ON THE STRUCTURE OF F-INDISTINGUISHABILITY OPERATORS

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ABSTRACT

Some properties concerning the structure of the F-indistinguishability operators are analyzed. It is shown that any of such operators is generated by a family of fuzzy subsets. This result, since it gives the way to construct F-indistinguishabilities, facilitates new applications of fuzzy relations. The links between F-indistinguishability operators and a kind of generalized metrics in the unit interval -which are also explored- are used to define the canonical generators of a F-indistinguishability operator that is, the fuzzy partition associated with the operator.

1. INTRODUCTION

Fuzzy relations and their applications have been largely studied in the recent past. Since (Zadeh,1971) a number of papers dealing with various aspects related with these relations have appeared. Fuzzy partition, distance, Cluster Analysis, Clustering, Pattern Recognition, Preference, etc., are common key words in these papers, and indicate the main topics with which fuzzy relations are concerned. In fact, fuzzy relations provide a unifying point of view for many concepts and techniques for categorization used in various fields (see, for instance, Trillas, 1982 and Trillas and Valverde,1983a). In addition, as it is shown in (Bezdek and Harris,1978; Ovchinnikov,1981; Ruspini, 1982 and others), new concepts and methods arise from the theory of fuzzy relations.

In this paper, it is shown that any F-indistinguishability operator on a set X , i.e. any reflexive, symmetric and F-transitive fuzzy binary relation, is generated by a family of fuzzy subsets of X. This result, which includes the representation theorem for probabilistic relations given in (Ovchinnikov,1982), allows the construction of F-indistinguishabilities in a more efficient way than the transitive closure method (Tamura et al. 1971; Kaufmann,1975) and the graph theoretical methods (see Dunn,1974) and, consequently, facilitates new applications of these relations.

The links between F-indistinguishability operators and a type of generalyzed pseudo-metrics (Trillas and Alsina, 1978; Schweizer and Sklar,1983) are also analyzed. These links are used to extend the definition and results of (Ruspini,1982) concerning fuzzy cluster coverages, i.e. the

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counterpart -for F-indistinguishability operators- of classical partitions.

For the sake of completeness, there is a preliminary section concerned with the definition and properties of the quasi-inverses of continuous t-norms and t-conorms, which are used throughout this paper.

2. PRELIMINARIES

The standard notations and conventions related with fuzzy binary relations, t-norms and t-conorms will be used in this paper. However, it is convenient to recall the following facts:

2.1.- For any continuous t-norm F, $ qF $ will denote the quasi-inverse of F, i.e.

It can be easily shown that the quasi-inverse of a continuous t-norm satisfies, among others, the following properties:

(2.1.1) If F is an *Archimedean* t-norm generated by f, i.e. f is a continuous and strictly decreasing bijection from [0,1] into [0, $ + inf $] with f(1)=0 such that

then

$ pf $ being the pseudo-inverse of f.

(2.1.2) $ qF $ is a continuous function if, and only if, F is *nilpotent,* i.e. F is Archimedean and such that $ f(0)^<^ + inf $ for any additive generator f of F.

(2.1.3) $ qF $ is non-increasing with respect to its first argument and non-decreasing with respect to the second argument.

(2.1.4) It is $ qF (x|y)^=^1 $ if, and only if, $ x^<=^y $.

(2.1.5) $ z^<=^ qF (x|y) $ if, and only if, $ F(z,x)^<=^y $.

(2.1.6) $ F(qF (x|y),x)^=^y $ if, and only if, $ x^>=^y $.

(2.1.7) $ qF (Max(x,y)|Min(x,y))^=^Min (qF (x|y), qF (y|x)) $.

(2.1.8) $ F(qF (x|y), qF (y|z))^<=^ qF (x|z) $.

(2.1.9) If $ F sub 1 ^<=^ F sub 2 $ then $ F sub 1 sup "^" ^>=^ F sub 2 sup "^" $

The following are examples of t-norms, with their quasi-inverses, used in this paper:

(E.2.1) If $ F(x,y)^=^Min(x,y) $, then

(E.2.2) If $ F(x,y)^=^xy $ then $ qF (x|y)^=^Min (1, y over x) $.

(E.2.3) If $ F(x,y)^=^Max(x+ y-1,0) $, then $ qF (x|y)^=^Min(1-x+ y,1) $.

