ABSOLUTE $\underset{\sim}{k}$-STABILITY OF LINEAR DISTRIBUTED n-PORTS

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Abstract

This paper considers exclusively linear time-invariant distributed n-ports specified by a convolution operator: $\check{b} = \check{S} * \check{a}$. It defines I/O stability in the time-domain and characterizes it for a broad class of such n-ports. It characterizes absolutely stable n-ports. Finally, it defines absolute $\underset{\sim}{k}$-stability — i.e. stability of given n-port under any loading by passive k_1-ports, k_2-ports, ⋯ , k_r-ports, where $k_1 + k_2 + \cdots k_r = n$. The necessary and sufficient conditions for absolute $\underset{\sim}{k}$-stability are obtained using Doyle's μ functional. The paper is self contained.

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1. Introduction

The problem of determining conditions under which a lumped active n-port is stable when each one of its ports is terminated by an arbitrary linear passive 1-port has long been studied. A considerable amount of the literature on this subject is devoted to the special case of linear lumped 2-ports [De. 1], [Woo. 1], [Kuh 1]. Youla in [You. 1] obtained necessary and sufficient conditions (n.a.s.c.) working with impedance matrices and later, in [You. 2], n.a.s.c. in terms of the scattering matrix of the n-port. [Zeh. 1] contains a different set of n.a.s.c. using the impedance matrix.

In this paper we generalize the classical problem in two directions by considering distributed n-ports and by allowing a less restrictive class of terminations. We consider exclusively linear, time-invariant, causal, active n-ports characterized by Laplace transformable convolution operators [Sch. 1], [Zem. 1], [Des. 1], [Vid. 1]; and define the I/O-stability in terms of I/O time-domain concepts in section 2. For a general class of such n-ports we characterize those that are stabilizable and those that are absolutely stable in section 3. In section 4 we define the new concepts of $\underset{\sim}{k}$-terminations and absolute $\underset{\sim}{k}$-stability and, finally, in section 5, using the function $\mu_{\underset{\sim}{k}}$ recently defined in [Doy. 1], we obtain necessary and sufficient conditions for the absolute $\underset{\sim}{k}$-stability of a class of distributed n-ports.

Notation

$a := b$ means a denotes b; $\mathbb{R}$ is the field of real numbers, $\mathbb{C}$ is the field of complex numbers; $\mathbb{R}_+$ is the set of non-negative real numbers; $\mathbb{C}_+$ ($\mathbb{C}_{\sigma,+}$) is the set of complex numbers such that Re $z \geq 0$ (Re $z \geq \sigma$, respectively). For any positive integer k, $\underline{k} := \{1,2,\cdots,k\}$. For any set A, A^{nxn} denotes the class of all nxn arrays with elements in A, and $\mathring{A}$ denotes the interior of A. $\mathbb{R}_p(s)$ denotes the class of all proper rational functions with coefficients in $\mathbb{R}$. For any $A \in \mathbb{C}^{mxn}$, $\bar{\sigma}[A]$ is the $\sigma_{max}[A]$, the maximum singular value of A. Given $\sigma \in \mathbb{R}$ (typically $\sigma > 0$), $f \in \mathcal{A}(\sigma)$ iff $f(t) = f_a(t) + \sum_0^\infty f_i\ \delta(t-t_i)$, where $f_a : \mathbb{R} \to \mathbb{R}$, with $f_a(t) = 0$ for $t < 0$ and $t \mapsto \exp(-\sigma t)\ f_a(t) \in L_1$; $t_o = 0$, $t_i > 0$, $\forall i > 0$; $\forall i$, $f_i \in \mathbb{R}$ and $i \mapsto f_i \exp(-\ t_i) \in \ell_1$; $f \in \mathcal{A}_-(\sigma)$ iff, for some $\sigma_1 < \sigma$, $f \in \mathcal{A}(\sigma_1)$. $\hat{f}$ denotes the Laplace transform of f. $\mathcal{A} := \mathcal{A}(0)$, $\mathcal{A}_- := \mathcal{A}_-(0)$ $\hat{\mathcal{A}} := \{\hat{f} : f \in \mathcal{A}\}$, $\hat{\mathcal{A}}_- := \{\hat{f} : f \in \mathcal{A}_-\}$. W.r.t. means "with respect to." U.t.c. means "under these conditions." W.l.o.g. means "without loss of generality."

2. Input-output -- stable linear time-invariant n-ports

2.1. Description of linear time-invariant n-ports and definition of I/O-stability.

We will view linear time-invariant n-ports as being represented by convolution operators. In order to do this, given an n-port $\mathcal{N}$, we choose n positive resistors $r_1,\cdots,r_n$ with respect to which the scattering matrix of $\mathcal{N}$, S, may be defined. To appreciate the generality of this point of view, recall that L. Schwartz has shown [Sch. 1, p. 162,197] that any linear time-invariant operator that satisfies some slight continuity properties has a representation of the form $a \mapsto \check{S}*a$, where $\check{S}(t)$ is a distribution. In the context of this paper, $S(s) = \mathcal{L}[\check{S}(t)]$ is the scattering matrix of the n-port under consideration, a the incident wave and $b = \check{S}*a$, the reflected wave. $\check{S}$ is a causal convolution kernel iff it is supported on $\mathbb{R}_+$. We make the following assumption:

A.1: The linear time-invariant n-port is causal and is represented by a convolution operator $\check{\underset{\sim}{S}}$, and $\check{\underset{\sim}{S}}$ is Laplace transformable.

We adopt the following definition of stability:

Definition: An n-port is said to be I/O-stable iff

i) for all $p \in [1,\infty]$, it takes an L_p-input, a, into an L_p-output, b, with a finite gain; equivalently for some $M < \infty$, $\|b\|_p < M\|a\|_p$;

ii) it takes continuous and bounded inputs (periodic inputs, almost periodic inputs, resp.) into outputs belonging to the same class, respectively.

Comment: In contrast to many authors [You. 1], [Zeh. 1], we do not define stability in terms of frequency domain concepts: first, stability is a time-domain concept; second, it is only for transfer functions that are known to be proper and rational that analyticity in the closed right half-plane ($\mathbb{C}_+$) is equivalent to exponential stability and to the requirements i) and ii) above. (For a proof of this fact, see [Cal. 1, p. 124]).

For example, the time-functions $f_1(t) := t^n e^t \sin(e^t)$; $f_2(t)$ $f_2(t) := t^n \sin(t^\alpha)$ — where $n \in \mathbb{N}$, $\alpha > 1$ — have Laplace transforms that are analytic everywhere in $\mathbb{C}$ (except at ∞). (Such time-functions have Laplace transforms that are not proper rational functions. The network functions of distributed circuits are, in general, not rational functions either). Since both $f_1(\cdot)$ and $f_2(\cdot)$ are unbounded on $\mathbb{R}_+$ and do not belong to $L_1(\mathbb{R}_+)$, these time-functions cannot be associated with "stable" circuits. Hence, for distributed circuits, the conventional definition of stability, based on analyticity in $\mathbb{C}_+$, is totally inappropriate.

To alleviate technical difficulties, we forego a slight extension and make the following assumption:

A2: The kernel, $\check{S}(\cdot)$, (equivalently, the measure) representing an I/O-stable n-port has no singular continuous part.

Fact 2.1: Let the n-port $\mathcal{N}$, described by S, be I/O-stable and let it satisfy A1, A2; then $\check{S}(\cdot) \in \mathcal{A}^{n \times n}$.

Proof: A1 and I/O-stability give that $b = \check{S}*a$ and $S*: a \mapsto b$ maps L_p to L_p, $\forall p \in [1,\infty]$. Thus, by the Riesz representation theorem [Rud. 1] S can be represented by a measure. This measure is supported on $\mathbb{R}_+$, by A1. From A2, $\check{S}(t) \in \mathcal{A}^{n \times n}$.

2.2. Interconnection of n-ports and transfer functions

Let an n-port $\mathcal{N}$ be loaded by an n-port, $\mathcal{N}_\ell$. If the interconnection of $\mathcal{N}$ and $\mathcal{N}_\ell$ is driven in parallel (series) by current (voltage) sources, it is called $\mathcal{N}_{t_i}$ ($\mathcal{N}_{t_e}$). See Fig. 1 (Fig. 2).

