TOPOLOGICAL TRANSFORMATIONS

OF ELECTRICAL NETWORKS

by

H. Narayanan

(cover)

TOPOLOGICAL TRANSFORMATIONS

OF ELECTRICAL NETWORKS

by

H. Narayanan

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

ECV

Topological Transformations of Electrical Networks

by

H. Narayanan

ABSTRACT

This paper deals with effects of modifications of network structure that may be studied without reference to the type of devices present in the network. We introduce and make systematic use of the notion of a generalized minor of a vector space. This operation generalizes the usual short and open circuit operations for a graph. Using the generalized minor operation we show how to make the equations of a given network appear to be the "bordered version" of the equations of some other specified network. We also consider the decomposition of a network into several "multiports" and a "port connection diagram." We show that some of the properties of the original network are retained by a reduced network that can be defined on the port connection diagram. In each case we present efficient algorithms wherever appropriate. While the paper makes use of ideas from elementary matroid theory it is entirely self contained and requires no more than the knowledge of elementary linear algebra from the reader.

Address: Department of Electrical Engineering and Computer Sciences
            University of California
            Berkeley, CA 94720 USA

on leave from: Department of Electrical Engineering
            I. I. T. Bombay
            400076, INDIA

1. Introduction

In this paper we deal with structural modifications of electrical networks and their effects that can be studied ignoring the type of devices present in the network. Such modifications arise, quite naturally, in both practical and theoretical network analysis. In practical network analysis it may be desirable to split the given network into several subnetworks. In theoretical network analysis splitting the original network into multiports of various types is quite common: for instance, the given nonlinear network may thought of as a linear network terminated by nonlinear elements, or the given dynamic network may be thought of as a resistive network terminated by reactive elements. The network theorist who is faced with such problems handles them intuitively using his experience with the *type* of network being studied. A formal study would, at the least, clarify the fundamental ideas underlying such techniques.

Consider for instance the most natural topological transformations that every network theorist uses -- open and short circuits. It is easy to show that such operations are special cases of the following operation (see sec. 4): Let the original KCL, KVL equations of the network be

$$\mathbf{A}_S \mathbf{i}_S^T = 0$$

$$\mathbf{B}_S \mathbf{v}_S^T = 0$$

We derive complementary orthogonal equations on $\mathbf{i}_{S-P}$, $\mathbf{v}_{S-P}$ (associated with subset S-P of S) by imposing the conditions (on $\mathbf{i}_P$, $\mathbf{v}_P$)

$$\mathbf{A}_P \mathbf{i}_P^T = 0$$

$$\mathbf{B}_P \mathbf{v}_P^T = 0 , \quad \textbf{where} \quad \mathbf{A}_P , \mathbf{B}_P$$

generate complementary orthogonal spaces. Observe that to open circuit the edges P we can choose $A_P = [U]$ and to short circuit the edges P we can choose $B_P = [U]$. In general however the complementary orthogonal equations we obtain on i_{S-P}, v_{S-P} would not be the KCL and KVL equations associated with a graph. One is therefore forced to study topological problems not in terms of graphs but in terms of complementary orthogonal spaces. However the results one obtains are relevant to networks based on graphs. Indeed, one can construct general procedures of topological transformations and, plugging in the fact that the vector space we are working with is the space of coboundaries of a graph, derive additional advantages.

In this paper we use the generalized minor operation (which has been informally defined in the preceding paragraph) to solve the following problems:

(1) Given a network N_1 on graph G_1 on set of edges S what is the least number of extra variables required to "convert" it into network N_2 on graph G_2 on the same set of edges and the same device characteristic? The "conversion" here refers to obtaining a vector space over a larger set which has the coboundary spaces of the graphs of N_1 and N_2 as generalized minors. We can use this technique to obtain the equations of N_1 as "bordered versions" of the equations of N_2. (The "thickness" of the border is equal to the number of extra variables). We give efficient alghorithms for doing the "conversion" in general and better algorithms for certain important special cases.

(2) Given a network N with graph G on S and a partition $\{S_1, \cdots, S_n\}$ of S how to decompose it into multiports on $S_1 \cup P_1, \cdots, S_n \cup P_n$ and a "port connection diagram" on $\cup P_i$ such that $|\cup P_i|$ is a minimum? For this problem we give an algorithm which makes the port connection diagram into a graph, while the multiports are "generalized." We also show through

an example that solutions of the original network, in which devices lying in different S interact, map to solutions of a reduced network defined on the port connection diagram.

The ideas in this paper have arisen from an attempt to understand the essence of Kron's ideas [1], [2]. It is hoped, however, that they have a right to independent existence.

The organization of the paper is as follows:

Section 2 contains mathematical preliminaries. We have outlined proofs for all the results from elementary matroid theory that we have used in this paper.

Section 3 introduces the generalized minor operation.

Section 4 gives a physical interpretation for the generalized minor operation in terms of ideal transformers.

Section 5 describes the notion of mutual extension of vector spaces and gives algorithms for the construction of minimal extensions. Application to practical network analysis is outlined by considering two special topological transformations.

Section 6 describes decomposition of a vector space into several component spaces (multiports) and a coupler space (port connection diagram). Application to network theory is indicated by considering the case of an RLMC network in some detail.

2. Preliminaries

We deal with finite sets throughout. If S is a set $|S|$ denotes its cardinality. A function $f: S \rightarrow F$ is said to be a *vector* on S over the field F. Unless otherwise stated F would be the real field R. If used in equations f refers to a row vector and $(f)^T$ refers to a column vector. If T is a set, f_T would denote a vector on T, 0_T the zero vector on T. We define *restriction* of a vector f on S to a subset T of

S in the usual way and denote it by f / T. *Scalar multiplication* and *addition* of two vectors on the same set are defined in the usual way. However we permit *addition* of two vectors on *different* sets as follows: Let f be on S, g on T. Then $f + g$ is defined on $S \cup T$ and agrees with f on $S - T$, with g on $T - S$ and on $S \cap T$ with the usual addition of vectors on the same set. The Σ *notation* would be also used for the extended notion of addition. When $T \cap R = \varphi$ we may write $f_T \oplus f_R$ to emphasize the fact that T, R are disjoint. When addition of several vectors over disjoint sets is involved we may use $\overset{n}{\underset{i=1}{\oplus}}$ in place of $\overset{n}{\underset{i=1}{\sum}}$. A collection of vectors on S closed under addition and scalar multiplication is called a vector space on S. V_S would denote a vector space on S. *Independence* and *rank* of a collection of vectors on S are defined in the usual way. *Rank* of a collection of vectors P is denoted by $r(P)$. We define $V_P + V_T$ in the obvious way as the collection of all sums of vectors one in V_P and the other in V_T. $V_P + V_T$ would therefore be on $P \cup T$. We use $\oplus$ when P, T are disjoint. $V_1 - V_2$ refers to set theoretic difference. If g, f are on S, $\langle g , f \rangle \equiv \underset{e \in S}{\sum} g(e) \cdot f(e)$. If $\langle g , f \rangle = 0$ then we say g, f are *orthogonal* to each other. V^* is the vector space of all vectors orthogonal to vectors in V. We would call it the *dual* of V. Let $T \subseteq S$ and let V be on S. Then

$$V \times T \equiv \{g_T = f/ T , f \in V \text{ and } f(e) = 0 , e \in S - T\}$$

$V \cdot T$ is the collection of all restrictions of vectors of V to T. When $R \subseteq T \subseteq S$ we write $V \times T \cdot R$ for $(V \times T) \cdot R$ and $V \cdot T \times R$ for $(V \cdot T) \times R$. Such spaces are referred to as minors of V. We say $T \subseteq S$ is a *separator* of V iff $V \times T = V \cdot T$. Observe that we then have $V = (V \times T) \oplus (V \times (S - T))$. If K is a matrix, $(K)^T$ is its transpose. The symbol $[U]$ refers to an identity matrix whose order would be clear from the context. A *positive (negative) definite matrix* K is a symmetric real matrix such that $x(K)x^T > 0$ $(x(K)x^T < 0)$ for all nonzero real x. A

permutation matrix is a square matrix whose columns are obtained by permuting the columns of an identity matrix of the same order.

The *generator matrix* A_V of a vector space V is a matrix whose rows form a maximal linearly independent set of vectors of V (basis of V). We say that A_V generates V. Observe the $g \in V^*$ iff $(A_V)(g)^T = 0$. If A_V^1, A_V^2 are generator matrices of V observe that a set of columns of A_V^1 are linearly independent iff the corresponding set of columns of A_V^2 are linearly independent. Let V be a vector space on S. Let $T \subseteq S$. We say that T is a *circuit* of $M(V)$ iff the set of columns of a generator matrix A_V of V corresponding to T are minimal linearly dependent. T is a *bond of $M(V)$ iff it is a circuit of $M(V^*)$*. ($M(V)$ stands for "matroid associated with V." However we do not use the idea of a matroid explicity anywhere in this paper).

