MODEL REFERENCE ADAPTIVE CONTROL - STABILITY, PARAMETER CONVERGENCE AND ROBUSTNESS

by

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Abstract

We study stability, parameter convergence and robustness aspects of single input-single output model reference adaptive systems. We begin by establishing a framework for studying parametrizable and unparametrizable uncertainty in the plant to be controlled. Using the standard assumptions on the parametrizable part of the plant dynamics we give a corrected proof (of Narendra, Lin and Valavani) of the stability of the nominal adaptive scheme. Next, we give conditions on the exogenous input to the adaptive loop, the reference signal, to guarantee exponential parameter and error convergence. Using our framework for studying unmodelled (unparametrized) dynamics; we show how the model should be chosen, and the update law modified (by a deadzone in the update law) to preserve stability of the adaptive loop in the presence of output disturbances and unmodelled dynamics. Finally, we compare adaptive and non-adaptive control and list directions of ongoing research.

Key words: Model Reference Adaptive Systems, Control Systems Design, Unmodelled Dynamics, Robustness, Stability

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Section I. Introduction

Most currently used techniques for the design of control systems are based on a good understanding of the plant under study and its environment. However, when the plant is too complex and the physical processes in it are not fully understood these techniques need to be modified. A common method of attacking the problem is to apply some system identification technique to obtain a model of the process and its environment from some input-output experiments. The controller design is then based on the resulting model. Further, the parameters of the controller are adjusted during the operation of the plant as the amount of data for plant identification increases. For a number of PID controllers this is in fact done manually - however, when the number of parameters is larger than three or four, automatic adjustment is required. Adaptive control refers to a procedure for automatically tuning controllers. Two of the most popular schemes are self tuning regulators (STRs) and model reference adaptive systems (MRAS). The design techniques, in theory, are meant for unknown but fixed (i.e., time invariant) plants; in practice they work for slowly-varying (in time) and unknown plants.

For STRs the starting point is a design method for a known plant. Given that the plant is cncompletely known the parameters of the controller cannot be determined. They are, instead, obtained from a recursive parameter estimator. Since a separation between identification and control is assumed - this is referred to as an indirect methodology for adaptive control.

For MRAS, the starting point is a reference model which represents the performance desired of the control system after feedback. If the plant were known a suitable precompensator and feedback compensator could be used to obtain the closed loop transfer function equal to that of the model. Since the plant is unknown, one starts with an initial guess for the parameter of the compensator. This initial guess is updated (adapted) based on the error between the output of the model and the closed-loop plant when driven by a reference signal. The aim of the adaptation is to drive the error between the model output and plant output to zero. Since, no explicit separation between identification and control is assumed - this is referred to as a direct control methodology.

In this paper we study in detail model reference adaptive systems and their properties in regard to:

(1) Overall stability of the scheme,

(2) Convergence of the adaptive scheme - parameter and error convergence,

(3) Properties of the possible limiting adaptive controllers, and

(4) Sensitivity of the adaptive scheme to plant output disturbances and unmodelled (unparametrizable) dynamics.

The study of model reference adaptive systems is not new. They have been studied extensively by Narendra and co workers [1,2,5,12], Morse [3], Goodwin, Ramadge and Caines[4]. The papers [1,2,5] and Anderson [6], Kreisselmeier [7] in particular give a complete solution to questions (1), (2) and (3) above. Rohrs [10] discusses thoroughly the sensitivity of MRAS to unmodelled dynamics and output disturbances and shows by way of simulation evidence that unmodelled and unparametrized dynamics can result in the loss of stability of the MRAS schemes discussed in [1,2,5]. Ioannou and Kokotovic [11,21] represent the unmodelled dynamics as being singularly perturbed 'fast' or 'parasitic' dynamics and suggest a modification in the update law of [1,2,5] to ensure stability of the adaptive scheme. Narendra and Peterson [14], Kreisselmeier and Narendra [15] suggest the inclusion of a dead zone in the update law of [1,2,5] to preserve stability of the MRAS.

Our contribution and the layout of the paper is as follows: In Section 2, we discuss concepts of structured (parametrizable) and unstructured (unparametrizable) uncertainty in the model of a plant to be controlled. Such a discussion is novel in an adaptive control context. We discuss here when it might be preferrable to use adaptive rather than robust non-adaptive control as is used, say, in Doyle and Stein [18] or Desoer and Gustafson [22]. We introduce the information needed about the structured uncertainties in the plant to do adaptive control. Section 3 gives the controller structure and the parametrization of the plant uncertainty through controller parameters. In Section 4 and Appendix 1 we rederive a proof of stability and convergence of the error to zero for the MRAS - while our proof is very similar to that of [2], we take this opportunity to correct the errors in the proof of [2], (Lemmas 4 and 5 esp. of Section IV in [2]). In Section 5 we give explicit conditions on the exogenous reference signal input into the adaptive system to guarantee exponential convergence of the parameter error to zero.* The conditions available in the literature [5,6,7] were in terms of the sufficient richness of a certain signal inside the time-varying adaptive loop and so were not explicitly verifiable. In Section 6 we study robustness aspects of adaptive control - we first survey briefly the results in [15] to show that a deadzone in the parameter update law stabilizes the MRAS system against output disturbances. We use this same machinery to

*These results were derived jointly with S. Boyd, see [23].

show a suitably chosen deadzone can also stabilize the adaptive system against the effects of unmodelled dynamics. The price to be paid is that the error between plant and model output does not converge to zero but rather to a magnitude less than the size of the deadzone (off a set of finite measure). We again compare the robustified adaptive controller with a non-adaptive controller.

In Section 7, we discuss briefly the work in multivariable adaptive control along with research directions for further work. While the present paper discusses only continuous time model reference adaptive control, the modification of the results on stability and parameter convergence for the discrete time case are procedural. The modification of the robustness results to the discrete-time case are not easy or obvious to this author (for lack of a good notion of unstructured uncertainty).

Section 2. Structured and Unstructured Uncertainty

In a large number of control systems design problems, the designer is not furnished with a detailed state space model of the plant to be controlled either because it is too complex or because its dynamics are incompletely understood. Consider first the kind of prior information available to control a stable plant: by performing input-output experiments (usually applying a succession of sinusoidal signals at the input), the designer obtains a Bode diagram of the form shown in Figure 1. Typically, an inspection of the Bode diagram shows that the data obtained beyond a certain frequency, ω_H is unreliable because the measurements are poor, corrupted by noise, etc. What is available then is essentially no phase information and only an 'envelope' of the magnitude response beyond ω_H. The dashed lines in the magnitude and phase response correspond to the assumption that there are no dynamics at frequencies beyond ω_H. For frequencies below ω_H it is easy to 'guess' the presence of a zero in the neighborhood of ω_1 (shown dashed in Fig. 1(a)), poles in the neighborhood of ω_2, ω_3 and pole pairs in the neighborhood of ω_4, ω_5.

2.1. Model Reference Control Adpative and Non-Adaptive

To keep the design goal specific we will assume that the designers goal is model following: the designer is furnished with a reference model with transfer function $\hat{M}(s) \in \mathbb{R}(s)$ by the user and told to design a control system so as to get the plant output denoted $y_P(t)$ to track the model output, denoted $y_M(t)$, in response to reference signals $r(t)$ driving the model. This is shown pictorially in Figure 2. The controller generates the input $u(t)$ to the plant using $y_M(t)$, $y_P(t)$, $r(t)$ so that asymptotically the error between the plant and model, $e_1(t) = y_P(t) - y_M(t) \to 0$. Two options are available to the designer at this point:

(1) Non-adaptive Control. Here one uses as model for the plant a nominal transfer function with a zero at $s = -\omega_1$, poles at $s = -\omega_2, -\omega_3$; pole pairs at $s = -\sigma_4 \pm j\omega_4$, $s = -\sigma_5 \pm j\omega_5$ (estimates for ω_4, ω_5 are obtained from the heights of the magnitude peaks in Figure 1a at ω_4, ω_5), so as to obtain a 'nominal' rational transfer function $\hat{P}(s) \in \mathbb{R}(s)$

$$\hat{P}(s) = \frac{k_p(s+\omega_1)}{(s+\omega_2)(s+\omega_3)(s+\sigma_4+j\omega_4)(s+\sigma_4-j\omega_4)(s+\sigma_5+j\omega_5)(s+\sigma_5-j\omega_5)} \qquad (2.1)$$

The gain k_p in (2.1) is obtained from the nominal high frequency asymptote of Fig. 1(a) (i.e., the dashed line). One then estimates the modelling error due to inaccuracies in the postulated pole-zero locations and poor data and represents the plant transfer function as

$$\hat{P}(s)[1+\hat{U}_1(s)] \tag{2.2}$$

$\hat{U}_1(s) \in \mathbb{R}(s)$ is referred to as the multiplicative (loosely speaking 'percentage') uncertainty and $|\hat{U}_1(j\omega)|$ is typically as shown in Fig. 3 (see [18] for a good discussion of the form (2.2)). One treats $\hat{M}(s)$ as the desired input-output transfer function for the control system over the frequency range of the r(t). Then, one attempts to build a linear, time-invariant controller of the form shown in Fig. 4, with feedforward compensator $\hat{C}(s) \in \mathbb{R}(s)$ and feedback compensator $\hat{F}(s) \in \mathbb{R}(s)$ so that the nominal closed loop transfer function

$$\hat{P}\hat{C}(1+\hat{P}\hat{C}\hat{F})^{-1} \tag{2.3}$$

is equal to $\hat{M}$ over the range of reference inputs r(t). Further $\hat{C}$, $\hat{F}$ are chosen so as to preserve stability in the presence of the unmodelled dynamics represented by $\hat{U}_1$ and reduce sensitivity of the actual closed loop transfer function to the modelling errors represented by $\hat{U}_1$; i.e., if the actual closed transfer function is

$$\hat{P}(1+\hat{U}_1)\hat{C}[1+\hat{P}(1+\hat{U}_1)\hat{C}\hat{F}]^{-1} = \hat{M}(1+\hat{U}_2) \tag{2.4}$$

then $\hat{C}$, $\hat{F}$ are chosen so that $\hat{U}_2$ is smaller than $\hat{U}_1$ over the frequency range of the reference signal, r(t).

(2) Adaptive Control. The distinction is made in this methodology between the two kinds of uncertainty present in the description of Fig. 1 - parametric or structured uncertainty in the pole-zero locations and inherent

or unstructured uncertainty beyond ω_H. Rather than postulate a nominal transfer function for the plant, the designer decides to identify the pole zero locations on-line, i.e., during the operation of the plant. This on-line 'tuning' corresponds to the reduction of structured uncertainty during plant operation. The aim is to obtain asymptotically as $t \to \infty$ is a match between $\hat{M}(s)$ and the controlled plant that is better than in option 1 above for frequencies below ω_H. A key feature of this on-line tuning approach is that the controller is generally non-linear and time-varying. The added complexity of adaptive control is made worthwhile when the performance achieved by non-adaptive control is inadequate.

In summary, the plant model for adaptive control is given by

$$\hat{P}_{\theta^*}(s)(1+\hat{U}_3(s)) \tag{2.4}$$

where $\hat{P}_{\theta^*}(s)$ stands for the plant indexed by the parameters θ^*, for true pole and zero locations and $\hat{U}_3(s)$ is the unstructured uncertainty. The difference between (2.2) and (2.4) lies in the fact that

$$|\hat{U}_3(j\omega)| \leqq |\hat{U}_1(j\omega)| \quad \forall\ \omega \tag{2.5}$$

with the inequality being strict at low frequencies, as shown for example in Fig. 3. While θ^* are the parameters corresponding to true pole and zero locations, these are apriori unknown and the aim of adaptive control is to obtain θ^* from an initial guess θ.

When the plant is unstable, a frequency response curve as shown in Fig. 1 can no longer be obtained, and a certain amount of off-line identification or detailed modelling needs to be performed. As before, however, the plant model will have better structured and unstructured uncertainty and the design options will be the same as above. The difference only arises in the representation of the uncertainty. In option 1, the uncertainty was modelled as (see Eq. (2.2)) $\hat{P}(s)(1+\hat{U}_1(s))$, with $\hat{U}_1(s)$ stable and with bounds on $|\hat{U}_1(j\omega)|$. When the plant is unstable, then since the nominal locations of the unstable poles may not be chosen exactly $\hat{U}_1(s)$ may be an unstable transfer function.

For adaptive control, we require that all unstable poles of the system be parametrized (of course their exact location is not essential) so that the description for the uncertainty is still given by (2.4) i.e.

$$\hat{P}_\theta(s)(1+\hat{U}_3(s))$$

where $\hat{U}_3(s)$ is stable, even though $\hat{P}_\theta(s)$ is unstable.

Example

Consider a plant with 'true' transfer function with ε small and $M > 0$ large

$$\frac{1 \cdot M}{(s-1+\varepsilon)(s+M)}$$

For non-adaptive control the nominal model is chosen to be $\hat{P}(s) = \frac{1}{s-1}$ so that

$$\hat{U}_1(s) = \frac{-s^2+s-\varepsilon s-\varepsilon M}{(s-1+\varepsilon)(s+M)} \tag{2.6}$$

For adaptive control on the other hand the model $\hat{P}_\theta(s)$ of the form $\frac{1}{s+\theta}$ is chosen and

$$\hat{U}_3(s) = \frac{-s}{s+M} \tag{2.7}$$

Note that (2.6) is unstable while (2.7) is not.

2.2. Prior Information for Adaptive Control

While there are several methods of adaptive control available, our focus in this paper is on direct model reference adaptive control. The plant to be controlled is of the form (2.4), i.e.,

$$\hat{P}_\theta(s)(1+\hat{U}_3(s)). \tag{2.4}$$

We make the parametrization more explicit; by writing the nominal plant $\hat{P}_\theta$ as

$$k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \tag{2.8}$$

where $\hat{n}_p(s), \hat{d}_p(s) \in \mathbb{R}[s]$ are coprime, monic polynomials of degrees m and n respectively. While the zeros of $\hat{n}_p$ and $\hat{d}_p$ (the zeros and poles of $\hat{P}_\theta(s)$ respectively) are not assumed to be known, we will assume that the following reasonable information about $\hat{P}_\theta$ is known:

A1. The number of poles of $\hat{P}_\theta$, i.e. n is known.

A2. The number of zeros of $\hat{P}_\theta$, i.e., m is known.

A3. The sign of the high frequency gain k_p is known.

A4. $\hat{P}_\theta$ is minimum phase, i.e., the zeros of $\hat{n}_p$ lie in the open left half plane ($\mathring{\mathbb{C}}_-$).

Comments

Assumptions 1, 2, 3 apply to the nominal plant $\hat{P}$; but not necessarily to the plant of (2.4). In particular $\hat{P}_\theta(1+U)$ (the subscript 3 is dropped for simplicity) can have many more stable poles and zeros than $\hat{P}_\theta$ (of the example of Section 2.1). Further, the sign of the high frequency gain of $\hat{P}_\theta(1+\hat{U})$ is usually indeterminate as indicated in Figure 1. Assumption 4 is the most stringent of the four assumptions and is needed since the adaptive control procedure of Sections 3 and 4 uses asymptotic zero cancellation.

The literature [2,3,4] lists assumptions 1 and 2 slightly differently: it asks for knowledge of an upper bound on n and the exact relative degree n-m of the plant. From the discussion of Section 2.1, the present form of the assumptions is adequate.

In Sections 3, 4, 5 we will discuss the structure and convergence properties of the error and parameters for the idealized model of the plant <u>with no unstructured uncertainty</u>, i.e., we will assume, <u>temporarily</u>, that the plant is modelled adequately as $\hat{P}_\theta$ with $\hat{P}_\theta$ satisfying (A1)-(A4). These assumptions will be relaxed in Section 6 which studies robustness aspects of adaptive control.

Section 3. Controller Structure

3.1. Problem Statement

The nominal plant transfer function $\hat{P}(s) \in \mathbb{R}(s)$ is described as in Section 2.2 by

$$\hat{P}(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \tag{3.1}$$

where $\hat{n}_p(s)$, $\hat{d}_p(s) \in \mathbb{R}[s]$ are monic, coprime polynomials of degree m, n respectively. The zeros of $\hat{n}_p$ are assumed to be in $\mathring{\mathbb{C}}_-$ and k_p is unknown but positive. The input and output of the plant are denoted $u(t)$ and $y_p(t)$ respectively.

The model $\hat{M}(s)$ represents the behavior expected from the plant after a suitable controller has been found and has transfer function

$$\hat{M}(s) = k_M \frac{\hat{n}_M(s)}{\hat{d}_M(s)} \tag{3.2}$$

with $\hat{d}_M(s)$ a monic, polynomial of degree n, $\hat{n}_M(s)$ a monic polynomial of degree r and k_M a positive constant. It is assumed that $\hat{n}_M$ and $\hat{d}_M$ both have zeros in $\mathring{\mathbb{C}}_-$. The input and output of the model are denoted $r(t)$ and $y_M(t)$ respectively.

The derivation of the model output from the plant output is given by $e_1(t) = y_p(t) - y_M(t)$. The aim of the adaptive control scheme is to find $u(t)$ so that $e_1(t) \to 0$ as $t \to \infty$.