We call the interconnection of $\mathcal{N}$ and $\mathcal{N}_\ell$, $\mathcal{N}_{t_i}$ in Fig. 1 and $\mathcal{N}_{t_e}$ in Fig. 2. S_ℓ will be the scattering matrix representing the "load" n-port $\mathcal{N}_\ell$ and S the scattering matrix of $\mathcal{N}$. For the $\mathcal{N}_{t_e}$ of Fig. 2 we may write the following equations in the frequency domain:

$$
\begin{aligned}
a_\ell + b_\ell + e &= a + b \\
a_\ell - b_\ell &= -a + b \\
b_\ell &= S_\ell a_\ell \\
b &= Sa
\end{aligned}
$$

In order to eliminate a_ℓ and b_ℓ; we add (subtract resp.) the first 2 eqns. to get

$$a = b - \frac{e}{2} \quad (b = a - \frac{e}{2}, \text{ resp.}).$$

Using the last 2 equations we get,

$$(a - \frac{e}{2}) = S_\ell(b - \frac{e}{2})$$

and finally,

$$a = (I - S_\ell S)^{-1} (I - S_\ell) \frac{e}{2} \tag{2.2.0a}$$

$$b = Sa = S(I - S_\ell S)^{-1} (I - S_\ell) \frac{e}{2} \tag{2.2.0b}$$

A similar exercise may be carried out for $\mathcal{N}_{t_i}$ of Fig. 1. In summary, for the circuits of Figs. 1 and 2 we obtain the following transfer functions:

$$\mathcal{N}_{t_i} \begin{cases} (I-S_\ell S)^{-1}(I+S_\ell) : \; i \mapsto a & (2.2.1a) \\ S(I-S_\ell S)^{-1}(I+S_\ell) : i \mapsto b & (2.2.1b) \end{cases}$$

$$\mathcal{N}_{t_e} \begin{cases} (I-S_\ell S)^{-1}(I-S_\ell) \; : e \mapsto a & (2.2.1c) \\ S(I-S_\ell S)^{-1}(I-S_\ell) : e \mapsto b & (2.2.1d) \end{cases}$$

We will study the I/O-stability of interconnected n-ports $\mathcal{N}_{t_i}$, $\mathcal{N}_{t_e}$ shown in Figs. 1 and 2. In order to do this, we make the following (technical) assumption: roughly speaking, we may state it as:

"For all $|s|$ "sufficiently large", $S(s)$ is analytic and bounded away from 1."

Let ρ be positive and large

and

$$M_\rho := \mathbb{C}_+ \cap \{s : |s| > \rho\}$$

Thus, we may state this assumption more precisely as follows:

<u>A3</u>: $\exists \rho > 0, \; \exists \varepsilon > 0$ s.t. $\forall s \in M_\rho$, $S(s)$ is analytic <u>and</u>

$$\|S(s)\|_2 \leq 1 - \varepsilon < 1$$

(Here $\|S(s)\|_2$ is the ℓ_2-induced norm of $S(s) \in \mathbb{C}^{n x n}$.

<u>Comment</u>: This is equivalent to assuming that the n-port represented by $S(s)$ is strictly passive in M_ρ [Kuh 1].

3. Stabilizability of linear time-invariant n-ports

3.1. Definition of stabilizability

Consider an n-port, $\mathcal{N}$, described by a scattering matrix S(s) (with respect to some choice of positive normalizing resistances, $r_i > 0$, $i \in \underline{n}$). Let assumptions A1, A2 and A3 hold. We say that such an n-port is stabilizable iff there exists a lumped "load" n-port, $\mathcal{N}_\ell$, (described by a scattering matrix $S_\ell(s) \in \mathbb{R}_p(s)^{nxn}$ with respect to (the same) $r_i > 0$, $i \in \underline{n}$), such that i) $\mathcal{N}_{t_e}$ and $\mathcal{N}_{t_i}$ are I/O-stable and ii) each of the four transfer functions of (2.2.1) belong to $\hat{\mathcal{A}}_-^{nxn}$.

Comment: This is a little stronger than just I/O-stability which would require that the four transfer functions of (2.2.1) each belong to $\hat{\mathcal{A}}^{nxn}$.

Consider:

$(I-S_\ell S)^{-1}$ (3.1.1.a)

$(I-SS_\ell)^{-1}$ (3.1.1b)

$S_\ell(I-SS_\ell)^{-1}$ (3.1.1c)

$S(I-S_\ell S)^{-1}$ (3.1.1d)

The following algebraic fact is true:

Fact 3.1.1: Let $\mathcal{M}$ be a commutative algebra. U.t.c. the four transfer functions of (2.2.1) belong to the matrix algebra $\mathcal{M}^{nxn}$ iff the four transfer functions of (3.1.1) belong to the same matrix algebra $\mathcal{M}^{nxn}$.

Note: In the context of this paper, $\mathcal{M}$ is $\hat{\mathcal{A}}$ or $\hat{\mathcal{A}}_-$.

Proof: Note that $S(I-S_\ell S)^{-1} = (I-SS_\ell)^{-1}S$ and $I-(I-SS_\ell)^{-1} = S(I-S_\ell S)^{-1}S_\ell$. Then note that each of the 4 transfer functions of (2.2.1) is a linear combination of the 4 transfer functions of (3.1.1), and conversely. ¤

Comments: a) Thus an n-port $\mathcal{N}$ satisfying A1, A2, A3 is stabilizable iff there exists $\mathcal{N}_\ell$ (described by some $S_\ell \in \mathbb{R}_p(s)^{n\times n}$) such that the four transfer functions of (3.1.1) belong to $\hat{\mathcal{A}}_-^{n\times n}$.

b) Note that some 1-ports are not stabilizable by any stable 1-ports: e.g., $S(s) = \frac{2(s-1)}{(s+1)(s-2)}$ describes a 1-port that is not stabilizable by a stable 1-port.

It is now possible to detail a consequence of stabilizability.

Fact 3.1.2: Let the n-port $\mathcal{N}$ satisfy A1, A2 and A3. U.t.c., if $\mathcal{N}$ is stabilizable, then $S(s)$, the scattering matrix of $\mathcal{N}$ w.r.t. some choice of normalizing resistors $r_i > 0$, $i \in \underline{n}$, has only a finite number of $\mathbb{C}_+$-poles.

Proof: $\mathcal{N}$ is stabilizable hence $\exists S_\ell(s) \in \mathbb{R}_p(s)^{n\times n}$ such that $(I-S_\ell S)^{-1}$, $(I-SS_\ell)^{-1}$, $S_\ell(i-SS_\ell)^{-1}$, $S(I-S_\ell S)^{-1} \in \hat{\mathcal{A}}_-^{n\times n}$. Let $H_1 := (I-S_\ell S)^{-1}$; $H_2 := S(I-S_\ell S)^{-1}$. Then $H_1, H_2 \in \hat{\mathcal{A}}_-^{n\times n}$ and $\det H_1 \in \hat{\mathcal{A}}_-$. Hence, for some $\varepsilon > 0$, H_1, H_2 and $\det H_1$ are analytic and bounded in $\mathbb{C}_{-\varepsilon,+}$.

$S_\ell(s)$ is bounded at ∞ since $S_\ell(s) \in \mathbb{R}_p(s)^{n\times n}$ and, by A3, $\exists \rho > 0$ such that $S(s)$ is bounded in M_ρ. Hence, $\det(I-S_\ell S) = \det[H_1^{-1}]$ is bounded in M_ρ. Since $\det[H_1^{-1}] = 1/\det H_1$ we conclude that $s \mapsto \det[H_1(s)]$ is bounded away from zero in M_ρ. We know that $s \mapsto \det[H_1(s)]$ is analytic in $C_{-\varepsilon,+}$ hence the zeros of $s \mapsto \det[H_1(s)]$ are isolated [Dieu. 1, Thm. 9.1.5] and do not belong to M_ρ. Now $\mathbb{C}_+ \backslash M_\rho$ is a compact set in the domain of analyticity of $\det H_1(s)$; consequently $\det H_1$ has only a finite number of zeros in $\mathbb{C}_+ \backslash M_\rho$. Now $(I-S_\ell S) = H_1^{-1}$ has a pole in $\mathbb{C}_+$ if and only if $\det[H_1(s)] = 0$. Since H_2 is analytic in $\mathbb{C}_{-\varepsilon,+}$, $S = H_2 \bullet H_1^{-1}$ has only a finite number of $\mathbb{C}_+$-poles. We may write

$$S(s) = \sum_{i=1}^{\ell} \sum_{k=0}^{m_i-1} r_{ik}(s-p_i)^{-m_i+k} + S_0(s), \text{ where } Re(p_k) > -\varepsilon < 0 \text{ and}$$

$S_0(s) \in \hat{\mathcal{A}}_-^{nxn}$.

Comment: Suppose that, in Definition 3.1, we did not require ii). Then by i) (I/O-stability), each of the four transfer functions of (3.3.1) would belong to $\hat{\mathcal{A}}^{nxn}$ (but not necessary to $\hat{\mathcal{A}}_-^{nxn}$) and we could then prove the following (weaker) version of Fact 3.

Let the n-port $\mathcal{N}$ satisfy A1, A2, and A3. U.t.c., if $\mathcal{N}$ is stabilizable, then, for any $\varepsilon > 0$, $S(s)$ the scattering matrix of $\mathcal{N}$ w.r.t. some choice of normalizing resistors $r_i > 0$, $i \in \underline{n}$, has only a finite number of $\mathbb{C}_{\varepsilon+}$-poles. The point is that there might be an infinite number of poles in $\mathbb{C}_+$ with an accumulation point in $\mathbb{C}_+$ on the $j\omega$-axis.

3.2. Absolutely stable n-ports -- a characterization

Definition: An n-port is said to be absolutely stable iff the interconnections $\mathcal{N}_{t_e}$, $\mathcal{N}_{t_i}$ are I/O-stable for all lumped passive "load" n-ports, $\mathcal{N}_\ell$.