(E.2.4) If $ F sub a (x,y)^=^ pf (f(x)+ f(y)-f(Max(x,y,a))) $, then

where f is any continuous and strictly decreasing function from [0,1] into [0, $ + inf $] and $ a epsilon [0,1] $. With respect to the family $ (F sub a) sub {a epsilon [0,1]} $ it should be emphasized that it is a family of continuous t-norms such that $ F sub a ^>=^ F sub b $ if $ a^<=^b $, and $ F sub 0 (x,y)^=^Min(x,y) $, $ F sub 1 (x,y)^=^ pf (f(x)+ f(y)) $, i.e. that family varies continuously with the parameter a, from the Archimedean t-norm $ F sub 1 $ to the t-norm Min, which is, as it is well known, the greatest t-norm.

2.2.- Analogously, if G is a continuous t-conorm, then qG will denote the quasi-inverse of G, i.e.

which satisfies, with the corresponding modifications, the properties (2.1.1) to (2.1.9), that is:

(2.2.1) If G is an Archimedean t-conorm generated by g, then

(2.2.2) qG is a continuous function if, and only if, G is *nilpotent*, i.e. G is Archimedean and such that $g(1) < +\infty$ for any additive generator g of G.

(2.2.3) If $x \leq x'$ then $qG(x|y) \geq qG(x'|y)$ and $qG(y|x) \leq qG(y|x')$.

(2.2.4) It is $qG(x|y) = 0$ if, and only if, $x \leq y$.

(2.2.5) It is $z \geq qG(x|y)$ if, and only if, $G(x,z) \geq y$.

(2.2.6) $G(qG(x|y),x) = y$ if, and only if, $x \geq y$.

(2.2.7) $qG(\mathrm{Min}(x,y)|\mathrm{Max}(x,y)) = \mathrm{Max}(qG(x|y), qG(y|x))$.

(2.2.8) $G(qG(x|y), qG(y|z)) \geq qG(x|z)$.

(2.2.9) If $G_1 \leq G_2$, then $G_1^{\wedge} \leq G_2^{\wedge}$.

The following are examples of t-conorms, with their quasi-inverses, used in this paper:

(E.2.5) If $G(x,y) = \mathrm{Max}(x,y)$, then

(E.2.6) If $G(x,y) = x+y-xy$, then $qG(x|y) = \mathrm{Max}(0,(y-x)/(1-x))$.

(E.2.7) If $G(x,y) = \mathrm{Min}(x+y,1)$, then $qG(x|y) = \mathrm{Max}(y-x,0)$.

(E.2.8) If $G_a(x,y) = pg(g(x)+g(y)-g(\mathrm{Min}(x,y,a)))$, then

2.3.- If G is a continuous t-conorm, then

is a continuous t-norm for any continuous and order reversing bijection φ from [0,1] into itself, and vice versa. When φ is an involution (i.e. a strong negation) then F_φ (resp. $G_\varphi(x,y) = \varphi^{-1}(G(\varphi(x),\varphi(y)))$) is termed the φ-dual t-norm of G (resp. the φ-dual t-conorm of F) and (F, G_φ, φ) is called a DeMorgan Triple. $F_\varphi^{\wedge}$ and G also satisfy

(2.3.1) $F_\varphi^{\wedge}(x|y) = \varphi^{-1}(qG(\varphi(x)|\varphi(y)))$, and

(2.3.2) If $G_1 \leq G_2$, then $F_{1\varphi} \geq F_{2\varphi}$ and vice versa.

Finally, let it be noticed that in (Schweizer and Sklar,1983) a detailed study of t-norms and t-conorms can be found. In addition, some specific properties of t-norms and t-conorms from the standpoint of their use in Fuzzy Set Theory can be found in (Klement, 1981; Alsina et al. 1983; Trillas and Valverde,1983b) and related papers.

3. ON F-INDISTINGUISHABILITY OPERATORS

Through out this section the basic concepts and properties related to F-indistinguishability operators are given, paying special attention to the most relevant facts for the subsequent development. However, the readers are referred to the basic papers on that topic already mentioned in the introduction.

In what follows X stands for a non-empty set and F for a continuous t-norm. F-indistinguishability operators are defined in the following way:

Definition 3.1. A map R from XxX into [0,1] is termed **F-indistinguishability operator** if the following properties hold for any x,y and z in X:

(3.1.1) $ R(x,x)^=^1 $, (Reflexivity)

(3.1.2) $ R(x,y)^=^R(y,x) $, (Symmetry)

(3.1.3) $ F(R(x,y),R(y,z))^<=^R(x,z) $ (F-transitivity).