3.2. System Structure

The basic controller structure is shown in Fig. 5. The configuration is similar to the non-adaptive scheme of Figure 4 with the F_1 block playing the role of the precompensator $\hat{C}(s)$ and the F_2 block that of the feedback controller $\hat{F}(s)$. The precompensator is realized in feedback form using an auxiliary signal generator with transfer function $(sI-\Lambda)^{-1}b \in \mathbb{R}^{n-1}(s)$ with input $u(t)$ and output $v^{(1)}(t) \in \mathbb{R}^{n-1}$. Here $\Lambda \in \mathbb{R}^{n-1 \times n-1}$ and $b \in \mathbb{R}^{n-1}$. The auxiliary signals $v^{(1)}$ are multiplied by the (n-1) adaptive gains of $c(t) \in \mathbb{R}^{n-1}$ to form part of the input $u(t)$. The F_2 block consists of a signal generator identical to that in F_1 with input $y_p(t)$ and output $v^{(2)}(t) \in \mathbb{R}^{n-1}$. The auxiliary signals

$v^{(2)}(t)$ are multiplied by the adaptive gains $d(t) \in \mathbb{R}^{n-1}$ and the plant output $y_p(t)$ by the adaptive gain $d_0(t)$ to obtain the plant input

$$u(t) = c_0 r + c^T v^{(1)} + d_0 y_p + d^T v^{(2)} \tag{3.3}$$

The adaptive controller structure should have enough degrees of freedom to that with the time varying parameters c_0, c^T, d, d^T set equal to constant; and suitable choice of Λ, b the overall plant and controller transfer function matches that of the specified model. This is verified in Theorem 3.1 under the <u>condition that the model has the same relative degree as the plant</u>. First, notice that Λ, b enter our calculations only through the transfer function $(sI-\Lambda)^{-1}b$ so that we may assume without loss of generality that Λ, b are in controllable form so that

$$(sI-\Lambda)^{-1}b = \frac{1}{\hat{\Lambda}(s)} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-2} \end{bmatrix} \tag{3.4}$$

with $\hat{\Lambda}(s) = \det(sI-\Lambda) \in \mathbb{R}(s)$. To keep the controller stable we will insist on $\hat{\Lambda}(s)$ being Hurwitz: from (3.4), we note that the choice of Λ, b is now equivalent to the choice of a stable $\hat{\Lambda}(s) \in \mathbb{R}(s)$.

<u>Theorem 3.1</u>

Given a model $\hat{M}$ of the form (3.2) and plant $\hat{P}$ of the form (3.1); then so long as $r = m$, there exists a choice of c_0^*, $d_0^* \in \mathbb{R}$, c^*, $d^* \in \mathbb{R}^{n-1}$, $\hat{\Lambda}(s) \in \mathbb{R}(s)$ and Hurwitz so that the closed loop transfer function of the plant and controller equals $\hat{M}$.

<u>Proof</u>: Verify that when the adaptive gains are constant with values

given by c_0^*, c^*, d_0^*, d^* the transfer function of the precompensation F_1 is given by

$$1/(1-\hat{C}^*(s)/\hat{\Lambda}(s)) \tag{3.5}$$

with $\hat{C}^*(s) = c_1^* + c_2^*s + \ldots + c_{n-1}^*s^{n-2}$ (using (3.4)) and the transfer function of the feedback controller F_2 is given by

$$\hat{D}^*(s)/\hat{\Lambda}(s) \tag{3.6}$$

where $D^*(s) = d_0\hat{\Lambda}(s) + d_1^* + \ldots + d_{n-1}^*s^{n-2}$. Note that $\hat{C}^*(s)$ has degree $n-2$ and $\hat{D}^*(s)$ degree $(n-1)$. Using (3.5) and (3.6) we see that the closed loop plant transfer function is given by

$$\frac{c_0^*\hat{P}(s)\hat{\Lambda}(s))}{\hat{\Lambda}(s)-\hat{C}^*(s)-\hat{D}^*(s)\hat{P}(s)} \tag{3.7}$$

To show that the expression in (3.7) can be made equal to $\hat{M}$ for suitable choice of $\hat{\Lambda}$ of degree n-1, $\hat{D}^*$ of degree (n-2) we need the following lemma.

<u>Lemma 3.2</u>

Given two coprime monic polynomials $\hat{n}_p$, $\hat{d}_p$ of degree m and n; and an arbitrary monic polynomial $\hat{X}$ of degree 2n-1, there exist unique polynomials $\hat{T}$ (monic) and $\hat{R}$ of degree (n-1) respectively so that

$$\hat{T}\hat{d}_p + \hat{R}\hat{n}_p \equiv \hat{X} \tag{3.8}$$

<u>Proof</u>: Since $\hat{d}_p$ and $\hat{n}_p$ are coprime, it follows from the Euclidean algorithm that there exist polynomials $\hat{A}$ of degree m-1 and $\hat{B}$ of degree

n-1 (e.g., Jacobson [19])

$$\hat{A}\hat{n}_p + \hat{B}\hat{d}_p \equiv 1. \tag{3.9}$$

Multiplying both sides of (3.9) by $\hat{X}$ we have

$$\hat{X}\hat{A}\hat{n}_p + \hat{X}\hat{B}\hat{d}_p \equiv \hat{X}. \tag{3.10}$$

Dividing $\hat{X}\hat{A}$ by $\hat{d}_p$ we get

$$\frac{\hat{X}\hat{A}}{\hat{d}_p} := \hat{Y} + \frac{\hat{R}}{\hat{d}_p} \tag{3.11}$$

where $\hat{R}$ is the remainder of degree $\leq$ n-1. Rewrite (3.10) as

$$(\hat{X}\hat{A}-\hat{Y}\hat{d}_p)\hat{n}_p + (\hat{X}\hat{B}-\hat{Y}\hat{n}_p)\hat{d}_p \equiv \hat{X}. \tag{3.12}$$

Using (3.11) we see that (3.12) is of the required form (3.8) since $\hat{X}\hat{A}-\hat{Y}\hat{n}_p = \hat{R}$ has degree $\leq$ n-1; thereby forcing $\hat{X}\hat{B}-\hat{Y}\hat{n}_p =: \hat{T}$ to have degree $\leq$ n-1. Further $\hat{X}$ monic implies that $\hat{T}$ is monic.

<u>Uniqueness</u>

If there exist other polynomials $\hat{T}_1$ (monic) and $\hat{R}_1$ of degree (n-1) so as to satisfy (3.8) we would have

$$(\hat{T}-\hat{T}_1)\hat{d}_p + (\hat{R}-\hat{R}_1)\hat{n}_p \equiv 0$$

so that $\dfrac{\hat{n}_p}{\hat{d}_p} = -\dfrac{(\hat{R}-\hat{R}_1)}{(\hat{T}-\hat{T}_1)}$

with $\hat{T}-\hat{T}_1$ of degree n-1, thereby contradicting the coprimeness of $\hat{n}_p$, $\hat{d}_p$. □

We will now use the lemma to show that for suitable choice of c_0^*, $\hat{C}^*$, $\hat{D}^*$, $\hat{\Lambda}$ the expression of (3.7) can be made equal to

$$k_M \frac{\hat{n}_M(s)}{\hat{d}_M(s)} \tag{3.13}$$

where $\hat{d}_M$, $\hat{n}_M$ are monic, Hurwitz polynomials of degree n, r respectively: Choose $c_0^* = \frac{k_M}{k_p}$; and choose $\hat{\Lambda}$ to be any monic Hurwitz polynomial so that $\hat{n}_M$ divides $\hat{\Lambda}$. Such a choise is always possible since $\hat{n}_M$ has degree $r = m \leq n-1$. Now, to apply the Lemma 3.2 choose

$$\hat{X} = \frac{\hat{d}_M \hat{n}_p \hat{\Lambda}}{\hat{n}_M} .$$

Verify that $\hat{X}$ is a monic polynomial of degree 2n-1. By Eq. (3.8) of Lemma 3.2, there exist polynomials $\hat{T}$ and $\hat{R}$ of degree n-1, with $\hat{T}$ monic so that

$$\hat{T}\hat{d}_p + \hat{R}\hat{n}_p = \frac{\hat{d}_M \hat{n}_p \hat{\Lambda}}{\hat{n}_M} \tag{3.14}$$

now choose

$$\hat{D}^*(s) = -\hat{R}(s)$$

and

$$\hat{C}^*(s) = -\hat{T}(s) + \hat{\Lambda}(s)$$

($\hat{D}^*(s)$ has degree n-1 since $\hat{R}$ is of degree n-1, $\hat{C}^*(s)$ has degree n-2 since it is the difference of two monic polynomials of degree n-1).
Verify now that the expression in (3.7) may be written as

$$\frac{c_0^* k_p \hat{n}_p \hat{\Lambda}}{(\hat{\Lambda}-\hat{C}^*)\hat{d}_p - D^* \hat{n}_p} \tag{3.16}$$

Using (3.14), (3.15) and $c_0^* = k_M/k_p$ we see that (3.16) is equal to $k_M\hat{n}_M/\hat{d}_M$ as required. Once c_0^*, $\hat{\Lambda}$ have been chosen, the choice of $\hat{C}^*$, $\hat{D}^*$ is unique by the Lemma 3.2. It follows that n_p divides $\hat{\Lambda}-C^*$. Also $\hat{\Lambda}$ has been chosen so that $\hat{n}_M$ divides $\hat{\Lambda}$. In the instance that $\hat{n}_p$ and $\hat{n}_M$ are of degree n-1 (the relative degree 1 case), $\hat{\Lambda}-C^* = \hat{n}_p$ and $\hat{\Lambda} = \hat{n}_M$ so that equation (3.16) simplifies to

$$\frac{k_M\hat{n}_M}{\hat{d}_p-\hat{D}^*} = \frac{k_M\hat{n}_M}{\hat{d}_M} \quad .$$

In this case then

$$\hat{n}_M - \hat{n}_p = \hat{C}^*$$

and

$$\hat{d}_p - \hat{d}_M^* = \hat{D}^*$$

so that the c_i^* and d_i^* are <u>related affinely to the numerator and denominator coefficients of the plant transfer function</u>. The relationship is more complex when the relative degree is greater than 1.

3.3. <u>The Error Equation</u>

From the proof of Theorem 3.1, we see that if $\hat{n}_p$ and $\hat{d}_p$ were known the precompensator and feedback compensator could be chosen to obtain model matching. Since $\hat{n}_p$ and $\hat{d}_p$ are not known exactly, $\hat{C}^*$ and $\hat{D}^*$ cannot be obtained. The purpose of adaptive control is to start with an initial guess for the parameter c_0, c, d_0, d and then modify the guess based on the output error $e_1 = y_p - y_M$. We derive here the equation relating output error to parameter error. Define the parameter vector $\theta \in \mathbb{R}^{2n}$ by

$$\theta^T = [c_0, c^T, d_0, d^T] \tag{3.17}$$

and the signal vector $w \in \mathbb{R}^{2n}$ by

$$w^T = [r, v^{(1)T}, y_p, v^{(2)T}]. \tag{3.18}$$

Now, for given choice of $\hat{\Lambda}$ let $\theta^* \in \mathbb{R}^{2n}$ represent the value of the parameters required to obtain the model transfer function. Then the parameter error ϕ is defined to be $\theta-\theta^*$. To obtain the error equation, let $c_p^T(sI-A_p)^{-1}b_p$ be a minimal realization of $k_p\hat{n}_p/\hat{d}_p$. Then the equations of the control system in state space form are

$$\begin{bmatrix} \dot{x}_p \\ \dot{v}^{(1)} \\ \dot{v}^{(2)} \end{bmatrix} = \begin{bmatrix} A_p & 0 & 0 \\ 0 & \Lambda & 0 \\ bc_p^T & 0 & \Lambda \end{bmatrix} \begin{bmatrix} x_p \\ v^{(1)} \\ v^{(2)} \end{bmatrix} + \begin{bmatrix} b_p \\ b \\ 0 \end{bmatrix} u \tag{3.19}$$

Since $u = \theta^T w$ and $\theta = \theta^* + \phi$, equation (3.19) can be written as

$$\begin{bmatrix} \dot{x}_p \\ \dot{v}^{(1)} \\ \dot{v}^{(2)} \end{bmatrix} = \underbrace{\left[\begin{array}{c|c|c} A_p + d_0^* b_p c_p^T & b_p c^{*T} & b_p d^{*T} \\ \hline b d_0^* c_p^T & \Lambda + b c^{*T} & b d^{*T} \\ \hline b c_p^T & 0 & \Lambda \end{array}\right]}_{\tilde{A}} \begin{bmatrix} x_p \\ v^{(1)} \\ v^{(2)} \end{bmatrix} + \underbrace{\begin{bmatrix} b_p \\ b \\ 0 \end{bmatrix}}_{\tilde{b}} (\phi^T w + c_0^* r)$$

The output equation is

$$y_p = \underbrace{[c_p \quad 0 \quad 0]}_{\tilde{c}} \begin{bmatrix} x_p \\ v^{(1)} \\ v^{(2)} \end{bmatrix} \tag{3.21}$$

The model can be similarly represented in non-minimal form (with suitable choice of initial conditions) as

$$\begin{bmatrix} \dot{x}_M \\ \dot{v}_M^{(1)} \\ \dot{v}_M^{(2)} \end{bmatrix} = \tilde{A} \begin{bmatrix} x_M \\ v_M^{(1)} \\ v_M^{(2)} \end{bmatrix} + \tilde{b}(c_0^* r) \tag{3.22}$$

$$y_M = \tilde{c} \begin{bmatrix} x_M \\ v_M^{(1)} \\ v_M^{(2)} \end{bmatrix} \tag{3.23}$$

where $\tilde{A} \in \mathbb{R}^{3n-2\times 3n-2}$, $\tilde{b}$, $\tilde{c}^T \in \mathbb{R}^{3n-2}$ are exactly the same matrices as in (3.20), (3.21). Though the realization (3.22), (3.23) for $\hat{M}$ is non-minimal it is easily verified to be <u>detectable</u> and <u>stabilizable</u> (i.e., no unstable modes are hidden). These facts will be useful in the stability proofs of Section 4. Subtracting Eq. (3.22) from (3.20) and (3.23) from (3.21) we get: with $e^T := [x_p^T, v^{(1)T}, v^{(2)T}] - [x_M^T, v_M^{(1)}, v_M^{(2)T}]$

$$\left.\begin{aligned} \dot{e} &= \tilde{A}e + \tilde{b}\phi^T w, \; e \in \mathbb{R}^{3n-2} \\ e_1 &= \tilde{c}e. \end{aligned}\right\} \tag{3.24}$$

Also it follows from Eqs. (3.22), (3.23) that

$$\tilde{c}(sI-A)^{-1}\tilde{b}c_0^* = \hat{M}$$

so that the equation for the error e_1 may be drawn as in Figure 6 and represented as

$$e_1 = \frac{1}{c_0^*}\hat{M}(\phi^T w) = \frac{k_p}{k_M}\hat{M}\cdot(\phi^T w) \tag{3.25}$$

In Eq. (3.25) $\hat{M}(\phi^T w)$ is to be understood as the usual convolution between the impulse response of $\hat{M}$ and $\phi^T w$. Further, the effect of the initial condition is not explicit in Eq. (3.24).

3.3. The Update Law

We have shown that when the adaptive parameters θ are not at the desired values θ^* the output error between plant and model is given by

$$e_1 = \frac{k_p}{k_M}\hat{M}(s)\cdot\phi^T w.$$

what remains to be specified is a mechanism for starting from an initial guess θ_0 of the parameter values and updating this on line so as to get e_1 to tend to zero as $t \to \infty$ and (if possible) $\theta \to \theta^*$. This is referred to as the update law. The update law is a formula for $\dot{\theta}$, or equivalent the parameter error $\dot{\phi}$.

Section 4. The Stability Proofs and Choice of Updata Law

The basic controller structure will need to be modified slightly during the course of this section. Rather than give the most general proof immediately, we will give a simple stability proof for the case when the relative degree of the plant i.e., n-m = 1 (Section 4.1), when n-m = 2

(Section 4.2) and finally when n-m $\geq$ 3 (Section 4.3).

4.1. The Relative Degree One Case

For the purpose of a stability proof we will need to assume that the model $\hat{M}(s)$ is strictly positive real.* Superficially, and certainly mathematically, this entails no loss of generality since given any stable, minimum phase $\hat{M}(s)$ of relative degree 1 (which are the only models that can be matched when the plant has relative degree 1 - see Theorem 3.1), there exists a proper, but not strictly proper, stable and stably invertible $\hat{G}(s) \in \mathbb{R}(s)$ wo that $\hat{G}(s)\hat{M}(s)$ is strictly positive real, (for example, $\hat{G}(s) = \hat{M}^{-1}(s)/s+1$). By using $\hat{G}^{-1}(s)$ as prefilter for the reference signal r(t) we can change the model matching problem from one of matching $\hat{M}(s)$ to one of matching the strictly positive real transfer function $\hat{G}(s)M(s)$. (It is important to note that the prefilter $G^{-1}(s)$ has not differentiators by choice of $\hat{G}(s)$ proper but not strictly proper; also $\hat{G}^{-1}$ is stable). The implications of assuming the model $\hat{M}$ to be positive real are considered in Section 6 when unmodelled dynamics are present. We now have

Theorem 4.1

For the adaptive system of Figure 5, consider the update law

$$\dot{\theta} = -e_1 w. \tag{4.1}$$

where θ and w are defined in (3.17), (3.18) respectively. Then, provided r(t) is bounded;

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*A transfer function $\hat{M}(s)$ is said to be strictly positive real if for some $\varepsilon > 0$ Re $M(-\varepsilon+j\omega) \geq 0 \quad \forall\ \omega \in \mathbb{R}$.

$$\lim_{t\to\infty} e_1(t) = 0 \tag{4.2}$$

Proof: Consider the error equations of (3.24), namely

$$\left.\begin{aligned} \dot{e} &= \tilde{A}e + \tilde{b}\phi^T w \\ e_1 &= \tilde{c}e \end{aligned}\right\} \tag{3.24}$$

where $\tilde{c}(sI-\tilde{A})^{-1}\tilde{b}$ is a nonminimal but detectable and controllable realization of the transfer function $\frac{k_p}{k_M}\hat{M}(s)$. Since $\hat{M}$ is strictly positive real (SPR) and k_p, $k_M > 0$; we have that $\tilde{c}(sI-\tilde{A})^{-1}b$ is SPR. Hence, by the PR Lemma (see for example [25]), there exist positive definite matrices P, Q such that

$$\left.\begin{aligned} \tilde{A}^T P + P\tilde{A} &= - Q \\ P\tilde{b} &= \tilde{c}^T. \end{aligned}\right\} \tag{4.3}$$

Now consider the (time-varying) dynamical system

$$\left.\begin{aligned} \dot{e} &= \tilde{A}e + \tilde{b}\phi^T w \\ \dot{\phi} &= -e_1 w \end{aligned}\right\} \tag{4.4}$$

with Lyapunov function

$$V(e,\phi) = e^T Pe + \phi^T\phi \tag{4.5}$$

Verify using Eqs. (4.3) that

$$\dot{V}(e,\phi) = -e^T Qe \leq 0. \tag{4.6}$$

Since $\dot{V} \leq 0$; we have that $e(t)$, $\phi(t)$ are bounded. Recall that

$e^T(t) = [x_p^T, v^{(1)T}, v^{(2)T}] - [x_M^T, v_M^{(1)T}, v_M^{(2)T}]$. Further note that $r(t)$ bounded and the realization (3.22), (3.23) of the model is stable so that x_M, $v_M^{(1)}$, $v_M^{(2)}$ are bounded. Consequently $[x_p^T, v^{(1)T}, v^{(2)T}]$ is bounded. From this it follows that $w^T = [r, v^{(1)T}, y_p, v^{(2)T}]$ is bounded. Using this fact in Eqs. (4.4) we may conclude that $\dot{e}$, $\dot{\phi}$ are bounded (so that e, ϕ are uniformly continuous). Now since $V(e,\phi) \geq 0$ we have that

$$\int_0^\infty \dot{V} dt < \infty. \tag{4.7}$$

Since $\dot{V} = -e^T Q e$ is uniformly continuous (e is uniformly continuous), as a function of time, we have that

$$\lim_{t\to\infty} \dot{V} = 0 \Rightarrow \lim_{t\to\infty} e = 0 \tag{4.8}$$

From (3.24) above it follows that $\lim_{t\to\infty} e_1(t) = 0$. □

Remarks: (1) We have shown above that $e(t)$, $\phi(t)$, $w(t)$, $\dot{e}(t)$, $\dot{\phi}(t)$ are bounded and that $\lim_{t\to\infty} e(t) = 0$. We see also from (4.4)

$$\lim_{t\to\infty} \dot{\phi} = 0$$

and from (4.5), (4.6) that

$$\lim_{t\to\infty} V(e,\phi) = \lim_{t\to\infty} \phi^T \phi = V^*$$

exists. Though $\|\phi(t)\|$ converges to $(V^*)^{1/2}$, we can say nothing about the convergence of $\phi(t)$ as $t\to\infty$ (leave alone about its convergence to zero).

