tilde{V}_{e}(t) = \hat{Z}(t)\tilde{j}_{e}(t), \tag{33}$$

where $\hat{Z}(t)$ is a general linear impedance operator that may be considered known once the external-circuit properties have been specified.

In order for (32) to be useful in the context of our preceding considerations (where the basic equations were solved for the uniform plasma region), $\Delta\tilde{V}_{sl}(t)$ and $\Delta\tilde{V}_{sr}(t)$ must be calculated. In a finite-sheath theory this would imply at least approximate integration of the Vlasov-Poisson system in the (non-uniform) sheath regions. However, in the thin-sheath approximation considered throughout this paper (cf. Subsec. II.A) we may, by definition, set

$$\Delta\tilde{V}_{sl}(t) = 0 = \Delta\tilde{V}_{sr}(t), \tag{34}$$

so that (32) reduces to

$$\Delta\tilde{V}_{e}(t) = \Delta\tilde{V}_{p}(t). \tag{35}$$

This means that the perturbations of the potential drops developing over the thin sheath regions are neglected against those building up over the much more extended plasma region. (This behavior is exactly opposed to equilibrium conditions, where the main potential drops occur over the sheath regions.)

The operator $\hat{Z}$ introduced in Eq. (32) is assumed to be linear but otherwise completely general. However, for many (if not all) cases of practical interest $\hat{Z}$ will be of such a form that the time Laplace transform of (33) becomes

$$\Delta\tilde{V}_e(\omega) = \tilde{V}_{e0}(\omega) + Z_e(\omega)\tilde{j}_e(\omega). \tag{36}$$

This class of external circuits (which is still very general in that it includes the usual RLC circuits with a.c. sources) is the one considered henceforth. It may be useful to note that $\tilde{V}_{e0}(\omega)$ reflects the effect of initial conditions and/or external sources, whereas $Z_e(\omega)$ is a generalized impedance. The initial state of the external circuit also specifies the initial potential-difference perturbations $\Delta\tilde{V}_{ei}$ and $\Delta\tilde{V}_{pi}$.

An explicit treatment of an exemplary external circuit is given in Part II,[8] where the above aspects are worked out in detail.

Equation (36), with $\Delta\tilde{V}_e(\omega) = \Delta\tilde{V}_p(\omega)$, closes the system of equations governing our perturbational problem, whose general structure is discussed in the following subsection.

F. General structure of the problem and basis-set expansion

Equations (36) (combined with (35) and (19)), (25), (30), and (31) form a closed set of $2+2n_\sigma$ coupled integral equations, from which the $2+2n_\sigma$ unknown time Laplace transforms $\tilde{j}_e(\omega)$, $\tilde{E}(x,\omega)$, $\tilde{f}_l^\sigma(v > 0,\omega)$, and $\tilde{f}_r^\sigma(v < 0,\omega)$ can, in principle, be determined as functions of x, v, and ω. (Once these solutions have been established, $\tilde{f}(x,v,\omega)$ could be calculated via (9).) The last step in solving the complete perturbational problem then consists in applying the inverse time Laplace transform (8b) to the above functions, which will yield the time-dependent perturbations $\tilde{j}_e(t)$, $\tilde{E}(x,t)$, $\tilde{f}_l^\sigma(v > 0,t)$, $\tilde{f}_r^\sigma(v < 0,t)$, and finally, via (6), $\tilde{f}^\sigma(x,v,t)$ $\quad(\sigma = 1,\ldots,n_\sigma)$.

In order to reduce the problem to its basic structure, we re-write the above equations more concisely as follows:

$$Z_e(\omega)\,\tilde{j}_e(\omega) + \int_0^L dx'\,\tilde{E}(x',\omega) = -\tilde{V}_{e0}(\omega) \tag{37}$$

$$\left.\begin{aligned} &-k_5(x,\omega)\,\tilde{j}_e(\omega) + \tilde{E}(x,\omega) + \mathbb{S}_0(x,[x'],\omega)\,\tilde{E}([x'],\omega) \\ &\quad + \sum_{\sigma=1}^{n_\sigma} \mathbb{V}_{0l}^\sigma(x,[v>0],\omega)\,\tilde{f}_l^\sigma([v],\omega) \\ &\quad\quad + \sum_{\sigma=1}^{n_\sigma} \mathbb{V}_{0r}^\sigma(x,[v<0],\omega)\,\tilde{f}_r^\sigma([v],\omega) = \tilde{k}_8(x,\omega) \end{aligned}\right\} \tag{38}$$

$$\begin{aligned} &\mathbb{S}_l^\sigma(v>0,[x'],\omega)\,\tilde{E}([x'],\omega) \\ &\quad + \tilde{f}_l^\sigma(v,\omega) + \sum_{\sigma'=1}^{n_\sigma} \mathbb{V}_l^{\sigma\sigma'}(v,[v'>0],\omega)\,\tilde{f}_l^{\sigma'}([v'],\omega) = \tilde{k}_{10}^\sigma(v,\omega) \end{aligned} \tag{39}$$

$$\begin{aligned} &\mathbb{S}_r^\sigma(v<0,[x'],\omega)\,\tilde{E}([x'],\omega) \\ &\quad + \tilde{f}_r^\sigma(v,\omega) + \sum_{\sigma'=1}^{n_\sigma} \mathbb{V}_r^{\sigma\sigma'}(v,[v'<0],\omega)\,\tilde{f}_r^{\sigma'}([v'],\omega) = \tilde{k}_{13}^\sigma(v,\omega) \end{aligned} \tag{40}$$

where S_0, S_l, S_r are known space-integral operators and $\mathcal{V}^{\sigma}_{0l}$, $\mathcal{V}^{\sigma}_{0r}$, $\mathcal{V}^{\sigma\sigma'}_{l}$, $\mathcal{V}^{\sigma\sigma'}_{r}$ are known velocity-integral operators. These operators are defined in Eqs. (A16) through (A22) of Appendix A. The square brackets indicate the variables of integration.

The integral equations (37)–(40) may be viewed as the bounded-system analogue to the infinite-system result (1), which answers the question raised in Sec.I. It thus turns out that in general (i.e., for sufficiently non-degenerate cases) it is no longer possible to obtain the time Laplace transforms of the perturbations explicitly as functions of ω. However, in Subsec. II.G it will turn out that there still exists a systematic way of analyzing stability and dispersion properties.

Depending on the degeneracy of a specific configuration considered, some of the equations (37)–(40) may be trivially satisfied (as is the case in the extended Pierce-type problem treated in Part II[8]), so that only an appropriate subset thereof must be actually considered.

In practical applications, when the coefficient functions and integral operators occurring in Eqs. (37)–(40) must be explicitly established, it is convenient to distinguish between four levels of quantities as illustrated by Table I. The level-one quantities, which determine the equilibrium, initial, boundary, and external-circuit conditions, must be specified, while any higher-level quantity can be constructed from lower-level ones. For each quantity is indicated in Table I an equation or section where it is defined or introduced. The level-four quantities are the functions and operators occurring in Eqs. (37)–(40).