Comments: a) Recall that the n-port $\mathcal{N}_\ell$ (described by $S_\ell(s)$) is lumped and passive $\Leftrightarrow$ $S_\ell(s) \in \mathbb{R}_p(s)^{nxn}$ and $\sup_{Re(s)\geq 0} \|S_\ell(s)\|_2 \leq 1$ [New. 1].

b) Consider the n-port $\mathcal{N}_0$, described by S_0. Then, $S_0(s) \in \mathbb{R}_p(s)^{nxn}$ and $S_0(s)$ analytic in $\mathring{\mathbb{C}}_+$ and bounded on the $j\omega$-axis imply that $\mathcal{N}_0$ is I/O-stable.

The following theorem characterizes absolutely stable n-ports.

Theorem 3.1: Let assumptions A1, A2, A3 hold for a distributed n-port, $\mathcal{N}$, described by a scattering matrix, $S(s)$. U.t.c. $\mathcal{N}$ is absolutely stable iff

i) $s \mapsto S(s)$ is analytic in $\mathring{\mathbb{C}}_+$ and bounded in $\mathbb{C}_+$;

ii) $\forall \omega \in \mathbb{R}_+$, $\|S(j\omega)\|_2 < 1$.

Comments: a) Recall that in Fact 2.1.1 above, we have shown that satisfies A1, A2 and is I/O-stable iff $S(s) \in \hat{\mathcal{A}}^{n \times n}$; thus $\omega \mapsto S(j\omega)$ is bounded but the bound may be larger than 1.

b) In appendix 2 we exhibit a class of linear time-invariant distributed n-ports (more concretely specified in appendix 2, p.) which have scattering matrix in $\hat{\mathcal{A}}^{n \times n}$.

c) Theorem 3.1 is a generalization, to the distributed case, of a known theorem for the lumped case.

Proof: (⇒) Consider the particular interconnection, $\mathcal{N}_{t_e}$, in which $\mathcal{N}_{\ell}^R$ is the n-port of normalizing resistors (hence $S_{\ell}^R = 0$). Then, from eqn. (2.2.0a)

$$a = \frac{1}{2} e \quad \text{and} \quad b = S\,a = \frac{1}{2} S\,e\,; \quad \text{or, } 2b = Se\,.$$

I/O-stability implies that

$$\forall e \in L_2^n, \ \frac{1}{2\pi} \int \|S(j\omega)\, e(j\omega)\|_2^2 \, d\omega = \|2b\|_2 \leq (\text{const.}) \|e\|_2,$$

hence $\omega \mapsto S(j\omega)$ is bounded on $\mathbb{R}_+$. Now, $\mathcal{N}$ absolutely stable implies, in particular, that $\mathcal{N}_{t_e}$ is I/O-stable. Thus, $\forall \check{e}(t)$ s.t. $\check{e} \in L_2^n$ and $\text{supp}[\check{e}] \subset \mathbb{R}_+$, by I/O-stability $b \in L_2$ and b is analytic in $\mathring{\mathbb{C}}_+$ so that S(s) must be analytic in $\mathbb{C}_+$. And, by the theorem of the maximum, S(s) is bounded in $\mathbb{C}_+$. This proves i)

To prove ii) we use contradiction. Suppose that $\exists\ \omega_0 \in \mathbb{R}_+$ s.t. $\|S(j\omega_0)\|_2 = 1$.[1] The numerical matrix $S(j\omega_0)$ can be written as UH, where

[1] A1, A2 and I/O-stab. for $\mathcal{N} \Rightarrow S(s) \in \hat{\mathcal{A}}^{n \times n} \Rightarrow \omega \mapsto S(j\omega)$ uniformly continuous on $\mathbb{R}$. Thus we can assume, $\exists \omega_0 \in \mathbb{R}_+$ s.t. $\|S(j\omega_0)\|_2 = 1$ w.l.o.g. for if $\exists \omega_1 \in \mathbb{R}_+$ s.t. $\|S(j\omega_1)\| > 1$, then, the (uniform) continuity of $\omega \mapsto S(j\omega)$ and A3, which says that for ω sufficiently large $\|S(j\omega)\|_2 \leq 1-\varepsilon < 1$, imply, by the intermediate value theorem, that $\exists \omega_0 \in \mathbb{R}_+$, s.t. $\|S(j\omega_0)\|_2 = 1$.

$U \in \mathbb{C}^{n\times n}$ is unitary and $H \in \mathbb{C}^{n\times n}$ is Hermitian. Note $\bar{\sigma}[S(j\omega_0)] = \bar{\sigma}(H) = 1$.

By a synthesis technique [Car. 1; p. 370, 412 ff.] we can construct a passive lossless "load" n-port $\mathcal{N}_\ell^o$ (scattering matrix S_ℓ^o, w.r.t. normalizing resistances $r_i > 0$ for S) such that $S_\ell^o(j\omega_0) = U^*$. Consequently, $S_\ell^o(j\omega_0)\, S(j\omega_0) = H$. H is Hermitian, so H has all eigenvalues real and the largest is 1. Consequently, $\det(I - S^o(j\omega_0)\, S(j\omega_0)) = 0$. Hence $s \mapsto (I-S^oS)^{-1}(s)$ has a pole at $j\omega_0$ and so $(I-S^oS)^{-1} \notin \hat{\mathcal{A}}^{n\times n}$. This contradicts I/O-stability, by Fact 2.1.1.

($\Leftarrow$) We know that i) and ii) hold and i) with A1, A2 gives that $S(s) \in \hat{\mathcal{A}}^{n\times n}$. Consider an arbitrary lumped passive "load" n-port $\mathcal{N}_\ell$, described by $S_\ell(s)$. Then, in particular, $S_\ell(s) \in \hat{\mathcal{A}}^{n\times n}$ and

$$\forall \omega \in \mathbb{R},\ \|S_\ell(j\omega)\|_2 \le 1. \quad (*)$$

Since $S_\ell(s)$, $S(s) \in \hat{\mathcal{A}}^{n\times n}$, $S(s)$ and $S_\ell(s)$ are analytic in $\mathring{\mathbb{C}}_+$ and bounded in $\mathbb{C}_+$. Now,

$$\begin{aligned}\sup_{\omega\in\mathbb{R}} \bar{\sigma}[S_\ell(j\omega)S(j\omega)] &\le \sup_{\omega\in\mathbb{R}} [\|S_\ell(j\omega)\|_2 \cdot \|S(j\omega)\|_2] \\ &\le \sup_{\omega\in\mathbb{R}} \|S_\ell(j\omega)\|_2 \sup_{\omega\in\mathbb{R}} \|S(j\omega)\|_2 \\ &< 1\end{aligned}$$

where in the last step (*) and A3 were used. The function $s \mapsto \bar{\sigma}[S_\ell(s)S(s)]$ is subharmonic, hence using a useful property of subharmonic functions [Rud. 1; Them. 17.4, p.362][2], we may write $\sup_{\mathrm{Re}\, s \ge 0} \bar{\sigma}[S_\ell(s)S(s)] < 1$ which implies that

$$\inf_{\mathrm{Re}\, s \ge 0} |\det(I-S_\ell S)(s)| > 0 \Rightarrow (I-S_\ell S)^{-1} \in \hat{\mathcal{A}}^{n\times n}$$

[2]We use theorem 17.4, p. 362, of [Rud. 1] with the $\le$ signs replaced by strict inequality signs, in which form (it is easy to see !) it is still true.

Since S_ℓ, $S \in \hat{\mathcal{A}}^{n\times n}$ (by i)), the 4 transfer functions of (3.1.1) and hence (Fact 3.1.1) those of (2.2.1) are in $\hat{\mathcal{A}}^{n\times n}$, so $\mathcal{N}_{t_e}$ and $\mathcal{N}_{t_i}$ are I/O-stable.

4. $\underset{\sim}{k}$-terminations and absolute $\underset{\sim}{k}$-stability

In this section we define $\underset{\sim}{k}$-terminations and absolute $\underset{\sim}{k}$-stability which is a generalization of absolute stability and then derive necessary and sufficient conditions for absolute $\underset{\sim}{k}$-stability.

4.1. Definitions

Let $\underset{\sim}{k} = (k_1,\cdots,k_r) \in \mathbb{N}_+^r$ and let the n ports of $\mathcal{N}$ be partitioned into r sets of $k_1,\cdots,k_r$ ports each, where $\sum_{i=1}^{r} k_i = n$ = number of ports of the given n-port, $\mathcal{N}$.

Definition 4.1.1: A $\underset{\sim}{k}$-termination of $\mathcal{N}$ is the following:

the first set of k_1 ports is terminated by a k_1-port, $\mathcal{N}_{k_1}$;

the next set of k_2 ports is terminated by a k_2-port, $\mathcal{N}_{k_2}$;

$\vdots$

the last set of k_r ports is terminated by a k_r-port, $\mathcal{N}_{k_r}$.

The n-port made up of $\mathcal{N}_{k_1},\cdots,\mathcal{N}_{k_r}$ is described by scattering matrix S_ℓ, w.r.t. $r_1,\cdots,r_n$ (which are the normalizing resistances for $\mathcal{N}$, which is then described by S). Note that S_ℓ is a block-diagonal matrix whose successive blocks are of size $k_1,\cdots,k_r$.