(2) The asymptotic stability of the system (4.4) has been studied

in [5,6,7]. For given $w(t)$, the system (4.4) is a linear time varying system. The linear time-varying system is uniformly asymptotically stable (and equivalently exponentially stable) if iff $\exists\ \alpha, \delta > 0$ such that $\forall\ s \in \mathbb{R}_+$

$$\int_s^{s+\delta} ww^T dt \geq \alpha I \qquad I \in \mathbb{R}^{2n\times 2n} \tag{4.9}$$

The condition (4.9), is referred to as a sufficient richness condition; since it requires that $w(t)$ be uniformly exciting in all directions over any interval of length δ. The principal drawback of the condition (4.9) is that is applies to a vector of signals $w(t) \in \mathbb{R}^{2n}$ inside the time-varying adaptive loop. In Section 5 we give conditions on the exogenous reference input $r(t)$ so that $w(t)$ satisfies (4.9).

(3) It is easy to verify that Theorem 4.1 still holds with the update law of (4.1) replaced by

$$\dot{\theta} = -\Gamma e_1 w$$

where Γ is any positive definite matrix. The corresponding Lyapunov function in the proof of Theorem (4.1) is $V(e,\phi) = e^T Pe + \phi^T \Gamma^{-1} \phi$, and so on.

4.2. The Relative Degree Two Case

When the model and plant have relative degree 2 (i.e., $n-m=2$), $\hat{M}$ cannot be assumed positive real. We can however assume (using suitable prefiltering, if necessary, as in Section 4.1) that there exists $\hat{L}(s) = (s+\delta)$ with $\delta > 0$ such that $\hat{M}\hat{L}$ is strictly positive real. The adaptive scheme of Fig. 5 is modified by replacing each of the adaptive gains θ_i, viz. c_0, c, d_0, d, by the gains $\hat{L}\theta_i\hat{L}^{-1}$, which in turn is given by

$$\hat{L}\theta_i\hat{L}^{-1} = \theta_i + \theta_i\hat{L}^{-1} \qquad i = 1, \ldots, 2n \tag{4.10}$$

To obtain the new error model, define the signal vector

$$\zeta^T(t) = [\hat{L}^{-1}r, \hat{L}^{-1}v^{(1)T}, \hat{L}^{-1}y_p, L^{-1}v^{(2)T}].$$

Then, from reasoning completely analogous to that of Section 3.3, we obtain that

$$e_1(t) = \frac{k_p\hat{M}(s)\cdot\hat{L}(s)}{k_M}\cdot\phi^T\zeta. \tag{4.11}$$

We now have

Theorem 4.2

For the adpative system of Fig. 5 modified by replacing each of the $\theta_i(i=1,\ldots,2n)$ by

$$\theta_i + \dot{\theta}_i\hat{L}^{-1}$$

as in Eq. (4.10), consider the update law

$$\dot{\theta} = -e_1\zeta, \tag{4.12}$$

Then, provided $r(t)$ is bounded,

$$\lim_{t\to\infty} e_1(t) = 0$$

Proof: Follows exactly along the lines of Theorem 4.1 since $\frac{k_p\hat{M}(s)\hat{L}(s)}{k_M}$ is SPR by choice of $\hat{L}$. Thus for the nonminimal realization of $\hat{M}\hat{L}$ given by the plant loop; the error equation (4.11), and the adaptive law (4.12) one writes down the same Lyapunov function $V(e,\phi)$ and checks that $\dot{V} \leqq 0$. To conclude that $\lim_{t\to\infty} \dot{V} = 0$ and hence $\lim_{t\to\infty} e_1(t) = 0$, one needs to show that $\zeta(\cdot)$ is bounded. This is established as follows

$$r(t) \text{ bounded} \Rightarrow y_M^{(\cdot)},\ v_M^{(1)}(\cdot),\ v_M^{(2)}(\cdot) \text{ bounded} \tag{4.13}$$

$$e(t) \text{ bounded and (4.13)} \Rightarrow y_p(\cdot),\ v^{(1)}(\cdot),\ v^{(2)}(\cdot) \text{ bounded} \tag{4.14}$$

Now $\hat{L}^{-1}$ stable and (4.14) $\Rightarrow \zeta(\cdot)$ bounded.

Remark: (1) As before Theorem 4.2 does not establish anything about the convergence of ϕ (and consequently θ).

(2) $e(t)$, $\phi(t)$ converge exponentially to zero iff the signal vector $\zeta(t)$ satisfies the sufficient richness condition, i.e., $\exists\ \alpha, \delta > 0$ such that $\forall\ s \in \mathbb{R}_+$

$$\int_s^{s+\delta} \zeta\zeta^T dt \geqq \alpha I \qquad I \in \mathbb{R}^{2n\times 2n}$$

4.3. The Relative Degree ≥ 3 Case

As in Section 4.2, pick a Hurwitz polynomial $\hat{L}$ so that $\hat{L}\hat{M}$ is SPR. The trick used in Section 4.2 in order to obtain the SPR error transfer function (4.11) is no longer possible since $\hat{L}\theta_i\hat{L}^{-1}$ depends on second (and possibly higher) derivatives of θ_i. However, to obtain the same error equation we could augment the model output by

$$\frac{k_p}{k_M}\hat{M}\hat{L}[\theta^T\hat{L}^{-1}-\hat{L}^{-1}\theta^T]w. \tag{4.16}$$

In Eq. (4.16) the notation $\hat{L}^{-1}w$ stands for each component of w ($\in \mathbb{R}^{2n}$) filtered by $\hat{L}^{-1}$. The difficulty in implementing (4.16) arises from the fact that k_p is unknown. Consequently, we augment the model output not by (4.16) but by

$$y_a(t) = \hat{M}\hat{L}\theta_{2n+1}(t)[\theta^T\hat{L}^{-1}-\hat{L}^{-1}\theta^T]w \tag{4.17}$$

with $\theta_{2n+1}(t)$ being a new adaptive parameter expected to converge to $\frac{k_p}{k_M}$. If $[\theta^T\hat{L}^{-1}-\hat{L}^{-1}\theta^T]w$ were to be denoted $\xi(t)$ the error equation would now read

$$e_1 = \frac{k_p}{k_M}\hat{M}\hat{L}\{\phi^T\zeta+\phi_{2n+1}\xi\} \tag{4.18}$$

where ϕ_{2n+1} is the parameter error in θ_{2n+1} given by

$$\phi_{2n+1} = 1 - \frac{k_M}{k_p}\theta_{2n+1}. \tag{4.19}$$

With this modification in hand it would appear from the same reasoning as in Theorems 2.1, 4.2 that the adaptive laws

$$\begin{aligned} \dot{\theta} &= \dot{\phi} = -e_1\zeta \\ \dot{\theta}_{2n+1} &= -\frac{k_p}{k_M}\dot{\phi}_{2n+1} = e_1\xi. \end{aligned} \tag{4.20}$$

would yield $e_1(t) \to 0$ as $t \to \infty$. Difficulty however arises in showing that $w(t)$ is bounded; $w(t)$ would be bounded if $y_p(t)$ were bounded. By assuming $r(t)$ bounded we have that $y_M(t)$ is bounded, and $e_1(t)$ is bounded by the Lyapunov function analysis. Now $y_p(t) = e_1(t) + y_M(t) + y_a(t)$; so that in the absence of information about $y_a(t)$ we cannot conclude that $y_p(t)$ is bounded. Further, even if we did conclude that $e_1(t) \to 0$ as $t \to \infty$; we could not conclude that $y_M(t) \to y_p(t)$ unless $y_a(t) \to 0$ as well.

We will now show that the conlcusions $y_M(t) \to y_p(t)$, $y_a(t) \to 0$ as $t \to \infty$ can be achieved if $\dot{\phi} \in L^2$. This in turn is assured if the augmented error signal $y_a(t)$ is changed from (4.17) to

$$y_a(t) = \hat{M}\hat{L}\theta_{2n+1}(t)\{[\theta^T\hat{L}^{-1}-\hat{L}^{-1}\theta^T]w-\alpha\zeta^T\zeta e_1\} \tag{4.21}$$

where $\alpha > 0$ is a positive number, along with the adaptive law (4.20), the resulting error equation given in (4.21) is represented in Fig. 1

$$e_1(t) = \frac{k_p}{k_M}[\hat{M}\hat{L}](\phi^T\zeta+\phi_{2n+1}\xi-\alpha e_1\zeta^T\zeta) \tag{4.22}$$

Note that no differentiators are used in the implementation of the control law since $\hat{M}\hat{L}$ is a proper rational function. The notation and preliminary lemmas used in the proof of the following theorem are collected in Appendix 1.

Theorem 4.3

The adaptive system of Section 4.3 with the error equation (4.22) and the adaptive law (4.20) yields when the reference signal r(t) is bounded that

$$\lim_{t\to\infty} y_M = y_p. \tag{4.23}$$

Proof: Let (A_1,b_1,c_1) be a realization of $\frac{k_p}{k_M}\hat{M}\hat{L}$ which is SPR by choice of $\hat{L}$. The realization is given by

$$\dot{e} = A_1e + b_1(\phi^T\zeta+\phi_{2n+1}\xi-\alpha e_1\zeta^T\zeta)$$

$$e_1 = c_1e$$

By the PR lemma, there exist P, Q > 0 such that

$$A_1^T P + PA_1 = - Q$$

$$Pb_1 = c_1^T$$

Now consider the Lyapunov function

$$\dot{V}(e,\phi,\phi_{2n+1}) = e^T Pe + \phi^T \phi + \frac{k_p}{k_M} \phi_{2n+1}^2$$

with time derivative

$$\dot{V}(e,\phi,\phi_{2n+1}) = - e^T Qe - 2\, e_1^2 \zeta^T \zeta \leqq 0$$

It follows that e, ϕ, ϕ_{2n+1} are bounded and since $\int \dot{V} dt < \infty$ that $e \in L^2$ and $e_1 \zeta \in L^2$. From the expression (4.20) for $\dot{\phi}$, we conclude that $\underline{\underline{\dot{\phi} \in L^2}}$. This fact is central to the remainder of the proof.

Recall that

$$\begin{aligned} y_p &= \hat{P}\theta^T w \\ &= \hat{M}r + \frac{k_p}{k_M} \hat{M}\phi^T w \\ &= \hat{M}r + \frac{k_p}{k_M} \hat{M}\hat{L}[\hat{L}^{-1}\phi^T \hat{L}]\zeta \end{aligned} \tag{4.24}$$

By Lemma A.4 of Appendix 1

$$[\hat{L}^{-1}\phi^T \hat{L}]\zeta = \phi^T \zeta + o(w) \tag{4.25}$$

By Lemma A.3 of Appendix 1 $w = O(y_p)$ so that (4.25) is rewritten as

$$[\hat{L}^{-1}\phi^T\hat{L}]\zeta = \phi^T\zeta + o(y_p). \tag{4.26}$$

Also from (4.26) it follows that

$$\begin{aligned}\xi &= [\hat{L}^{-1}\phi^T - \phi^T\hat{L}^{-1}]w \\ &= \hat{L}^{-1}\phi^T\hat{L}\zeta - \phi^T\zeta \\ &= o(y_p)\end{aligned} \tag{4.26}$$

Using (4.26) and the fact that $\hat{M}\hat{L}$ is stable we get

$$y_p = \hat{M}r + \frac{k_p}{k_M}\hat{M}\hat{L}\{\phi^T\zeta\} + o(y_p) \tag{4.27}$$

We use (4.22) to evaluate

$$\begin{aligned}\frac{k_p}{k_M}\hat{M}\hat{L}\{\phi^T\zeta\} &= e_1 - \frac{k_p}{k_M}\hat{M}\hat{L}\{\phi_{2n+1}\xi - \alpha e_1\zeta^T\zeta\} \\ &= e_1 - \frac{k_p}{k_M}\hat{M}\hat{L}\{\phi_{2n+1}\xi - \alpha_1\dot{\phi}^T\zeta\}\end{aligned} \tag{4.28}$$

Using (4.26) along with the fact that ϕ_{2n+1} is bounded for the second term on the R.H.S. of (4.28), and Lemma A.1 for the third we get

$$\begin{aligned}\frac{k_p}{k_M}\hat{M}\hat{L}\{\phi^T\zeta\} &= e_1 + o(y_p) + o(\zeta) \\ &= e_1 + o(y_p)\end{aligned}$$

Using this in (4.27) we have that

$$y_p = \hat{M}r + o(y_p)$$

Since r is bounded it follows that y_p is bounded and consequently w and

ζ are bounded. This implies that $\dot{V} \to 0$ as $t \to \infty$ so that e, e_1, $\dot{\phi} \to 0$ as $t \to \infty$. Note also that the inputs to the $\hat{M}\hat{L}$ block, viz. $(\hat{L}^{-1}\phi^T - \phi^T\hat{L}^{-1})w$ and $\alpha e_1\zeta^T\zeta$ tend to zero as $t \to \infty$, the former by (4.26) and the latter from $e_1 \to 0$. Consequently $y_a \to 0$ as $t \to \infty$ and we have

$$y_M \to y_P \text{ as } t \to \infty \qquad \square$$

A particularly neat form of the adaptive control scheme of Fig. 7 results from the choice of $\hat{L} = \hat{M}^{-1}$. In this case, $\hat{L}$ is not a polynomial, of course, but a non-proper rational function. In this case $\hat{M}\hat{L} = 1$, the augmented error signal $y_a(t)$ of (4.21) is replaced by

$$y_a = \theta_{2n+1}\{(\theta^T\hat{M}-\hat{M}\theta^T)w-\alpha\zeta^T\zeta e_1\} \tag{4.29}$$

with $\zeta = \hat{M}w$ and the error equation of (4.22) is replaced by

$$e_1 = \frac{k_P}{k_M}\,(\theta^T\zeta+\phi_{2n+1}\xi-\alpha e_1\zeta^T\zeta) \tag{4.30}$$

with $\xi = (\theta^T\hat{M}-\hat{M}\theta^T)w$. The adaptive law of (4.20) still yields that $e_1(t) \to 0$ as $t \to \infty$ since the transfer function $\hat{M}\hat{L} = 1$ is positive real. An interesting observation for this scheme is that the error equation (4.30) has no dynamics. The price that one pays is more integrators in the control law (since the degree of $\hat{M}^{-1}$ > minimal degree polynomial $\hat{L}$ needed to make $\hat{M}\hat{L}$ SPR). This form of the adaptive system is used in Section 6. Note that (4.30) may be rewritten as

$$e_1 = \frac{k_P}{k_M}\,(\phi^T\zeta+\phi_{2n+1}\xi)/(1+\alpha\zeta^T\zeta) \tag{4.31}$$

In this case, we may replace the update law of (4.20) by

$$\left.\begin{aligned}\dot{\phi} &= \frac{-e_1\zeta}{1+\alpha\zeta^T\zeta} \\ \dot{\phi}_{2n+1} &= \frac{e_1\zeta}{1+\alpha\zeta^T\zeta}\end{aligned}\right\} \tag{4.32}$$

replace the augmented output signal by

$$y_a = \theta_{2n+1}\{(\theta^T\hat{M}-\hat{M}\theta^T)w\} \tag{4.33}$$

$$e_1 = \frac{k_p}{k_M}(\phi^T\zeta+\phi_{2n+1}\xi) \tag{4.34}$$

Note the resemblance of Eqs. (4.32), (4.34) to the schemes of Goodwin, Ramadge and Caines [4].