Let G be an oriented graph on the set of edges S. Let $T \subseteq S$. Then $G \cdot T$ is the graph obtained by deleting the edges in S - T and any isolated vertices formed. $G \times T$ is the graph obtained by fusing the end vertices of each edge in (S-T) and deleting it. A *coboundary* of G is a vector on S that satisfies the Kirchhoff voltage (tension) equations of G. A *cycle* of G is a vector on S that satisfies the Kirchhoff current (flow) equations of G. $V_{cob}(G)$ ($V_{cy}(G)$) denotes the vector space of coboundaries (cycles) of G. Let G_1, G_2 be on disjoint sets of edges S_1, S_2 and vertices V_1, V_2. We construct $G_1 \oplus G_2$ on edges $S_1 \cup S_2$ and vertices $V_1' \cup V_2'$, where V_1', V_2' are disjoint copies of V_1, V_2, by making it agree with G_1 on S_1 and with G_2 on S_2. In the graph $G_1 \oplus G_2$ observe that S_1, S_2 are separators of both the coboundary as well as the cycle spaces. A vector space V_S is *graphic (cographic)* iff it is the coboundary (cycle) space of a graph.

A *generalized electrical network* N is a triple (S, V_S, D_S) where S is a finite set of "edges," V_S is a vector space on S over R and the *device characteristic D_S* is a collection of ordered pairs $(v_S(\cdot), i_S(\cdot))$ where for all $t \in \mathbb{R}$, $v_S(t)$, $i_S(t)$ are

each vectors on S. Usually D_S will be specified informally as $D(\mathbf{v}_S, \mathbf{i}_S) = 0$. A *solution* of **N** is a pair $(\mathbf{v}_S(\cdot), \mathbf{i}_S(\cdot))$ belonging to D_S where $\mathbf{v}_S(t) \in \mathbf{V}_s$, $\mathbf{i}_S(t) \in \mathbf{V}_S^\bullet$, for all t. We will refer to $\mathbf{V}_S$ as the coboundary space of **N** and $\mathbf{V}_S^\bullet$ as the cycle space of **N**. A generalized network is *Ordinary* when $\mathbf{V}_S$ is the coboundary space of a graph. Let S be partitioned into $S_1, S_2, ..., S_n$. We say $S_1, S_2, ..., S_n$ appear *decoupled* in D_S iff there exist collections D_{S_i} of ordered pairs $(\mathbf{v}_{S_i}(\cdot), \mathbf{i}_{S_i}(\cdot))$ such that $(\mathbf{v}_S(\cdot), \mathbf{i}_S(\cdot))$ belong to D_S iff $(\mathbf{v}_S(\cdot)/S_i, \mathbf{i}_S(\cdot)/S_i)$ belong to D_{S_i} $(i = 1, 2, ..., n)$. A *multiport* is a generalized network with a subset P of edges specified as *ports* such that on the ports there are no device characteristic constraints.

We now present a number of results which we use freely in the rest of this paper. In order to make this paper self contained we outline proofs for most of them. The results are standard in elementary linear Algebra and matroid theory. [3], [4].

Let $\mathbf{V}_S$ be a vector space on S.

Theorem P1.

    (a) $(\mathbf{V}_S^\bullet)^\bullet = \mathbf{V}_S$

    (b) $r(\mathbf{V}_S) + r(\mathbf{V}_S^\bullet) = |S|$

Outline of Proof: Observe that if $\mathbf{V}_S$ has a generating matrix $[UA_{12}]$, then $\mathbf{V}_S^\bullet$ has the generating matrix $[-A_{12}^T\, U]$.

                                                ■

Theorem P2. Let $\mathbf{V}_1$, $\mathbf{V}_2$ be vector spaces on S. Then $\mathbf{V}_1 \subseteq \mathbf{V}_2 <= \Rightarrow \mathbf{V}_1^\bullet \supseteq \mathbf{V}_2^\bullet$.

Theorem P3 Let $T \subseteq S$. Then

$$r(\mathbf{V}_S \cdot T) + r(\mathbf{V}_S \times (S-T)) = r(\mathbf{V}_S) \ .$$

Outline of Proof: Choose a generator matrix A for V_S of the form shown below, where A_{TT}, A_{22} have linearly independent rows

$$A = \begin{bmatrix} A_{TT} & A_{T2} \\ 0 & A_{22} \end{bmatrix}$$

Observe that A_{TT} is a generator matrix for $V_S \cdot T$ and A_{22} is a generator matrix for $V_S \times (S - T)$.

∎

Theorem P4 Let $T \subseteq S$. Then

    (a) $(V_S \cdot T)^\bullet = V_S^\bullet \times T$

    (b) $(V_S \times T)^\bullet = V_S^\bullet \cdot T$.

Proof: (a) Let $f_T \in V_S^\bullet \times T$. Then there exists $f_S \in V_S^\bullet$ such that $f_S / T = f_T$ and $f_S / (S - T) = 0$. If $g_S \in V_S$, it follows that $\langle f_S, g_S \rangle = 0$. Hence $\langle f_T, g_S / T \rangle = 0$. Hence $f_T \in (V_S \cdot T)^\bullet$. Next, let $f_T \in (V_S \cdot T)^\bullet$. Consider a vector f_S defined as before. Let $g_S \in V_S$. Then $\langle f_S, g_S \rangle = \langle f_T, g_S / T \rangle = 0$. Hence $f_S \in V_S^\bullet$ and $f_T \in V_S^\bullet \times T$.

(b) follows from (a) by application of Theorem P1(a).

Theorem P5 $r(V_1 \cap V_2) + r(V_1 + V_2) = r(V_1) + r(V_2)$.

Outline of Proof: Choose a basis B_n of $V_1 \cap V_2$. Extend it to a basis of V_1 by adjoining B_1, and a basis of V_2 by adjoining B_2. Observe that $B_n \cup B_1 \cup B_2$ is a basis of $V_1 + V_2$.

∎

Theorem P6 Let $T \subseteq S$. (a) The circuits of $M(V_S)$ contained in T and those of $M(V_S \cdot T)$ are identical. (b) The bonds of $M(V_S)$ contained in T and those of $M(V_S \times T)$ are identical.

Proof: Consider a generator matrix of V_S of the form shown in the proof of Theorem P3. Observe that minimal linearly dependent set of columns of A contained in T are the same as the minimal linearly dependent set of columns of A_{TT}. This proves (a). Bonds of $M(V_S)$ contained in T are the same as circuits of $M(V_S^{\bullet})$ contained in T and bonds of $M(V_S \times T)$ are the same as circuits of

$M(V_S^{\bullet} \cdot T)$ by Theorem P4. So (b) follows from (a).

∎

Theorem P7 Let $T \subseteq S$. (a) If $M(V_S)$ has circuits contained in T then $r(V_S \cdot T) < |T|$. (b) If $M(V_S)$ has bonds contained in T then $r(V_S \times T) > 0$.   -

Proof: If $M(V_S)$ has circuits contained in T then $M(V_S \cdot T)$ has circuits. Hence the columns of a generator matrix of $V_S \cdot T$ are linearly dependent. Since the rows are linearly independent it follows that the number of rows is less than the number of columns of the generator matrix. Hence $r(V_S \cdot T) < |T|$. By theorems P1, P4 $r(V_S^{\bullet} \cdot T) + r(V_S \times T) = |T|$. (b) now follows by Theorems P6(b), the definition of a bond and (a) above.

∎

Theorem P8 (Tellegen) $(V_{cob}(G))^{\bullet} = V_{cy}(G)$.

Theorem P9

  (a) $V_{cob}(G \times T) = (V_{cob}(G)) \times T$

  (b) $V_{cob}(G \cdot T) = (V_{cob}(G)) \cdot T$.

Outline of Proof: Construct a fundamental cutset matrix Q of G of the form shown below, i.e., choose maximum possible number of edges of T in the tree.

$$Q = \begin{matrix} & \overset{T}{} & & \overset{S-T}{} \\ & \begin{bmatrix} U & Q_{12} & : & 0 & Q_{14} \\ 0 & 0 & : & U & Q_{24} \end{bmatrix} \end{matrix}$$

Then $[U\ Q_{12}]$ is a fundamental cutset matrix of $G \cdot T$ and a generator matrix of

$(\mathbf{V}_{cob}(G))\cdot T$ while $[\mathbf{U}\ \mathbf{Q}_{24}]$ is a fundamental cutset matrix of $G \times (S-T)$ and a generator matrix of $(\mathbf{V}_{cob}(G)) \times (S-T)$.

■

Theorem P10

   (a) $\mathbf{V}_{cy}(G \times T) = (\mathbf{V}_{cy}(G)) \cdot T$

   (b) $\mathbf{V}_{cy}(G \cdot T) = (\mathbf{V}_{cy}(G)) \times T$.

Proof:

$$
\begin{aligned}
\mathbf{V}_{cy}(G \times T) &= (\mathbf{V}_{cob}(G \times T))^* \\
&= ((\mathbf{V}_{cob}(G)) \times T)^* \\
&= \mathbf{V}_{cy}(G) \cdot T \ ,
\end{aligned}
$$

by the use of Theorems P8, P9(a) and P4. (b) can be proved similarly.

■

3. The Generalized Minor

In this section we introduce an operation on vector spaces which we use throughout this paper.