Definition 4.1.2: A passive (resp. lumped, I/O-stable) $\underset{\sim}{k}$-termination is a $\underset{\sim}{k}$-termination with $\mathcal{N}_{k_i}$ passive (resp. lumped, I/O-stable) for all $i \in \underline{r}$.

Notation: Let $\mathcal{P}_{\ell_{\underset{\sim}{k}}}$ be the class of all lumped passive $\underset{\sim}{k}$-terminations.

Definition 4.1.3: An n-port, $\mathcal{N}$, is said to be absolutely $\underset{\sim}{k}$-stable iff the interconnections $\mathcal{N}_{t_e}$, $\mathcal{N}_{t_i}$ are I/O-stable for all lumped passive $\underset{\sim}{k}$-terminations, $\mathcal{N}_\ell$, (i.e., for all $\mathcal{N}_\ell \in \mathcal{P}\ell_k$).

4.2. Characterization of absolutely $\underset{\sim}{k}$-stable n-ports

The following characterization is an easy algebraic consequence of facts stated earlier:

Theorem 4.1: Let the distributed n-port, $\mathcal{N}$, satisfy A1, A2 and be I/O-stable. Let $S(s)$ be the scattering matrix of $\mathcal{N}$ w.r.t. some choice of normalizing resistances, $r_i > 0$, $i \in \underline{n}$. Let $S_\ell(s)$ be the scattering matrix of the n-port $\mathcal{N}_\ell \in \mathcal{P}\ell_{\underset{\sim}{k}}$ w.r.t. (the same) $r_i > 0$, $i \in \underline{n}$. U.t.c.,

$$\mathcal{N} \text{ is absolutely } \underset{\sim}{k}\text{-stable iff for all } \mathcal{N}_\ell \in \mathcal{P}\ell_k, \quad \inf_{\mathrm{Re}\, s \geq 0} |\det(1-S_\ell(s)S(s)| > 0.$$

Proof: ($\Rightarrow$) By definition $\mathcal{N}$ is absolutely $\underset{\sim}{k}$-stable iff the 4 transfer functions of (2.2.1) belong to $\hat{\mathcal{A}}^{n\times n}$ equivalently, by Fact 3.1.1 iff the 4 transfer functions of 3.1.1 belong to $\hat{\mathcal{A}}^{n\times n}$. In particular, $(I-S_\ell S)^{-1} \in \hat{\mathcal{A}}^{n\times n}$ which is true iff $\inf_{\mathrm{Re}\, s \geq 0} |\det(I-S_\ell(s)S(s))| > 0$ because both S and S_ℓ have elements in $\hat{\mathcal{A}}$.

($\Leftarrow$) $\mathcal{N}$ satisfies A1, A2 and is I/O-stable $\Leftrightarrow$ $S(s) \in \hat{\mathcal{A}}^{n\times n}$, (comment following thm. 3.1). Consider an arbitrary n-port $\mathcal{N}_\ell \in \mathcal{P}\ell_k$ then $S_\ell(s) \in \mathbb{R}_p^{n\times n}(s)$, (Comment a) preceding thm. 3.1). $\inf_{s \in \mathbb{C}_+} |\det(I-S_\ell(s)S(s))| > 0 \Leftrightarrow (I-S_\ell S)^{-1} \in \hat{\mathcal{A}}^{n\times n}$. Now, by closure under multiplication in the algebra $\hat{\mathcal{A}}^{n\times n}$, the 4 transfer functions of (2.2.1) belong to $\hat{\mathcal{A}}^{n\times n}$ which is true iff the 4 transfer functions of (3.1.1) belong to $\hat{\mathcal{A}}^{n\times n}$. Since the $\mathcal{N}_\ell$ in $\mathcal{P}\ell_{\underset{\sim}{k}}$ was arbitrary we have shown $\mathcal{N}$ to be absolutely $\underset{\sim}{k}$-stable.

¤

In section 5.2 we will use the tools developed in section 5.1 below and our knowledge of the structure of $S_\ell(s)$ to obtain a more useful characterization than the one above.

5. Characterization of absolute $\underset{\sim}{k}$-stability in terms of Doyles $\mu_{\underset{\sim}{k}}$ function

5.1. Definition and required properties of Doyle's function μ [Doy. 1]

Notation

$\mathbb{B}_\delta(\underset{\sim}{k})$:= {block diagonal matrices in $\mathbb{C}^{n\times n}$, with r square blocks $\in \mathbb{C}^{k_i \times k_i}$, $i \in \underline{r}$ and with $\|\cdot\|_2$-norm of each block $\leq \delta \in \mathbb{R}_+$}

Recall that $\underset{\sim}{k} := (k_1,\cdots,k_r) \in \mathbb{N}_+^r$ and that $\sum_{i=1}^{r} k_i = n$

$\mathbb{B}_\infty(\underset{\sim}{k}) := \bigcup_{\delta\in\mathbb{N}} \mathbb{B}_\delta(\underset{\sim}{k})$; in words, $\mathbb{B}_\infty(k)$ is the set of block diagonal matrices with structure determined by $\underset{\sim}{k}$

$\mathcal{U}(\underset{\sim}{k})$:= {unitary matrices} $\cap\, \mathbb{B}_1(\underset{\sim}{k})$; in words, $\mathcal{U}(\underset{\sim}{k})$ is the set of all unitary matrices with block-diagonal structure determined by $\underset{\sim}{k}$.

<u>Definition 5.1</u>: $\mu_{\underset{\sim}{k}} : \mathbb{C}^{n\times n} \to \mathbb{R}_+$ is a function defined as follows:

$$\forall M \in \mathbb{C}^{n\times n} \begin{cases} \mu_{\underset{\sim}{k}}(M) = 0 \quad \text{if } \exists\ \Delta \in \mathbb{B}_\infty(k) \text{ such that } \det(I-M\Delta) = 0 \\ \dfrac{1}{\mu_{\underset{\sim}{k}}(M)} = \min\limits_{\Delta\in\mathbb{B}_\infty(\underset{\sim}{k})} \{\bar{\sigma}(\Delta) \text{ such that } \det(I-M\Delta) = 0\} \end{cases}$$

<u>Comment</u>: Intuitively if we think of $\bar{\sigma}(\Delta)$ as measuring the "size" of $\Delta \in \mathbb{B}_\infty(k)$ then $1/\mu_{\underset{\sim}{k}}(M)$ is the minimum size of $\Delta \in \mathbb{B}_\infty(\underset{\sim}{k})$ such that $\det(I-M\Delta) = 0$ (for all "smaller" $\Delta \in \mathbb{B}_\infty(k)$, $\det(I-M\Delta) \neq 0$).

From Definition 5.1 the following proposition is immediate.

Proposition 5.1:

$$\forall M \in \mathbb{C}^{n\times n},\ \forall \Delta \in \mathbb{B}_\delta(\underset{\sim}{k}),\ \det(I-M\Delta) = \det(I-\Delta M) \neq 0 \ \Leftrightarrow\ \delta\cdot\mu_{\underset{\sim}{k}}(M) < 1$$

Proposition 5.2: Given $s \mapsto M(s) \in \mathbb{C}^{n\times n}$ analytic in $\mathbb{C}_+$,

$$\forall \Delta \in \mathbb{B}_1(\underset{\sim}{k}),\ \inf_{s\in\mathbb{C}_+} |\det(I-\Delta M(s))| > 0 \ \Leftrightarrow\ \sup_{s=j\omega} \mu_{\underset{\sim}{k}}(M(s)) < 1$$

Proof: Appendix

The following proposition, first stated in [Doy. 1], is the last item we need:

Proposition 5.3: $\forall M \in \mathbb{C}^{n\times n}$, $\max\limits_{U\in\mathcal{U}(k)} \rho(UM) = \mu_{\underset{\sim}{k}}(M)$, where $\rho(A)$:= spectral radius of $A = \max\limits_{\lambda\in\sigma(A)} \{|\lambda|\}$.

Proof: Appendix

5.2. A characterization of absolute $\underset{\sim}{k}$-stability

Using Proposition 5.2, the following equivalent formulation of Theorem 4.1 is immediate.

Theorem 5.1: Let the distributed n-port, $\mathcal{N}$, satisfy A1, A2 and be I/O-stable. Let $S(s)$ be its scattering matrix of $\mathcal{N}$ w.r.t. some choice of positive normalizing resistances $r_i > 0$, $i \in \underline{n}$. U.t.c.

$$\mathcal{N} \text{ is absolutely } \underset{\sim}{k}\text{-stable} \ \Leftrightarrow\ \sup_{s=j\omega} \mu_{\underset{\sim}{k}}(S(s)) < 1$$

Proof: Immediate from Theorem 3.1 and Proposition 5.2 with $M = S(s)$ and the fact that $\mathcal{N}_\ell \in \mathcal{P}\ell_{\underset{\sim}{k}} \Rightarrow \forall s \in \mathbb{C}_+,\ S_\ell(s) =: \Delta \in \mathbb{B}_1(\underset{\sim}{k})$ (see comment a) following definition 3.2). ¤

Now we use Proposition 5.3 to arrive at the most useful (from a computational point of view) equivalent formulation of Theorem 4.1.