Section 5. Parameter Convergence*

In the preceding section, we showed that the adaptive laws (4.1) (in the relative degree 1 case), (4.12) (in the relative degree 2 case), and (4.20) (in the relative degree $\geq$ 3 case) yield that $e_1(t) \to 0$ as $t \to \infty$, no matter what the reference signal, $r(t)$ is. It was remarked that nothing could be said about the convergence of $\theta(t)$, without further conditions. It has been shown in [5,6,7] that both $e_1(t)$, and the parameter error $\theta(t)$ converge exponentially to zero iff $\exists\ \delta, \alpha > 0$ such that $\forall\ s \in \mathbb{R}_+$

$$\int_s^{s+\delta} ww^T dt \geq \alpha I \tag{5.1}$$

in the relative degree 1 case, and

*The results of this section were derived jointly with S. Boyd, see [23].

$$\int_s^{s+\delta} \zeta\zeta^T dt \geq \alpha I \tag{5.2}$$

in the relative degree ≥ 2 case. The conditions (5.1), (5.2) are referred to as sufficient richness conditions on w, ζ. As is widely recognized, the conditions (5.1) and (5.2) are not explicit since the signals w, ζ are generated inside the time-varying plant loop. In this section, we give explicit conditions on the reference signal input to the adaptive loop to guarantee that $e_1(t)$, $\phi(t)$ converge exponentially to zero. The intuition for our results is as follows:

Consider, first, the case when $r(t)$ is a step. In this case, if the parameter error vector converges, it converges to a value such that the (asymptotic) closed loop plant transfer function matches the model transfer function at D.C. (0 rad./sec). This observation suggests the following argument: assuming that the parameter vector does converge, the plant loop is "asymptotically time-invariant." If the input r has spectral lines at frequencies $\nu_1, \ldots, \nu_N$ we expect that y_p will also; since $y_p \to y_M$ we "conclude" that the asymptotic closed loop plant transfer function matches the model transfer function at $s = j\nu_1, \ldots, j\nu_N$. If N is large enough, this implies that the asymptotic closed loop transfer function is precisely the model transfer function so that the parameter error converges to zero. We make this intuition precise, for the relative degree 1 case. The extension of the following proof to the higher relative degree cases is straightforward:

By way of notation we define the signal $w_M \in \mathbb{R}^{2n}$ as

$$w_M^T = [r, v_M^{(1)T}, y_M, v_M^{(2)T}]. \tag{5.3}$$

w_M is the vector of signals in the model loop corresponding to the vector of signals, w in the plant loop as shown in Fig. 8.

Recall now that in the proof of Theorem 4.1 we showed that

$$\int_0^\infty \dot{V}dt = -\int_0^\infty e^TQedt < \infty \tag{4.7}$$

thereby implying that e, defined by

$$e^T = [x_p^T, v^{(1)T}, v^{(2)T}] - [x_M^T, v_M^{(1)T}, v_M^{(2)T}] \in L^2.$$

Consequently, we can conclude that $w - w_M \in L^2$ (note that $y_p - y_M = c_p(x_p - x_M)$). We denote the L^2 norm of $w-w_M$ by $\|w(\cdot)-w_M(\cdot)\|_2$. It is easy to see from the Lyapunov argument of Theorem 4.1 that

$$\|w(\cdot)-w_M(\cdot)\|_2 \leqq K_0\{\|\theta(0)-\theta^*\| + \|x_M(0)-x_p(0)\| + \|v^{(1)}(0)-v_M^{(1)}(0)\|$$
$$+ \|v^{(2)}(0)-v_M^{(2)}(0)\|\} \tag{5.4}$$

Thus, a bound on the L^2 norm of $w(\cdot)-w_M(\cdot)$ is obtained from <u>prior bounds on the parameter error, initial state errors</u>. Also in the proof of Theorem 4.1 we had that if r(t) was bounded,

$\|w(t)\|, \|w_M(t)\| \leq K_1$ for all t.

The <u>first step</u> in our proof is to obtain from (5.1) and (5.4) a condition on $w_M(t)$ for parameter convergence. The advantage of doing this is that the vector of signals $w_M(t)$ is generated by a linear time-invariant loop rather than by a linear time-varying loop.

<u>Theorem 5.1</u>

Suppose that $\|w(t)\|, \|w_M(t)\| \leq K_1$ for all t, and $\|w(\cdot)-w_M(\cdot)\|_2 \leqq K_2$.

Then, $w(t)$ is sufficiently rich $\Leftrightarrow w_M(t)$ is sufficiently rich.

Proof: The argument is symmetric between w and w_M. Hence, we only prove $\Rightarrow$. w sufficiently rich implies that $\exists\ \alpha, \delta > 0$ such that $\forall\ s \in \mathbb{R}_+,\ z \in \mathbb{R}^{2n}$

$$z^T[\int_s^{s+\delta} ww^T dt]z \geqq \alpha z^T z \tag{5.5}$$

Iterating on (5.5) p times, we get that $\forall\ p \in \mathbb{Z}_+$

$$z^T[\int_s^{s+p\delta} ww^T dt]z \geqq \alpha p z^T z \tag{5.6}$$

Now, note that

$$(z^T w)^2 - (z^T w_M)^2 = z^T(w-w_M)z^T(w+w_M) \leqq z^T z \cdot 2K_1 \cdot \|w-w_M\|$$

Hence,

$$\int_s^{s+p\delta} (z^T w)^2 - (z^T w_M)^2 dt \leqq 2K_1 z^T z \int_s^{s+p\delta} \|w-w_M\| dt \tag{5.7}$$

By Cauchy Schwarz,

$$\int_s^{s+p\delta} \| w - w_M \| \, dt \leq (p\delta)^{\frac{1}{2}} \int_s^{s+p\delta} \| w - w_M \|^2 \, dt \leq K_2 (p\delta)^{\frac{1}{2}} \tag{5.8}$$

Using (5.8) in (5.7) and (5.6) we have that $\forall\ p \in \mathbb{Z}_+$

$$z^T[\int_s^{s+p\delta} w_M w_M^T dt]z \geq z^T z(\alpha p - 2K_1 K_2 (p\delta)^{1/2})$$

Choose p_0 sufficiently large so that

$$\bar{\alpha} := \alpha p_0 - 2K_1 K_2 (p_0 \delta)^{1/2} > 0$$

and define $\bar{\delta} = p_0 \delta$. Then, we have that $\forall\ s \in \mathbb{R}_+$

$$\int_s^{s+\bar{\delta}} w_M w_M^T dt \geq \bar{\alpha} I.$$

Thus w_M is sufficiently rich. □

The <u>second step</u> consists of giving conditions on $r(t)$ to ensure that $w_M(t)$ is sufficiently rich. For this we need the following definition (Wiener [8]).

<u>Definition 5.2</u>. A function $u(t): \mathbb{R}_+ \to \mathbb{R}^n$ is said to have a <u>spectral line at frequency ν of amplitude</u> $\hat{u}(\nu)$ $(\neq 0) \in \mathbb{C}^n$ iff

$$\frac{1}{T} \int_s^{s+T} u(t) e^{-j\nu t} dt$$

converges to $\hat{u}(\nu)$ as $T \to \infty$, uniformly in s. □

The following lemma is immediate:

<u>Lemma 5.3</u>

Let $u(t)$, $y(t)$ be the input and output respectively of an exponentially stable, linear, time invariant system with transfer function $\hat{L}(s)$ (and arbitrary initial condition). If u has a spectral line of frequency ν then so does y, with amplitude

$$\hat{y}(\nu) = \hat{L}(j\nu)\hat{u}(\nu) \tag{5.8}$$

<u>Remark</u>: Since the initial condition contributes a decaying exponential to $y(t)$, it does not appear in (5.8).

The next lemma is key to our main result:

<u>Lemma 5.4</u>

Let $x(t) \in \mathbb{R}^N$ have spectral lines at frequencies $\nu_1, \nu_2, \ldots, \nu_N$.

Further let $\{\hat{x}(\nu_1),\hat{x}(\nu_2),\ldots\hat{x}(\nu_N)\}$ be linearly independent in $\mathbb{C}^N$. Then, $x(t)$ is sufficiently rich i.e., $\exists\ \alpha,\ \delta > 0$ such that $\forall\ s$

$$\int_s^{s+\delta} xx^T dt \geq \alpha I \tag{5.9}$$

<u>Proof</u>: Define the N×N matrix $X(s,\delta)$ by

$$X(s,\delta) := \frac{1}{\delta}\int_s^{s+\delta} \begin{bmatrix} e^{-j\nu_1 t} \\ \vdots \\ e^{-j\nu_N t} \end{bmatrix} x^T(t)dt$$

and the N N matrix X_0 which is the uniformlimit (in s) as $T \to \infty$ of $X(s,\delta)$

$$X_0 = \begin{bmatrix} \hat{x}^T(j\nu_1) \\ \vdots \\ \hat{x}^T(j\nu_N) \end{bmatrix}$$

By hypothesis, X_0 is non-singular. Hence, for δ sufficiently large ($\geq \delta^*$, say) $X(s,\delta)$ is invertible and

$$\|X(s,\delta)\|^{-1} \leq 2\|X_0\|^{-1}\ \forall\ s.$$

Now for $z \in \mathbb{R}^n$ and any $\nu \in \mathbb{R}$

$$\frac{1}{\delta}\int_s^{s+\delta} (x^T z)^2 dt = \frac{1}{\delta}\int_s^{s+\delta} |x^T z e^{-j\nu t}|^2 dt$$

$$\geq \left|\frac{1}{\delta}\int_s^{s+} x^T z e^{-j\nu t} dt\right|^2 \quad \text{(by Jensen's inequality)} \tag{5.10}$$

Using (5.10) for $\nu_1, \ldots, \nu_N$ we have

$$\frac{1}{\delta}\int_s^{s+\delta}(x^Tz)^2dt \geq \frac{1}{N}\sum_{k=1}^{N}\left|\frac{1}{\delta}\int_s^{s+\delta}(x^Tze^{-j\nu_k t}dt\right|^2$$

$$= \frac{1}{N}\|X(s,\delta)z\|^2$$

$$\geq \frac{1}{N}\|X(s,\delta)^{-1}\|^{-2} \qquad \text{for } \delta \geq \delta^*$$

$$\geq \frac{1}{4N}\|X_0^{-1}\|^{-2}$$

Equation (5.9) now holds with $\delta = \delta^*$, $\alpha = \frac{\delta^*}{4N}\|X_0^{-1}\|^{-2}$. □

We now use Lemmas 5.3, 5.4 to prove the main result of this section:

Theorem 5.5

Suppose $r(t)$ has spectral lines at $\nu_1, \nu_2, \ldots, \nu_{2n}$. Then $w_M(t)$ is sufficiently rich.

Proof: Recall that $w_M^T = [r, v_M^{(1)T}, y_M, v_M^{(2)T}]$. The transfer function from $r(t)$ to $w_M^T(t)$ is given by

$$\hat{Q}^T(s) = [1, \frac{\hat{M}}{\hat{p}}\cdot\frac{1}{\hat{n}_M}, \frac{\hat{M}}{\hat{p}}\cdot\frac{s}{\hat{n}_M}, \ldots, \frac{\hat{M}}{\hat{p}}\cdot\frac{s^{n-2}}{\hat{n}_M}, \hat{M}, \frac{\hat{M}s}{\hat{n}_M}, \ldots, \frac{\hat{M}s^{n-2}}{\hat{n}_M}] \quad (5.11)$$

Since the plant is minimum phase and the model is exponentially stable $\hat{Q}(s)$ in (5.11) is exponentially stable. Simplifying (5.11) we have

$$\hat{\theta}^T = \frac{k_M}{k_p\hat{n}_p\hat{d}_M}\left[\frac{k_p\hat{n}_p\hat{d}_M}{k_M}, \hat{d}_p, \hat{d}_ps, \ldots, \hat{d}_ps^{n-2}, k_p\hat{n}_p\hat{n}_M, k_p\hat{n}_p, \ldots, k_p\hat{n}_ps^{n-2}\right] \quad (5.12)$$

The (n+1)th entry of $\hat{Q}$ has numerator polynomial $\hat{n}_p\hat{n}_M$ with $\hat{n}_M$ of degree (n-1). Further the first entry of $\hat{Q}$ has numerator polynomial $\hat{n}_p\hat{d}_M$ with $\hat{d}_M$ of degree n. Compare these terms with the last (n-1) entries of

$\hat{\theta}$, viz. $\hat{n}_p$, $\hat{n}_p s, \ldots, \hat{n}_p s^{n-1}$. Using constant row operations then we can write

$$w_M^T = (T\bar{w})^T = \frac{1}{\hat{n}_p \hat{d}_M} [\hat{d}_p, \ldots, \hat{d}_p s^{n-2}, \hat{n}_p, \ldots, \hat{n}_p s^{n-2}, \hat{n}_p s^{n-1}, \hat{n}_p s^n]$$

for some $T \in \mathbb{R}^{2n \times 2n}$, a non-singular matrix. It follows that w_M is sufficiently rich iff $\bar{w}$ is sufficiently rich. By Lemma 5.3, $\bar{w}$ has spectral lines at $\nu_1, \nu_2, \ldots, \nu_{2n}$ of amplitude

$$\hat{w}_M(\nu_i)^T = \frac{1}{\hat{n}_p(j\nu_i)\hat{d}_M(j\nu_i)} [\hat{d}_p(j\nu_i), \ldots, \hat{d}_p(j\nu_i)(j\nu_i)^{n-2}, \hat{n}_p(j\nu_i),$$

$$\ldots, \hat{n}_p(j\nu_i)(j\nu_i)^n] \quad \text{for } i = 1, 2, \ldots, 2n. \tag{5.13}$$

By Lemma 5.4 if the $\hat{w}_M(\nu_i)$ are linearly independent, w_M is sufficiently rich. If they were not $\exists$ a row vector $[\beta \vdots \gamma]$; $\beta^T \in \mathbb{R}^{n-1}$, $\gamma^T \in \mathbb{R}^{n+1}$ such that

$$[\beta \vdots \gamma] \begin{bmatrix} \hat{w}_M^T(\nu_1) \\ \vdots \\ \hat{w}_M^T(\nu_{2n}) \end{bmatrix} = 0 \tag{5.14}$$

Using (5.13) and defining $\hat{\beta}(s) = \beta_1 + \beta_2 s + \ldots + \beta_{n-1} s^{n-2}$; $\hat{\gamma}(s) = \gamma_1 + \gamma_2 s + \ldots + \gamma_{n+1} s^n$, we may write (5.14) as

$$\hat{\beta}(s)\hat{d}_p(s) + \hat{\gamma}(s)\hat{n}_p(s) = 0 \quad \text{at} \quad s = j\nu_1, \ldots, j\nu_{2n} \tag{5.15}$$

The polynomial in (5.15) has degree (2n-1) so we conclude that it is identically zero and

$$\hat{\beta}\hat{d}_p \equiv - \hat{\gamma}\hat{n}_p.$$

Since $\hat{n}_p$, $\hat{d}_p$ are coprime by assumption, the zeros of $\hat{\beta}$ must include those of $\hat{n}_p$. But this is impossible since $\hat{\beta}$ has degree n-2 and $\hat{n}_p$ has degree n-1. This establishes the contradiction and proves that the $\hat{w}_M(\nu_i)$ are linearly independent and shows that w_M is sufficiently rich. □

Comments

(1) We have shown that when r(t) has spectral lines at $j\nu_1$, $j\nu_2$, ..., $j\nu_{2n}$ that $w_M(t)$ and consequently w(t) are sufficiently rich, thereby yielding parameter convergence.

(2) r(t) need not be almost periodic [9] to satisfy the conditions of Theorem 5.5. Also the strength of the spectral lines at ν_1 , ..., ν_{2n} appear only in an estimate of the rateof exponential convergence.

(3) An estimate for the rate of convergence of the parameter and output error would proceed as follows: use the estimates of Lemma 5.4 to obtain the α,δ in the definition of sufficient richness for w_M. Then use prior bounds on parameter error and initial error to bound the L^2 norm $\|w(\cdot)-w_M(\cdot)\|_2$. Now, use Theorem 3.1 to find the α and δ in the definition of sufficient richness for w. Then use the techniques of [6] to obtain a (conservative!) rate of convergence estimate.

(4) For the higher relative degree cases the proof is a minor modification of the above - the statement of the result is exactly the same: when r(t) has 2n spectral lines, then w_M and consequently ζ_M are sufficiently rich. Since $\zeta(\cdot)-\zeta_M(\cdot) \in L^2$, ζ is also sufficiently rich yielding exponential output error and parameter error convergence.

Section 6. Robustness Issues in Adaptive Control

The development of Sections 3, 4 and t assumed that the plant, given by $\hat{P}_\theta$ had no unstructured uncertainty and satisfied the assumptions A1-A4. These assumptions are, as expected, dubious in practice. In this section we remove these assumptions and explicitly take into account unmodelled dynamics. Some of the machinery we need is developed in Section 6.1 which deals with the robustness of the adaptive system to output disturbances.

6.1. Robustness to Output Disturbances

In Section 4, we saw that the parameter update laws always read, as

$$\dot{\phi} = e_1 \zeta \tag{6.1}$$

with ϕ, the vector of parameter errors, e_1 the error signal between the plant and model output and ζ a vector of signals derived from r, y_p. The formula (6.1) shows that adaptation ceases when $e_1 = 0$. Consider a scenario in which adaptation is complete, i.e., $e_1 = 0$; when a disturbance signal enters the output of the plant. This will cause parameter value to change from their converged values resulting in further error e_1 and potential loss of stability as has been verified in several simulation studies [10,11,15]. We would like adaptive laws to be robustified to output disturbances - guarantee boundedness of y_1, e_1 in the presence of some output disturbance in the plant. The price we shall pay is that the output error e_1 will be non-zero as $t \to \infty$. The basic idea is simple: turn off the adaptation when the error is smaller than some small constant. The details are as follows.