Definition 3.1   Let $\mathbf{V}_S$ be a vector space on S. Let $P \subseteq S$. Let $\mathbf{V}_P$ be a vector space on P. Then the *generalized minor* of $\mathbf{V}_S$ with respect to $\mathbf{V}_P$ is denoted $\mathbf{V}_S \leftarrow \mathbf{V}_P$ and is defined as follows:

$$
\mathbf{V}_S \leftarrow \mathbf{V}_P \equiv \{\mathbf{f}_{S-P} : \text{there exist} \mathbf{f}_S \in \mathbf{V}_s \ , \ \mathbf{f}_P \in \mathbf{V}_P
$$
$$
\text{such that } \mathbf{f}_S / P = \mathbf{f}_P \ , \ \mathbf{f}_S / S - P = \mathbf{f}_{S-P}\} \ .
$$

We now describe a convenient way of constructing a generating matrix A for a space $\mathbf{V}_S$ so that the generating matrix for $\mathbf{V}_S \leftarrow \mathbf{V}_P$, $\mathbf{V}_S \times (S-P)$, $\mathbf{V}_S \times P$, $\mathbf{V}_P \cap (\mathbf{V}_S \times P)$, $\mathbf{V}_P \cap (\mathbf{V}_S \cdot P)$ appear as a submatrix. We would say that these subspaces are *visible* in such a generating matrix. Construct the basis $\mathbf{B}_X^{S-P}$ for

$V_S \times (S-P)$. Let us suppose that these vectors form the row vectors of the submatrix $A_{X(S-P)}$ of A shown below.

$$A = \begin{bmatrix} A_{X(S-P)} & 0 \\ A_{2(S-P)} & A_{2P} \\ A_{3(S-P)} & A_{3P} \\ 0 & A_{\cap XP} \\ 0 & A_{5P} \end{bmatrix} \begin{matrix} \overset{S-P}{} & \overset{P}{} \end{matrix}$$

Next choose a basis $B_{\cap X}^P$ for $(V_S \times P) \cap V_P$. Let the vectors of $B_{\cap X}^P$ form the row vectors of $A_{\cap XP}$. Extend this to a basis $B_{\cap \bullet}^P$ of $(V_S \cdot P) \cap V_P$. Let the additional vectors form row vectors of the matrix A_{3P}. Extend $B_{\cap X}^P$ to a basis B_X^P of $V_S \times P$. Let the additional vectors form the row vectors of the matrix A_{5P}. Extend $B_{\cap \bullet}^P \cup B_X^P$ to a basis $B_\bullet^P$ of $V_S \cdot P$. Let the additional vectors form row vectors of the matrix A_{2P}. The row vectors of A_{2P} and A_{3P} are restrictions of certain vectors from V_S. Restricting these vectors to (S-P), we obtain the row vectors of

$$\begin{bmatrix} A_{2(S-P)} \\ A_{3(S-P)} \end{bmatrix}$$

We now list the set of properties for this matrix.

Property 1. The rows of $\begin{bmatrix} A_{\cap XP} \\ A_{5P} \end{bmatrix}$ form a basis for $V_S \times P$.

Property 2. The rows of $A_{X(S-P)}$ form a basis for $V_S \times (S-P)$.

Property 3. The rows of $\begin{bmatrix} A_{2P} \\ A_{3P} \\ A_{\cap XP} \\ A_{5P} \end{bmatrix}$ form a basis for $V_S \cdot P$.

Property 4. The rows of $\begin{bmatrix} A_{3P} \\ A_{\cap XP} \end{bmatrix}$ form a basis for $(V_S \cdot P) \cap V_P$.

-12-

Property 5. Rows of $\begin{bmatrix} A_{X(S-P)} \\ A_{2(S-P)} \\ A_{3(S-P)} \end{bmatrix}$ form a basis for $V_S \cdot (S-P)$.

Proof: The rows must be linearly independent as otherwise by a linear combination of row vectors of $\begin{bmatrix} A_{2P} \\ A_{3P} \end{bmatrix}$ we would get a vector of $V_S \times P$.

■

Property 6. The rows of $\begin{bmatrix} A_{X(S-P)} \\ A_{3(S-P)} \end{bmatrix}$ form a basis for $V_S \leftarrow V_P$.

Proof: Let f belong to $V_S \leftarrow V_P$. Then there exist vectors $f_S \in V_S$, $f_P \in V_P$ such that $f_S / (S-P) = f$, $f_S / P = f_P \cdot f_S / P$ is linearly dependent on the rows of $\begin{bmatrix} A_{3P} \\ A_{\cap XP} \end{bmatrix}$ and hence f is linearly dependent on rows of $\begin{bmatrix} A_{X(S-P)} \\ A_{3(S-P)} \end{bmatrix}$. Conversely if f is linearly dependent on rows of $\begin{bmatrix} A_{X(S-P)} \\ A_{3(S-P)} \end{bmatrix}$, it can be expressed as $f_1 + f_2$ where f_1 is linearly dependent on rows of $A_{X(S-P)}$ and f_2 is linearly dependent on rows of $A_{3(S-P)}$. Let $f_1 = (\sigma_1) (A_{X(S-P)})$ and let $f_2 = (\sigma_2) (A_{3(S-P)})$. Choose $f_s = (\sigma_1 \ \sigma_2) \begin{bmatrix} A_{X(S-P)} & 0 \\ A_{3(S-P)} & A_{3P} \end{bmatrix}$ and $f_P = (\sigma_2) (A_{3P})$. Since rows of A_{3P} belong to V_P it follows that $f \in V_S \leftarrow V_P$.

■

The generator matrix for V_S in which $V_S \times P$, $V_S \times (S-P)$, $(V_S \cdot P) \cap V_P$ and $V_S \leftarrow V_P$ are visible can be used to derive the following results.

Theorem 3.1. $V_{(S-P)}$ is a g-minor of V_S iff $V_S \times (S-P) \subseteq V_{(S-P)} \subseteq V_S \cdot (S-P)$.

Proof: The construction of the appropriate generator matrix for V_S described earlier shows that if $V_{(S-P)}$ is a g-minor of V_S then

$V_S \times (S-P) \subseteq V_{(S-P)} \subseteq V_s \cdot (S-P)$. Conversely if $V_{(S-P)}$ satisfies the condition of the theorem we could build a generator matrix A for V_S as shown below where the rows of $A_{X(S-P)}$ from a basis for $V_S \times (S-P)$, the rows of $\begin{bmatrix} A_{X(S-P)} \\ A_{3(S-P)} \end{bmatrix}$

form a basis for $V_{(S-P)}$ and the rows of $\begin{bmatrix} A_{X(S-P)} \\ A_{2(S-P)} \\ A_{3(S-P)} \end{bmatrix}$ form a basis for $V_S \cdot (S-P)$.

$$A = \begin{array}{cc} S-P & P \\ \begin{bmatrix} A_{X(S-P)} & 0 \\ A_{2(S-P)} & A_{2P} \\ A_{3(S-P)} & A_{3P} \\ 0 & A_{XP} \end{bmatrix} \end{array}$$

Observe that the rows of $\begin{bmatrix} A_{2P} \\ A_{3P} \\ A_{XP} \end{bmatrix}$ are linearly independent. If now we choose V_P as the space generated by the rows of A_{3P}, it can be seen that $V_S \leftarrow V_P = V_{(S-P)}$.

∎

Theorem 3.2.

$$\tau(V_S \leftarrow V_P) = \tau(V_S X(S-P)) + \tau((V_S \cdot P) \cap V_P)$$
$$- \tau((V_S \times P) \cap V_P) \ .$$

Proof: This follows immediately from the construction of the generator matrix in which $V_S \leftarrow V_P$, $V_S \times (S-P)$, $V_S \times P$, $V_P \cap (V_S \times P)$, $V_P \cap (V_S \cdot P)$ are visible.

∎

The next Lemmas are needed for the proof of Theorem 3.3. Lemma 3.1 is a standard result from Linear Algebra and can be proved by routine use of Theorems P1 and P5. Lemma 3.2 is merely a restatement of Theorem P3.

Lemma 3.1. Let V_1, V_2 be on S. Then $(V_1 \cap V_2)^* = V_1^* + V_2^*$.

Lemma 3.2. Let V_S be on S. Let $P \subseteq S$. Then

$$r(\mathbf{V}_S \cdot P) - r(\mathbf{V}_S \times P) = r(\mathbf{V}_s \cdot (S-P)) - r(\mathbf{V}_S \times (S-P)).$$

Theorem 3.3. Let $\mathbf{V}_S$, $\mathbf{V}_P$ be spaces on S, P respectively where $P \subseteq S$. Then $(\mathbf{V}_S \leftarrow \mathbf{V}_P)^* = \mathbf{V}_S^* \leftarrow \mathbf{V}_P^*$.

