STRONG LAW OF LARGE NUMBERS FOR RANDOM

VARIABLES WITH VALUES IN THE FUZZY REAL LINE

by

Erich Peter Klement

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Erich Peter Klement*

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

We introduce random variables mapping into the fuzzy real line and their expected value, and prove a Strong Law of Large Numbers.

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*Permanent address: Institut für Mathematik, Johannes Kepler Universität, A-4040 Liwz, Austria.

INTRODUCTION

After the introduction of the concept of a fuzzy set by ZADEH [24] much research has been done to establish a fuzzy real line. Its purpose is to provide arithmetics for "numbers" such as "approximately five" or "much larger than ten" where the vagueness is not inherited from randomness.

An interesting problem is now the estimation of parameters which are vague in the sense as described above. As a first step into this direction we introduce random variables mapping into the fuzzy real line (which is surveyed in Section 1) and their expected value. We prove a Strong Law of Large Numbers for independent, identically distributed fuzzy-valued random variables with finite second moment.

1. THE FUZZY REAL LINE

Our concept of fuzzy numbers follows the ideas of HÖHLE [9-11] being somewhat different from other ideas in this context, as summarized in DUBOIS and PRADE [4]. The fuzzy real line has been introduced independently (and in different forms) by HUTTON [12], GANTNER, STEINLAGE and WARREN [5], and HÖHLE [10], all of which focussed on its topological aspects. More recently, additional research in this topic has been done by LOWEN [15] and RODABAUGH [12-21]. In particular, the multiplication of fuzzy numbers has been developed in [21] most recently.

Throughout this paper, $\mathbb{R}$ will denote the real line, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ the two-point compactification of $\mathbb{R}$. For the unit interval [0,1] we shall write I.

The (extended) <u>fuzzy real line</u> $\overline{\mathbb{R}}(I)$ is the set of all functions $\rho : \overline{\mathbb{R}} \to I$ such that

$$\rho(-\infty) = 0, \ \rho(+\infty) = 1; \tag{1.1}$$

$$\forall \ r \in \mathbb{R} : \rho(r) = \sup\{\rho(s) | s < r\}. \tag{1.2}$$

Note that these are precisely the cumulative distribution functions on $\overline{\mathbb{R}}$. In a more particular setting (usually assuming $\rho(r) = 0$ for $r < 0$) they have been studied extensively in the context of probabilistic metric spaces (see SCHWEIZER and SKLAR [22]).

A natural interpretation of a fuzzy number ρ is the following: $\rho(r)$ is the degree to which ρ is less than the (nonfuzzy) number r, or, the truth value of the statement "ρ is less than r." This degree and truth value were suggested by ZADEH [24] to take on any value in I rather than 0 and 1 only. The letter I in $\overline{\mathbb{R}}(I)$ referes to the fact that more general objects (such as lattices) were considered instead of I in the literature (see GOGUEN [6], but also [15] and [19-21]).

Of course, any nonfuzzy real number r is identified with $\delta_r = 1_{(r,+\infty]}$, where 1_A denotes the characteristic function of the set A. The fuzzy numbers corresponding to $-\infty$ and $+\infty$ are $\delta_{-\infty} = 1_{(-\infty,+\infty]}$ and $\delta_{+\infty} = 1_{\{+\infty\}}$. A fuzzy number ρ is said to be finite if $\inf \{\rho(r) | r \in \mathbb{R}\} = 0$ and $\sup\{\rho(r) | r \in \mathbb{R}\} = 1$. The set $\{\rho_{/\mathbb{R}} | \rho \in \overline{\mathbb{R}}(I), \rho\}$ finite will be denoted $\mathbb{R}(I)$.

Next consider closed intervals $[a,b]$, $[c,d]$ (whose bounds may be infinite) and a nondecreasing function $f : [a,b] \to [c,d]$ which is left-continuous in (a,b) and satisfies $f(a) = c$. Then its <u>quasi-inverse</u> $[f]^q : [c,d] \to [a,b]$ defined by

$$[f]^q(s) = \sup\{r \in [a,b] | f(r) < s\} \tag{1.3}$$

is again nondecreasing, left-continuous in (c,d), and satisfies

$[f]^q(c) = a$. This concept was introduced by SHERWOOD and TAYLOR [23] and has been studied by HÖHLE [11] in a more general setting.