Once established explicitly, Eqs. (37)–(40) will in many cases be too complicated to allow for closed analytical solutions. It may then be convenient to expand the functions and operators involved in suitable sets of basis functions. This permits one to transform the integral equations into a system of linear algebraic equations for the expansion coefficients of the unknown functions. The coordinate ranges to be covered are $0 < v \leq v_{max}$ and $v_{min} \leq v < 0$, where the cutoff velocities v_{max} and v_{min} must be chosen so as to cover all phenomena of interest in velocity space. Let there be given three denumerably infinite, complete sets of linearly independent (but not necessarily orthogonal and/or normalized) basis functions $\{\varphi_\kappa(x)\}$, $\{\varphi_\lambda^+(v > 0)\}$, and $\{\varphi_\mu^-(v < 0)\}$, defined in the above intervals, respectively, with $\kappa, \lambda, \mu = \ldots 1, 2, \ldots$. Any function involved may then be expanded in one of the forms

$$f(x) = f(\kappa)\ \varphi_\kappa(x) \tag{41a}$$

$$f_l(v > 0) = f_l(\lambda)\ \varphi_\lambda^+(v) \tag{41b}$$

$$f_r(v < 0) = f_r(\mu)\ \varphi_\mu^-(v) \tag{41c}$$

or straightforward extensions thereof, e.g.

$$g(x, v < 0) = g(\kappa, \mu)\ \varphi_\kappa(x)\ \varphi_\mu^-(v). \tag{42}$$

Note that we have adopted the Einstein convention (double indeces imply summation) for κ, λ, and μ (but not for the species index σ). As usual, these expansions permit us to represent continuous functions and operators in terms of their respective coefficient vectors or matrices. Generally, the latter are infinite but in practice must always be truncated, the number of terms retained usually being a compromise between accuracy requirements and technical limitations.

The coefficient equations corresponding to (37)–(40) are respectively given by

$$Z_e(\omega)\,\tilde{j}_e(\omega) + \left[\int_0^L dx'\ \varphi_{\kappa'}(x')\right]\tilde{E}(\kappa',\omega) = -\tilde{V}_{e0}(\omega) \tag{43}$$

$$\begin{aligned} &-k_5(\kappa,\omega)\,\tilde{j}_e(\omega) + [\delta_{\kappa\kappa'} + S_0(\kappa,\kappa',\omega)]\,\tilde{E}(\kappa',\omega) \\ &+ \sum_{\sigma=1}^{n_\sigma} V_{0l}^{\sigma}(\kappa,\lambda',\omega)\,\tilde{f}_l^{\sigma}(\lambda',\omega) + \sum_{\sigma=1}^{n_\sigma} V_{0r}^{\sigma}(\kappa,\mu',\omega)\,\tilde{f}_r^{\sigma}(\mu',\omega) = \tilde{k}_8(\kappa,\omega) \end{aligned} \tag{44}$$

$$\begin{aligned} &S_l^{\sigma}(\lambda,\kappa',\omega)\,\tilde{E}(\kappa',\omega) \\ &\quad + \tilde{f}_l^{\sigma}(\lambda,\omega) + \sum_{\sigma'=1}^{n_\sigma} V_l^{\sigma\sigma'}(\lambda,\lambda',\omega)\,\tilde{f}_l^{\sigma'}(\lambda',\omega) = \tilde{k}_{10}^{\sigma}(\lambda,\omega) \end{aligned} \tag{45}$$

$$\begin{aligned} &S_r^{\sigma}(\mu,\kappa',\omega)\,\tilde{E}(\kappa',\omega) \\ &\quad + \tilde{f}_r^{\sigma}(\mu,\omega) + \sum_{\sigma'=1}^{n_\sigma} V_r^{\sigma\sigma'}(\mu,\mu',\omega)\,\tilde{f}_r^{\sigma'}(\mu',\omega) = \tilde{k}_{13}^{\sigma}(\mu,\omega), \end{aligned} \tag{46}$$

where the matrix elements representing the above integral operators are defined in a straightforward manner. As an example consider the operator S_0, whose matrix elements may be written in the form

$$\begin{aligned} &S_0(\kappa,\kappa',\omega) = \\ &-4\pi \sum_{\sigma=1}^{n_\sigma} \frac{(e^{\sigma})^2}{m^{\sigma}} \int_{-\infty}^{0} \frac{dv}{v}\, k_6(\kappa,v,\omega)\,\bar{f}_v^{\sigma}(v) \int_0^L dx'\ \exp(-i\omega\frac{x'}{v})\,\varphi_{\kappa'}(x'). \end{aligned} \tag{47}$$

Using matrix notation, we may re-write Eqs. (43)–(46) in the form

$$\underline{D}(\omega)\cdot\tilde{\mathbf{u}}(\omega) = \tilde{\mathbf{k}}(\omega). \tag{48}$$

The (known) infinite coefficient matrix $\underline{D}(\omega)$, the (known) infinite column vector $\tilde{\mathbf{k}}(\omega)$, and the (unknown) infinite column vector $\tilde{\mathbf{u}}(\omega)$ introduced here are given by

$$\underline{D}(\omega) =$$

$$\begin{pmatrix}
Z_e & L\underline{\varphi}^T & \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\
-\mathbf{k}_5 & \underline{1}+\underline{S}_0 & \underline{V}_{0l}^1 & \underline{V}_{0l}^2 & \cdots & \underline{V}_{0l}^{n_\sigma} & \underline{V}_{0r}^1 & \underline{V}_{0r}^2 & \cdots & \underline{V}_{0r}^{n_\sigma} \\
\mathbf{0} & \underline{S}_l^1 & \underline{1}+\underline{V}_l^{11} & \underline{V}_l^{12} & \cdots & \underline{V}_l^{1n_\sigma} & \underline{0} & \underline{0} & \cdots & \underline{0} \\
\mathbf{0} & \underline{S}_l^2 & \underline{V}_l^{21} & \underline{1}+\underline{V}_l^{22} & \cdots & \underline{V}_l^{2n_\sigma} & \underline{0} & \underline{0} & \cdots & \underline{0} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\mathbf{0} & \underline{S}_l^{n_\sigma} & \underline{V}_l^{n_\sigma 1} & \underline{V}_l^{n_\sigma 2} & \cdots & \underline{1}+\underline{V}_l^{n_\sigma n_\sigma} & \underline{0} & \underline{0} & \cdots & \underline{0} \\
\mathbf{0} & \underline{S}_r^1 & \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{1}+\underline{V}_r^{11} & \underline{V}_r^{12} & \cdots & \underline{V}_r^{1n_\sigma} \\
\mathbf{0} & \underline{S}_r^2 & \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{V}_r^{21} & \underline{1}+\underline{V}_r^{22} & \cdots & \underline{V}_r^{2n_\sigma} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\mathbf{0} & \underline{S}_r^{n_\sigma} & \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{V}_r^{n_\sigma 1} & \underline{V}_r^{n_\sigma 2} & \cdots & \underline{1}+\underline{V}_r^{n_\sigma n_\sigma}
\end{pmatrix} \tag{49a}$$