Proof: We will first show that the two spaces are orthogonal to each other and then show that their ranks add up to $|S-P|$. Let $\mathbf{f} \in \mathbf{V}_S \leftarrow \mathbf{V}_P$. Then there exist vectors $\mathbf{f}_S$, $\mathbf{f}_P$ belonging to $\mathbf{V}_S$, $\mathbf{V}_P$ respectively such that $\mathbf{f}_S / P = \mathbf{f}_P$, $\mathbf{f} = \mathbf{f}_{S/(S-P)}$. Let $\mathbf{g} \in \mathbf{V}_S^* \leftarrow \mathbf{V}_P^*$. Then there exist vectors $\mathbf{g}_S$, $\mathbf{g}_P$ belonging to $\mathbf{V}_S^*$, $\mathbf{V}_P^*$ such that $\mathbf{g}_S / P = \mathbf{g}_P$, $\mathbf{g} = \mathbf{g}_{S/(S-P)}$. We now have $\langle \mathbf{f}, \mathbf{g} \rangle = -\langle \mathbf{f}_S / P, \mathbf{g}_S / P \rangle = -\langle \mathbf{f}_P, \mathbf{g}_P \rangle = 0$. Next consider $r(\mathbf{V}_S \leftarrow \mathbf{V}_P) + r(\mathbf{V}_S^* \leftarrow \mathbf{V}_P^*)$. By Theorem 3.2,

$$r(\mathbf{V}_S \leftarrow \mathbf{V}_P) = r(\mathbf{V}_S \times (S-P)) + r((\mathbf{V}_S \cdot P) \cap \mathbf{V}_P) - r((\mathbf{V}_S \times P) \cap \mathbf{V}_P) \ .$$

$$r(\mathbf{V}_S^* \leftarrow \mathbf{V}_P^*) = r(\mathbf{V}_S^* \times (S-P)) + r((\mathbf{V}_S^* \cdot P) \cap \mathbf{V}_P^*) - r((\mathbf{V}_S^* \times P) \cap \mathbf{V}_P^*) \ .$$

By application of Lemma 3.1 and Theorems P1, P4 we have

$$r((\mathbf{V}_S^* \cdot P) \cap \mathbf{V}_P^*) = |P| - r((\mathbf{V}_S \times P) + \mathbf{V}_P) \ .$$

and

$$r((\mathbf{V}_S^* \times P) \cap \mathbf{V}_P^*) = |P| - r((\mathbf{V}_S \cdot P) + \mathbf{V}_P) \ .$$

So

$$r(\mathbf{V}_S \leftarrow \mathbf{V}_P) + r(\mathbf{V}_S^* \leftarrow \mathbf{V}_P^*) = r(\mathbf{V}_S \times (S-P)) + r(\mathbf{V}_S^* \times (S-P))$$
$$+ r((\mathbf{V}_S \cdot P) \cap \mathbf{V}_P) + r((\mathbf{V}_S \cdot P) + \mathbf{V}_P)$$
$$- |P| - r((\mathbf{V}_S \times P) \cap \mathbf{V}_P) - r((\mathbf{V}_S \times P) + \mathbf{V}_P) + |P| \ .$$
$$= r(\mathbf{V}_S \times (S-P)) + r(\mathbf{V}_S^* \times (S-P)) + r(\mathbf{V}_S \cdot P) + r(\mathbf{V}_P)$$
$$- r(\mathbf{V}_S \times P) - r(\mathbf{V}_P) \quad \text{by Theorem P5} \ .$$
$$= |S-P| - [r(\mathbf{V}_S \cdot (S-P)) - r(\mathbf{V}_S \times (S-P))]$$
$$+ r(\mathbf{V}_S \cdot P) - r(\mathbf{V}_S \times P)$$
$$= |S-P| \quad \text{by Lemma 3.2}$$

This proves the theorem.

■

4. Ideal Transformers

In this section we dwell briefly on the concept of an ideal transformer. Using ideal transformers we give a simple physical interpretation for the notion of a generalized minor. Generalized networks' may be thought of as being constructed by plugging 2-terminal electrical devices to the ports of ideal transformers. The g-minor operation is therefore natural for generalized networks. Ordinary networks obtained by connecting 2-terminal electrical devices according to a graph are a special case of generalized networks. The g-minor operation is therefore applicable to ordinary networks also. We show in this section that the g-minor operation generalizes the short circuit and open circuit operations.

Definition 4.1. An ideal transformer $\mathbf{I}_S$ on S is a "black box" with S as its set of ports, and satisfying the following condition: Let $\mathbf{V}_S^v$, $\mathbf{V}_S^i$ be the sets of all voltage vectors and current vectors that can exist at S. Then $\mathbf{V}_S^v = (\mathbf{V}_S^i)^*$.

Since an ideal transformer is fully characterized by the vector space $\mathbf{V}_S^v$ on S we will identify $\mathbf{I}_S$ with the pair $(S, \mathbf{V}_S^v)$. We will refer to $\mathbf{V}_S^v$ as the space of coboundaries of $\mathbf{I}_S$ and $(\mathbf{V}_S^v)^*$ as the space of cycles of $\mathbf{I}_S$.

Example 4.1.

Consider the 3-winding ideal transformer of Fig. 4.1. Here $S = \{1,2,3\}$ $v_2 = (n_2/n_1)\,v_1$, $v_3 = (n_3/n_1)v_1$. The rows of matrix $Q = [\,1\ \overset{1}{}\ \overset{2}{n_2/n_1}\ \overset{3}{n_3/n_1}]$ generate $\mathbf{V}_S^v$. The rows of matrix $B = \begin{bmatrix} -n_2/n_1 & 1 & 0 \\ -n_3/n_1 & 0 & 1 \end{bmatrix}$ generate $\mathbf{V}_S^i$.

Example 4.2.

Let G be an oriented graph on S. Tellegen's Theorem states that the space of coboundaries of G are complementary orthogonal to the space of cycles of G. It follows that $(S,\mathbf{V}_S)$ where $\mathbf{V}_S$ is the coboundary space of G may be regarded as an ideal transformer. In other words a graph is a special case of an ideal transformer.

Definition 4.2. Let $\mathbf{I}_S = (S,\mathbf{V})$. Then the *dual of* $\mathbf{I}_S$ denoted $\mathbf{I}_S^*$ is the pair $(S,\mathbf{V}^*)$.

Observe that in our notation the dual of a nonplanar graph would be an ideal transformer.

Definition 4.3. Let $\mathbf{I}_{S_1} = (S,\mathbf{V}_1)$ ..... $\mathbf{I}_{S_n} = (S_n,\mathbf{V}_n)$. be ideal transformers on pairwise disjoint sets $S_1, ..., S_n$. Then their direct sum $\mathbf{I}_{S_1} \oplus \cdots \oplus \mathbf{I}_{S_n}$ is the ideal transformer $\left[\bigcup_{i=1}^{n} S_i,\ \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_n \right]$.

The following simple lemma is useful. Its routine proof is omitted.

Lemma 4.1. $(\mathbf{V}_{S_1} \oplus \cdots \oplus \mathbf{V}_{S_n})^* = (\mathbf{V}_{S_1}^* \oplus \cdots \oplus \mathbf{V}_{S_n}^*)$.

An immediate consequence is

Theorem 4.1. $(\mathbf{I}_{S_1} \oplus \cdots \oplus \mathbf{I}_{S_n})^* = (\mathbf{I}_{S_1}^* \oplus \cdots \oplus \mathbf{I}_{S_n}^*)$.

Definition 4.4. Let S be a set and let $P \subseteq S$. Let $I_S = (S, V_S)$, $I_P = (P, V_P)$. Then the g-minor of I_S with respect to I_P is denoted $I_S \leftarrow I_P$ and is defined by

$$I_S \leftarrow I_P = ((S-P), V_S \leftarrow V_P).$$

The g-minor operation gives us a simple way of deriving new ideal transformers from old.

Example 4.3.

Let $I_S = (S, V_S)$. Then $I^1_{(S-P)} \equiv ((S-P), V_S \times (S-P)) = I_S \leftarrow I^1_P$, where $I^1_P = (P, 0_P)$ and $I^2_{(S-P)} \equiv ((S-P), V_S \cdot (S-P)) = I_S \leftarrow I^2_P$ where $I^2_P = (P, \mathbb{R}^P)$ where $\mathbb{R}^P$ is the space of all vectors on P over $\mathbb{R}$. By Theorem P9 $V_{cob}(G \times (S-P)) = (V_{cob}(G)) \times (S-P)$ and $V_{cob}(G \cdot (S-P)) = (V_{cob}(G)) \cdot (S-P)$. Hence, for graphs, the operations of short circuiting edges and of open circuiting edges can be achieved by the g-minor operation. Let $I^3_{(S-P)} = ((S-P), V_S \times (S-P_1) \cdot (S-P))$, $P_1 \subseteq P$. Then $I^3_{(S-P)} = I_S \leftarrow I^3_P$, where $I^3_P = (P, \{0_{P_1}\} \oplus \mathbb{R}^{(P-P_1)})$. The g-minor operation was introduced to generalize the ordinary minor operations. The above illustrate this fact.

Theorem 3.3 permits us to give a simple physical interpretation of the operation of g-minor. Let I_S, I_P be ideal transformers on S, P with $P \subseteq S$. Let $I_S = (S, V_S)$, $I_P = (P, V_P)$. Let us identify the ports P in both transformers as in Fig. 4.2. Consider the current and voltage constraints on the exposed ports $(S-P)$. A vector f_{S-P} can be a voltage (current) vector on the exposed ports iff there exist voltage (current) vectors $f_S \in V_S(V_S^*)$, $f_P \in V_P(V_P^*)$ such that $f_S / P = f_P$, i.e., iff $f_{S-P} \in V_S \leftarrow V_P(V_S^* \leftarrow V_P^*)$. By Theorem 3.3 $(V_S \leftarrow V_P)^* = (V_S^* \leftarrow V_P^*)$. It follows that on the exposed ports (S-P) we have an ideal transformer. Thus if we "plug" the ports P in I_S by I_P the ideal transformer $I_S \leftarrow I_P$ results. The g-minor operation can be used to prove the following standard result.