If $\overline{\mathbb{R}}^q(I)$ denotes the set of all quasi-inverses of fuzzy numbers ρ in $\overline{\mathbb{R}}(I)$ then it is easily seen that the mapping $q : \rho \to [\rho]^q$ is an involution from $\overline{\mathbb{R}}(I)$ onto $\overline{\mathbb{R}}^q(I)$. It is therefore possible to introduce an algebraic structure on $\overline{\mathbb{R}}(I)$ as follows (of course only if the right-hand sides of (1.5) and (1.6) make sense):

$$\rho \leq \rho \Leftrightarrow [\phi]^q(\alpha) \leq [\phi]^q(\alpha) \quad \text{for all } \alpha \in I; \tag{1.4}$$

$$[\rho \oplus \rho]^q(\alpha) = [\rho]^q(\alpha) + [\phi]^q(\alpha); \tag{1.5}$$

$$[\rho \odot \phi]^q(\alpha) = \sup\{[\rho^+]^q(\beta)\cdot[\phi^+]^q(\beta)+[\rho^+]^q(1-\beta)\cdot[\phi^-]^q(\beta)$$
$$+ [\rho^-]^q(\beta)\cdot[\rho^+]^q(1-\beta)+[\phi^-]^q(1-\beta)\cdot[\phi^-]^q(1-\beta)\,|\,\beta<\alpha\},$$

where in the latter formula ρ^+ and ρ^- are fuzzy numbers as follows:

$$\rho^+(r) = \begin{cases} 0 & \text{if } r \leq 0, \\ \rho(r) & \text{if } r > 0; \end{cases} \tag{1.7}$$

$$\rho^-(r) = \begin{cases} \rho(r) & \text{if } r \leq 0, \\ 1 & \text{if } r > 0. \end{cases} \tag{1.8}$$

It is clear that ρ is finite if and only if $[\rho]^q(\alpha)$ is finite for any $\alpha \in (0,1)$. $(\overline{\mathbb{R}}(I), \leq)$ is a complete lattice, $(\mathbb{R}(I), +, \leq)$ is a partially ordered abelian semigroup with neutral element δ_0, $(\mathbb{R}(I), \odot)$ is an abelian semigroup with neutral element δ_1, and $\odot$ is distributive over $\oplus$. Altogether, $(\mathbb{R}(I), \oplus, \odot, \leq)$ has been called a fuzzy hyperfield in [21]. Of course, $i : r \to \delta_r$ is an order preserving monomorphism from ($\overline{\mathbb{R}}, \oplus, \odot, \leq$) into ($\overline{\mathbb{R}}(I), \oplus, \odot, \leq$). It should be mentioned that ($\mathbb{R}(I), \oplus, \dot{\odot}$) is not a ring; therefore it does not coincide with PR as considered in [23]. A sequence $(\rho_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}}(I)$ converges to if for all $r \in C(\rho)$

$$\rho(r) = \lim_{n\to\infty} \rho_n(r), \tag{1.9}$$

where $C(\rho)$ is the set of continuity points of ρ. This is equivalent to the weak convergence of the corresponding probability measures (see BILLINGSLEY [2], PARTHASARATHY [17]). It is clear that $(\rho_n) \to \rho$ if and only if $[\rho_n]^q \to [\rho]^q$.

2. FUZZY-VALUED RANDOM VARIABLES

Let (Ω,A,P) be a probability space. Since $\overline{\mathbb{R}}(I)$ can be embedded naturally into $[0,1]^{\overline{\mathbb{R}}}$ (equipped with the product σ-algebra) it makes sense to consider measurable functions $X:\Omega \to \overline{\mathbb{R}}(I)$ which we shall call fuzzy-valued random variables. Obviously, this generalizes the concept of real-valued random variables. However, they are different from the fuzzy random variables as studied by PURI and RALESCU [18] and KLEMENT, PURI and RALESCU [14] where the values are fuzzy subsets of $\mathbb{R}^p$ rather than fuzzy numbers, thus generalizing the idea of random sets (see MATHERON [16]).