$$\tilde{\mathbf{k}}(\omega) = \begin{pmatrix} -\tilde{V}_{s0} \\ \tilde{\mathbf{k}}_8 \\ \tilde{\mathbf{k}}_{10}^1 \\ \tilde{\mathbf{k}}_{10}^2 \\ \vdots \\ \tilde{\mathbf{k}}_{10}^{n_\sigma} \\ \tilde{\mathbf{k}}_{13}^1 \\ \tilde{\mathbf{k}}_{13}^2 \\ \vdots \\ \tilde{\mathbf{k}}_{13}^{n_\sigma} \end{pmatrix} \qquad \tilde{\mathbf{u}}(\omega) = \begin{pmatrix} \tilde{j}_e \\ \tilde{\mathbf{E}} \\ \tilde{\mathbf{f}}_l^1 \\ \tilde{\mathbf{f}}_l^2 \\ \vdots \\ \tilde{\mathbf{f}}_l^{n_\sigma} \\ \tilde{\mathbf{f}}_r^1 \\ \tilde{\mathbf{f}}_r^2 \\ \vdots \\ \tilde{\mathbf{f}}_r^{n_\sigma} \end{pmatrix}, \tag{49b,c}$$

where 0, $\tilde{\varphi}$, $\mathbf{k}_5$, $\tilde{\mathbf{k}}_8$, $\tilde{\mathbf{k}}_{10}^{\sigma}$, $\tilde{\mathbf{k}}_{13}^{\sigma}$, $\tilde{\mathbf{E}}$, $\tilde{\mathbf{f}}_l^{\sigma}$, $\tilde{\mathbf{f}}_r^{\sigma}$ are infinite column (sub)vectors with components 0, $L^{-1}\int_0^L dx\,\varphi_\kappa(x)$, $k_5(\kappa,\omega)$, $\tilde{k}_8(\kappa,\omega)$, $\tilde{k}_{10}^{\sigma}(\lambda,\omega)$, $\tilde{k}_{13}^{\sigma}(\mu,\omega)$, $\tilde{E}(\kappa,\omega)$, $\tilde{f}_l^{\sigma}(\lambda,\omega)$, $\tilde{f}_r^{\sigma}(\mu,\omega)$, respectively, the superscript T indicates transposed vectors, and $\underline{0}$, $\underline{1}$, $\underline{S}_0$, $\underline{S}_l^{\sigma}$, $\underline{S}_r^{\sigma}$, $\underline{V}_{0l}^{\sigma}$, $\underline{V}_{0r}^{\sigma}$, $\underline{V}_l^{\sigma\sigma'}$, $\underline{V}_r^{\sigma\sigma'}$ are infinite (sub)matrices with elements 0, $\delta_{\kappa\kappa'}$, $S_0(\kappa,\kappa',\omega)$, $S_l^{\sigma}(\lambda,\kappa',\omega)$ $S_r^{\sigma}(\mu,\kappa',\omega)$, $V_{0l}^{\sigma}(\kappa,\lambda',\omega)$, $V_{0r}^{\sigma}(\kappa,\mu',\omega)$, $V_l^{\sigma\sigma'}(\lambda,\lambda',\omega)$, $V_r^{\sigma\sigma'}(\mu,\mu',\omega)$, respectively. Let us recall that in practice these vectors and matrices must always be truncated.

As already mentioned, chances are that in a specific application some of the coefficient equations (43)–(46) are trivially satisfied. In order to obtain a well-posed problem it is then necessary to eliminate the trivial equations from the system and to accordingly reduce the matrix $\underline{D}(\omega)$ as well as the vectors $\tilde{\mathbf{k}}(\omega)$ and $\tilde{\mathbf{u}}(\omega)$.

G. Formal solution and calculation of eigenfrequencies

The last steps toward the solution of our perturbational problem are now clearly prescribed. Equation (48) has to be solved for the unknown expansion coefficients as functions of ω:

$$\tilde{\mathbf{u}}(\omega) = \underline{D}^{-1}(\omega)\cdot\tilde{\mathbf{k}}(\omega) \tag{50}$$

where $\underline{D}^{-1}(\omega)$ is the inverse matrix to $\underline{D}(\omega)$. Application of the inverse time Laplace transformation (8b) to this equation yields the time-dependent coefficient vector

$$\tilde{\mathbf{u}}(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\,\tilde{\mathbf{u}}(\omega) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\,\underline{D}^{-1}(\omega)\cdot\tilde{\mathbf{k}}(\omega), \tag{51}$$

i.e.,

$$\tilde{j}_e(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\,\tilde{j}_e(\omega) \tag{52}$$

$$\tilde{E}(\kappa, t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \tilde{E}(\kappa, \omega) \tag{53}$$

$$\tilde{f}_l^\sigma(\lambda, t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \tilde{f}_l^\sigma(\lambda, \omega) \tag{54}$$

$$\tilde{f}_r^\sigma(\mu, t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \tilde{f}_r^\sigma(\mu, \omega). \tag{55}$$

The final solution functions are then given by (52) and

$$\tilde{E}(x, t) = \sum_\kappa \tilde{E}(\kappa, t)\, \varphi_\kappa(x) \tag{56}$$

$$\tilde{f}_l^\sigma(v > 0, t) = \sum_\lambda \tilde{f}_l^\sigma(\lambda, t)\, \varphi_\lambda^+(v) \tag{57}$$

$$\tilde{f}_r^\sigma(v < 0, t) = \sum_\mu \tilde{f}_r^\sigma(\mu, t)\, \varphi_\mu^-(v). \tag{58}$$

As in Landau's infinite-plasma case,[1] some general properties of these solution functions can be conveniently retrieved by choosing an appropriate contour of integration in the complex ω plane, cf. Eqs. (52)– (55) and Fig. 2. Let the "original" contour C_1 be an infinite straight line with $\mathrm{Im}\,\omega = (\mathrm{Im}\,\omega)_1 = const.$, where $(\mathrm{Im}\,\omega)_1$ must be sufficiently positive for C_1 to lie above all singularities (poles and branch cuts) of the integrand vector, $\underline{D}^{-1}(\omega) \cdot \tilde{\mathrm{k}}(\omega)$. As is well known from function theory, the same result $\tilde{\mathrm{u}}(t)$ will follow for any other contour that is obtained by deforming C_1 without crossing the singularities of the integrand. Let us construct a specific contour C_2 by shifting C_1 down to $\mathrm{Im}\,\omega = (\mathrm{Im}\,\omega)_2 < 0$ but excluding the singularities as shown in Fig 2. The solution (51) may then be written

$$\begin{aligned}\tilde{\mathbf{u}}(t) = &\sum_{\nu}\left[-i\,\mathrm{Res}_{\omega_\nu}\left(\underline{D}^{-1}\cdot\tilde{\mathbf{k}}\,e^{-i\omega t}\right)\right] + \sum_{\nu'}\left[-i\,\mathrm{Res}_{\omega_{\nu'}}\left(\underline{D}^{-1}\cdot\tilde{\mathbf{k}}\,e^{-i\omega t}\right)\right] \\ &+\sum_{b}\int_{B_b}\frac{d\omega}{2\pi}\underline{D}^{-1}\cdot\tilde{\mathbf{k}}\,e^{-i\omega t} + \sum_{b'}\int_{B_{b'}}\frac{d\omega}{2\pi}\underline{D}^{-1}\cdot\tilde{\mathbf{k}}\,e^{-i\omega t} \\ &+\int_{(\mathrm{Im}\,\omega)_2}\frac{d\omega}{2\pi}\underline{D}^{-1}\cdot\tilde{\mathbf{k}}\,e^{-i\omega t}\end{aligned} \tag{59}$$