In Figure 9, we have redrawn the adaptive loop of Figure 7, along with a disturbance term $v_1(t)$ at the plant output so that

$$y_p = \hat{P}\theta^T w + v_1. \tag{6.2}$$

with $|v_1(t)| < \Delta$ for all t. Note that the signal w is derived from r and the (new) y_P of equation (6.2). To derive the error equation we wewrite (6.2) using the techniques of Section 3.3

$$y_P = \hat{M}r + \frac{k_P}{k_M}(\hat{M}\phi^T w+v) \tag{6.3}$$

with

$$v = \frac{\hat{M}}{\hat{P}}(1-\frac{\hat{C}^*}{\hat{\Lambda}})v_1 \tag{6.4}$$

The transformation is shown pictorially in Fig. 10. Note that the transfer function of Eq. (6.4) is proper and stable. Given that $|v_1(t)| < \Delta$ for all t; then, it follows that

$$|v(t)| \leq \Delta_0$$

for some Δ_0 which depends on the unknown parameters of the plant through dependence on $\hat{C}^*$ and k_P. Since the transfer function

$$\frac{\hat{M}}{\hat{P}}(1-\frac{\hat{C}^*}{\hat{\Lambda}}) = \frac{k_M}{k_P}\frac{\hat{n}_M\hat{d}_P}{\hat{n}_P\hat{d}_M}\frac{[\hat{\Lambda}(s)-\hat{C}^*(s)]}{\hat{\Lambda}(s)} \tag{6.5}$$

is stable; <u>prior bounds on parameter values of the plant</u> could be used to obtain an estimate of the L^∞ frequency norm of (6.5); which is an estimate of the ratio Δ_0/Δ. The details of this procedure are not insightful and yield extremely conservative bounds. We feel that the size of the bound Δ_0 is best decided by the control engineer in a specific application. Of course from the discussion following Eq. (3.16) it follows that $\hat{n}_P$ divides $\hat{\Lambda}-\hat{C}^*$ and $\hat{n}_M$ divides $\hat{\Lambda}$ so that (6.5) can be simplified. When both the (nominal) plant and the model have relative degree 1 then, as in Section 3.2 we have that $\hat{n}_M = \hat{\Lambda}$, $\hat{n}_P = \hat{\Lambda} - \hat{C}^*$ and (6.5) simplifies to

$$\frac{k_M}{k_P}\,\frac{\hat{d}_P}{\hat{d}_M}\,. \tag{6.6}$$

Equation (6.6) illustrates the point that the estimate Δ_0 of the disturbance v depends on the choice of the model in an essential way, since

$$\Delta_0 \leqq \Delta \sup_{\omega} \left| \frac{\hat{d}_P(j\omega)}{\hat{d}_M(j\omega)} \right| \tag{6.7}$$

$\hat{d}_P(j\omega)$ in (6.7) above is not known; however, from prior parameter bounds (6.7) can be estimated. Equation (6.7) could also be sharpened if the disturbances v(t) were localized in a small frequency range.

The adaptive system that we will study is the one described at the end of Section 4, corresponding to the choice $\hat{L} = \hat{M}^{-1}$. The error equation in this case is given by

$$e_1 = \frac{k_P}{k_M}(\phi^T\zeta+\phi_{2n+1}\xi) \tag{6.8}$$

and with update law

$$\begin{aligned} \dot{\phi} &= -e_1\zeta/(1+\alpha\zeta^T\zeta) \\ \dot{\phi}_{2n+1} &= -e_1\xi/(1+\alpha\zeta^T\zeta) \end{aligned} \tag{6.9}$$

we have that $e_1(t) \to 0$ as $t \to \infty$ for bounded r. With the output disturbance included, (6.8) is modified to

$$e_1 = \frac{k_p}{k_M}(\phi^T\zeta+\phi_{2n+1}\xi+v) \tag{6.10}$$

the update law (6.9) cannot even guarantee that y_p is bounded. However, when (6.9) is modified to

$$\begin{aligned} \dot{\phi} &= - e_1\zeta/1+\alpha\zeta^T\zeta & \text{for} \quad |e_1| > \Delta_0 + \delta \\ \dot{\phi}_{2n+1} &= - e_1\xi/1+\alpha\zeta^T\zeta & \text{for} \quad |e_1| > \Delta_0 + \delta \\ \dot{\phi},\dot{\phi}_{2n+1} &= 0 & \text{for} \quad |e_1| \leq \Delta_0 + \delta \end{aligned} \tag{6.11}$$

(for some $\delta > 0$) we show that the adaptive system yields y_p bounded and other desirable properties. The adaptation has a deadzone of size $\Delta_0+\delta$; i.e., no adaptation takes place when we cannot distinguish between the error signal and the disturbance.

For the error system (6.10) with update law (6.11) consider the Lyapunov function

$$V(\phi,\phi_{2n+1}) = \frac{1}{2}\,\phi^T\phi + \frac{1}{2}\,\phi_{2n+1}^2 \tag{6.12}$$

Then we have

$$\begin{aligned} \dot{V}(\phi,\phi_{2n+1}) &= \frac{-k_p}{k_M} \cdot \frac{(\phi^T\zeta+\phi_{2n+1}\xi)(\phi^T\zeta+\phi_{2n+1}\xi+\dot{v})}{1+\alpha\zeta^T\zeta} & \text{when } |e_1| > \Delta_0+\delta \\ &= 0 & \text{when } |e_1| \leq \Delta_0+\delta \end{aligned} \tag{6.13}$$

From (6.13) we have that $\dot{V}(\phi,\phi_{2n-1}) \leq 0$ and that ϕ, ϕ_{2n+1} are bounded. Consequently $\dot{\phi}$, $\dot{\phi}_{2n+1}$ are also bounded. Let $\Omega_1 = \{t : |e_1| < \Delta_0+\delta\}$ and $\Omega_2 = \{t : |e_1| \geq \Delta_0+\delta\}$, be the time intervals of no adaptation and adaptation respectively. Then

$$\int_0^\infty \dot{V}dt = -\int_{\Omega_2} \frac{k_p}{k_M} \frac{(\phi^T\zeta+\phi_{2n+1}\xi)(\phi^T\zeta+\phi_{2n+1}\xi+v)}{1+\alpha\zeta^T\zeta} dt < \infty \tag{6.14}$$

For ease of notation in what follows we define $\bar{\phi}$, $\bar{\zeta} \in \mathbb{R}^{2n+1}$ as $\bar{\phi}^T = [\phi^T, \phi_{2n+1}]$ and $\bar{\zeta}^T = [\zeta^T, \xi]$. Then

$$e_1 = \frac{k_p}{k_M} (\bar{\phi}^T\bar{\zeta}+v_1) \tag{6.10}$$

and

$$\dot{V}(\bar{\phi}) = \frac{-k_p}{k_M} \frac{(\bar{\phi}^T\bar{\zeta})(\bar{\phi}^T\bar{\zeta}+v)}{1+\alpha\zeta^T\zeta} \qquad \text{when } |e_1| > \Delta_0+\delta$$

$$= 0 \qquad \text{when } |e_1| \leq \Delta_0+\delta \tag{6.13}$$

Using the inequalities

$$\frac{\delta}{v_0+\delta} |\bar{\phi}^T\bar{\zeta}+v| \leq |\bar{\phi}^T\bar{\zeta}| \leqq \frac{2v_0+\delta}{v_0+\delta} |\bar{\phi}^T\bar{\zeta}+\bar{v}| \text{ for } t \in \Omega_2$$

in Eq. (6.14) we obtain

$$\int_{\Omega_2} \frac{(\bar{\phi}^T\bar{\zeta})^2}{1+\zeta^T\zeta} dt < \infty \quad \text{and} \quad \int_{\Omega_2} \frac{(\bar{\phi}^T\bar{\zeta}+v)^2}{1+\zeta^T\zeta} dt < \infty \tag{6.15}$$

Now

$$|\dot{\bar{\phi}}|^2 = \frac{\bar{\zeta}^T\bar{\zeta}(\bar{\phi}^T\bar{\zeta}+v)^2}{(1+\zeta^T\zeta)^2} \quad \text{for } t \in \Omega_2$$

$$= 0 \qquad \text{for } t \in \Omega_1.$$

so that $\dot{\bar{\phi}} \in L^2$, which is key to the following theorem.

Theorem 6.1

The adaptive system of Fig. 7 with $\hat{L} = \hat{M}^{-1}$, error equation (6.8) and the modified adaptive law (6.11) yields that y_p is bounded when r is bounded.

Proof: Case 1. $t \in \Omega_1$ for $t \geq t_1$. In this case $|\bar{\phi}^T\bar{\zeta}+v| \leq \Delta_0$ for $t \geq t_1$ and $\bar{\phi}^T\bar{\zeta}$ is bounded. Further,

$$y_p = \frac{k_p}{k_M}(\hat{M}\phi^T w+v) + \hat{M}r \tag{6.16}$$

Also as in the proof of Theorem 4.3, $\dot{\phi} \in L^2$ implies that

$$\hat{M}\phi^T w = \phi^T\zeta + o(w)$$

$$= \phi^T\zeta + o(y_p) \tag{6.17}$$

(since $w = O(y_p)$). Also $\xi = o(y_p)$ as before and we have $\phi^T\zeta = \bar{\phi}^T\bar{\zeta} + o(y_p)$. Using this fact along with (6.17) in (6.16) we conclude that y_p is bounded. This implies that $y_a \to 0$ as $t \to \infty$ and

$$|y_p - y_M| < \Delta_o + \delta \text{ as } t \to \infty.$$

Case 2. $t \in \Omega_2$ for $t \geq t_2$. From Eq. (6.15) we have

$$\int_{\Omega_2} \frac{(\bar{\phi}^T\bar{\zeta})^2}{1+\zeta^T\zeta}\, dt < \infty$$

Also since $\|\dot{\bar{\phi}}\|$ is bounded and $\|\dot{\zeta}\| \leq M_1\|\zeta\| + M_2$, it follows that $(\bar{\phi}^T\bar{\zeta})^2/1+\zeta^T\zeta$ is uniformly continuous on Ω_2 and we have

$$\lim_{t\to\infty} \frac{\bar{\phi}^T\bar{\zeta}}{1+\zeta^T\zeta} = 0 \Rightarrow \bar{\phi}^T\bar{\zeta} = o(\zeta) \tag{6.18}$$

Also $\dot{\phi} \in L^2$ yields (6.17) again, i.e.

$$\hat{M}\phi^T w = \phi^T \zeta + o(y_p) \tag{6.17}$$

and

$$\xi = o(y_p). \tag{6.19}$$

Conbining (6.19) and (6.18) we get $\phi^T\zeta = o(\zeta) + o(y_p) = o(y_p)$. Using this in (6.17) we get for y_p

$$y_p = o(y_p) + \frac{k_p}{k_M} v + \hat{M}r$$

thereby guaranteeing that y_p is bounded.

Case 3. Both Ω_1 and Ω_2 are unbounded sets. The estimates (6.17), (6.19) still stand. Further, we have

$$|\phi^T\zeta| < 2\Delta_0+\delta \quad t \in \Omega_2$$

$$\phi^T\zeta = o(y_p) \qquad t \in \Omega_2 \text{ from } (6.18, 6.19).$$

Combining these estimates and using them in (6.16) yields that y_p is bounded. □

Remarks: (1) We have shown that the modified adaptive law yields that the signals y_p, w, ζ, ξ are bounded.

(2) In Case 1, i.e. $t \in \Omega_1$ for $t \geq t_1$ we showed that $\lim_{t\to\infty} |y_p - y_M| < \Delta_0+\delta$. In the other cases we cannot guarantee that the difference between y_p and y_M is asymptotically less than the size of the deadzone, without further information about the nature of the disturbance. However, the following remark can be made: since $\|\zeta\| \leq A$, for some A,

$$\dot{V} = \bar{\phi}^T \dot{\phi} \le \frac{-\delta(\Delta_0+\delta)}{1+A^2} < 0 \quad \text{for} \quad t \in \Omega_2$$

i.e., $\dot{V}$ is bounded away from zero when adaptation does take place. Consequently, the measure of set Ω_2 has to be finite; i.e., adaptation ceases on all but a set of finite measure and $|y_p - y_M| < \Delta_0 + \delta$ except on the finite measure set Ω_2.

6.2. Robustness to Unstructured Uncertainty

In this section we revert to the description in Section 2 of the plant model incorporating both structured and unstructured uncertainty, i.e., the plant model is given as in (3.4) by the rational transfer function $\hat{P}_a(s)$:

$$\hat{P}_a(s) = \hat{P}(1+\hat{u}) \tag{6.20}$$

where the transfer function $\hat{P}$ satisfies the assumptions A1, A2, A3 and A4, $\hat{U}$ is stable and the magnitude bound on $|\hat{U}(j\omega)|$ is known. To derive the error equation for the adaptive system constructed using $\hat{P}$ as the actual plant model, we use the techniques of Section 3.3, to get

$$\begin{aligned} y_p &= \hat{P}(1+\hat{U})\theta^T w \\ &= \hat{M}(1+\hat{U}) \left\{ \frac{(\hat{C}^* - \hat{\Lambda})\hat{d}_p - k_p \hat{D}^* \hat{n}_p}{(\hat{C}^* - \hat{\Lambda})\hat{d}_p - k_p \hat{D}^* \hat{n}_p (1+\hat{U})} \right\} (r + \frac{k_p}{k_M} \phi^T w) \end{aligned} \tag{6.21}$$

as shown pictorially in Fig. 11. Define $\hat{U}_1(s)$ by the equation

$$1 + \hat{U}_1(s) = (1+\hat{U}) \left\{ \frac{(\hat{C}^* - \hat{\Lambda})\hat{d}_p - k_p \hat{D}^* \hat{n}_p}{(\hat{C}^* - \hat{\Lambda})\hat{d}_p - k_p \hat{D}^* \hat{n}_p (1+\hat{U})} \right\} \tag{6.22}$$

$\hat{U}_1$ has the interpretation of being the closed-loop unmodelled dynamics resulting from the open-loop unmodelled dynamics $\hat{U}$. Now a short calculation yields the error equation for the adaptive system in this case to be

$$e_1 = \hat{M}\hat{U}_1 r + \frac{k_p}{k_M}\hat{M}\hat{L}\bar{\phi}^T\bar{\zeta} + \frac{k_p}{k_M}\hat{M}\hat{U}_1\phi^T w$$

$$= \frac{k_p}{k_M}\hat{M}\hat{L}\bar{\phi}^T\bar{\zeta} + \hat{M}\hat{U}_1 r + \frac{k_p}{k_M}\hat{M}\hat{U}_1\phi^T w \tag{6.23}$$

The idea now is to treat $\hat{M}\hat{U}_1 r + \frac{k_p}{k_M}\hat{M}\hat{U}_1\phi^T w$ as a disturbance and to establish a bound Δ_0 on it. Once this is done, a deadzone of size Δ_0 can be used to obtain the conclusions of Section 6.1 above. Also, as in Section 6.1 above the choice of $\hat{L} = \hat{M}^{-1}$ will be made. The most important step in establishing the bound Δ_0 is obtaining a bound on $\hat{U}_1(j\omega)$. We first establish conditions under which $\hat{U}_1$ is stable.

Proposition 6.2

The closed loop uncertainty transfer function $\hat{U}_1$ is stable if the model $\hat{M}$ and the adaptive system are chosen so that

$$|\hat{U}(j\omega)| < \left| \frac{k_M\hat{\Lambda}(j\omega)}{k_p\hat{M}(j\omega)\hat{D}^*(j\omega)} \right| \tag{6.24}$$

Proof: The explicit expression for $\hat{U}_1$ from (6.22) is

$$\hat{u}_1 = \frac{\hat{U}(C^*-\hat{\Lambda})\hat{d}_p}{(\hat{C}^*-\hat{\Lambda})\hat{d}_p - k_p\hat{D}^*\hat{n}_p(1+\hat{u})} \tag{6.25}$$

Now from the results of Doyle and Stein [18] $\hat{U}_1$ is stable iff the non-adaptive loop of Fig. 11 is stable. This in turn is guaranteed when

$$|\hat{U}| < \left|1-\left(\frac{k_p\hat{n}_p}{\hat{d}_p}\frac{\hat{\Lambda}}{\hat{C}^*-\hat{\Lambda}}\cdot\frac{\hat{D}^*}{\hat{\Lambda}}\right)^{-1}\right| \quad \forall\ s = j\omega \tag{6.26}$$

Since by choice of $\hat{C}^*$ and $\hat{D}^*$

$$M = \frac{k_M\hat{n}_p\hat{\Lambda}}{d_p(\hat{C}^*-\hat{\Lambda})-k_p\hat{n}_p\hat{D}^*}$$

we have that (6.26) is equivalent to (6.24). □

Remarks: (1) It appears that in order to check Condition (6.24), the parameters d_i^*, k_p need to be known. However, prior bounds on these parameters are adequate to verify this condition.

(2) We remarked in Section 3 that the only restrictions regarding the choice of the model $\hat{M}$ are that it be stable, minimum phase and have the same relative degree as $\hat{P}$. Now, we see that the choice of the model $\hat{M}$ is also restricted by the condition that it satisfy (6.24). Consider, for example, the case when $\hat{M}$, $\hat{P}$ both have relative degree 1. From Section 3.2, we have $\hat{\Lambda} = \hat{n}_M$ and $\hat{D}^* = \hat{d}_p-\hat{d}_M$ so that (6.24) simplifies to

$$|\hat{U}(j\omega)| < \left|\frac{\hat{d}_M(j\omega)}{k_p(\hat{d}_p(j\omega)-\hat{d}_M(j\omega))}\right| \tag{6.27}$$

(3) The right hand side of (6.24) is asymptotically of the form $\frac{|j\omega|^{n-m}}{k_p}$; so that $|\hat{U}(j\omega)| < \frac{1}{k_p}|j\omega|^{n-m}$. Thus, the relative degree places a bound on the rate of growth of $\hat{U}$ that can be tolerated by the adaptive system. □

Once condition (6.24) has been verified, we can estimate $\hat{U}_1(j\omega)$ from Eq. (6.25). It is easy to see that for high frequencies (asymptotically as $\omega \to \infty$) $\hat{U}_1(j\omega) = \hat{U}(j\omega)$ and for low frequencies when

$$\hat{U} << 1 - \frac{d_p(\hat{C}^* - \hat{\Lambda})}{k_p \hat{n}_p \hat{D}^*}$$

that

$$\hat{U}_1 = \frac{\hat{U} k_p \hat{M}}{d_M \hat{P}} \cdot \frac{(\hat{C}^* - \hat{\Lambda})}{\hat{\Lambda}} \tag{6.29}$$

As before, prior bounds on the plant parameters are used to obtain bounds on $\hat{U}_1(j\omega)$. We use these bounds along with bounds on the reference signal spectrum $\hat{r}(j\omega)$ to estimate the disturbance term $\hat{M}\hat{U}_1 r$ in (6.23). Typically, the frequency content of r is in the low frequencies where $\hat{U}_1$ is small so that $\hat{M}\hat{U}_1 r$ is small. From prior parameter bounds we establish also bounds on ϕ in the second term (the Lyapunov argument guarantees that ϕ decreases along trajectories). Further as in Section 5, prior parameter bounds on w can be established by first calculating w_M (as in Fig. 8) and the error bound $w - w_M$. Since the frequency content of $\phi^T w$ is unknown apriori we have

$$\left| \frac{k_p}{k_M} \hat{M}\hat{U}_1 \phi^T w \right| \leq \sup_{\omega} \left| \frac{k_p}{k_M} \hat{M}(j\omega)\hat{U}_1(j\omega) \right| \sup_{t} |\phi^T w|$$

By this reasoning, a bound on the disturbance term is obtained and treated using a deadzone as in Section 6.1.