Theorem 5.2: Let the distributed n-port $\mathcal{N}$ satisfy A1, A2 and be

I/O-stable. Let $S(s)$ be its scattering matrix w.r.t. some choice of positive normalizing resistances, $r_i > 0$, $i \in \underline{n}$. U.t.c.

$$\mathcal{N} \text{ is absolutely } \underset{\sim}{k}\text{-stable} \Leftrightarrow \sup_{s=j\omega} \left(\max_{U \in \mathcal{U}(\underset{\sim}{k})} \rho(US) \right) < 1$$

Proof: Immediate from Theorem 5.1 and Proposition 5.3.

Comments: a) A reactance n-port $\mathcal{N}_\ell$ (made up of k_1-,$\cdots$,k_r-ports with $\sum_{i=1}^{r} k_i = n$) has a scattering matrix $S_\ell(j\omega) \in \mathcal{U}(\underset{\sim}{k})$, $\forall\omega$. Thus, Theorem 5.2 is a precise statement and proof of the conjecture that the absolute $\underset{\sim}{k}$-stability of an n-port can be checked by closing its ports, partitioned according to $\underset{\sim}{k}$, on reactive k_1-,k_2-,$\cdots$,k_r-ports. In particular, the absolute $(\overbrace{1,\cdots,1}^{n \text{ times}})$-stability (simply called absolute stability in the literature) of an n-port can be checked by closing each of its ports on reactive one-ports. This special case of Theorem 5.2 above was proved for lumped LT-I n-ports by Youla [You. 2].

b) See [Doy. 1] for some details on the computation of $\mu_{\underset{\sim}{k}}$.

Appendix 1

Proposition 5.2: Given $s \mapsto M(s) \in \mathbb{C}^{n\times n}$, analytic in $\mathbb{C}_+$,

$$\forall\Delta \in \mathbb{B}_1(\underset{\sim}{k}),\ \inf_{s\in\mathbb{C}_+} |\det(I-\Delta M(s))| > 0 \ \Leftrightarrow\ \sup_{\omega} \mu_{\underset{\sim}{k}}(M(j\omega)) < 1$$

Proof: ($\Rightarrow$) By assumption $\inf_{s\in\mathbb{C}_+} |\det(I-\Delta M(s))| > 0$, hence $\forall\Delta \in \mathbb{B}_1(\underset{\sim}{k})$, $\forall s \in \mathbb{C}_+$ $\det(I-\Delta M(s)) \neq 0$; consequently $\forall s \in \mathbb{C}_+$, $\mu_{\underset{\sim}{k}}(M(s)) < 1$ (from Proposition 1, with $\delta = 1$, since $\Delta \in \mathbb{B}_1(\underset{\sim}{k})$). Furthermore, $\sup_{\omega} \mu_{\underset{\sim}{k}}(M(j\omega)) < 1$, for if $\exists\ \{s_i\}$, $s_i \in \mathbb{C}_+$, $\forall i$ s.t. $\lim_{i\to\infty} \mu_{\underset{\sim}{k}}(M(s_i)) = 1$ then $\exists\ \Delta \in \mathbb{B}_1(\underset{\sim}{k})$ s.t. $\lim_{i\to\infty} |\det(I-\Delta M(s_i))| = 0$ contradicting the hypothesis.

($\Leftarrow$) As shown in (A.1.2) (see the proof of Proposition 5.3 below) $\forall A \in \mathbb{C}^{n\times n}$, $\rho(A) \leq \mu(A)$. Thus,

$$\sup_{\omega} \rho(\Delta M(j\omega)) \leq \sup_{\omega} \mu(\Delta M(j\omega)) = \sup_{\omega} \mu(M(j\omega)) < 1$$

where the equality holds because $\Delta \in \mathbb{B}_1(\underset{\sim}{k})$. By the maximum modulus theorem, $\sup_{s\in\mathbb{C}_+} \rho(\Delta M(s)) = \sup_{\omega} \rho(\Delta M(j\omega))$. Thus $\sup_{s\in\mathbb{C}_+} \rho(\Delta M(s)) < 1$ which implies that

$$\inf_{s\in\mathbb{C}_+} |\det(I-\Delta M(s))| > 0 \ .$$ ∎

Lemma A1 (Block diagonal singular value decomposition (SVD))

$\forall\Delta \in \mathbb{B}_\infty(\underset{\sim}{k})$, $\exists$ U, $V \in \mathcal{U}(\underset{\sim}{k})$, $\exists$ Σ a diag. matrix with diag. entries in $\mathbb{R}_+$, s.t. $\Delta = U\Sigma V^*$

Proof: Clear from standard SVD and definitions of $\mathcal{U}(\underset{\sim}{k})$, $\mathbb{B}_\infty(\underset{\sim}{k})$.

Lemma A2 (Doyle, [Doy. 1])

Let $f : \mathbb{C}^p \to \mathbb{C}$ polynomial in p complex variables, of degree no more than q in each variable.

Let $\hat{y} := \arg\min \{\|y\|_\infty : y \in \mathbb{C}^p \text{ and } f(y) = 0\}$. U.t.c.

$\exists\, v \in \mathbb{C}^p : f(v) = 0$ and $|v_j| = \|\hat{y}\|_\infty \quad \forall j \in \underline{p}$

<u>Proof by contradiction</u>: If for some minimizer $\hat{y} \in \mathbb{C}^p$ defined above, $|y_j| = \|\hat{y}\|_\infty$, $\forall j \in \underline{p}$, $\hat{y}$ satisfies the theorem and there is nothing to prove. (Introduce the notation $y =: (z,\omega)$ with $z \in \mathbb{C}^{p-1}$, $\omega \in \mathbb{C}$.) If not, choose the smallest component of $\hat{y}$, say $\hat{\omega} \in \mathbb{C}$, let $\hat{y} = (\hat{z},\hat{\omega})$ and we have

$$|\hat{\omega}| < \|\hat{z}\|_\infty = \|\hat{y}\|_\infty \quad .$$

Abusing notation we write $f(y) = f(z,\omega)$, $z \in \mathbb{C}^{p-1}$, $\omega \in \mathbb{C}$. We also have $f(\hat{y}) = f(\hat{z},\hat{\omega}) = 0$. Now view f as a polynomial in ω with coefficients that are polynomials in z; $f : \omega \mapsto f(z,\omega)$. By the Weierstrass preparation theorem [Die. 1], [Rud. 1] there exist an integer r and r functions $h_j(z)$, analytic in a neighborhood V of $\hat{z} \in \mathbb{C}^{p-1}$ such that

$$f(z,\omega) = (\omega^r + h_1(z)\omega^{r-1} + \cdots + h_r(z)) \quad g(z,\omega)$$

for all ω in some disc $D(\hat{\omega};\varepsilon)$ centered on $\hat{\omega}$ with radius ε; g is analytic in $V \times D(\hat{\omega};\varepsilon)$. For ε sufficiently small, $f(z,\omega)$ has exactly r solutions $z = \phi_k(\omega)$ in $V \times D(\hat{\omega},\varepsilon)$. Choose $z_1 \in V$ and ω_1 in $D(\hat{\omega};\varepsilon)$ such that $u_1 := (z_1,\omega_1) \in \mathbb{C}^p$ is a zero of f, $f(u_1) = f(z_1,\omega_1) = 0$, and $\|u_1\|_\infty < \|\hat{y}\|_\infty$. Thus f has a zero at u_1 <u>of smaller norm than</u> $\hat{y}$ which contradicts the definition of $\hat{y}$. Hence there is always a minimizer $\hat{y}$ with the property claimed.

<u>Corollary to Lemma A.2</u>: If $\hat{y}$ is real and nonnegative [$\mathrm{Im}(\hat{z}_j) = 0$ and $\mathrm{Re}(\hat{z}_j) \geq 0$, $\forall j$], then $\exists\, v \in \mathbb{R}^p$ nonnegative s.t. $f(v) = 0$ and $v_j = \|\hat{y}\|_\infty \quad \forall j \in \underline{p}$

Proposition 5.3: $\forall M \in \mathbb{C}^{n\times n}, \quad \max_{U \in \mathcal{U}(\underset{\sim}{k})} \rho(UM) = \mu_{\underset{\sim}{k}}(M)$.