where the sums over ν and b respectively contain the contributions from the poles ω_ν and branch cuts B_b of $\underline{D}^{-1}(\omega)$ (with $\mathrm{Im}\,\omega > (\mathrm{Im}\,\omega)_2$; Res_{ω_ν} indicates the residue at a pole ω_ν), the sums over ν' and b' contain the contributions from the poles $\omega_{\nu'}$ and branch cuts $B_{b'}$ of $\tilde{\mathbf{k}}(\omega)$ (with $\mathrm{Im}\,\omega > (\mathrm{Im}\,\omega)_2$), and the last integral represents the contributions from the straight portions with $\mathrm{Im}\,\omega = (\mathrm{Im}\,\omega)_2$.

As usual, we call the system unstable if the solution (59) contains components growing in time after the initial transient has decayed. Such contributions can arise from poles and branch-cut portions with $\mathrm{Im}\,\omega \geq 0$. An nth-order pole ω_p of the r.h.s. of Eq. (50) gives rise to a contribution of the form $P_{n-1}(t)\exp(-i\omega_p t)$, where $P_{n-1}(t)$ is a polynomial of degree $n-1$. In particular, for the most common case of a first-order pole ($n = 1$) we obtain a purely exponential time behavior.

The matrices $\underline{D}(\omega)$ and $\underline{D}^{-1}(\omega)$ do not depend on the perturbations but are solely determined by "intrinsic" equilibrium, boundary, and external-circuit properties. The poles ω_ν of $\underline{D}^{-1}(\omega)$ are the roots of the "characteristic equation",

$$|\underline{D}(\omega_\nu)| = 0 \qquad (\nu = \ldots 1, 2, \ldots) \tag{60}$$

where $|\underline{D}(\omega)|$ is the determinant of $\underline{D}(\omega)$. In accordance with common terminology, the roots ω_ν are the eigenfrequencies, and the related oscillation patterns in x, v, and t are the eigenmodes of our system. Thus, the first term on the r.h.s. of Eq. (59) represents the eigenmode contributions to the whole solution.

In a given application it may often be sufficient to study the dispersion and stability behavior, rather than finding the detailed solution functions (52) and (56)–(58). One then still has to determine the uppermost poles and branch-cut portions of both $\underline{D}^{-1}(\omega)$ and $\tilde{k}(\omega)$. However, in many cases it will be sufficient to consider the uppermost pole, and it may also be possible to obtain the determinant $|\underline{D}(\omega)|$ in analytical form, as is the case for the problem treated in Part II.[8]

Due to lack of sufficient experience with the present method, an exhaustive discussion of the possible forms of behavior of these singularities cannot be given as yet and is therefore deferred to an appropriate future date.

III. DISCUSSION AND COMPARISON WITH PREVIOUS LITERATURE

A. Discussion

The formalism developed in Sec. II is intended as a first step toward a comprehensive kinetic treatment of bounded plasma systems. By "comprehensive" we mean that plasma, boundary, and external-circuit effects are to be taken into account simultaneously and realistically.

As the first step it represents, the present work is still subject to some restrictions which, however, may possibly be relaxed at a later stage, cf. below. The most restrictive assumptions are those concerning the linearity of the problem (which means that we restrict ourselves to small perturbations about a given equilibrium state) and the one-dimensional system geometry involving a uniform plasma region and thin sheaths (whose relevance has been discussed in Subse. II.A). Within this framework, however, the admissible equilibrium, initial, boundary, and external-circuit conditions are certainly suitable to model a wide range of realistic situations. No kinetic treatment whatsoever is known to us which includes these latter features in comparable generality, cf. Subsec. III.B.

The key result of the present paper is the system of integral equations (37)–(40), which is essentially equivalent to all the basic and auxiliary relations entering the theory but, at the same time, much more concise and "advanced" than these. This is because in deriving it we have already performed, in general terms, many of the mathematical manipulations that would be required anyway when solving for any non-trivial special case. We may state that, within the limits of the model chosen, the treatment of the perturbational problem has been formalized to the extent that seems possible, so that in practice the "real" work needs to set in only at a level where the peculiar details of the special case considered become important.

As already mentioned, we bear some hope that it may become possible at a later stage to relax the major restrictions still inherent to the present work. This hope is founded in the fact that integral-equation approaches, which tend to be more compact than the equivalent differential-equation formulations, seem to be widely (if not generally) applicable to the comprehensive kinetic description of bounded plasma systems which is attempted here. For, the sources of integral terms spotted in the present paper, namely the trajectory integral (6), the current-conservation relation (15) (with (16)), and the boundary conditions (26), will also persist in more general models. In spite of the added complexity to be expected, extensions of the present work (e.g, to two dimensions, non-uniform equilibria, and/or collisional plasma behavior) seem to be worthwhile and will be attempted. Specific examples involving the solution of integral equations are provided by Refs. 23–25 (where the stability of non-uniform diode equilibria was studied) or Refs. 26 and 27 (where path-integral formulations of the linearized Boltzmann equation were used to solve collisional swarm-drift problems).

B. Comparison with previous literature

In order to locate the position of the present work within the pertinent literature, we now present a comparison with previous contributions to the kinetic theory of linear, longitudinal, collisionless plasma oscillations in one-dimensional geometry. An extremely large number of relevant treatments, henceforth referred to as class A, has accumulated over the years.[2] Hence, the list of references to be quoted here cannot claim completeness, but we nevertheless believe it to be representative.

Let us first note that the overwhelming majority of class-A treatments are those concerned with Landau's infinite-plasma, initial-value problem,[1-6] cf. Sec. I. Within the framework of the present method, this case is the most degenerate one and can be readily retrieved, e.g., from Eq. (11) as described there. What then remains to be considered is the much smaller number of those class-A treatments that actually involve boundaries and will henceforth be referred to as class B.

Table II provides a fairly detailed comparison between 17 pertinent treatments, namely Landau's classical infinite-plasma problem (Ref. 1, §1), 15 published class-B treatments, and the preesent work. Each treatment is evaluated with respect to seven basic criteria, and the results of the comparison may be summarized as follows.

Criterion 1 (model). Only nine references, including the present work, employ truly kinetic theory, whereas the other eight are based on the cold-fluid approach but are still included here because this type of analysis is most frequently encountered and easily reproducible from kinetic theory, cf. Part II.[8] The fact that the cold-fluid case can be retrieved from the present method as a special case is indicated by the symbol (+).

Criterion 2 (number of boundaries). As the only representative of the huge number of infinite-plasma treatments[2,3] we have included Landau's classical initial-value analysis (Ref.1, §1). Four references are concerned with semi-infinite plasmas, whereas true diode configurations are dealt with in twelve treatments, including the present one. The latter covers the infinite and semi-infinite geometries as special cases.