Section 6.3. Comparison Between Adaptive and Non-adaptive Schemes

We have discussed one method of robustifying the model reference adaptive scheme to unmodelled dynamics - namely, a deadzone in the parameter update law. The crux of the method lay in neglecting the model-plant mismatch error signal when it was comparable to the "noise" generated by output disturbances and unmodelled dynamics. The price to be paid is that we are no longer guaranteed convergence of the error to zero but only to the interior of the deadzone (off a time period of finite measure). We discussed techniques for estimating the size of the deadzone. Typically, estimates of the kind given in the previous section will be too conservative (because of their generality). In an application the designer would pick a deadzone - depending on his estimate of the "noise" in the error signal - such an estimate could be obtained in the initial time period of the transient in adaptation when the error is large. Now, however, if a robust, non-adaptive control methodology, say, of the form suggested in [22] (using a nominal model for the plant), could give model matching to an accuracy greater than the size of the deadzone, then, of course, the choice of feedback law is the non-adaptive one.

Other techniques for robustifying adaptive schemes have been suggested: [11,21] suggest the use of a forgetting factor in the update law (say, $\dot{\theta} = -\alpha\theta - e_1\zeta$); [26] suggests that when the nominal adaptive loop is exponentially stable (i.e., when the reference signal is sufficiently rich - has sufficiently many spectral lines), its stability is robust to the presence of unmodelled dynamics. The analytic details of these approaches to robustness are as yet incomplete.

Section 7. Concluding Remarks

We have given in this paper a corrected set of stability proofs to the adaptive schemes of [1], [2], explicit conditions on the reference signal for parameter convergence and a framework to discuss the sensitivity of adaptive systems to unmodelled dynamics. The use of a deadzone to suppress the effects of unmodelled dynamics was considered in Section 6. It is clear that further work needs to be done in understanding the role of unmodelled dymamics in adaptive control, along lines considered by other researchers - Ioannou and Kokotovic [11,21], Anderson, Kosut and others [20,26].

The present paper is devoted entirely to single input-single output systems. For multi-input, multi-output systems, parametrization of the plant to be adaptively controlled is well understood only when the Hermite form of the plant is diagonal (see e.g., [27,28,4]). In these cases the adaptive scheme of the present paper can be modified to obtain a scheme where the controlled plant matches a model chosen to have a stable Hermite form identical to that of the plant. As is expected questions of sensitivity and robustness of those schemes are much more complicated. Further work is necessary both in the area of parametrization and robustness properties of multiinput-multioutput systems.

Appendix

Proof of Stability of the Adaptive Scheme when Relative Degree ≥ 3

Preliminary Definitions

We will need to compare the asymptotic magnitudes of time signals. For this we develop some notation. Let $x(\cdot) \in L_n^{\infty e}$ and $y(\cdot) \in L_m^{\infty e}$ (the extended L_∞ spaces with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$ for all t (see [24]). Define their truncated norms by

$$|x|_t := \sup_{\tau \leq t} |x(\tau)|,$$

$$|y|_t := \sup_{\tau \leq t} |y(\tau)|.$$

Both $|x|_t$, $|y|_t$ are monotone non-decreasing functions of time.

Def. 1. $y = O(x)$ if $\exists$ $\beta(\cdot)$ continuous with $\beta(t) \to 0$ as $t \to \infty$, such that $y(t) = \beta(t)x(t)$.

Def. 2. $y = O(x)$ if $\exists$ M a constant such that

$$|y|_t \leq M|x|_t.$$

Def. 3. $y \sim x$ (equivalent) if $y = O(x)$ and $x = O(y)$.

Definition 1, 2, 3 compare the asymptotic magnitudes of x, y. To get a feel for these definitions consider

$$L_+ = \overline{\lim_{t\to\infty}} \frac{|y|_t}{|x|_t} \quad \text{and} \quad L_- = \underline{\lim_{t\to\infty}} \frac{|y|_t}{|x|_t} \qquad \text{(A.1)}$$

$$\text{(with } L_- \leq L_+)$$

L_+ and L_- are extended real valued (with $L_- \leq L_+$). The following table shows the relation between L_+, L_- and the Defs. 1-3 when $x(\cdot)$ and $y(\cdot)$ are both continuous

L_+	L_-	Conclusions
0	0	$y = o(x)$, $y = O(x)$
Finite*	0	$y = O(x)$
Finite	Finite	$y \sim x$
∞	Finite	$x = O(y)$
∞	∞	$x = o(y)$, $x = O(y)$
∞	0	No conclusion

Remarks: 1. The table is symmetric about the $y \sim x$ entry; except for the case when $L_+ = \infty$ and $L_- = 0$ when nothing can be said about x, y in terms of Def. 1, 2, 3.

2. Note that $y = o(x) \Rightarrow y = O(x)$ and $x = o(y) \Rightarrow x = O(y)$ when x, y are continuous.

Lemma A.1

Let $x(\cdot) \in L^1 \cup L^2$, $\zeta(\cdot) \in L^e$ both be scalar functions and $\hat{H}(s)$ be the rational transfer function of a strictly proper, exponentially stable linear system. Denote by $y(\cdot)$ the output of this system when the input is $x(\cdot)\zeta(\cdot)$. Then,

$$y = o(\zeta) \tag{A.2}$$

Proof: Let $h(t)$ be the convolution kernel corresponding to $\hat{H}(s)$. Then

*Finite is to be interpreted as finite, non-zero in the table.

$h(\cdot) \in L^1$ and

$$y(t) = \int_0^t h(t-\tau)x(\tau)\zeta(\tau)d\tau \tag{A.3}$$

(Initial conditions will contribute a decaying exponential to (A.3) which will not affect asymptotic conclusions like those of (A.2). From (A.3) we have

$$|y(t)| \le |\zeta|_t \int_0^t |h(t-\tau)x(\tau)|d\tau.$$

Since $\hat{H}(s)$ is exponentially stable and $x(\cdot) \in L^1 \cup L^2$, we have

$$\int_0^t |h(t-\tau)x(\tau)|d\tau \to 0 \text{ as } t \to \infty.$$

This establishes A.2. □

Lemma A.2 establishes conditions under which the input x of a minimum phase (not necessarily stable) linear system is $O(y)$, with y the output of the linear system:

<u>Lemma A.2</u>

Let $x(\cdot)$, $y(\cdot)$ be the input and output respectively on a single input-single output, proper minimum phase transfer function $\hat{H}(s)$ with zero initial condition.* Let $x(\cdot) \in L^{\infty e}$ satisfy

$$\dot{x} = O(x) \tag{A.3}$$

Then

$$x = O(y) \tag{A.4}$$

*We assume this for simplicity of proof alone. It is easy to see from the proof that this assumption is unnecessary.

<u>Proof</u>: Write $\hat{H} = \hat{H}_1\hat{H}_2$ where $\hat{H}_1$ has the same number of poles and zeros and $\hat{H}_2$ only has poles. Define $x_1 = \hat{H}_2(s)x$; then $y = \hat{H}_1(s)x_1$. Since $\hat{H}_1^{-1}$ is proper and stable;

$$x_1 = 0(y) \tag{A.5}$$

The proposition is trivial if x is bounded so we assume that x is unbounded. Let t_i be a sequence $\uparrow \infty$. We will show that it is impossible to have

$$\lim_{i\to\infty} |x_1|_{t_i}/|x|_{t_i} = 0 \tag{A.6}$$

We will assume that $x(t_i) > 0$ (this is possible since x is unbounded). Now we show that (A.6) implies that for δ sufficiently small

$$\lim_{i\to\infty} |x_1|_{t_i+\delta}/|x|_{t_i+\delta} = 0 \tag{A.7}$$

This follows from (A.3) which guarantees that $\dot{x}_1 = 0(x_1)$ so that $\exists\ c_1, c_2, \delta > 0$, independent of i (by the Bellman Gronwall lemma),

$$\left.\begin{aligned} c_2|x|_{t_i} &> x(\tau) > c_1|x|_{t_i} \\ c_2|x_1|_{t_i} &> |x_1(\tau)| > c_1|x_1|_{t_i} \end{aligned}\right\} \quad \tau \in [t_i, t_i+\delta] \tag{A.8}$$

We now establish a contradiction to (A.7). Note that

$$x_1(t_i+\Delta) = x_1(t_i) + \int_{t_i}^{t_i+\Delta} h_2(t_i+\Delta-\tau)x(\tau)d\tau$$

Further, since $h_2(t)$ the convolution kernel corresponding to $\hat{H}_2(s)$ is

> 0 for t small enough (due to the fact that $H_2(s)$ only has poles) we have that

$$x_1(t_i+\Delta) > x_1(t_i) + c_1 c_3 |x|_{t_i}$$

with $c_3 = \int_0^{\Delta} h_2(\tau)d\tau > 0$ for Δ small. This yields

$$\frac{|x_1|_{t_i+\Delta}}{|x|_{t_i+\Delta}} \geq \frac{x_1(t_i+\Delta)}{|x|_{t_i+\Delta}} \geq \frac{x_1(t_i)}{c_2 |x|_{t_i}} + \frac{c_1 c_3}{c_2}$$

so that

$$\lim_{t\to\infty} \frac{|x_1|_{t_i+\Delta}}{|x|_{t_i+\Delta}} \geq \frac{c_1 c_3}{c_2} > 0$$

contradicting (A.7).

Thus, $L_- = \underline{\lim}_{t\to\infty} \frac{|x_1|_t}{|x|_t} \neq 0$ and $L_+ = \overline{\lim}_{t\to\infty} \frac{|x_1|_t}{|x|_t} \geq L_- \neq 0$

From the table we see that when $L_+ \neq 0$ and $L_- \neq 0$ the only possibilities that remain are $x \sim x_1$ or $x = O(x_1)$ or $x = o(x_1)$ (which implies $x = O(x_1)$). At any rate, $L_+ + 0$, $L_- \pm 0 \Rightarrow x = O(x_1)$. Combining this with (A.5) we obtain (A.4). □

Lemma A.3

In the adaptive scheme (for relative degree ≥ 3) of Section 4,

$$u,\ v^{(1)},\ v^{(2)} = O(y_p) \qquad (A.9)$$

Proof: Since $v^{(2)} = (sI-\Lambda)^{-1}\, b y_p$ and $v^{(1)} = (sI-\Lambda)^{-1} bu$, with Λ

exponentially stable we have that

$$v^{(2)} = O(y_p) \text{ and } v^{(1)} = O(u) \tag{A.10}$$

To prove (A.9) it needs to be shown that $u = O(y_p)$. By reasoning as in the preceding lemma, we will prove this by establishing that there is no sequence $t_i \uparrow \infty$ such that

$$\lim_{i\to\infty} \frac{|y_p|_{t_i}}{|u|_{t_i}} = 0. \tag{A.11}$$

Consider,

$$u = \theta^T w + r \tag{A.12}$$

The lemma is trivial if u is bounded, so assume that u is unbounded. Now, r is bounded and θ is bounded. Since (A.11) holds along the sequence $\{t_i\}$ we have that $|y_p|_{t_i} = o(|u|_{t_i})$ and by (A.10) that $|v^{(2)}|_{t_i} = O(|u|t_i)$. Hence (A.12) yields that

$$|u|_{t_i} = O(|v^{(1)}|t_i) \tag{A.13}$$

Thus, for some component i_0 of $v^{(1)}$, we have that

$$|u|_{t_i} = O(|v_{i_0}^{(1)}|_{t_i})$$

Now, since

$$\dot{v}^{(1)} = \Lambda v^{(1)} + bu$$

we have that

$$|\dot{v}_{i_0}^{(1)}|_{t_i} = 0(|v_{i_0}^{(1)}|_{t_i})$$

Further,

$$v_{i_0}^{(2)}(t) = \hat{P}v_{i_0}^{(1)}(t)$$

Since $\hat{P}$ has no zeros in the right half plane we have from Lemma A.2 that for the sequence $\{t_i\}$

$$\lim_{i\to\infty} \frac{|v_{i_0}^{(2)}|_{t_i}}{|v_{i_0}^{(1)}|_{t_i}} \neq 0.$$

This establishes a contradiction as follows: By assumption (A.11), $|y_p|_{t_i}/|u|_{t_i} \to 0$, which implies by (A.10) that $|v_{i_0}^{(2)}|_{t_i}/|u|_{t_i} \to 0$. Now by (A.13), we have that $|v_{i_0}^{(2)}|_{t_i}/|v_{i_0}^{(1)}|_{t_i} \to 0$ as $i \to \infty$. □

To prove Lemma A.4 we need the following preliminaries concerning $\hat{L}(s)$: For simplicity we will assume that $\hat{L}(s)$ is a polynomial and has only real zeros, i.e.,

$$\hat{L}(s) = \prod_{i=1}^{r} (s+a_i) \tag{A.14}$$

with $a_i > 0$ (the following development can be repeated with only minor changes in notation when $\hat{L}$ has complex zeros. Recall that $\hat{L}(s)$ is Hurwitz and define

$$\hat{A}_i(s) = \prod_{j=i+1}^{r} (s+a_j), \qquad i = 0, 1, \ldots, 4-1$$

Then if $\zeta(t)$ is a C^r function,

$$[\hat{L}^{-1}\phi^T\hat{L}]\zeta = [\phi^T - \sum_{i=0}^{r-1}\hat{A}_i^{-1}\phi^T\hat{A}_{i+1}]\zeta \tag{A.15}$$

neglecting the effects of exponentially decaying initial conditions. The proof of (A.15) follows by recursion on

$$\frac{1}{s+a_r}\,\phi^T(s+a_r)\zeta = \phi^T\zeta + \frac{1}{s+a_r}\,\dot{\phi}^T\zeta. \tag{A.16}$$

Lemma A.4

With $\hat{L}$ defined as in (A.14), and $w(t) = \hat{L}(s)\zeta(t)$, $w(\cdot) \in L_{\infty e}^{2n}$ and $\dot{\phi} \in L^2$, we have

$$[\hat{L}^{-1}\phi^T\hat{L}]\zeta = \phi^T\zeta + o(w) \tag{A.17}$$

Proof: From (A.15) we h-ve

$$(\hat{L}^{-1}\phi^T\hat{L}]\zeta = \phi^T\zeta - \sum_{i=0}^{r-1}\hat{A}_i^{-1}\dot{\phi}^T\hat{A}_{i+1}(s)\hat{L}^{-1}(s)w(t) \tag{A.18}$$

Now

$$\hat{A}_{i+1}\hat{L}^{-1}w = O(w)$$

and $\hat{A}_i^{-1}$ is stable for $i = 0, \ldots, r-1$. Using the fact that $\dot{\phi} \in L^2$ and Lemma A.1 we have that

$$\hat{A}_i^{-1}\dot{\phi}\hat{A}_{i+1}\hat{L}^{-1}w = o(w) \qquad i = 0, \ldots, r-1$$

Using this in (A.18) establishes (A.17). □

Remark: Lemma A.4 was derived for the case when $\hat{L}(s)$ was a Hurwitz polynomial with real roots. It is straightforward, though notationally cumbersome, to verify that when $\hat{L}(s)$ is a (improper) rational function with Hurwitz numerator and denominator (possibly with complex zeros) Lemma A.4 still holds.

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Figure Captions

Fig. 1. Bode plots of the system to be controlled.

Fig. 2. Model following control system.

Fig. 3. Typical plot of multiplicative uncertainty $|\hat{U}_1(j\omega)|$, $|\hat{U}_3(j\omega)|$ for adaptive control.

Fig. 4. Non-adaptive controller structure.

Fig. 5. Basic adaptive controller structure.

Fig. 6. Error equation for the basic adaptive system.

Fig. 7. Schematic of adaptive system when relative degree ≥ 3.

Fig. 8. The adaptive system of Figure 5 with a new representation for the model.

Fig. 9. The adaptive loop of Figure 7 with an output disturbance v_1.

Fig. 10. Showing the procedure for factoring the disturbance $v_1(t)$ out of the adaptive loop.

Fig. 11. Showing the procedure for finding the effect of the unmodelled dynamics on the adaptive loop.