Proof: We first show:

$$\max_{U \in \mathcal{U}(\underset{\sim}{k})} \rho(UM) \leq \mu_{\underset{\sim}{k}}(M) \tag{A.1.1}$$

If $\underset{\sim}{k} = (\underbrace{1,\cdots,1}_{n \text{ times}})$ then $\mathbb{B}_{\delta}(\underset{\sim}{k}) = \{\lambda I : \lambda \in C, |\lambda| \leq \delta\}$ and then $\mu_{\underset{\sim}{k}}(M)$ $= \rho(M) (:= \max_{\lambda \in \sigma(M)} |\lambda|)$, and $\det(I - \lambda M) \neq 0, \forall |\lambda| \leq \delta$ iff $\delta\rho(M) < 1$. But δ is the solution to a constrained minimization problem and $\mu_{\underset{\sim}{k}}(M) = \frac{1}{\delta}$ and since $\mu_{\underset{\sim}{k}}(M) = \rho(M)$ for $\underset{\sim}{k} = (1,\cdots,1)$ it follows that, for general $\underset{\sim}{k}$,

$$\rho(M) \leq \mu_{\underset{\sim}{k}}(M) \;. \tag{A.1.2}$$ [3]

$\Delta \in \mathbb{B}_{\delta}(\underset{\sim}{k})$, $U \in \mathcal{U}(\underset{\sim}{k})$ hence $U\Delta, \Delta U \in \mathbb{B}_{\delta}(\underset{\sim}{k})$. Also, $\det(I-UM\Delta)$ $= \det(UU^*-UM\Delta) - \det(U(I-M\Delta U)U^*) = \det(I-M\Delta U)$

Hence,

$$\mu_{\underset{\sim}{k}}(MU) = \mu_{\underset{\sim}{k}}(UM) = \mu_{\underset{\sim}{k}}(M) \tag{A.1.3}$$

From (A.1.2) and (A.1.3) we have $\max_{U \in \mathcal{U}(k)} \rho(UM) \leq \mu(M)$ which is (A.1.1). We now prove the proposition.

If $\mu_{\underset{\sim}{k}}(M) = 0$ then the result follows immediately from (A.1.1).

Otherwise, let $\mu_{\underset{\sim}{k}}(M) = (1/\delta) > 0$. Then $\exists\, \Delta \in \mathbb{B}_{\delta}(\underset{\sim}{k})$ such that $\bar{\sigma}(\Delta) = \delta$ and $\det(I-\Delta M) = 0$. By lemma A.1 $\exists$ U, $V \in \mathcal{U}(\underset{\sim}{k})$ and $\exists\, \Sigma$, a diagonal nxn matrix with diagonal entries in $\mathbb{R}_+$ such that $\Delta = U\Sigma V^*$ then $\det(I-\Delta M) = 0$ iff $\det(I-U\Sigma V^*M) = 0$ which is a polynomial in the diagonal elements of Σ and by definition of $\mu_{\underset{\sim}{k}}$, Σ is the minimum norm solution to this polynomial equation. By the corollary to lemma A.2, Σ may be replaced by δI with $\delta = \|\Sigma\|_{\infty}$. Thus,

[3] *Equation (A.1.2) was used in the proof of proposition 5.2.*

$$\det(I-U\Sigma V^*M) = 0 \;\Rightarrow\; \det(I-\delta UV^*M) = 0$$

$$\Rightarrow\; \rho(UV^*M) \geq \frac{1}{\delta} = \mu_{\underset{\sim}{k}}(M)$$

But, by (A.1.1), we also have $\rho(UV^*M) \leq \mu_{\underset{\sim}{k}}(M)$, since $U, V^* \in \mathcal{U}(\underset{\sim}{k})$. Hence

$$\max_{U\in\mathcal{U}} \rho(UM) = \mu_{\underset{\sim}{k}}(M).$$

Appendix 2

Theorem A1: Consider an n-port $\mathcal{N}$ made up of linear time-invariant passive R, L, C elements, gyrators and voltage-controlled current-sources (VCCS's). Let the "gains" of the VCCS's have the following form: $g_m\left(\frac{1}{1+j\frac{\omega}{\omega_m}}\right)$ where $g_m \in \mathbb{R}$, $\omega_m \in \mathbb{R}_+$ ("large"). Let $\mathcal{N}$ be described by scattering matrix S(s) w.r.t. $r_i > 0$, $i \in \underline{n}$ and let S(s) be analytic in $\mathbb{C}_+$. Then $S(s) \in \hat{\mathcal{Q}}^{n\times n}$.

Comments: a) It is sufficient to consider only VCCS's since it can be shown that one can represent any other kind of controlled source by pre- and/or port-cascading a VCCS with gyrator(s). It should also be noted that ideal transformers can be represented by controlled sources and hence by a suitable combination of VCCS's and gyrators.

b) It is the nature of the elements (passive R, L, C, gyrators, VCCS's with gains $g_m\left(\frac{1}{1+j\frac{\omega}{\omega_m}}\right)$) that ensures that $S(s) \in \hat{\mathcal{Q}}^{n\times n}$, so that Theorem A1 is not a mathematical fact. In fact if the gain of the VCCS's were a constant, Theorem A1 would be false and, indeed, some entries of S(s) could be made to behave like polynomials in s.

Proof: Since $\mathcal{N}$ is made up of lumped linear time-invariant elements, its scattering matrix $S(s) \in \mathbb{R}(s)^{n\times n}$. Further, since S(s) is analytic in C_+, it only remains to show that S(s) is bounded at ∞ on the jω-axis to conclude that $S(s) \in \hat{\mathcal{Q}}^{n\times n}$.

Let $\mathcal{N}$ "contain" k VCCS's. The first step is to "extract" all the VCCS 2-ports: after extracting the "voltage-sensing" ports and the corresponding controlled current-sources, an (n+2k)-port, $\mathcal{N}_e$, is created. Since $\mathcal{N}_e$ "contains" only passive R, L, C elements and gyrators we know that [New. 1, p.98]

$$S(s) \in \mathbb{R}_p(s)^{n\times n} \text{ is analytic in } \mathbb{C}_+ \tag{A.2.1}$$

$$\forall\omega \in \mathbb{R}\,,\ I - S^*(j\omega)\,S(j\omega) \text{ is positive semi-definite} \tag{A.2.2}$$

Inspection of Fig. A.1 shows that we need to examine 3 kinds of transfer functions for boundedness

i) the transfer function from an "original" port such as ① to a voltage-sensing "created" open-circuit port such as ②;

ii) the transfer function from a (controlled) current-source terminated "created" port, such as ③, to an "original" port such as ⑤; and

iii) the transfer function from a CCS-terminated "created" port, such as ③, to a "created" o.c. port, such as ④.

Inspection of Figs. A.2., A.3, and A.4 gives the following expressions for the transfer functions in terms of scattering-parameters and impedance parameters

$$V_2 = \frac{2s_{21}}{1-s_{22}}\, I_{s_1} \tag{A.2.3}$$

$$E_2 = \frac{s_{21}}{1-s_{11}}\, I_{s_1} \tag{A.2.4}$$

$$V_2 = z_{21}\, I_{s_1} \tag{A.2.5}$$

It now remains to show that the transfer functions $\frac{s_{21}}{1-s_{22}}$ and $\frac{s_{21}}{1-s_{11}}$ of (A.2.3), (A.2.4) are bounded; and that under the assumption on the behavior of the "gains" of the controlled-current-sources the RHS of (A.2.5) is bounded at ∞.

We first prove ii)

Claim: $\frac{s_{21}}{1-s_{11}}$ is bounded at ∞ on the $j\omega$-axis.

Proof: Equation (A.2.2) implies that the s_{ij}'s are bounded and specifically that:

$$1 - (|s_{11}|^2) + |s_{21}|^2) \geq 0 \tag{A.2.6}$$

$\frac{s_{21}}{1-s_{11}}$ can only be unbounded if $s_{11} \to 1$ faster than $s_{21} \to 0$ (which is required by (A.2.6) if $s_{11} \to 1$). We must therefore examine rates of convergence: s_{11} is rational, hence a Taylor expansion (evaluated at $s = \infty$) gives

$$s_{11} = (1 + \frac{\beta}{\omega^2} + \cdots + \cdots +) + j(\frac{\alpha}{\omega} + \frac{\gamma}{\omega^3} + \cdots) \tag{A.2.7}$$

From (A.2.7),

$$|s_{11}|^2 = 1 + \frac{\alpha^2+2\beta}{\omega^2} + 0(\frac{1}{\omega^4}) \tag{A.2.8}$$

Using (A.2.8) in (A.2.6) gives, $1 - [1 + \frac{\alpha^2+2\beta}{\omega^2} + 0(\frac{1}{\omega^4})] \geq |s_{21}|^2$

$$\Rightarrow |s_{21}|^2 \leq \frac{\delta}{\omega^2} \quad ,$$

$$\Rightarrow \quad s_{21} = 0(\frac{1}{j\omega})$$

From (A.2.7) $1 - s_{11} = 0(\frac{1}{j\omega})$

Thus, $\frac{s_{21}}{1-s_{11}}$ tends to a finite constant as $\omega \to \infty$.

We now prove i)

<u>Claim</u>: $\frac{s_{21}}{1-s_{22}}$ is bounded at ∞ on the $j\omega$-axis.

<u>Proof</u>: (by contradiction) For $\frac{s_{21}}{1-s_{22}}$ to be unbounded at ∞ we must have <u>s_{21} bounded away from 0 at ∞</u> while $s_{22} \to 1$ (since by passivity, all the s_{ij}'s of $\mathcal{N}_o$ are bounded). Thus, in terms of Taylor expansions,

$$s_{21} = \tau_o + \frac{\tau_1}{j\omega} + \cdots + \cdots; \tag{A.2.9}$$

$$s_{22} = 1 + \frac{\alpha}{j\omega} + \cdots \tag{A.2.10}$$

where we assume $\tau_o \neq 0$.