Criterion 3 (degree of non-uniformity of the d.c. plasma equilibrium). As already stated, the present version of our method is restricted to uniform plasmas with thin sheaths and thus does not include the problems of Refs. 23, 24, 25, 31, and 33 as special cases.

Criterion 4 (description of spatial behavior). In five treatments, the spatial eigenmode profiles are constructed by suitable superposition of infinite-plasma modes. References 13 and 30 start out with general Fourier-series expansions, whereas spatial Laplace transforms are employed in Refs. 18, 19, 20 and in the present work. In the remaining six references, the integral equations governing the spatial eigenmode structures are directly derived from the basic equations.

Criterion 5 (description of time behavior). Eleven of the 17 references considered only deal with eigenmodes or driven modes characterized by a time dependence of the form $\exp(\alpha t)$. Reference 33 in addition considers the effect of an externally applied delta pulse, whereas Laplace transformations, covering the full initial-value problem, are used in six references including the present work.

Criterion 6 (boundary conditions on particles). None of the previous treatments allows for boundary conditions other than constant emission, plain absorption (i.e., absorption without the release of secondary particles), and specular reflection. These are all special cases of the more general relations (26). The boundary conditions assumed in Refs. 18 and 20 are not immediately clear.

Criterion 7 (a.c. external-circuit properties). Ten previous treatments deal with the external short circuit, and two are concerned with the open circuit. No explicit statement about external circuits is made in Refs. 1 (§1 and §2), 13, 18 19, 20, and for Refs. 13, 18, and 20 it would require some detailed work to reconstruct the external-circuit conditions that were implicitly assumed. Interpreting Landau's infinite-plasma problem (Ref. 1, §1) in terms of the diode picture as discussed after Eq.(11), the conditions $\tilde{f}^{\sigma}(\pm\infty, v, t) = 0$ and $\tilde{V}(\pm\infty, t) = 0$ imply constant emission (criterion 6) and external short-circuit (present criterion). Of the treatments considered, only Ref. 31 and the present method allow for non-trivial external circuits of some generality.

In summarizing the foregoing comparison we may state that the present method covers the problems treated in Refs. 1 (§1 and §2), 13, 18, 19, 20, 28, 29, 30, 32, and 34 as special cases. This is not true of Refs. 23, 24, 25, 31, and 33 because these are concerned with truly non-uniform plasma equilibria. Apart from this, all previous treatments are less general than the present one in that they start out with more restricted specifications and thus cover, *a priori*, a narrower range of the physical effects to be expected in real systems. In particular, it appears that the effects of realistic boundary conditions and external circuits have not been treated to a significant extent, and that the full perturbational problem for the diode (i.e., the problem involving initial and external perturbations) has not been theoretically analyzed at all. We thus believe that the present method represents, in several aspects, a suitable starting point for more comprehensive, realistic, and systematic kinetic descriptions of bounded plasma systems than have been given so far.

IV. CONCLUSIONS

We have proposed a theoretical method for studying linear longitudinal perturbations in one-dimensional collisionless plasma diodes with thin sheaths. The key result is the set of $2 + 2n_\sigma$ integral equations (37)–(40) for the $2 + 2n_\sigma$ time Laplace transforms $\tilde{j}_e(\omega)$, $\tilde{E}(x, \omega)$, $\tilde{f}_l^\sigma(v > 0, \omega)$, and $\tilde{f}_r^\sigma(v < 0, \omega)$. Upon expanding the functions involved in suitable basis sets, these equations transform into the matrix equation (48) for the unknown- coefficient vector **u**. In principle, the full solution of the perturbational problem can be found by solving these equations, in either representation, for the above functions of ω, and by then Laplace inverting the latter into the corresponding time-dependent perturbations.

However, as discussed in Subsec. II.C, the linear dynamics of the system is basically governed by the singularities of the coefficient determinant $|\underline{D}(\omega)|$, so that it may often be sufficient to study these, rather than evaluating all the details of the full time-dependent solution.

Apart from the geometrical simplicity of the model, the present method is comprehensive in that it allows for very general equilibrium, initial, boundary, and external-circuit conditions and thus combines a good deal of the essential elements present in real systems. The literature survey of Subsec. III.B suggests that there exists no other kinetic treatment which simultaneously includes all of these aspects in comparable generality.

Thus, the present method may well represent a suitable starting point for more complete, systematic, and detailed kinetic studies of bounded plasma systems. A first application is provided by Part II of the present work,[8] where it is demonstrated that the Pierce instability[28] can be drastically modified by a non-trivial external circuit.

As discussed in Subsec. III.A, the integral equations (37)–(40) are equivalent to all the basic and auxiliary equations entering the theory, but at the same time much more concise and "advanced" than these. Due to the relative geometrical simplicity of the one-dimensional diode model considered, it has been possible here to perform a major portion of the necessary mathematical manipulations in general terms. This means that the whole approach has been formalized to a large extent, so that, in a given application, the "real" work has to set in only at a level where the peculiar details of the special case considered become important.

According to Subsec. III.A, it seems plausible that integral-equation formulations of the kinetic perturbational problem can also be found for bounded plasma systems that are more general than the one considered here. In spite of the added complexity to be expected, it may well be possible and worthwhile to extend the present approach to some of those.

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APPENDIX: KNOWN FUNCTIONS AND INTEGRAL OPERATORS

$$\tilde{k}_1^\sigma(x,v,\omega)=\int \frac{dq}{2\pi}\frac{\exp(-iqx)}{q(q+\frac{\omega}{v})}\frac{\tilde{f}_i^\sigma(q,v)}{D(q,\omega)} \tag{A1}$$

$$k_2(x,\omega)=\int \frac{dq}{2\pi}\frac{\exp(-iqx)}{q\,D(q,\omega)} \tag{A2}$$

$$k_3(x,v,\omega)=\int \frac{dq}{2\pi}\frac{\exp(-iqx)}{q\,(q+\frac{\omega}{v})D(q,\omega)} \tag{A3}$$

$$\tilde{k}_4(x,\omega)=-\tilde{E}_{li}\frac{k_2(x,\omega)}{\omega}-4\pi\sum_{\sigma=1}^{n_\sigma}e^\sigma\int_{-\infty}^{\infty}\frac{dv}{v}\tilde{k}_1^\sigma(x,v,\omega) \tag{A4}$$

$$k_5(x,\omega)=-\frac{4\pi}{\omega}k_2(x,\omega) \tag{A5}$$

$$k_6(x,v,\omega)=\frac{v}{\omega}\int \frac{dq}{2\pi}\frac{\exp(-iqx)}{(q+\frac{\omega}{v})D(q,\omega)} \tag{A6}$$

$$\tilde{k}_7^\sigma(v,\omega)=\int_0^L dx'\,\exp\left(-i\omega\frac{x'}{v}\right)\tilde{f}_i^\sigma(x',v) \tag{A7}$$

$$\begin{aligned}\tilde{k}_8(x,\omega)=&-\tilde{E}_{li}\frac{k_2(x,\omega)}{\omega}\\&-4\pi\sum_{\sigma=1}^{n_\sigma}e^\sigma\int_{-\infty}^{\infty}\frac{dv}{v}\left[\tilde{k}_1^\sigma(x,v,\omega)+U(-v)\,k_6(x,v,\omega)\,\tilde{k}_7^\sigma(v,\omega)\right]\end{aligned} \tag{A8}$$