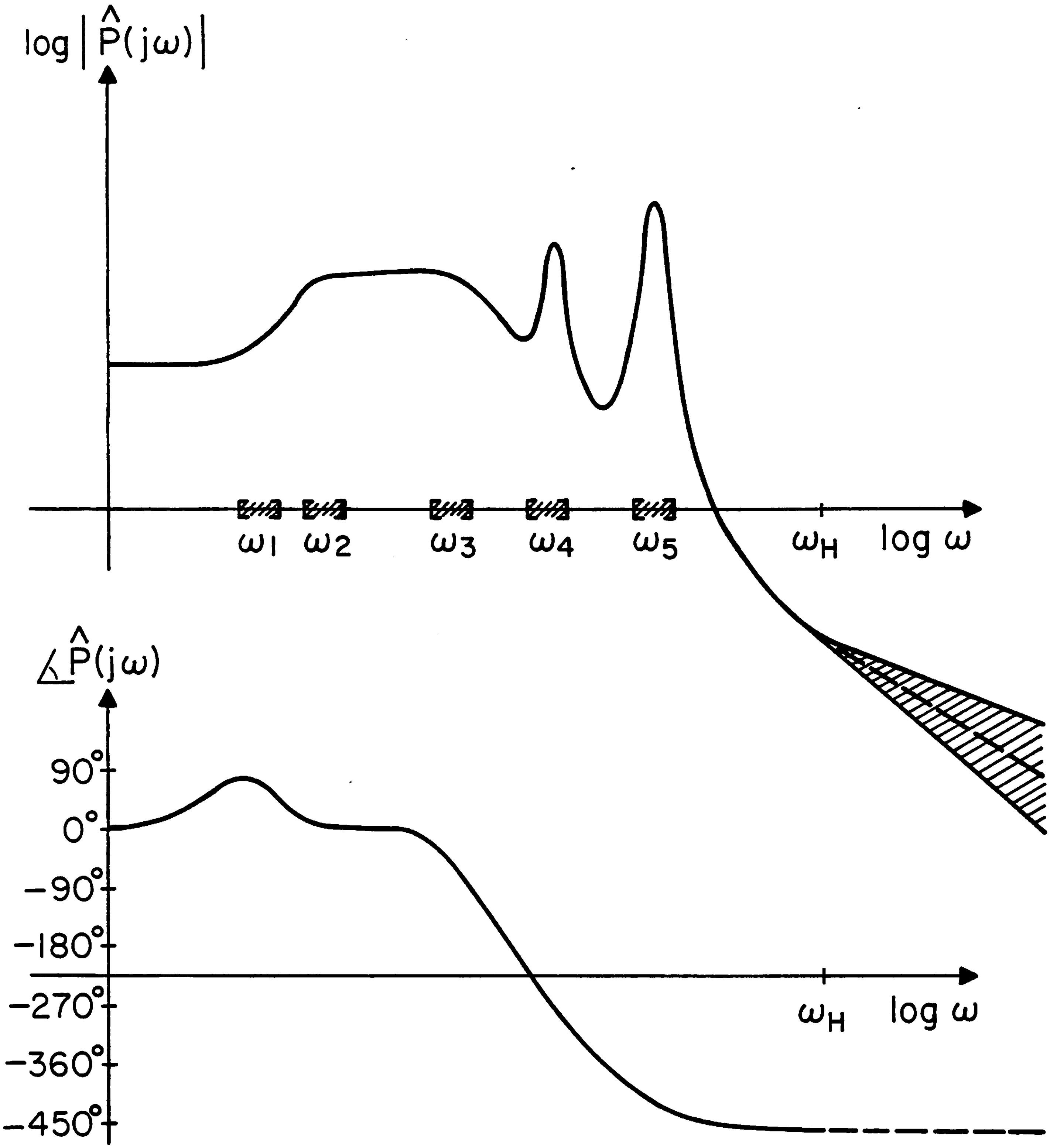


Fig. 1

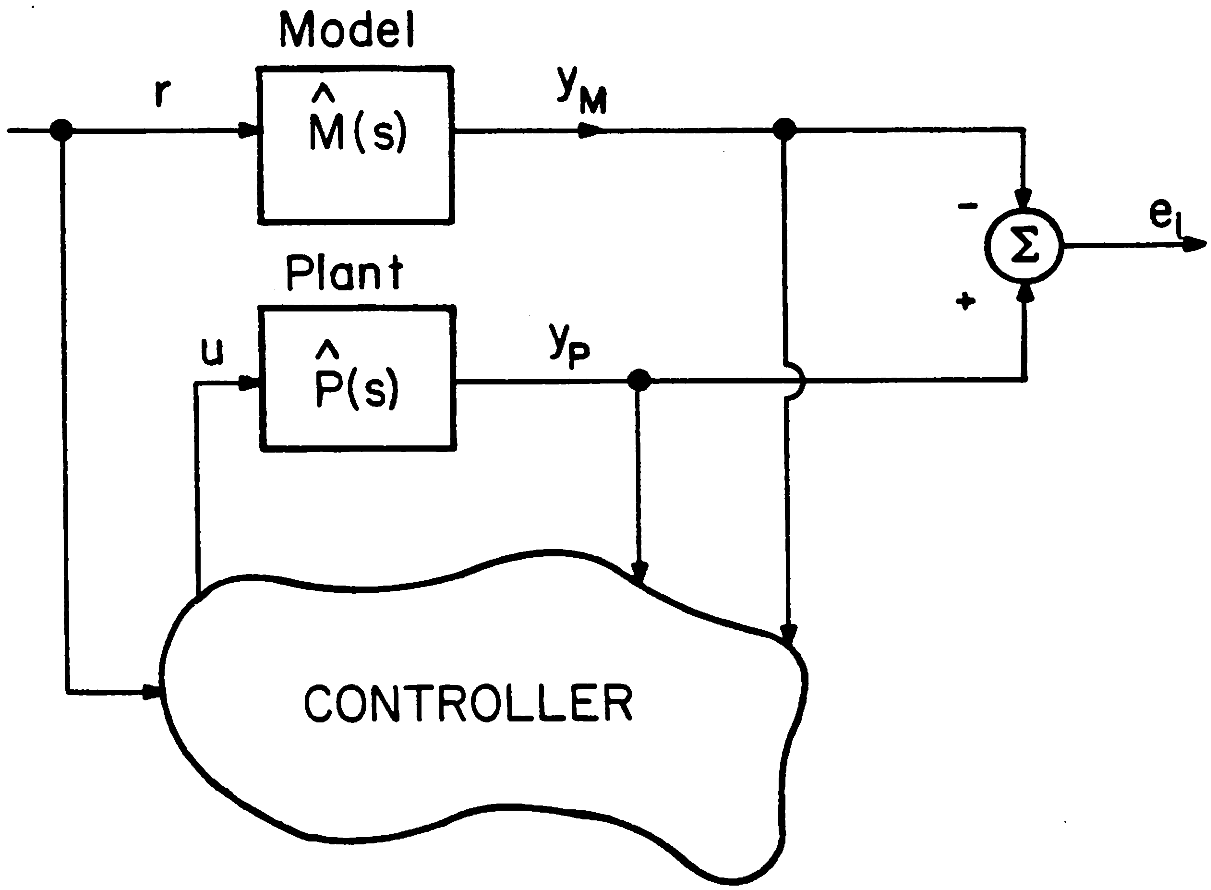

Model
r
$\hat{M}(s)$
y_M
e_I
Plant
u
$\hat{P}(s)$
y_P
CONTROLLER

Fig. 2

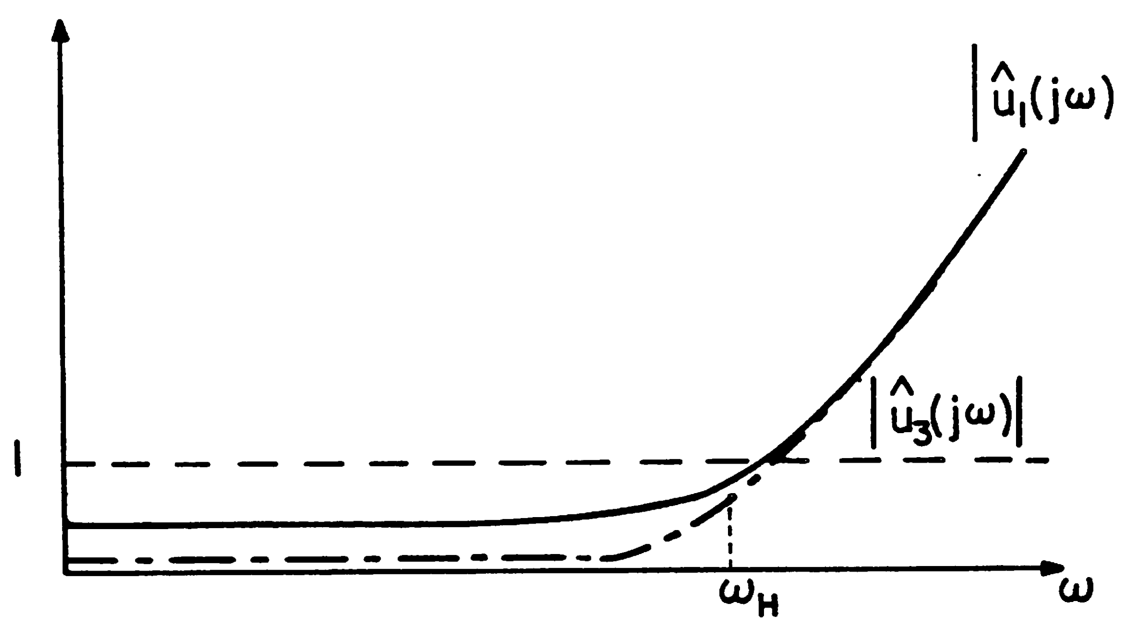

$|\hat{u}_1(j\omega)|$
$|\hat{u}_3(j\omega)|$
1
ω_H
ω

Fig.3

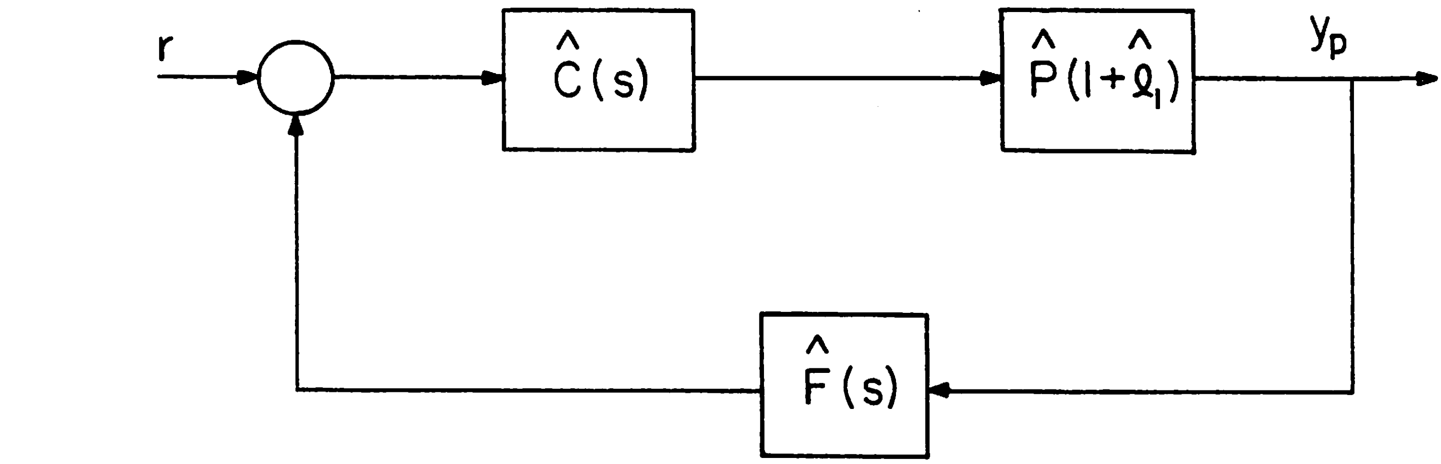

r
Ĉ(s)
P̂(1+ℓ̂₁)
y_p
F̂(s)

Fig. 4

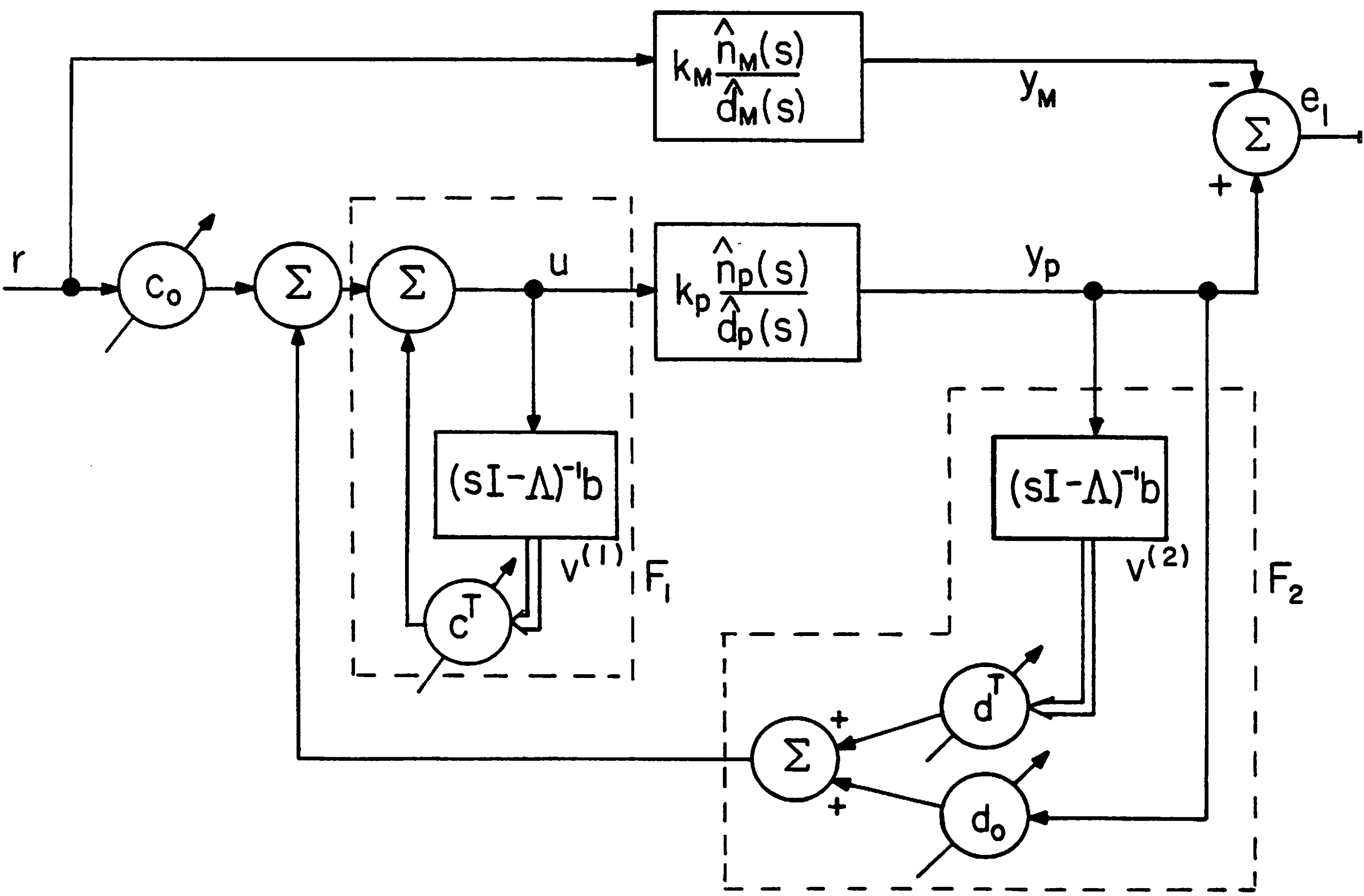
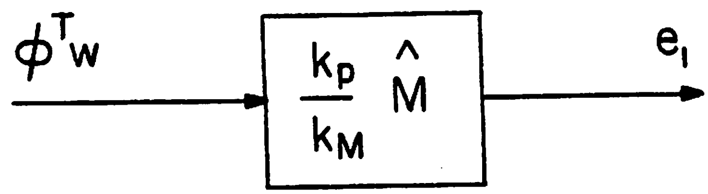
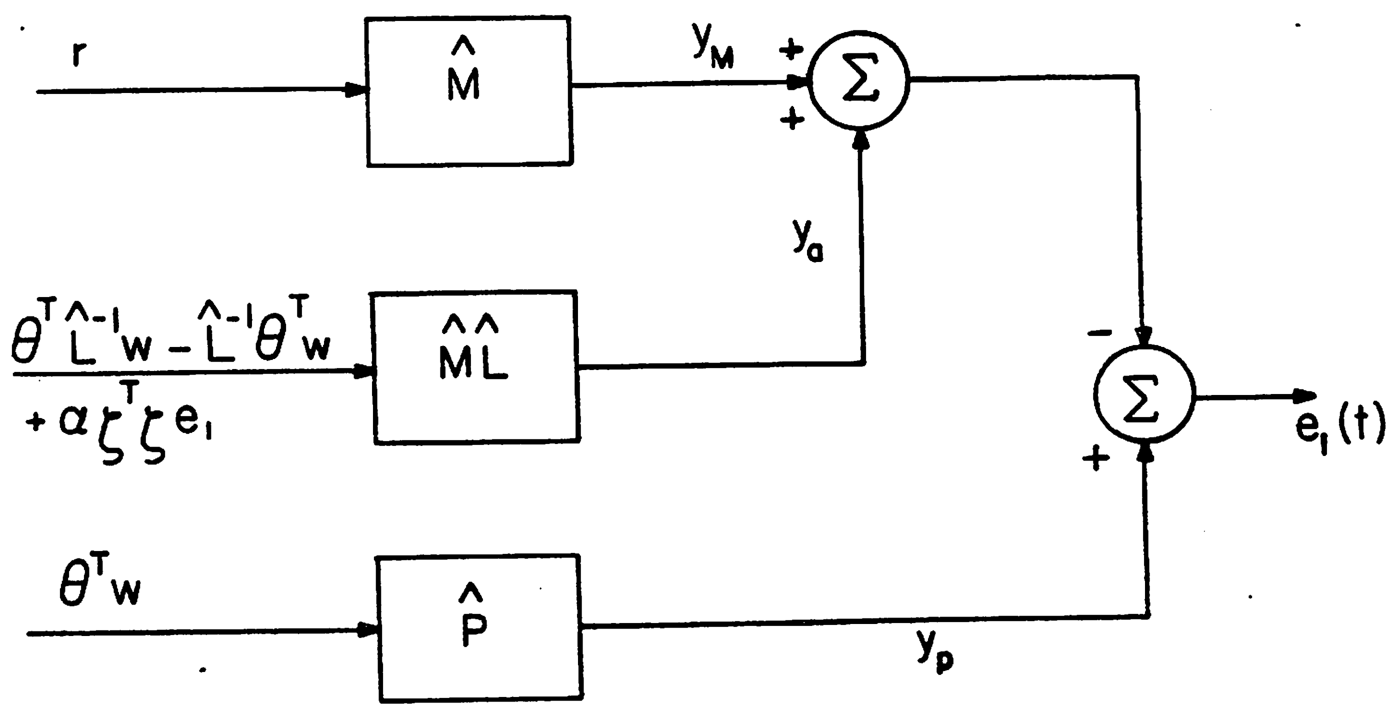

k_M n̂_M(s)/d̂_M(s)
y_M
e_1
r
c_o
Σ
u
k_p n̂_p(s)/d̂_p(s)
y_p
(sI−Λ)^{-1}b
v^{(1)}
F_1
c^T
(sI−Λ)^{-1}b
v^{(2)}
F_2
d^T
d_o