Theorem 4.2. (BELEVITCH [5]) Physical connection of ideal transformers results in an ideal transformer.

Proof: Let $I_{S_1}, ..., I_{S_n}$ be ideal transformers on pairwise disjoint sets of ports $S_1, ..., S_n$. Let us suppose that the ports $S_{1c} \subseteq S_1, ..., S_{nc} \subseteq S_n$ are connected according to a graph G. The remaining ports $S = (\cup S_i - \cup S_{ic})$ are exposed. The graph may be treated as an ideal transformer $I_{S_c} \equiv (\cup S_{ic}, V_{cob}(G))$. Connection of ports $\cup S_{ic}$ according to graph G is equivalent to imposing the KCL and KVL conditions of the graph G on the ports $\cup S_{ic}$ of the ideal transformer $I_{S_1} \oplus \cdots \oplus I_{S_n}$. The result of this operation is the ideal transformer $(I_{S_1} \oplus \cdots \oplus I_{S_n}) \leftarrow I_{S_c}$.

∎

5. Extension of Vector Spaces

5.1. In this section we introduce the "inverse" operation of g-minor, namely "extension" of a vector space. While analyzing a given network we can use the topology of a different network by constructing a mutual extension of the coboundary spaces of the two networks. These ideas are detailed in subsection 5.2 and exemplified in subsection 5.3.

5.2. Definitions and Theorems on Extension.

Definition 5.1. Let V_S, V_{SP} be vector spaces on S, $S \cup P$ respectively. We say V_{SP} is an *extension* of V_S iff V_S is a g-minor of V_{SP}.

Definition 5.2. Let $V_S^1, ..., V_S^n$ be vector spaces on S. V_{SP} is a minimal extension of $\{V_S^1, ..., V_S^n\}$ iff V_{SP} is an extension of V_S^i (i = 1,...,n) and if $V_{SP'}^i$ is any other extension of V_S^i (i=1,...,n) then $|P| \leq |P'|$.

Theorem 5.1. Let V_S^1, V_S^2 be vector spaces on S. Let V_{SP} on $S \cup P$ be an extension of V_S^1 and V_S^2. V_{SP} is a minimal extension of V_S^1 and V_S^2 iff

$$|P| = r(\mathbf{V}_S^1 + \mathbf{V}_S^2) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2) \ .$$

Proof: Let $\mathbf{V}_{SP}$ be an extension of $\mathbf{V}_S^1$, $\mathbf{V}_S^2$. Then $\mathbf{V}_{SP} \cdot S \supseteq \mathbf{V}_S^1$ and $\mathbf{V}_{SP} \cdot S \supseteq \mathbf{V}_S^2$ by Theorem 3.1. Hence $\mathbf{V}_{SP} \cdot S \supseteq \mathbf{V}_S^1 + \mathbf{V}_S^2$. Also $\mathbf{V}_{SP} \times S \subseteq \mathbf{V}_S^1$ and $\mathbf{V}_{SP} \times S \subseteq \mathbf{V}_S^2$. by Theorem 3.1. Hence $\mathbf{V}_{SP}^* \cdot S \supseteq (\mathbf{V}_S^1)^*$ and $\mathbf{V}_{SP}^* \cdot S \supseteq (\mathbf{V}_S^2)^*$ by Theorems P4 and P2. Hence $\mathbf{V}_{SP}^* \cdot S \supseteq (\mathbf{V}_S^1)^* + (\mathbf{V}_S^2)^*$ and hence $\mathbf{V}_{SP} \times S \subseteq \mathbf{V}_S^1 \cap \mathbf{V}_S^2$ by Theorems P 4 and P2. By Lemma 3.2,

$$r(\mathbf{V}_{SP} \cdot S) - r(\mathbf{V}_{SP} \times S) = r(\mathbf{V}_{SP} \cdot P) - r(\mathbf{V}_{SP} \times P) \ .$$

Hence $r(\mathbf{V}_{SP} \cdot P) - r(\mathbf{V}_{SP} \times P) \geq r(\mathbf{V}_S^1 + \mathbf{V}_S^2) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2)$. Since $0 \leq r(\mathbf{V}_{SP} \times P) \leq r(\mathbf{V}_{SP} \cdot P) \leq |P|$, it follows that if $|P| = r(\mathbf{V}_S^1 + \mathbf{V}_S^2) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2)$, $\mathbf{V}_{SP}$ is a minimal extension of $\mathbf{V}_S^1$ and $\mathbf{V}_S^2$. We now show how to construct a minimal extension of given spaces $\mathbf{V}_S^1$ and $\mathbf{V}_S^2$. Construct a basis $B_\cap$ of $\mathbf{V}_S^1 \cap \mathbf{V}_S^2$. Extend it to a basis B_1 of $\mathbf{V}_S^1$ and a basis B_2 of $\mathbf{V}_S^2$. Clearly $B_1 \cup B_2$ is a basis of $\mathbf{V}_S^1 + \mathbf{V}_S^2$. Let $B_\cap$, $B_1 - B_\cap$, $B_2 - B_\cap$ form the sets of row vectors of $\mathbf{A}_{\cap S}$, $\mathbf{A}_{1S}$, $\mathbf{A}_{2S}$ respectively. The matrix $\mathbf{A}$ shown below is taken to be the generator matrix for $\mathbf{V}_{SP}$. (Here $P_1 \cup P_2 = P$).

$$\mathbf{A} = \begin{array}{c} \quad S \quad P_1 \ P_2 \\ \begin{bmatrix} \mathbf{A}_{1S} & \mathbf{U} & 0 \\ \mathbf{A}_{\cap S} & 0 & 0 \\ \mathbf{A}_{2S} & 0 & \mathbf{U} \end{bmatrix} \end{array}$$

If $\mathbf{V}_P^1$ has the generator matrix $\begin{pmatrix} P_1 & P_2 \\ \mathbf{U} & 0 \end{pmatrix}$ and $\mathbf{V}_P^2$ has the generator matrix $\begin{pmatrix} P_1 & P_2 \\ 0 & \mathbf{U} \end{pmatrix}$ then it is easy to see that $\mathbf{V}_{SP} \leftarrow \mathbf{V}_P^1 = \mathbf{V}_S^1$ and $\mathbf{V}_{SP} \leftarrow \mathbf{V}_P^2 = \mathbf{V}_S^2$. Note that

$$\begin{aligned} |P| &= |P_1| + |P_2| \\ &= r(\mathbf{V}_S^1) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2) + r(\mathbf{V}_S^2) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2) \\ &= r(\mathbf{V}_S^1 + \mathbf{V}_S^2) - r(\mathbf{V}_S^1 \cap \mathbf{V}_S^2) \ , \end{aligned}$$

by Theorem P5. It follows that $\mathbf{V}_{SP}$ is a minimal extension of $\mathbf{V}_S^1$ and $\mathbf{V}_S^2$. $\blacksquare$

In Theorem 5.1 observe that we are able to obtain V_S^1 and V_S^2 as ordinary minors (as opposed to g-minors) of V_{SP} since $V_S^1 = V_{SP} \cdot (S \cup P_2) \times S$ and $V_S^2 = V_{SP} \cdot (S \cup P_1) \times S$. If we have to construct a minimal extension of $\{V_S^1 \cdots V_S^n\}$ when $n > 2$, ordinary minors would be inadequate. By using the ideas of the first half of the proof of Theorem 5.1 we can show

$$|P| \geq r(\sum_{i=1}^{n} V_S^i) - r(V_S^1 \cap \cdots \cap V_S^n) .$$

The matrix $\mathbf{A}$ shown below can be taken as the generator matrix of V_{SP}

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{+S} & \mathbf{U} \\ \mathbf{A}_{\cap S} & \mathbf{0} \end{bmatrix}$$

(Rows of $\mathbf{A}_{\cap S}$ form a basis for $(V_S^1 \cap \cdots \cap V_S^n)$, while rows of $\begin{bmatrix} \mathbf{A}_{+S} \\ \mathbf{A}_{\cap S} \end{bmatrix}$ form a basis for $\sum_{i=1}^{n} V_S^i$.) Let V_S^i be generated by the matrix $[\mathbf{K}_+ \ \mathbf{K}_\cap] \begin{bmatrix} \mathbf{A}_{+S} \\ \mathbf{A}_{\cap S} \end{bmatrix}$. Let V_P^i be the space generated by the rows of $[\mathbf{K}_+]$. Clearly

$$V_{SP} \leftarrow V_P^i = V_S^i .$$

We summarize these results in Theorem 5.2 below:

Theorem 5.2   Let V_{SP} on $S \cup P$ be an extension of $\{V_S^1 \cdots V_S^n\}$. It is a minimal extension of $\{V_S^1 \cdots V_S^n\}$ iff $|P| = r(\sum_{i=1}^{n} V_S^i) - r(\bigcap_{i=1}^{n} V_S^i)$.

We next prove a simple but useful result.