$$k_9(x,x',\omega)=4\pi\sum_{\sigma=1}^{n_\sigma}\frac{(e^\sigma)^2}{m^\sigma}\int_{-\infty}^{0}\frac{dv}{v}k_6(x,v,\omega)\,\bar{f}_v^\sigma(v)\exp\left(-i\omega\frac{x'}{v}\right) \tag{A9}$$

$$\begin{aligned}\tilde{k}_{10}^\sigma(v>0,\omega)=&\tilde{f}_{lg}^\sigma(v,\omega)-\sum_{\sigma'=1}^{n_\sigma}\int_{-\infty}^{0}dv'\,b_l^{\sigma\sigma'}(v,v')\left\{\frac{1}{v'}\tilde{k}_7^{\sigma'}(v',\omega)\right.\\&-\exp\left(-i\omega\frac{L}{v'}\right)\sum_{\sigma''=1}^{n_\sigma}\int_0^{\infty}\frac{dv''}{v''}b_r^{\sigma'\sigma''}(v',v'')\exp\left(i\omega\frac{L}{v''}\right)\tilde{k}_7^{\sigma''}(v'',\omega)\\&\left.-\exp\left(-i\omega\frac{L}{v'}\right)\tilde{f}_{rg}^{\sigma}(v',\omega)\right\}\end{aligned} \tag{A10}$$

$$k_{11}^{\sigma}(x, v > 0, \omega) = \sum_{\sigma'=1}^{n_\sigma} \int_{-\infty}^{0} dv'\, b_l^{\sigma\sigma'}(v, v') \left\{ \frac{e^{\sigma'}}{m^{\sigma'}} \frac{1}{v'} \bar{f}_v^{\sigma'}(v') \exp\left(-i\omega \frac{x}{v'}\right) \right.$$

$$\left. - \exp\left(-i\omega \frac{L}{v'}\right) \sum_{\sigma''=1}^{n_\sigma} \frac{e^{\sigma''}}{m^{\sigma''}} \int_0^{\infty} \frac{dv''}{v''} b_r^{\sigma'\sigma''}(v', v'') \bar{f}_v^{\sigma''}(v'') \exp\left(i\omega \frac{L-x}{v''}\right) \right\} \tag{A11}$$

$$k_{12}^{\sigma\sigma'}(v > 0, v' > 0, \omega)$$

$$= \exp\left(i\omega \frac{L}{v'}\right) \sum_{\sigma''=1}^{n_\sigma} \int_{-\infty}^{0} dv''\, b_l^{\sigma\sigma''}(v, v'') \exp\left(-i\omega \frac{L}{v''}\right) b_r^{\sigma''\sigma'}(v'', v') \tag{A12}$$

$$\tilde{k}_{13}^{\sigma}(v < 0, \omega) = \tilde{f}_{rg}^{\sigma}(v, \omega)$$

$$+ \sum_{\sigma'=1}^{n_\sigma} \int_0^{\infty} dv'\, b_r^{\sigma\sigma'}(v, v') \exp\left(i\omega \frac{L}{v'}\right) \left\{ \frac{1}{v'} \tilde{k}_7^{\sigma'}(v', \omega) \right.$$

$$\left. - \sum_{\sigma''=1}^{n_\sigma} \int_{-\infty}^{0} \frac{dv''}{v''} b_l^{\sigma'\sigma''}(v', v'')\, \tilde{k}_7^{\sigma''}(v'', \omega) + \tilde{f}_{lg}^{\sigma'}(v', \omega) \right\} \tag{A13}$$

$$k_{14}^{\sigma}(x, v < 0, \omega)$$

$$= \sum_{\sigma'=1}^{n_\sigma} \int_0^{\infty} dv'\, b_r^{\sigma\sigma'}(v, v') \exp\left(i\omega \frac{L}{v'}\right) \left\{ -\frac{e^{\sigma'}}{m^{\sigma'}} \frac{1}{v'} \bar{f}_v^{\sigma'}(v') \exp\left(-i\omega \frac{x}{v'}\right) \right.$$

$$\left. + \sum_{\sigma''=1}^{n_\sigma} \frac{e^{\sigma''}}{m^{\sigma''}} \int_{-\infty}^{0} \frac{dv''}{v''} b_l^{\sigma'\sigma''}(v', v'') \bar{f}_v^{\sigma''}(v'') \exp\left(-i\omega \frac{x}{v''}\right) \right\} \tag{A14}$$

$$k_{15}^{\sigma\sigma'}(v < 0, v' < 0, \omega)$$

$$= \exp\left(-i\omega \frac{L}{v'}\right) \sum_{\sigma''=1}^{n_\sigma} \int_0^{\infty} dv''\, b_r^{\sigma\sigma''}(v, v'') \exp\left(i\omega \frac{L}{v''}\right) b_l^{\sigma''\sigma'}(v'', v') \tag{A15}$$

$$\mathrm{S}_0(x,[x'],\omega)\,\tilde{E}([x'],\omega)=\left\{-\int_0^L dx'\,k_0(x,x',\omega)\right\}\tilde{E}(x',\omega)$$

$$=\left\{-4\pi\sum_{\sigma=1}^{n_\sigma}\frac{(e^\sigma)^2}{m^\sigma}\int_0^L dx'\int_{-\infty}^0\frac{dv}{v}k_6(x,v,\omega)\,\bar{f}_v^\sigma(v)\exp\left(-i\omega\frac{x'}{v}\right)\right\}\tilde{E}(x',\omega) \tag{A16}$$

$$\vdots$$

$$\mathrm{S}_l^\sigma(v>0,[x'],\omega)\,\tilde{E}([x'],\omega)=\left\{-\int_0^L dx'\,k_{11}^\sigma(x',v,\omega)\right\}\tilde{E}(x',\omega)$$

$$\begin{aligned}&=\left\{\int_0^L dx'\sum_{\sigma'=1}^{n_\sigma}\int_{-\infty}^0 dv'\,b_l^{\sigma\sigma'}(v,v')\left[-\frac{e^{\sigma'}}{m^{\sigma'}}\frac{1}{v'}\bar{f}_v^{\sigma'}(v')\exp\left(-i\omega\frac{x'}{v'}\right)\right.\right.\\&+\exp\left(-i\omega\frac{L}{v'}\right)\sum_{\sigma''=1}^{n_\sigma}\frac{e^{\sigma''}}{m^{\sigma''}}\int_0^\infty\frac{dv''}{v''}\exp\left(i\omega\frac{L-x'}{v''}\right)\\&\times\left.\left.b_r^{\sigma'\sigma''}(v',v'')\bar{f}_v^{\sigma''}(v'')\right]\right\}\tilde{E}(x',\omega)\end{aligned} \tag{A17}$$

$$\mathrm{S}_r^\sigma(v<0,[x'],\omega)\,\tilde{E}([x'],\omega)=\left\{-\int_0^L dx'\,k_{14}^\sigma(x',v,\omega)\right\}\tilde{E}(x',\omega)$$