A **F-preorder** is a map R satisfying both (3.1.1) and (3.1.3).

In other words, an F-indistinguishability operator is simply a reflexive and symmetric fuzzy relation which satisfies some kind of weak transitive property or, if it is preferred, a F-indistinguishability operator is a symmetric F-preorder.

If, as usual, R(x,y) is assumed to be the strength of the relationship between the elements x and y, then (3.1.3) gives a threeshold which should be attained by the strength of the relationship between x and z given both the strengths of the relationship between x and y and y and z. Since $ F(x,y)^<=^Min(x,y) $, this threeshold is less than both R(x,y) and R(y,z).

Later we will come back to the meaning of that property, but now let be noticed that the similarity relations (Zadeh,1971) are F-indistinguishability operators with $ F(x,y)^=^Min(x,y) $; the same applies, for instance, to the probabilistic relations of indistinguishability (Menger,1951) with $ F(x,y)^=^xy $ and likeness relations (Bezdek and Harris,1978; Ruspini,1982) with $ F(x,y)^=^Max(x+y-1,0) $. On the other hand, any (classical) equivalence relation is a F-indistinguishability operator with respect to any t-norm F so, in that sense, the concept of F-indistinguishability operator is a generalization of the concept of equivalence relation. This fact may also be justified by the following interpretation of the meaning of the F-transitive property: assume that R(x,y) is the truth-value, $ v(p(x,y)) $ of the proposition p(x,y): "x and y are similar". If we suppose that the truth-values of the compound propositions "p(x,y) and p(x',y')",'p(x,y) or p(x',y')", "not p(x,y)" and "If p(x,y) then p(x',y')" are given, respectively, by

v(p(x,y)$ inter $p(x',y'))=F(R(x,y),R(x',y')),

v(p(x,y)$ union $p(x',y'))=G(R(x,y),R(x',y')),

v(~p(x,y))=n(v(p(x,y))), and

v(p(x,y)$ -> $p(x',y'))= $ I sub F (R(x,y),R(x',y'))^=^ qF (R(x,y)|R(x',y')),$

where (F,G,n) is a DeMorgan Triple and $ I sub F $ is the R-implication associated with F (see Trillas and Valverde,1983b), then the F-transitive property is equivalent to the assertion of the statement "*If p(x,y) and p(y,z) then p(x,z)*" i.e.

is equivalent to the inequality

>From this standpoint, the F-indistinguishability operators may be viewed as the equivalences associated to that kind of multivalued systems, i.e. in that systems they play the same role played by the equivalence relations in classical logic.

F-transitivity may also be formulated in terms of the Max-F composition noted here as $ o sub F $, that is, a relation R is F-transitive if, and only if,

When R is a reflexive relation, then F-trasitivity is equivalent to the equality

i.e. if R is a reflexive relation, then R is F-transitive if, and only if,

It is also well known (Zadeh,1971; Kaufmann,1975) that similarity relations can be constructed by means of the transitive closure of reflexive and symmetric relations: the same method may be used to construct F-indistinguishability operators. To show this, first we have the

Proposition 3.1. Let $ (R sub i) sub {ieI} $ be a family of F-indistinguishability operators over the same set X. Then

is a F-indistinguishability operator.

Proof. Obviously R satisfies both reflexivity and symmetry. The F-transitivity of R can be stated as follows:

and that inequality holds for any $ ieI $, hence

Since the relation defined by T(x,x)=1, for all x in X, is a F-indistinguishability operator for any t-norm F, it turns out that, given a reflexive and symmetric fuzzy relation R, the set $ B sub R $ of F-indistinguishability operators R' such that $ R prime ^<=^R $, is non-empty; consequently

is a F-indistinguishability operator containing R. $ R sup F $ is the so-called *F-transitive closure* of R. Using similar arguments as for similarity relations (see Kaufmann, 1975) it can be shown that

where $ R sup n ^=^R^ o sub F ~ sup n) ~ o sub F R $. When X is finite of cardinality p, then

result which allows the effective construction of the F-transitive closure of finite reflexive and symmetric fuzzy relations.