Let $I - S_o^*(j\omega)\, S_o(j\omega) =: \begin{bmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{bmatrix}$

Then,

$$a_{11} := 1 - (|s_{11}|^2 + |s_{21}|^2) \qquad \text{(A.2.11a)}$$

$$a_{12} := s_{11}^* s_{12} + s_{21}^* s_{22} \qquad \text{(A.2.11b)}$$

$$a_{22} := 1 - (|s_{12}|^2 + |s_{22}|^2) \qquad \text{(A.2.11c)}$$

and passivity of $\mathcal{N}_o$ (see Fig. A.2) implies that $I - S_o^*(j\omega)\, S_o(j\omega) \geq 0$ equivalently

$$a_{11} \geq 0 \qquad \text{(A.2.12a)}$$

$$a_{22} \geq 0 \qquad \text{(A.2.12b)}$$

$$a_{11}a_{22} - |a_{12}|^2 \geq 0 \qquad \text{(A.2.12c)}$$

From (A.2.10), (A.2.11.c) and (A.2.12b)

$$s_{12} = 0(\frac{1}{j\omega}) \qquad \text{(A.2.13)}$$

and thus from (A.2.10) and (A.2.13)

$$a_{22} = 1 - (|s_{12}|^2 + |s_{22}|^2) = 0(\frac{1}{\omega^2}) \qquad \text{(A.2.14)}$$

From (A.2.9) and (A.2.11a)

$$a_{11} = 0(1) \qquad \text{(A.2.15)}$$

Now, from (A.2.15), (A.2.14) and (A.2.12c) we must have

$$a_{12}^* = s_{12}^*\, s_{11} + s_{22}^*\, s_{21} = 0(\frac{1}{j\omega}) \qquad \text{(A.2.16)}$$

From (A.2.9), (A.2.10), $s_{22}^* s_{21} = 0(1)$ and hence

$$a_{12}^* = s_{12}^* s_{11} + s_{22}^* s_{21} = 0(1) \qquad \text{(A.2.17)}$$

(A.2.16) and (A.2.17) give the required contradiction.

<u>Claim iii)</u>: z_{21} is, at worst, $O(j\omega)$.

<u>Proof</u>: $\mathcal{N}_i$ (see Fig. A.4) has a positive-real Z matrix and by the direct PR test [New. 1, p. 117]

$$Z(j\omega) + Z^*(j\omega) \geq 0 . \tag{A.2.18}$$

Let $Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$

The positive-real property implies that z_{11}, z_{22} are $O(j\omega)$ at most; using this information in (A.2.18) gives that z_{21} is, at worst, $O(j\omega)$.

Thus, in eqn. (A.2.5), (since the extracted controlled current-sources I_{s_1} are assumed to be $O(\frac{1}{j\omega})$) the RHS $z_{21}\ I_{s_1}$ is bounded at ∞.

Since all transfer functions can be obtained by a suitable combination of type i), ii), iii) functions we have shown that S(s) is bounded at ∞.

We now generalize theorem A1 to include a class of distributed circuits.

<u>P.A.2</u>: Consider an n-port $\mathcal{N}$ made up of <u>passive</u> R, L, C elements, gyrators, VCCS's whose "gains" are of the form $g_m\left(\frac{1}{1 + j(\frac{\omega}{\omega_m})}\right)$ and a finite number k of uniform, lossless transmission lines. Let the k transmission lines be attached to m "internal" ports (m is less than 2k by the number of open- and short-circuited "stubs").

Extracting the m ports to which the k transmission lines are connected, an (n+m)-port, $\mathcal{N}_\ell$, (with corresponding scattering matrix S_ℓ) is created. Note that $\mathcal{N}_\ell$, which is lumped, satisfies the hypotheses of Theorem A.1. Hence $S_\ell \in \hat{\mathcal{A}}^{(n+m)x(n+m)}$. The distributed m-port of transmission lines, $\mathcal{N}_d$, has a scattering matrix $S_d \in \hat{\mathcal{A}}^{mxm}$ (this is easily seen for characteristic impedance terminations on all ports; entries are then either zero or of the form $e^{-\tau_i s}$, $i \in \underline{k}$. For other choices of normalizing resistances, the scattering matrix S_d' is similar to S_d, see [New. 1, p. 74]).

We now partition the (n + m) ports of $\mathcal{N}_\ell$ according to "original" or external variables, subscripted with an e; and "created" or internal variables, subscripted with an i. Thus,

$$b_e = S_{ee}a_e + S_{ei}a_i \tag{A.2.19}$$

$$b_i = S_{ie}a_e + S_{ii}a_i \tag{A.2.20}$$

Note that: i) Since the partitioned matrix $\begin{matrix} & n & m \\ n & S_{ee} & S_{ei} \\ m & S_{ie} & S_{ii} \end{matrix}$ is a permutation of $S_\ell \in \hat{\mathcal{A}}^{(n+m)\times(n+m)}$ we have $S_{ee} \in \hat{\mathcal{A}}^{n\times n}$, $S_{ei} \in \hat{\mathcal{A}}^{m\times n}$, $S_{ie} \in \hat{\mathcal{A}}^{m\times n}$, $S_{ii} \in \hat{\mathcal{A}}^{m\times m}$

ii) To arrive at S(s) we seek the relation between b_e and a_e when $\mathcal{N}_\ell$ is loaded at its m "internal" ports by the m-port $\mathcal{N}_d$. Denoting port-variables of $\mathcal{N}_d$ with a ~, we have,

$$\tilde{b}_i = S_d\tilde{a}_i \tag{A.2.21}$$

The interconnection equations are:

$$b_i = \tilde{a}_i \tag{A.2.22a}$$

$$a_i = \tilde{b}_i \tag{A.2.22b}$$

We may now state theorem A.2.

Theorem A2: Consider an LT-I n-port $\mathcal{N}$ made up of passive R, L, C elements, gyrators, VCCS's whose "gains" are of the form $g_m(\frac{1}{1 + j\frac{\omega}{\omega_m}})$ and a finite number, k, or uniform lossless transmission lines attached to m "internal" ports. Assume that,

$$\forall\, \omega \in \mathbb{R}\,,\ \|S_{ii}(j\omega)\|_2 < 1 \qquad \text{(see eqn, A.2.20 above for defn. of } S_{ii})$$

Then the scattering matrix of $\mathcal{N}$, $S(s) \in \hat{\mathcal{A}}^{n\times n}$.

Proof: Solving equations (A.2.19), (A.2.20), (A.2.21) and (A.2.22) for b_e in terms of a_e gives

$$S = S_{ee} + s_{ei}(I - S_dS_{ii})^{-1}\, S_dS_{ie}$$

Thus if $(I - S_dS_{ii})^{-1} \in \hat{\mathcal{A}}^{m\times m}$ then closure properties of the algebra $\hat{\mathcal{A}}^{n\times n}$ imply that $S \in \hat{\mathcal{A}}^{n\times n}$. $(I - S_dS_{ii})^{-1} \in \hat{\mathcal{A}}^{m\times m}$ iff $\inf_{\mathrm{Re}\, s \geq 0} |\det(I - S_dS_{ii})| > 0$. Since $\mathcal{N}_d$ is made up of lossless transission lines S_d is unitary on the $j\omega$-axis, i.e. $\|S_d(j\omega)\|_2 = 1$. Now,

$$\begin{aligned}\sup_{\omega\in\mathbb{R}} \bar{\sigma}\,[S_d(j\omega)\ S_{ii}(j\omega)] &\leq \sup_{\omega\in\mathbb{R}} [\|S_d(j\omega)\|_2 \cdot \|S_{ii}(j\omega)\|_2] \\ &\leq \sup_{\omega\in\mathbb{R}} \|S_d(j\omega)\|_2 \sup_{\omega\in\mathbb{R}} \|S_{ii}(j\omega)\|_2 \\ &< 1\end{aligned}$$

By subharminicity of $s \mapsto \bar{\sigma}[S_d(s)S_{ii}(s)]$ we may write $\sup_{\mathrm{Re}\, s>0} \bar{\sigma}[S_d(s)S_{ii}(s)] < 1$ which implies that $\inf_{s\in\mathbb{C}_t} |\det(I - S_dS_{ii})| > 0$. (For details see pf. of thm. 3.1)

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Figure Captions

Fig. 1. The circuit $\mathcal{N}_{ti}$ consists of the given n-port $\mathcal{N}$ terminated by the load n-port $\mathcal{N}_{\ell}$ and driven by the current-source i.