$$\begin{aligned}&=\left\{\int_0^L dx'\sum_{\sigma'=1}^{n_\sigma}\int_0^\infty dv'\,b_r^{\sigma\sigma'}(v,v')\exp\left(i\omega\frac{L}{v'}\right)\right.\\&\times\left[\frac{e^{\sigma'}}{m^{\sigma'}}\frac{1}{v'}\bar{f}_v^{\sigma'}(v')\exp\left(-i\omega\frac{x'}{v'}\right)\right.\\&-\left.\left.\sum_{\sigma''=1}^{n_\sigma}\frac{e^{\sigma''}}{m^{\sigma''}}\int_{-\infty}^0\frac{dv''}{v''}b_l^{\sigma'\sigma''}(v',v'')\bar{f}_v^{\sigma''}(v'')\exp\left(-i\omega\frac{x'}{v''}\right)\right]\right\}\tilde{E}([x'],\omega)\end{aligned} \tag{A18}$$

$$\mathcal{V}_{0l}^\sigma(x,[v'>0],\omega)\,\tilde{f}_l^\sigma([v'],\omega)=\left\{-4\pi e^\sigma\int_0^\infty dv'\,k_6(x,v',\omega)\right\}\tilde{f}_l^\sigma(v',\omega) \tag{A19}$$

$$\begin{aligned}&\mathcal{V}_{0r}^\sigma(x,[v'<0],\omega)\,\tilde{f}_r^\sigma([v'],\omega)\\&\quad=\left\{-4\pi e^\sigma\int_{-\infty}^0 dv'\,k_6(x,v',\omega)\exp\left(-i\omega\frac{L}{v'}\right)\right\}\tilde{f}_r^\sigma(v',\omega)\end{aligned} \tag{A20}$$

$$
\mathcal{V}_l^{\sigma\sigma'}(v>0,[v'>0],\omega)\,\tilde{f}_l^{\sigma'}([v'],\omega)=\left\{-\int_0^{\infty}dv'\,k_{12}^{\sigma\sigma'}(v,v',\omega)\right\}\tilde{f}_l^{\sigma'}(v',\omega)
$$

$$
=\left\{-\int_0^{\infty}dv'\,\exp\left(i\omega\frac{L}{v'}\right)\sum_{\sigma''=1}^{n_\sigma}\int_{-\infty}^{0}dv''\exp\left(-i\omega\frac{L}{v''}\right)\right.
$$

$$
\left.\times\,b_l^{\sigma\sigma''}(v,v'')\,b_r^{\sigma''\sigma'}(v'',v')\right\}\tilde{f}_l^{\sigma'}(v',\omega)
$$

(A21)

$$
\mathcal{V}_r^{\sigma\sigma'}(v<0,[v'<0],\omega)\,\tilde{f}_r^{\sigma'}([v'],\omega)=\left\{-\int_{-\infty}^{0}dv'\,k_{15}^{\sigma\sigma'}(v,v',\omega)\right\}\tilde{f}_r^{\sigma'}(v',\omega)
$$

$$
=\left\{-\int_{-\infty}^{0}dv'\,\exp\left(-i\omega\frac{L}{v'}\right)\sum_{\sigma''=1}^{n_\sigma}\int_0^{\infty}dv''\exp\left(i\omega\frac{L}{v''}\right)\right.
$$

$$
\left.\times\,b_r^{\sigma\sigma''}(v,v'')\,b_l^{\sigma''\sigma'}(v'',v')\right\}\tilde{f}_r^{\sigma'}(v',\omega)
$$

(A22)

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Tables

TABLE I. Level scheme for establishing the coefficient functions and integral operators of Eqs. (37)–(40).

LEVEL 1		LEVEL 2		LEVEL 3		LEVEL 4	
L	II.A			$\tilde{E}_{li}$	(18)	$\tilde{V}_{eo}(\omega)$	(36)
n_σ	(1)					$Z_e(\omega)$	(36)
m^σ	(2)	$D(q,\omega)$	(2)	$\tilde{k}_1^\sigma$	(A1)		
e^σ	(1)					k_5	(A5)
$\bar{f}^\sigma(v)$	(2)			k_2	(A2)	S_0	(A16)
						$\mathcal{V}_{0l}^\sigma$	(A19)
$\tilde{f}_i^\sigma(x,v)$	I	$\tilde{f}_i^\sigma(q,v)$	(9)	k_6	(A6)	$\mathcal{V}_{0r}^\sigma$	(A20)
						$\tilde{k}_8$	(A8)
$\tilde{f}_{lg}^\sigma(v>0,t)$	(26a)	$\tilde{f}_{lg}^\sigma(v>0,\omega)$	(27a)	$\tilde{k}_7^\sigma$	(A7)		
$b_l^{\sigma\sigma'}$	(26a)					S_l^σ	(A17)
$\tilde{f}_{rg}^\sigma(v<0,t)$	(26b)	$\tilde{f}_{rg}^\sigma(v<0,\omega)$	(27b)			$\mathcal{V}_l^{\sigma\sigma'}$	(A21)
$b_r^{\sigma\sigma'}$	(26b)					$\tilde{k}_{10}^\sigma$	(A10)
external-circuit condition	(33)	Laplace-transformed external-circuit condition	(36)			S_r^σ $\mathcal{V}_r^{\sigma\sigma'}$ $\tilde{k}_{13}^\sigma$	(A18) (A22) (A13)
initial state of external circuit [$\tilde{j}_{ei}$, $(d\tilde{j}_e/dt)_i$, *etc.*]	II.E	$\Delta\tilde{V}_{pi}$	(18)				

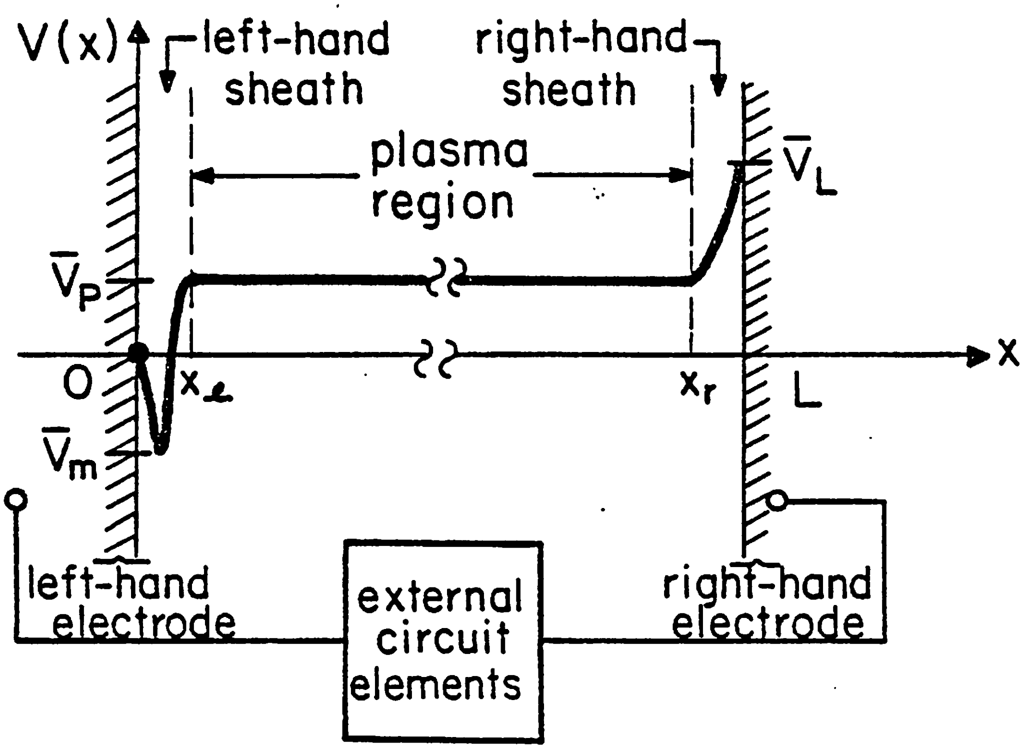
TABLE II. Comparison of some representative contributions to the kinetic theory of linear longitudinal plasma oscillations in one-dimensional geometry.