So far, in most algorithms which use fuzzy relations in Pattern Classification and Cluster Analysis, the indistinguishability operator is obtained computing the F-transitive closure of a given reflexive and symmetric fuzzy relation. That process requires a large number of operations and, in each step of the calculation, storage for the upper triangular part of three matrices is required. In the next section it will be shown that, for any given reflexive and symmetric fuzzy relation R and for any continuous t-norm F, a F-indistinguuishability operator $ R sub F $ exists such that $ R sub F ^<=^R $ and $ R sub F $ is the greatest F-indistinguishability operator which satisfies both for all x,y and z in X, i.e. R_F may be viewed as a sort of "lower" F-transitive closure of R. A method will be also given which allows R_F to be computed in just one step, which means that less operations will be used and only storage for the upper triangular part of two matrices is required.

F-indistinguishability operators may also be constructed from F-preorders in the following way:

Proposition 3.2. Let U be a F-preorder, then

is a F-indistinguishability operator for any t-norm F' such that $F \le F'$.

Thus, since $F(x,y) \le \mathrm{Min}(x,y)$ for any t-norm F, it turns out that, given a F-preorder U, the fuzzy relation

is the greatest F-indistinguishability operator contained in that F-preorder.

Moreover, a straigthforward calculation shows that a F-transitive fuzzy relation is F'-transitive for any other t-norm F' such that $F' \le F$ as well. So, any similarity relation is F-transitive for any t-norm F, as well as any F_a-indistinguishability operator, F_a being the t-norm of the example (E.2.4), is a F_1-indistinguishability operator.

It should be noticed that, when X has only two elements, the set of F-indistinguishability operators coincides with the set of reflexive and symmetric fuzzy relations, i.e. such relations are F-transitive for any t-norm F.

4. THE REPRESENTATION THEOREM

In this section it will be proven that any F-indistinguishability operator on a set X is generated by a family of fuzzy subsets of X. That result, which also provides an effective method to construct F-indistinguishability operators, works as follows:

Theorem 4.1. Let U be a map from XxX into [0,1] and let F be a continuous t-norm. U is a F-preorder if, and only if, there exists a family $\{h_j\}_{j \in J}$ of fuzzy subsets of X such that

Proof. Let U be a F-preorder. The F-transitivity of U entails

for any x,y and z in X. Thus

since U is reflexive, it follows that $U(x,y) = \overrightarrow{F}(U(y,y)|U(x,y))$, and therefore

where $h_z(x) = U(x,z)$ that is, the condition is necessary.

Now, let $\{h_j\}_{j \in J}$ be an arbitrary family of functions from X into [0,1], since $q_F(x|x) = 1$, the fuzzy relation

is reflexive. In order to prove that U is F-transitive, for a given x,y and z in X and for each $t \in J$, let be

>From the definition of U it follows that

thus

for any $t \in J$. Consequently, it is

To complete the proof, it suffices to show that

and this can be done in the following way: since $F(aft, h_t(z)) \leq h_t(x)$ and F is associative it is

consequently, we have

and therefore

Taking into account the proposition 3.1 and the property (2.1.7) fulfilled by any continuous t-norm F, the following theorem can be easily shown

Theorem 4.2.(Representation theorem). Let R be a map from XxX into [0,1] and let F be a continuous t-norm. Then R is an F-indistinguishability operator if, and only if, there exist a family $\{h_j\}_{j \in J}$ of fuzzy subsets of X such that

(RT)

for all x,y in X.

It is worth noting that one of the main features of this theorem is that it allows to compute the F-indistinguishability in just one step. Consequently it requires both less storage and less calculations than the traditional methods. In addition, since no constraints to the fuzzy subsets generating the indistinguishability, it turns out that in Fuzzy Cluster Analysis, for instance, this result is a very useful tool: assume that X is a set of elements which should be classified according to some prefixed criteria (i.e. prototypes); after the evaluation of the degree of similarity of each element $x \in X$ to the criteria j (i.e. the similarity between x and the prototype j) which is given by $h_j(x)$, what formula (RT) provides is a way to gather all these evaluations to get a F-indistinguishability operator and therefore, as will be shown in the last section, a structural description of the data sample.

EXAMPLES

(E.4.1) If $F(x,y) = \mathrm{Min}(x,y)$, then the F-indistinguishability operator (the similarity relation) associated to $\{ h_j \}_{j \in J}$, $h_j \in [0,1]^X$, is given by

where $J_{xy} = \{ j \in J \mid h_j(x) \neq h_j(y) \}$.