Theorem 5.3 V_{SP} is a minimal extension of $\{V_S^1, \cdots V_S^n\}$ iff V_{SP}^* is a minimal extension of $\{(V_S^1)^*, \cdots (V_S^n)^*\}$.

Proof: By Theorem 3.3 V_S^i is a g-minor of V_{SP} iff $(V_S^i)^*$ is a g-minor of V_{SP}^*. We next observe that

$$\tau\left(\sum_{i=1}^{n} V_S^i\right) = |S| - \tau\left(\bigcap_{i=1}^{n} (V_S^i)^*\right)$$

$$\tau\left(\bigcap_{i=1}^{n} V_S^i\right) = |S| - \tau\left(\sum_{i=1}^{n} (V_S^i)^*\right)$$

by Theorem P1 and Lemma 3.1. Hence

$$|P| = \tau\left(\sum_{i=1}^{n} \overline{V_S^i}\right) - \tau\left(\bigcap_{i=1}^{n} V_S^i\right)$$

iff

$$|P| = \tau\left(\sum_{i=1}^{n} (V_S^i)^*\right) - \tau\left(\bigcap_{i=1}^{n} (V_S^i)^*\right) \ .$$

∎

5.3. Application to Network Analysis

In this subsection we show the relevance of the notion of extension of vector spaces to network analysis. We will show that if we are allowed to increase the number of variables it is possible to solve a given network utilizing the topology of a different network. These ideas do not depend upon the types of devices present in the network.

Suppose we have to solve a network N_1 on the set of edges S i.e., solve Equations (5.1)

$$\left.\begin{array}{l} (A_{1S})\, i_S^T = 0 \\ (B_{1S})\, v_S^T = 0 \quad or \quad (A_{1S}^T)e_n^T = v_S^T \\ D(v_S, i_S) = 0 \end{array}\right\} \tag{5.1}$$

We wish to utilize the topology of a different network say N_2 on S with reduced incidence matrix A_{2S} but the same device characteristic $D(v_S, i_S) = 0$. Let V_S^1, V_S^2 be the spaces generated by matrices A_{1S}, A_{2S} respectively. We can then construct a generator matrix $\begin{bmatrix} A'_{1S} \\ A'_{nS} \\ A'_{2S} \end{bmatrix}$ for the space $V_S^1 + V_S^2$ such that $\begin{bmatrix} A'_{1S} \\ A'_{nS} \end{bmatrix}$, $\begin{bmatrix} A'_{nS} \\ A'_{2S} \end{bmatrix}$ are generator matrices for V_S^1, V_S^2 respectively. We next choose A'_{SP} shown below as a generator matrix for V_{SP}

$$A'_{SP} = \begin{bmatrix} A'_{1S} & U & 0 \\ A'_{nS} & 0 & 0 \\ A'_{2S} & 0 & U \end{bmatrix}$$

Here rows of A'_{1S}, A'_{2S} may be taken from rows of A_1, A_2. Then $V_{SP} \leftarrow V_P^1 = V_S^1$ and $(V_{SP} \leftarrow V_P^2) = V_S^2$ where V_P^1, V_P^2 are generated respectively by $\overset{P_1 \quad P_2}{[U \quad 0]}$, $\overset{P_1 \quad P_2}{[0 \quad U]}$. By a suitable row transformation we can choose A_{SP} as a generator matrix for V_{SP}, where

$$A_{SP} = \begin{bmatrix} U & 0 & 0 \\ 0 & K_{22} & K_{23} \end{bmatrix} \begin{bmatrix} A'_{1S} & U & 0 \\ A'_{nS} & 0 & 0 \\ A'_{2S} & 0 & U \end{bmatrix} = \begin{bmatrix} A'_{1S} & U & 0 \\ A_{2S} & 0 & K_{23} \end{bmatrix}$$

Equations (5.1) may now be rewritten as

$$\begin{bmatrix} A'_{1S} & U & 0 \\ A_{2S} & 0 & K_{23} \end{bmatrix} \begin{bmatrix} i_S^T \\ i_{P_1^T} \\ i_{P_2^T} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.2(a)}$$

$$[U \quad 0] \begin{bmatrix} i_{P_1^T} \\ i_{P_2^T} \end{bmatrix} = 0 \tag{5.2(b)}$$

$$D(\mathbf{v}_S, \mathbf{i}_S) = 0 \qquad\qquad (5.2(c))$$

$$\begin{bmatrix} (\mathbf{A}'_{1S})^T & \mathbf{A}^T_{2S} \\ \mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^T_{23} \end{bmatrix} \begin{matrix} \mathbf{e}^T_1 \\ \mathbf{e}^T_2 \end{matrix} = \begin{bmatrix} \mathbf{v}^T_S \\ \mathbf{v}^T_{P_1} \\ \mathbf{v}^T_{P_2} \end{bmatrix} \qquad\qquad (5.2(d))$$

$$[0 \ \ \mathbf{U}] \begin{matrix} \mathbf{v}^T_{P_1} \\ \mathbf{v}^T_{P_2} \end{matrix} = 0 \qquad\qquad \mathbf{or} \qquad\qquad (\mathbf{K}^T_{23}) \, \mathbf{e}^T_2 = 0. \qquad\qquad (5.2(e))$$

Equations (5.2(a)), (5.2(b)) follow $\mathbf{V}^1_S = \mathbf{V}_{SP} \leftarrow \mathbf{V}^1_P$. Equations (5.2(d)), (5.2(e)) follow from $(\mathbf{V}^1_S)^\bullet = \mathbf{V}^\bullet_{SP} \leftarrow (\mathbf{V}^1_P)^\bullet$. Equations (5.2) may be rewritten as (5.3) below:

$$(\mathbf{A}_{2S}) \mathbf{i}^T_s = - (\mathbf{K}_{23}) \mathbf{i}^T_{P_2} \qquad\qquad (5.3(a))$$

$$- \mathbf{v}^T_S + (\mathbf{A}^T_{2S}) \mathbf{e}^T_2 = - (\mathbf{A}'_{1S})^T \mathbf{v}_{P_1^T} \qquad\qquad (5.3(b))$$

$$D(\mathbf{v}_S, \mathbf{i}_S) = 0 \qquad\qquad (5.3(c))$$

$$(\mathbf{K}^T_{23}) \mathbf{e}^T_2 = 0 \qquad\qquad (5.3(d))$$

$$\mathbf{A}'_{1S} \mathbf{i}^T_S = 0 \qquad\qquad (5.3(e))$$

Equations (5.3) are equivalent to the equations (5.2) whatever be the device characteristic $D(\mathbf{v}_S, \mathbf{i}_S) = 0$. However, when networks $\mathbf{N}_1$ and $\mathbf{N}_2$ have unique solutions the following convenient procedure may be adopted for solving $\mathbf{N}_1$.

Observe that Equations (5.3 (a), (b), (c)) would reduce to the equations of $\mathbf{N}_2$ if the right side were zero. Let us for simplicity suppose that (5.3(c)) has the form

$$\mathbf{E} \, \mathbf{i}^T_S + \mathbf{F} \, \mathbf{v}^T_S = \mathbf{s} \qquad\qquad (5.4)$$

In order to solve $\mathbf{N_1}$ we could first solve i_S, e_2, $\mathbf{v}_S$ in terms of i_{P_2}, $\mathbf{v}_{P_1}$, and s. In the linear steady state case this is equivalent to solving $\mathbf{N_2}$, $|P|$ + 1 times. The equations (5.3 (d), (e)) can then be converted to an equation involving i_{P_2}, $\mathbf{v}_{P_1}$ and s. Solving this equation would yield i_{P_2}, $\mathbf{v}_{P_1}$ and s. Back substitution would yield i_S, $\mathbf{v}_S$. A slight modification of this technique will permit us to handle non-linear, dynamic networks also. It is clear from the construction that K_{23}, $(A'_{1S})^T$ have linearly independent columns. It follows that be addition of a suitable sub-set of rows of (5.3 (a)) to (5.3 (d)) and a suitable subset of (5.3 (b)) to (5.3 (e)) we get equations (5.5) below with the coefficient matrix on the right hand side being nonsingular.

$$\begin{bmatrix} H_{11} & H_{12} & 0 \\ H_{21} & H_{22} & H_{23} \end{bmatrix} \begin{matrix} i_S^T \\ e_2^T \\ v_S^T \end{matrix} = \begin{bmatrix} H_{14} & 0 \\ 0 & H_{25} \end{bmatrix} \begin{matrix} i_{P_2}^T \\ v_{P_1}^T \end{matrix} \tag{5.5}$$

Assume that we know $i_{P_2}^k(t_1)$, $v_{P_1}^k(t_1)$, $s(t_1)$ (where s is the source vector). We then solve the nonlinear resistive network $N_2^R(t_1)$ (obtained by using a non-dynamic approximation of $D(\mathbf{v}_S, i_S) = 0$ at t_1) iteratively in terms of $i_{P_2}^k(t_1)$, $v_{P_1}^k(t_1)$, $s(t_1)$ and obtain $i_S^k(t_1)$, $v_S^k(t_1)$, $e_2^k(t_1)$ and thence using (5.5) obtain $i_{P_2}^{(k+1)}(t_1)$, $v_{P_1}^{k+1}(t_1)$. The procedure can be repeated at $(t_1+\Delta)$ using $i_{P_2}^\infty(t_1)$ as $i_{P_2}^0(t_1+\Delta)$, $v_{P_1}^\infty(t_1)$ as $v_{P_1}^0(t_1+\Delta)$.