		Buts 1974	Derfler... 1966	Dobrowolny... 1969	Ender... 1979	Evans... 1970	Evans 1970	Evans... 1971	Faulkner... 1969	Iizuka... 1979	Kuznetsov... 1980	Landau (§1) 1946	Landau (§2) 1946	Montgomery... 1962	Pierce 1944	Pierce 1948	Rosenbluth... 1963	present
CRITERION↓	REFERENCE→	33	31	13	24	19	18	20	32	34	25	1a	1b	30	28	29	23	P
1. MODEL	cold-fluid	+	+			+		+	+	+					+	+		(+)
	kinetic			+	+		+				+	+	+	+			+	+
2. NUMBER OF BOUNDARIES	zero											+						(+)
	one					+	+	+					+					(+)
	two	+	+	+	+				+	+	+			+	+	+	+	+
3. PLASMA EQUILIBRIUM	uniform					+	+	+	+	+		+	+	+	+	+		(+)
	unif./thin sh.			+														+
	non-uniform	+	+		+						+						+	
4. SPATIAL DEPEND-ENCE	inf.-syst. modes								+	+		+			+	+		(+)
	Fourier series			+										+				(+)
	Laplace transf.					+	+	+										+
	integral eq.	+	+		+						+		+				+	
5. TIME DE-PENDENCE	exp(αt)	+	+	+	+				+	+	+		+		+	+	+	(+)
	δ pulse	+																(+)
	Laplace transf.					+	+	+				+		+				+
6. PARTICLE BOUNDARY CONDITIONS	const. emission	+	+	+	+	+			+	+	+	+	+		+	+	+	(+)
	plain absorption	+	+	+	+	+	?	?	+	+	+				+	+	+	(+)
	spec. reflection			+									+	+				(+)
	other																	+
7. A.C. EXTERNAL CIRCUIT	short	+			+				+	+	+	+		+	+	+	+	(+)
	open			?		+	?	?					+					(+)
	passive		+															(+)
	active	+	+															+

Figure Captions

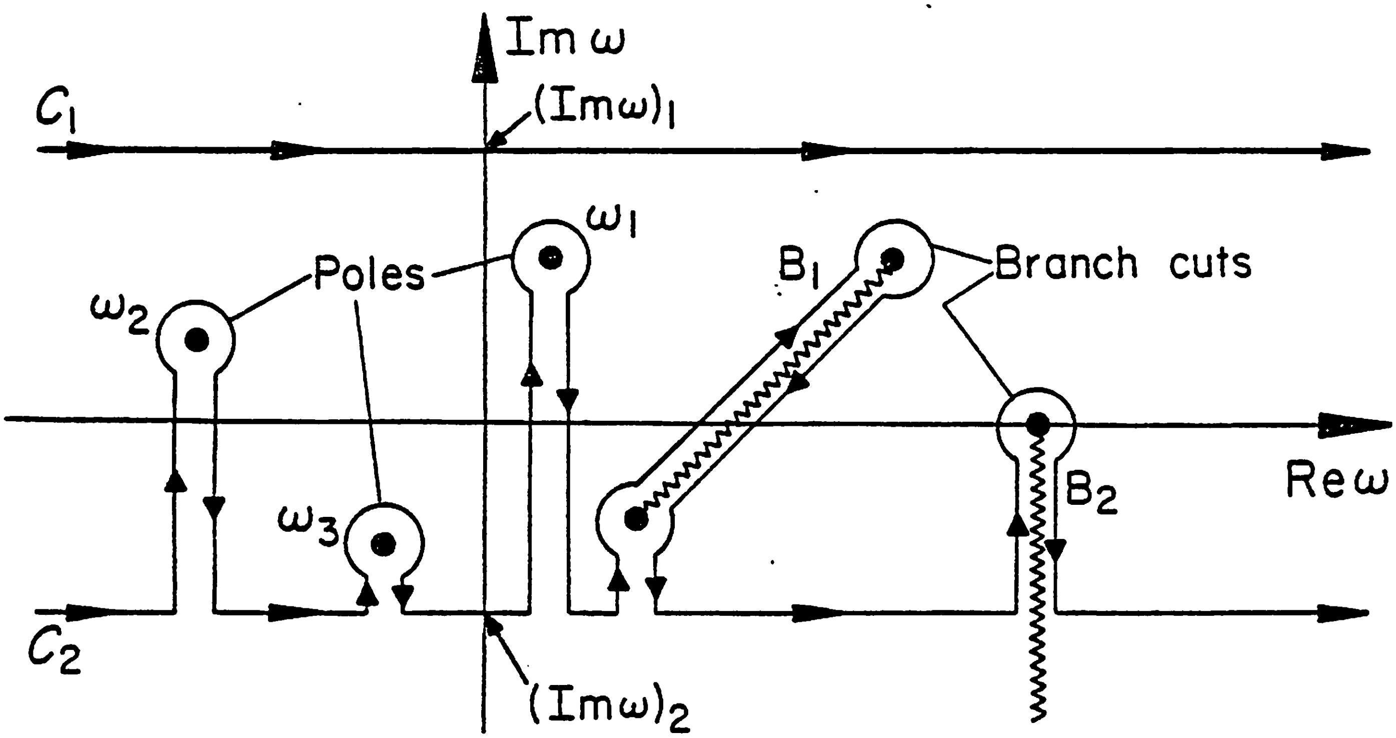
FIG. 1. Model geometry, with one-minimum equilibrium potential distribution as an example.

FIG. 2. Typical contours of integration for the inverse time Laplace transformation.



V(x)
left-hand sheath
right-hand sheath
plasma region
$\bar{V}_P$
$\bar{V}_L$
$\bar{V}_m$
O
x_ℓ
x_r
L
x
left-hand electrode
external circuit elements
right-hand electrode

Fig. 1



Im ω
C_1
$(\mathrm{Im}\,\omega)_1$
ω_1
Poles
ω_2
B_1
Branch cuts
ω_3
Re ω
B_2
C_2
$(\mathrm{Im}\,\omega)_2$

Fig. 2