If X is a fuzzy-valued random variable then its quasi-inverse $X^q:\Omega \to \overline{\mathbb{R}}^q(I)$ is defined by $X^q(\omega) = [X(\omega)]^q$. It is clear that X^q is measurable if and only if X is measurable.

Generalizing proposition 3.1 in [13] it is readily seen that the mapping

$$\alpha \to E(X^q)(\alpha) = \int [X(\omega)]^q(\alpha)dP(\omega) \tag{2.1}$$

belongs to $\bar{\mathbb{R}}^q(I)$ provided the Lebesque integral on the right hand side exists for each $\alpha \in I$. We therefore can consider the expected value of the fuzzy-valued random variable X given by

$$EX = [E(X^q)]^q, \tag{2.2}$$

which, of course, is a fuzzy number again. This extends the classical expected value of a random variable $X : \Omega \to \bar{\mathbb{R}}$ in the sense that

$$E(i \cdot X) = \delta_{EX}, \tag{2.3}$$

i being the embedding from $\bar{\mathbb{R}}$ into $\bar{\mathbb{R}}(I)$.

It is clear from the definition that $E(X + Y) = EX + EY$. Moreover, for any fuzzy number ρ we also have $E(\rho \odot X) = \rho \odot EX$, which follows from (1.6) and the Lebesque Monotone Convergence Theorem. The expected value is therefore a linear functional in the sense that

$$E(\rho \odot X \oplus \phi \odot Y) = \rho \odot EX \oplus \phi \odot EY. \tag{2.4}$$

3. LAW OF LARGE NUMBERS

Now consider fuzzy-valued random variables which are independent and identically distributed. Since it is not clear what the variance should be (there is no proper subtraction in $\bar{\mathbb{R}}(I)$ and no obvious matrix) we impose a finiteness condition upon the second moments in order to state a Strong Law of Large Numbers.

THEOREM. Let X_1, X_2, ... be a sequence of independent, identically distributed random variables with values in $\mathbb{R}(I)$ and $E(X_1^2)$ finite. Then

$$\frac{1}{n} \cdot (X_1 \oplus X_2 \oplus \ldots \oplus X_n) \to E(X_1).$$

Proof: First we claim that for each $\alpha \in I$ the real random variables $[X_1(\cdot)]^q(\alpha)$, $[X_2(\cdot)]^q(\alpha)$, ... are also independent and identically distributed. Fix $r \in \mathbb{R}$, $\alpha \in I$ and put $A = \{\rho \in \overline{\mathbb{R}}(I) | \rho(r) > \alpha\}$. Then we have

$$P\{X \in A\} = P\{\omega \in \Omega | X(\omega)(r) > \alpha = P\{\omega \in \Omega | [\overline{X}(\omega)]^q(\alpha) < r\},$$

from which our claim follows immediately.

Next we show that $E([X_1(\cdot)]^q(\alpha))^2 < +\infty$ for any $\alpha \in (0,1)$. Assume that $E([X_1(\cdot)]^q(\alpha_0))^2 = +\infty$ for some $\alpha_0 \in (0,1)$ and put $\Omega_1 = \{\omega \in \Omega | [X_1(\omega)]^q(\alpha_0) > 0\}$, $\Omega_2 = \{\omega \in \Omega | [X_1(\omega)]^q(\alpha_0 < 0\}$. Then, by (1.6) and the finiteness of $E(X_1^2)$ we get

$$\int_{\Omega_1} ([X_1(\omega)]^q(\alpha_0))^2 dP(\omega) \leq E[X_1^2(\cdot)]^q(\alpha_0) < +\infty,$$

thus implying $\int_{\Omega_2} ([X_1(\omega)]^q(\alpha_0))^2 dP(\omega) = +\infty$ which is impossible because of

$$\int_{\Omega_2} ([X_1(\omega)]^q(1-\alpha_0))^2 dP(\omega) \leq E[X_1^2(\cdot)]^q(1-\alpha_0) < +\infty.$$

Therefore for any $\alpha \in (0,1)$ the variance of the (real) random variable $[X_1(\cdot)]^q(\alpha)$ is finite and by the classical Strong Law of Large Numbers we get

$$[\frac{1}{n} \cdot (X_1 + X_2 + \ldots + X_n)]^q \quad [EX_1]^q,$$

thus completing the proof.

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