5.4 Two special topological transformations

The derivation of subsection 5.3 glosses over practically important details such as the construction of A'_{1S}, A'_{2S}, K_{23} and the size of the border $|P|$. Our aim there has been merely to show that such a procedure is possible rather than that it is efficient. The method can compete with other methods only if the required matrices can be computed efficiently, $|P|$ is small and N_2 has a very

"desirable" structure. We next present two cases where N_2 has some specified sets $S_1, \ldots, S_n$ as separators. We assume that these sets have been chosen in such a way that $|P|$ would be small ie we will assume that in the original network $S_1, \ldots, S_n$ are "loosely" connected. (This problem can only be handled heuristically). In the cases presented the required matrices can be computed efficiently and we present algorithms for doing so.

Case I   New graph $\equiv (G \times S_1) \oplus \cdots \oplus (G \times S_n)$.

Let $N_1 = (S, V_S, D_s)$ and let S_i ($i = 1, \ldots, n$) be a partition of S. We will assume that V_S is the coboundary space of a graph G. Choose $N_2 = (S, \overset{n}{\underset{1=1}{\oplus}} V_S \times S_i, D_S)$. In this case A'_{nS} would simply be the reduced incidence matrix of $(G \times S_1 \oplus \cdots \oplus G \times S_n)$. A'_{2S} would have no rows. K_{23} would have no columns. Algorithm I describes how to construct A'_{1S}. In this case $P_1 = P$ and $|P| = r(G) - \overset{n}{\underset{i=1}{\sum}} r(G \times S_i)$. Observe that N_2 is easier to solve if in the device characteristic $D(v_S, i_S) = 0$, the sets S_i, $i = 1, \ldots, $ n appear decoupled. Solving N_2 would then be equivalent to solving n smaller networks.

Algorithm I.   To construct A'_{1S} when $N_2 = (S, \overset{n}{\underset{i=1}{\oplus}} V_S \times S_i, D_S)$

Step I   Select trees t_i of graphs $G \times S_i$ (i = 1,...,n).

Step II   Construct the reduced incidence matrix of the graph $G \times (S - (\overset{n}{\underset{i=1}{\cup}} t_i))$.

Adjoin zero columns corresponding to $\overset{n}{\underset{i=1}{\cup}} t_i$. A'_{1S} is the resulting matrix.

END

Justification for Algorithm I.   By   Theorem   P9, $V_{cob}(G \times (S - \overset{n}{\underset{i=1}{\cup}} t_i)) = (V_{cob}(G)) \times (S - \overset{n}{\underset{i=1}{\cup}} t_i)$. The rows of A'_{1S} are linearly

independent and belong to $\mathbf{V}_{cob}(G)$ since they are obtained by padding vectors of $\mathbf{V}_{cob}(G) \times (S - \overset{n}{\underset{i=1}{\cup}} t_i)$ with zeros corresponding to $\overset{n}{\underset{i=1}{\cup}} t_i$. Since t_i forms a forest of $G \times S_i$ (i = 1,....,n) in the reduced incidence matrix of $\overset{n}{\underset{i=1}{\oplus}} G \times S_i$ the columns corresponding to $\overset{n}{\underset{i=1}{\cup}} t_i$ from a linearly independent set. In the matrix $\mathbf{A}'_{1S}$ we have only zeroes corresponding to these columns. Further the number of rows of $\mathbf{A}'_{1S}$ is $\tau(\mathbf{V}_{cob}(G)) - |\overset{n}{\underset{i=1}{\cup}} t_i|$. Hence the rows of $\mathbf{A}'_{1S}$ together with the rows of a reduced incidence matrix of $\overset{n}{\underset{i=1}{\oplus}} G \times S_i$ form a basis of $\mathbf{V}_{cob}(G)$.

Example 5.1 Let G in Fig 5.1 (a) be the graph of $\mathbf{N}_1$. Let $S_1 = \{1, \cdots 5\}$, $S_2 = \{6, \cdots 10\}$, $S_3 = \{11, \cdots 17\}$, $S_4 = \{18, \cdots 24\}$. The graph $\oplus$ $G \times S_i$ is shown in Fig 5.1(b). Here $t_1 = \{1, 2\}$, $t_2 = \{6, 7\}$, $t_3 = \{11, 12\ 13\ 14\}$, $t_4 = \{18, 19, 20, 21\}$. $\tau(G) - \overset{4}{\underset{i=1}{\sum}} \tau(G \times S_i) = 3$. So we need 3 extra variables for acquiring the advantages of working with S_1, S_2, S_3, S_4 as separators - $\mathbf{v}_{P_1}$ has three variables and $\mathbf{A}'_{1S}$ has 3 rows. The rewritten equations are as follows:

$$(\mathbf{A}_{211})\ \mathbf{i}_{S_1}^T = 0$$

$$(\mathbf{A}_{222})\ \mathbf{i}_{S_2}^T = 0$$

$$(\mathbf{A}_{233})\ \mathbf{i}_{S_3}^T = 0$$

$$(\mathbf{A}_{244})\ \mathbf{i}_{S_4}^T = 0$$

($\mathbf{A}_{2ii}$ is a reduced incidence matrix of $G \times S_i$)

$$
\begin{pmatrix}
-\mathbf{v}_{S_1}^T + (A_{211}^T)\mathbf{e}_{21}^T \\
-\mathbf{v}_{S_2}^T + (A_{222}^T)\mathbf{e}_{22}^T \\
-\mathbf{v}_{S_3}^T + (A_{233}^T)\mathbf{e}_{23}^T \\
-\mathbf{v}_{S_4}^T + (A_{244}^T)\mathbf{e}_{24}^T
\end{pmatrix}
= -
\begin{pmatrix}
(A_{1S}')^T
\end{pmatrix}
\mathbf{v}_{P_1^T}
$$

$$D(\mathbf{v}_S, \mathbf{i}_S) = 0$$

$$(A_{1S}')\mathbf{i}_S^T = 0$$

A_{1S}' may be chosen as the reduced incidence matrix of the graph obtained by adding $\cup\, t_i$ as self loops to $G \times (S - \overset{4}{\underset{i=1}{\cup}} t_i)$ shown in Fig. 5.1 (c). In this case, since the coboundary space of the graph of $\mathbf{N}_2$ is contained in the coboundary space of the graph of $\mathbf{N}_1$, the matrix K_{23}^T has no rows.

Case 2. New graph $= \overset{n}{\underset{i=1}{\oplus}} G \cdot S_i$

Let $\mathbf{N}_1 = (S, \mathbf{V}_S, \mathbf{D}_S)$ and let S_i ($i = 1,...n$) be a partition of S. We will assume that $\mathbf{V}_S$ is the coboundary space of a graph G. Choose $\mathbf{N}_2 = (S, \overset{n}{\underset{i=1}{\oplus}} \mathbf{V}_S \cdot S_i, \mathbf{D}_S)$. In this case A_{nS}' would simply be the reduced incidence matrix of G. Algorithm II describes how to construct K_{23}^T. In this case $P_2 = P$ and $|P| = (\overset{n}{\underset{i=1}{\sum}} r(G \cdot S_i) - r(G))$. Observe that $\mathbf{N}_2$ is easier to solve if in the device characteristic $\mathbf{D}(\mathbf{v}_S, \mathbf{i}_S) = 0$, S_i ($i = 1,...n$) appear decoupled. Solving $\mathbf{N}_2$ would then be equivalent to solving n smaller networks.

Algorithm II. To construct K_{23}^T when $\mathbf{N}_2 = (S, \overset{n}{\underset{i=1}{\oplus}} \mathbf{V}_S \cdot S_i, \mathbf{D}_S)$.

Let $\mathbf{n_b}$ be the set of boundary nodes of G where edges of more than one S_i are incident. Let the copy of node e_k of $\mathbf{n_b}$ in the graph $G \cdot S_i$ be named e_{ki}. Let

m_i be the number of components of $G \cdot S_i$ (i = 1,...n). For j = 1,...m_i and i = 1, ... n do the following. If the jth component of $G \cdot S_i$ has boundary nodes select a boundary node $e_{k_{ij} i}$ as a datum node. Construct the graph G_b by adding an edge e_{ki} directed from e_k to $e_{k_{ij}}$, for each boundary node e_{ki} in the jth component of $G \cdot S_i$. Let B_b be a cycle matrix of G_b. To this matrix add zero columns corresponding to nonboundary nodes of each $G \cdot S_i$ The resulting matrix is K_{23}^T.

■

Justification of Algorithm II. The condition $K_{23}^T e_2^T = 0$ represents the maximal linearly independent voltage constraints on all the nodes of $G \cdot S_i$ (i = 1,...n) in order to connect the $G \cdot S_i$ to makeup the graph G. The KVL conditions of G_b also represent the voltage constraints on all the boundary nodes of $G \cdot S_i$ (i = 1,...n) in order to connect the $G \cdot S_i$ to make up the graph G. If B_b is a cycle matrix of G_b then $B_b e^T = 0$ is a maximal linearly independent set of constraints among them. Padding the matrix B_b with zero columns corresponding to internal nodes of $G \cdot S_i$ (i = 1,...n) therefore yields (a candidate for) K_{23}^T.