(E.4.2) If $F(x,y) = xy$, then

(E.4.3) If $F(x,y) = \mathrm{Max}(x+y-1,0)$, then

(E.4.4) If $F_a(x,y) = pf\,(f(x)+f(y)-f(\mathrm{Max}(x,y,a)))$, then

Finally, let it be noticed that, when the representation theorem is applied to construct the F-indistinguishability operator R_F generated by a reflexive and symmetric fuzzy relation R, i.e. when the functions h_j are the rows (or the columns) of R, then it is

that is, $R_F(x,y)$ is either $R(x,y)$ or the greatest number among those which satisfy both

for all z in X. The point is that, from the representation theorem follow both the existence of such indistinguishability operator and the method to compute it. Moreover, the use of the representation theorem no longer requires a complete fuzzy binary relation, neither reflexivity nor symmetry are required. The initial data may be just a few (even one!) arbitrary functions from the set X into [0,1].

5. F-INDISTINGUISHABILITIES AND G-METRICS

Now we turn our attention to one of the most importants features of the F-indistinguishability operators, which is their relationship with metrics. It is well known that if R is a likeness relation, then $d(x,y) = 1-R(x,y)$ is a normalized pseudo-metric on X; in fact that property is used as one of the major arguments to introduce likeness relations because the triangle inequality refines the ultrametric inequality given by the similarity relations.

First of all, let it be noticed that any pseudo-metric may be transformed into a F-indistinguishability operator that is, we have the following

Theorem 5.1. Let d be a pseudo-metric on X and let f be a continuous and strictly decreasing bijection from $[0, +\infty]$ into [0,1], then

is a F-indistinguishability, where $F(x,y) = f(f^{-1}(x)+f^{-1}(y))$. Conversely, if R is a indistinguishability operator with respect to an Archimedean t-norm F, then is a pseudo-metric, f being any additive generator of F.

Thus, if d is a pseudo-distance, then

is a probabilistic relation, as well as

is a likeness relation, and so forth. That is, when there is Archimedeanity, just by reversing the "scale" -through the order reversing bijection f- a pseudo-metric is obtained and vice versa. F-indistinguishability operators with respect non Archimedean t-norms do not give pseudo-distances, but G-pseudometrics:

Definition 5.1. Given a continuous t-conorm G, a map m from XxX into [0,1] will be termed **G-pseudometric** if the following properties hold for all x,y and z in X:

(5.1.1) $m(x,x) = 0$

(5.1.2) $m(x,y) = m(y,x)$

(5.1.3) $G(m(x,y),m(y,z)) >= m(y,z)$, (G-triangular inequality).

A **G-metric** is a G-pseudometric such that m(x,y)=0 if, and only if, x=y.

G-pseudometrics are simply an special type of Generalized Metrics introduced by Trillas (see Trillas and Alsina,1978; Schweizer and Sklar,1983). It can be easily shown that, for any continuous t-norm G and any continuous and order reversing bijection ϕ from [0,1] into itself, the map

is a G-metric in [0,1]. The following are examples of that kind of G-metrics:

EXAMPLES.

(E.5.1) Let be $G(x,y) = Min(x+y,1)$ and $\phi(x) = 1-x$, then

In general, If G is a nilpotent t-conorm generated by g and $\phi(x) = pg(g(1)-g(x))$, then

(E.5.2) If $G(x,y)) = xy$ and $\phi(x) = 1-x$, then

(E.5.3) If $G(x,y) = Max(x,y)$ and ϕ is any continuous and order-reversing bijection from [0,1] into itself, then

i.e. mgp is an ultrametric.

(E.5.4) If $ G sub a (x,y)^=^ pg (g(x)+ g(y)-g(Min(x,y,a))) $, then

When $ g(x)^=^x $ and $ phi (x)^=^1-x $, we have

G-pseudometrics and F-indistinguishability operators are dual concepts in the following sense:

Theorem 5.2. Let R be a F-indistinguishability operator and let $ phi $ be a continuous and order-reversing bijection from [0,1] into itself. Then

is a G-pseudometric and vice versa, where $ G(x,y)^=^ phi sup {-1} (F(phi (x), phi (y))) $.