Fig. 5



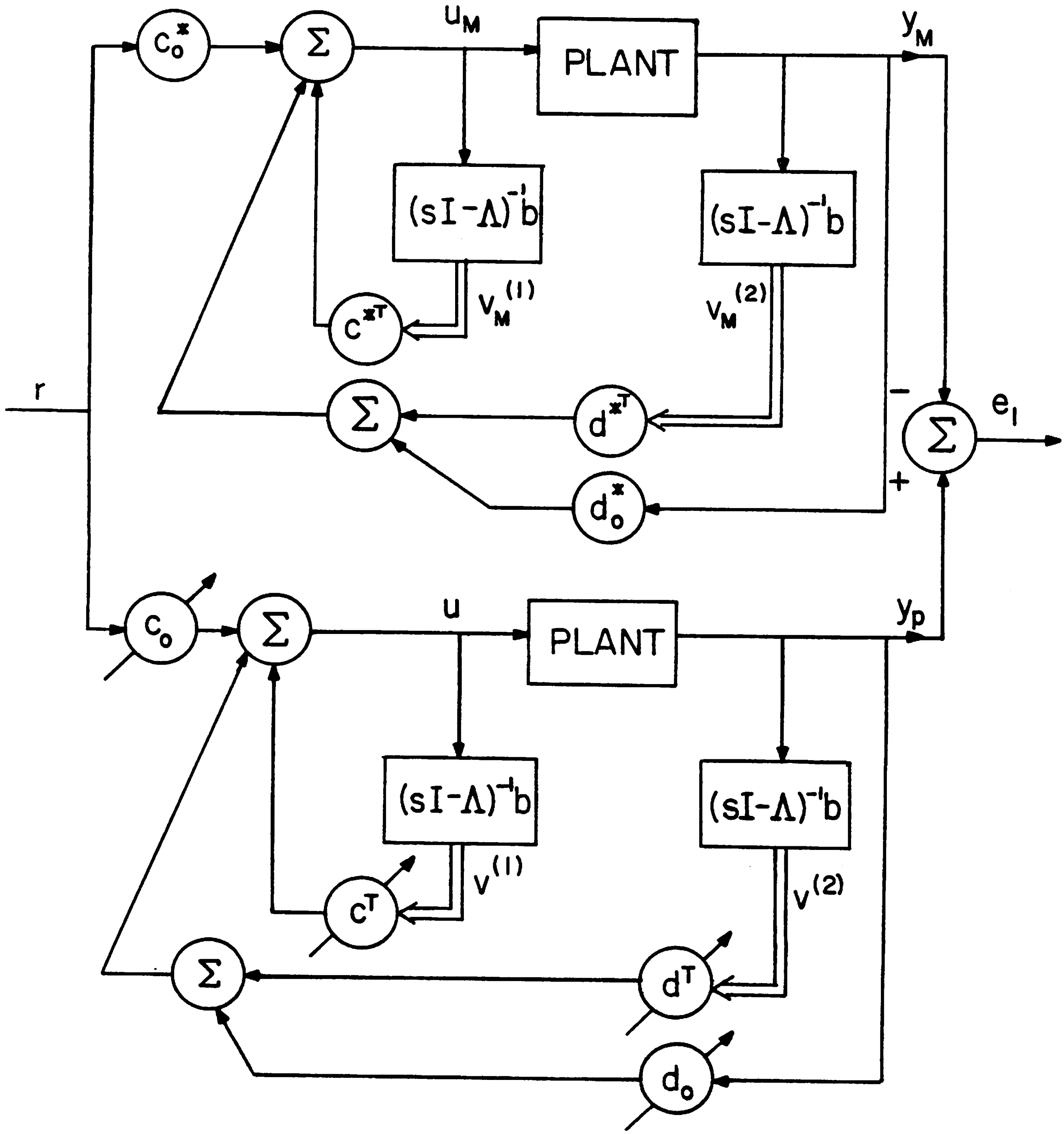
$\phi^T w$
$\frac{k_p}{k_M}\hat{M}$
e_1

Fig. 6



r
$\hat{M}$
y_M
+
+
Σ
y_a
$\theta^T \hat{L}^{-1} w - \hat{L}^{-1} \theta^T w$
$+ \alpha \zeta^T \zeta e_1$
$\hat{M}\hat{L}$
−
+
Σ
$e_1(t)$
$\theta^T w$
$\hat{P}$
y_p

Fig. 7



r
c_o^*
Σ
u_M
PLANT
y_M
$(sI-\Lambda)^{-1}b$
$(sI-\Lambda)^{-1}b$
c^{*T}
$v_M^{(1)}$
$v_M^{(2)}$
Σ
d^{*T}
d_o^*
−
+
Σ
e_1
c_o
Σ
u
PLANT
y_p
$(sI-\Lambda)^{-1}b$
$(sI-\Lambda)^{-1}b$
c^T
$v^{(1)}$
$v^{(2)}$
Σ
d^T
d_o

Fig. 8

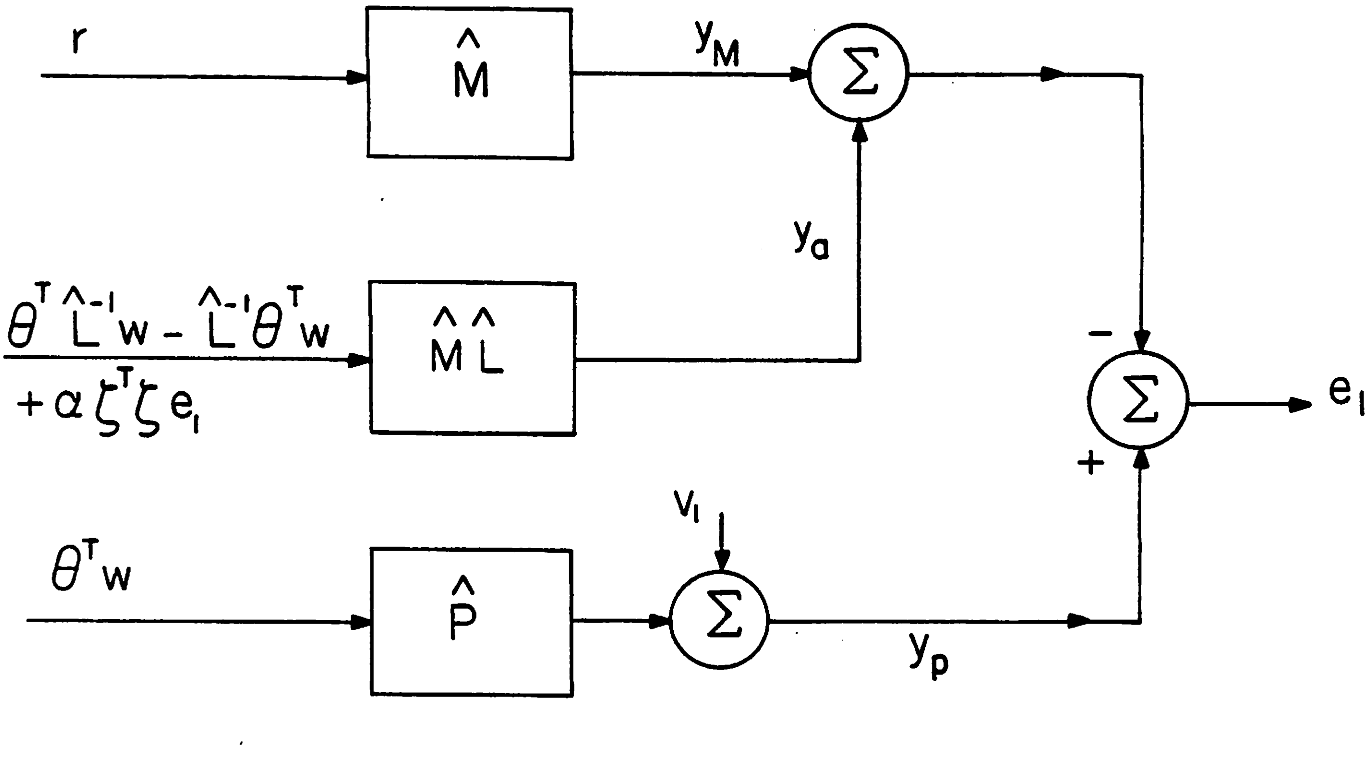

r
$\hat{M}$
y_M
Σ
y_a
$\theta^T \hat{L}^{-1} w - \hat{L}^{-1} \theta^T w$
$+ \alpha \zeta^T \zeta e_1$
$\hat{M}\hat{L}$
$-$
Σ
e_1
$+$
v_1
$\theta^T w$
$\hat{P}$
Σ
y_p

Fig. 9

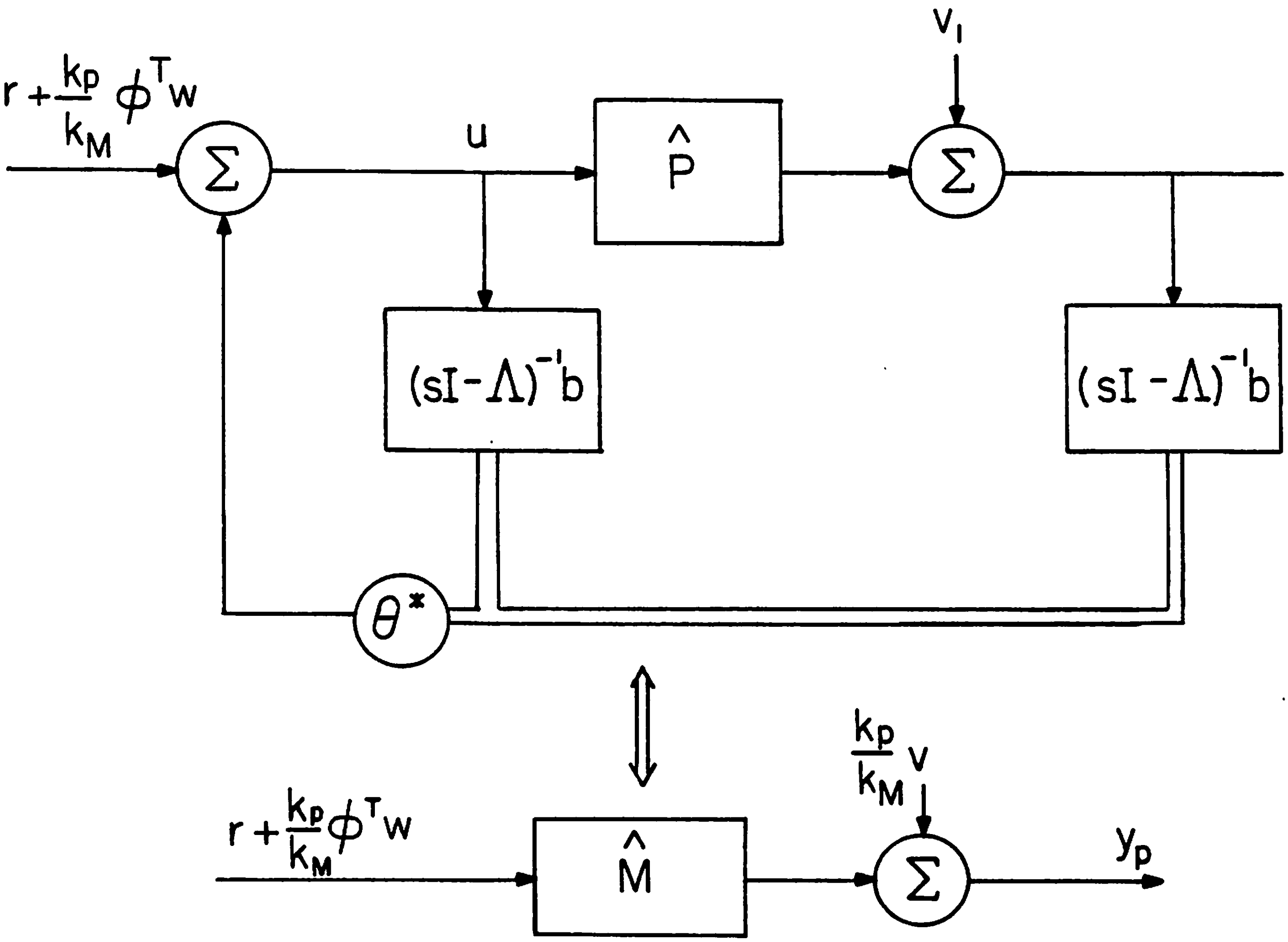
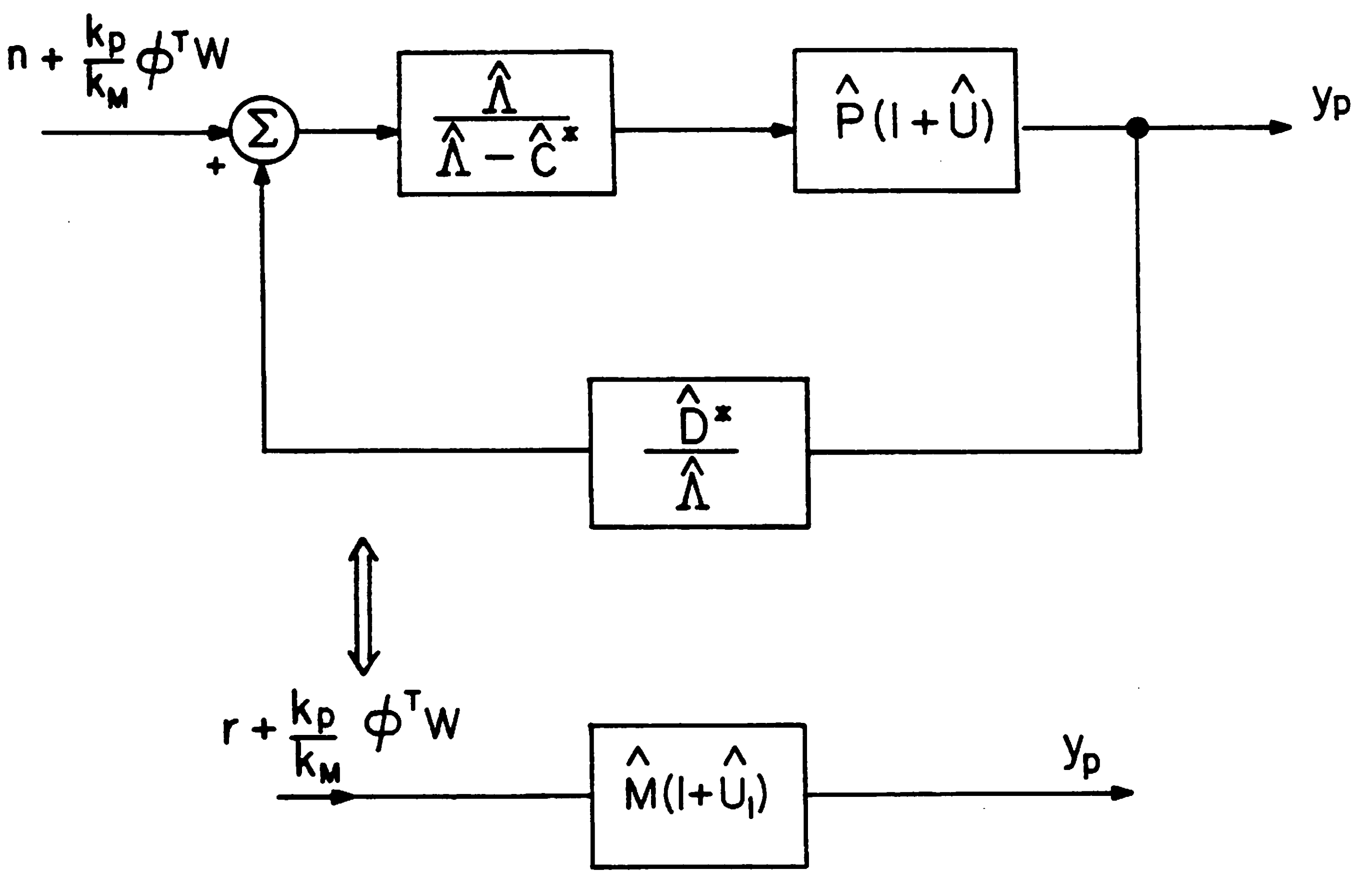

v_1
$r + \frac{k_p}{k_M} \phi^T w$
Σ
u
$\hat{P}$
Σ
$(sI - \Lambda)^{-1} b$
$(sI - \Lambda)^{-1} b$
θ^*
$\frac{k_p}{k_M} v$
$r + \frac{k_p}{k_M} \phi^T w$
$\hat{M}$
Σ
y_p

Fig. 10



$n + \frac{k_p}{k_M}\phi^T W$
Σ
+
$\frac{\hat{\Lambda}}{\hat{\Lambda} - \hat{C}^*}$
$\hat{P}(I + \hat{U})$
y_p
$\frac{\hat{D}^*}{\hat{\Lambda}}$
$r + \frac{k_p}{k_M}\phi^T W$
$\hat{M}(I + \hat{U}_I)$
y_p

Fig. 11