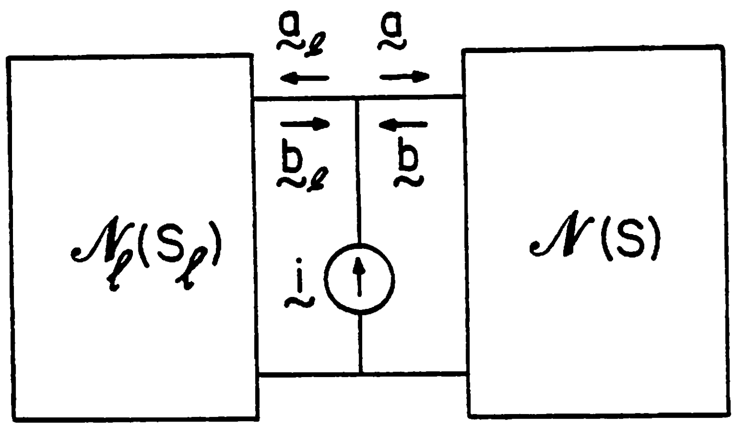
Fig. 2. The circuit $\mathcal{N}_{te}$ consists of the given n-port $\mathcal{N}$ terminated by the load n-port $\mathcal{N}_{\ell}$ and driven by the voltage-source e.

Fig. A.1. The figure shows the n-port $\mathcal{N}$ after the extraction of the VCCS's.

Fig. A.2. The type (i) transfer function relates V_2 to I_{S1}.

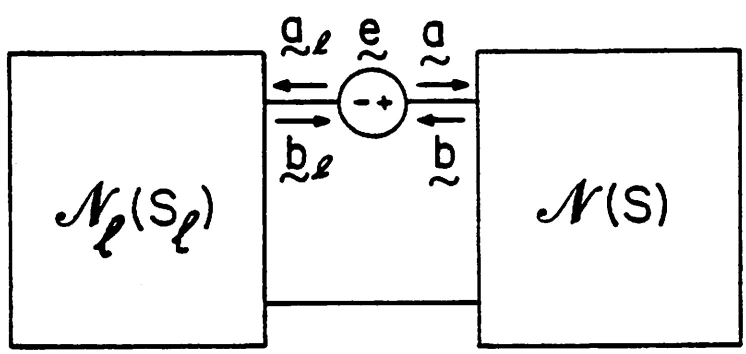
Fig. A.3. The type (ii) transfer function relates E_2 to I_{S1}.

Fig. A.4. The type (iii) transfer function relates V_2 to I_{S1}.



$\underset{\sim}{a}_\ell$
$\underset{\sim}{a}$
$\underset{\sim}{b}_\ell$
$\underset{\sim}{b}$
$\mathcal{N}_\ell(S_\ell)$
$\underset{\sim}{i}$
$\mathcal{N}(S)$

Fig. 1



$\underset{\sim}{a}_\ell$
$\underset{\sim}{e}$
$\underset{\sim}{a}$
− +
$\underset{\sim}{b}_\ell$
$\underset{\sim}{b}$
$\mathcal{N}_\ell(S_\ell)$
$\mathcal{N}(S)$

Fig. 2

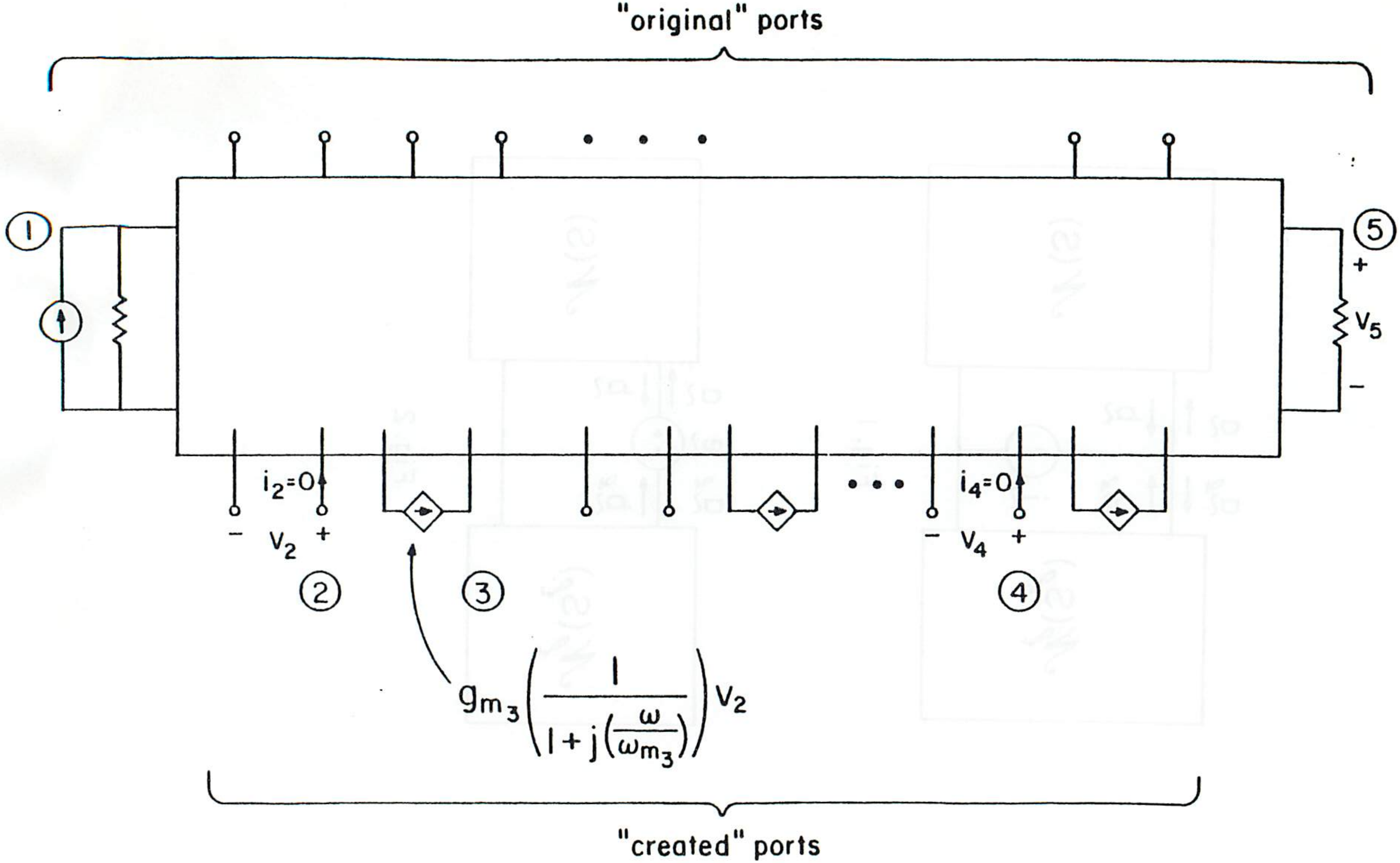

"original" ports
1
5
+
V5
−
i2=0
−
V2
+
2
3
i4=0
−
V4
+
4
$g_{m_3}\left(\frac{1}{1+j\left(\frac{\omega}{\omega_{m_3}}\right)}\right)V_2$
"created" ports

Fig. A.I.

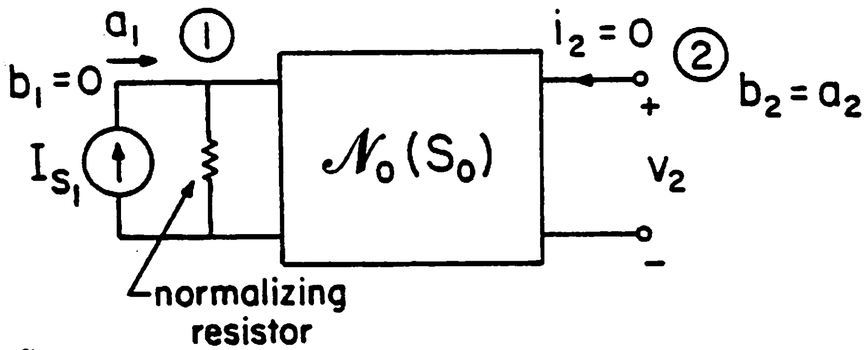


Fig. A.2.

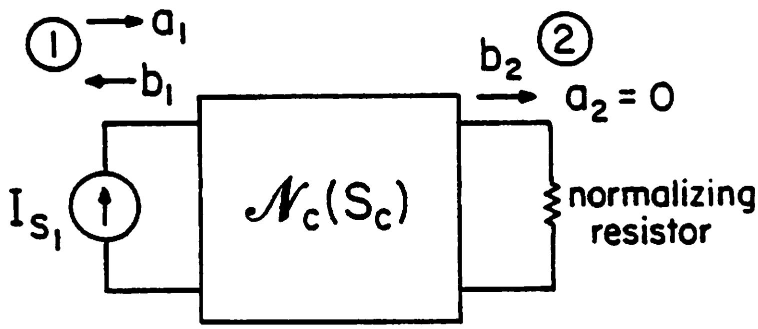


Fig. A.3.

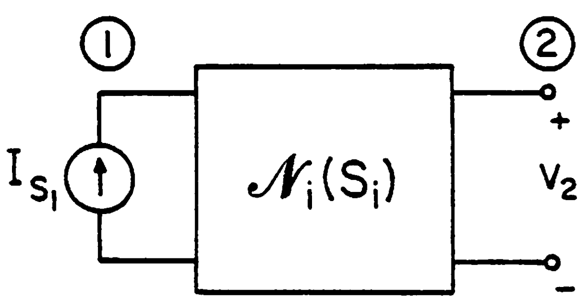


Fig. A.4.