It should be noticed that when $ phi $ is a strong negation, the G-pseudometric $ m(x,y)^=^ phi (R(x,y)) $ may be viewed as the degree of truth of the proposition ~p(x,y): "x and y are disimilar", in that case, the "disimilarity" between x and y is measured by a distance. From that point of view, the G-triangular inequality is equivalent to the assertion of the proposition "*If x and z are disimilar then x and y are disimilar or y and z are disimilar*".

Finally, it worth noting that, since F-indistinguishability operators and G-pseudometrics are dual concepts, it turns out that the theorem 4.2 also provides a representation theorem for any G-pseudometric that is, we have the following

Theorem 5.3. Let be G a continuous t-conorm and m a map from XxX into [0,1]. Then m is a G-pseudometric if, and only if, there exists a family of functions from X into [0,1] $ "{" hj "}" sub {jeJ} $, such that

Thus, for instance, any ultrametric m on a set X is determined by such a family of functions in the following way:

Analogously, any G-pseudometric m with respect to the t-conorm $ G sub a $ defined in the example (E.2.8) is determined by

6. FUZZY CLUSTER COVERAGES

Several authors have been dealt with the problem of how to extend to the fuzzy framework the concept of *partition*. Thus, several attempts to define the so-called *fuzzy partitions* have been made and the properties of the fuzzy relations associated with these fuzzy partitions (when that makes sense) have been characterized (see, for instance, Bezdek and Harris,1978; Ovchinnikov and Riera,1982). Here the converse way will be followed as, for instance, it is done in (Ovchinnikov,1981; Ruspini,1982), that is, a fuzzy partition will be defined as a family of fuzzy subsets satisfying some requirements which allow to characterize univocally a F-indistinguishability operator and we will be mainly concerned with the approach given in the last of the aforementioned papers, where the concept of *fuzzy r-cluster* is introduced in the following way: given a reflexive and symmetric fuzzy relation r on X, a fuzzy subset c of X with non-empty *core* (i.e. the set $\{x \in X \mid c(x) = 1\}$ is non-empty) is called *fuzzy r-cluster* if the following properties hold for any x,y and z in X:

(6.1) If c(x)=1, then c(y)=r(x,y),

(6.2) $|c(x)-c(y)| \leq 1-r(x,y)$.

It is pointed out that the family of "similarity classes" given by any *likeness relation* r (i.e. the fuzzy subsets of X defined by $c_x(y) = r(x,y)$) satisfies the two following properties:

(6.3) $\{c_x\}_{x \in X}$ is a **fuzzy coverage** of X (i.e. $\operatorname{Sup}_{x \in X} c_x(y) = 1 = \operatorname{Sup}_{y \in X} c_x(y)$),

(6.4) For any $x \in X$, c_x is a fuzzy r-cluster.

Conversely, if for a given reflexive and symmetric relation r, the family $\{c_x\}_{x \in X}$ defined as above is a fuzzy coverage of X satisfying the property (6.4), then r should be a likeness relation. Thus, fuzzy r-clusters are the counterpart, for likeness relations, of classical clusters (equivalence classes with respect to an equivalence relation).

In order to extend these results to any F-indistinguishability operator, let be noticed that Ruspini's definition involves a particular metric in the unit interval, the restriction to that interval of the Euclidean distance. In fact, this definition may be viewed as a generalization of the definition of classical clusters because, as it is easy to show, classical clusters can be characterized by means of

(6.5) If $\mu_A = 1$, then $\mu_A(y) = \mu_R(x,y)$,

(6.6) $d_0(\mu_A(x), \mu_A(y)) \leq 1 - \mu_R(x,y)$,

where μ_A is the characteristic function of the set A and d_0 is the discrete distance in the two-point set $\{0,1\}$.

Consequently, it may be expected that, by taking different metrics, different kinds of fuzzy r-clusters will be obtained and, therefore, different kinds of F-indistinguishability operators. In other words, it should be expected that a "metric" m in [0,1] may be associated with any t-norm F, in such a way that m allows , as the Euclidean distance does for likeness relations, the characterization of the similarity classes of any F-indistinguishability operator. It will be shown that the G-metrics mgp defined in section 5, play that role. To this end, the following definition is given:

Definition 6.1. Let r be a reflexive and symmetric fuzzy relation on X. For a given G-metric mgp, a fuzzy subset c of X with non-empty core will be termed **fuzzy r-cluster with respect to** mgp, if the following properties hold for any x and y in X:

(6.1.1) If c(x)=1 then c(y)=r(x,y),

(6.1.2) $\mathrm{mgp}(c(x),c(y)) \leq \varphi(r(x,y))$.

A **fuzzy r-cluster coverage** of X will be a fuzzy coverage of X such that each of its elements is a fuzzy r-cluster.

It is easy to show that if C is a fuzzy r-cluster coverage of X, then any two elements of C either coincide or have disjoint cores, i.e. the family of cores of the elements of C is a classical partition of X. Obviously, any one of such a partition is a fuzzy r-cluster coverage with respect to any G-metric that is, the above definition extends, in that sense, the concept of classical partition. Moreover, we have

Theorem 6.1. Let R be a F-indistinguishability operator on X and $C_R = \{g_x \in [0,1]^X \mid x \in X\}$, where $g_x(y) = R(x,y)$. For any continuous and order-reversing bijection φ from [0,1] into itself, C_R is a fuzzy R-cluster coverage with respect to $ mgp $, where $ G(x,y)^=^ phi sup {-1} (F(phi (x), phi (y))) $

Proof. >From the reflexivity of R it follows that $ C sub R $ is a fuzzy coverage of X. On the other hand, if $ g sub z (x)^=^1 $, since R is F-transitive, we have

that is, $ g sub z (y)^=^R(x,y) $.

Finally, again from the F-transitivity, both

and

follow, hence

Consequently, for any given continuous and order reversing bijection $ phi $ from the unit interval into itself, it is

that is,

where $ G(x,y)^=^ phi sup {-1} (F(phi (x), phi (y)) $.

Similar arguments can be used in order to prove the converse, that is the

Theorem 6.2. Let be R a reflexive and symmetric fuzzy relation on X and $ C sub R $ the family of fuzzy subsets of X defined as in the above theorem. If $ C sub R $ is a fuzzy R-cluster coverage of X with respect to some G-metric $ mgp $, then R is F-transitive, where $ F(x,y)^=^ phi sup {-1} (G(phi (x), phi (y))) $.

That is, with respect to the generalized metric spaces determined by the G-metrics in [0,1], F-indistinguishability operators are the same as classical equivalence relations with respect to the discrete distance. Thus, likeness relations are the indistinguishability operators associated with the restriction to [0,1] of the Euclidean distance, as well as probabilistic relations are the indistinguishability operators associated to the G-metric given in the example (E.5.2), and so forth.

Let it be noticed that the fuzzy cluster coverages may also be defined without any reference to the relation r in the following way

Definition 6.2. A fuzzy coverage $ "{" g sub x "}" x epsilon X $ of a given set X is called $ mgp **$-fuzzy cluster coverage** if

(6.7)

>From this definition it also follows that the fuzzy relation

is a F-indistinguishability operator, where $ F(x,y)^=^ phi sup {-1} (G(phi (x), phi (y))) $.

The requirement (6.7) is equivalent to the following equality

which also characterizes the fuzzy coverages associated with the F-indistinguishability operators (at is has been shown in (Ovchinnikov,1981)). Although definition (6.2) looks more elegant than definition (6.1), the latter has the advantadge of showing the metric interpretation of F-indistinguishability operators better.

Finally, it is interesting to note that the representation theorem may be formulated in the following way: with any family of fuzzy subsets of a given set X a fuzzy cluster coverage may be associated for any continuous t-norm F, or if it is preferred, for any G-metric in the unit interval $ mgp $. In general, the fuzzy cluster will be different for different t-norms but, it is easy to show that the classical partitions yielded by the cores of all fuzzy cluster coverages obtained from a given family $ "{" hj "}" sub {j epsilon J} $ will coincide. In fact, that common partition is the partition associated with the equivalence relation defined by $ x == y $ if, and only if, $ hj(x)^=^hj(y) $ for all $ jeJ $.

Consequently, the use of different t-norms makes differences only in the membership values to the fuzzy clusters which are different from 1 that is, since if $ F sub 1 ^<=^ F sub 2 $ then $ F sub 1 sup "^" ^>=^F sub 2 sup "^" $, the greater the t-norm the smaller those values. From that standpoint it makes sense to look for the "better" t-norm F (the better indistinguishability) associated with a given family of fuzzy subsets. That question is partially analyzed in (López de Mántaras and Valverde, 1983).

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