Example 5.2. Let G in Fig 5.1 (a) be the graph of N_1. Let $S_1 = \{1,...5\}$, $S_2 = \{6,...10\}$, $S_3 = \{11, \cdots 17\}$, $S_4 = \{18, \cdots 24\}$. The graph $\bigoplus_{i=1}^{4} G \cdot S_i$ is shown in Fig 5.2 (a). In this case $(\sum_{i=1}^{4} r(G \cdot S_i) - r(G)) = 1$. So we need one extra variable for acquiring the advantages of working with S_1, S_2, S_3, S_4 as separators. i_{P_2} has one variable and K_{23}^T has one row. The rewritten equations are as follows:

$$\left.\begin{array}{c} (A_{211})i_{S_1}^T \\ (A_{222})i_{S_2}^T \\ (A_{233})i_{S_3}^T \\ (A_{244})i_{S_4}^T \end{array}\right\} = \left[K_{23}\right]i_{P_2^T}$$

$$- v_{S_1}^T + A_{211}^T\, e_{21}^T = 0$$

$$- v_{S_2}^T + A_{222}^T\, e_{22}^T = 0$$

$$- v_{S_3}^T + A_{233}^T\, e_{23}^T = 0$$

$$- v_{S_4}^T + A_{244}^T\, e_{24}^T = 0$$

$$D(v_S, i_S) = 0$$

$$(K_{23}^T)e_2^T = 0 \ .$$

Here A_{2ii} is the reduced incidence matrix of $G \cdot S_i$. For the jth component of $G \cdot S_i$ (if it has a boundary node) the node $e_{k_{ij}}i$ must be chosen as the datum node i.e., the datum nodes selected in the construction of G_b (see Fig. 5.2 (b)) and the datum nodes selected for constructing reduced incidence matrix of $G \cdot S_i$ must be the same. In this example $e_{k_{11}} = e_2$, $e_{k_{21}} = e_3$, $e_{k_{31}} = e_1$, $e_{k_{41}} = e_3$. Different k_{ij} can turn out to be identical. Here, for instance, $k_{21} = k_{41} = 3$. It simply means that copies of the same boundary node (e_3 in this case) have been chosen as datum nodes in components of different $G \cdot S_i$. The matrix K_{23}^T has a single row in this example. It has entries for each non-datum node of each component of $G \cdot S_i$ (i = 1,...n). In this case it has entries corresponding to all the nodes of $\bigoplus\limits_{i=1}^{4} G \cdot S_i$ except e_{21}, e_{32}, e_{13} and e_{34} i.e., it has 16 entries. Of these entries all entries except those corresponding to edges of G_b are zero. Corresponding to edges of G_b we have the entries

$$(\cdots \overset{\overset{e_{11}\ e_{24}\ e_{42}\ e_{43}}{}}{1\quad 1\quad -1\quad 1}\ \cdots).$$ (This is a circuit vector of G_b). In this case A'_{1S} has no rows since V_S is a subspace of $\sum\limits_{i=1}^{4}(V \cdot S_i)$.

6. Decomposition of a vector space.

6.1. In this section we introduce another vector space notion based on the g-minor operation, namely decomposition of a vector space. The notion of decomposition arises when we decompose an electrical network into several submultiports and a port connection diagram. For theoretical network analysis the notion of decomposition provides a convenient means of network reduction. Properties of the original network, pertaining to interaction between different component multiports can be transferred to a reduced network based on the port connection diagram. A more detailed study of vector space decomposition is available in [6].

6.2. Definitions and Theorems

Definition 6.1. Let V_S be a vector space on S. Let S be partitioned into $S_1, \cdots S_n$. Let sets $P_1, \cdots P_n$ be pairwise disjoint and disjoint from S. Let $P = \bigcup\limits_{i=1}^{n} P_i$. Let $V_{S_iP_i}$ (i = 1,...n), V_P be vector spaces on $S_i \cup P_i$ (i = 1,...n), P respectively. We say that $\{V_P, V_{S_1P_1}, \cdots V_{S_nP_n}\}$ is an *n-decomposition of* V_S with respect to $S_1, \cdots S_n$ iff $V_S = (V_{S_1P_1} \oplus \cdots \oplus V_{S_nP_n}) \leftarrow V_P$. $V_{S_iP_i}$ (i = 1,...n) will be called *components* of the decomposition. P_i will be called the *sets of ports* of the component $V_{S_iP_i}$. V_P will be called the *coupler* of the decomposition. The decomposition is said to be *minimal* iff whenever $\{V_{P'}, V_{S_1P'_1}, \cdots V_{S_nP'_n}\}$ is an n-decomposition of V_S with respect to $S_1, \ldots S_n$, $|P'| \geq |P|$.

Example 6.1. Consider the graph G in Fig. 6.1. Let S be the set of edges of G.

Let $S_1 = \{1,..5\}$, $S_2 = \{6,..12\}$, $S_3 = \{13,..19\}$, $P_1 = \{p_1\}$, $P_2 = \{p_2\}$, $P_3 = \{p_3\}$. We see that the graph G has been broken up into three multiports and a port connection diagram G_p. Let V_S, $V_{S_i P_i}$ (i = 1, 2, 3), V_P be the coboundary spaces of G, $G_{S_i P_i}$ (i = 1, 2, 3), G_p respectively. Then it is possible to show that $V_S = (\sum_{i=1}^{3} V_{S_i P_i}) \leftarrow V_P$.

We begin with a result which gives characteristic properties that a vector space must possess in order to be a component of a decomposition of a given space V_S.

Theorem 6.1. Let $V_{S_i P_i}$ (i = 1,....n) be spaces on $S_i \cup P_i$, i = 1, ... n, where $S_i \cap P_j = \varphi$ for all i,j, $S_i \cap S_j = \varphi$ for $i \neq j$, $P_i \cap P_j = \varphi$, $i \neq j$. Let $\cup S_i = S$, $\cup P_i = P$. Then, there exists V_P on P such that $V_S = (V_{S_1 P_1} \oplus \cdots \oplus V_{S_n P_n}) \leftarrow V_P$ iff $V_S \times S_i \supseteq V_{S_i P_i} \times S_i$, $V_S \cdot S_i \subseteq V_{S_i P_i} \cdot S_i$, (i = 1,....n).

Proof: V_S can be a g-minor of $(V_{S_1 P_1} \oplus \cdots \oplus V_{S_n P_n})$ iff $V_S \subseteq (V_{S_1 P_1} \oplus \cdots \oplus V_{S_n P_n}) \cdot S$ and $V_S \supseteq (V_{S_1 P_1} \oplus \cdots V_{S_n P_n}) \times S$ by Theorem 3.1 We will show that these two conditions are equivalent to the ones in the statement of the theorem. If $V_S \subseteq (V_{S_i P_i} \oplus \cdots \oplus V_{S_n P_n}) \cdot S$ then clearly $V_S \cdot S_i \subseteq V_{S_i P_i} \cdot S_i$ (i = 1,....n). Next let $V_S \cdot S_i \subseteq V_{S_i P_i} \cdot S_i$ (i = 1, ...n). Let $f_S \in V_S$. Then $f_S / S_i \in V_{S_i P_i} \cdot S_i$ (i = 1,....n). Hence $f_S \in V_{S_1 P_1} \cdot S_1 \oplus \cdots \oplus V_{S_n P_n} \cdot S_n$. But $(V_{S_1 P_1} \cdot S_1) \oplus \cdots \oplus (V_{S_n P_n} \cdot S_n) = (V_{S_1 P_1} \oplus \cdots \oplus V_{S_n P_n}) \cdot S$. Thus it is clear that $V_S \subseteq (V_{S_1 P_1} \oplus \cdots \oplus V_{S_n P_n}) \cdot S$. Next $V_S \supseteq (V_{S_1 P_1} \oplus \cdots \oplus V_{S_n P_n}) \times S$ iff $V_S^{\bullet} \subseteq (V_{S_i P_i} \oplus \cdots \oplus V_{S_n P_n})^{\bullet} \cdot S$ by Theorems P2, P4, i.e., iff $V_S^{\bullet} \cdot S_i \subseteq V_{S_i P_i}^{\bullet} \cdot S_i$ (i = 1,...n) i.e., iff $V_S \times S_i \supseteq V_{S_i P_i} \times S_i$ (i = 1,...n) by Theorems P2, P4.

Theorem 6.2. Let $\{V_P, V_{S_1 P_1}, \cdots V_{S_n P_n}\}$ be a decomposition of V_S. Then $\{V_P^*, V_{S_1 P_1}^*, \cdots V_{S_n P_n}^*\}$ is a decomposition of V_S^*.

Proof:
$$V_S = (V_{S_1 P_1} \oplus \cdots \oplus V_{S_n P_n}) \leftarrow V_P \qquad \text{iff}$$
$V_S^* = (V_{S_1 P_1} \oplus \cdots \oplus V_{S_n P_n})^* \leftarrow V_P^*$ by Theorem 3.3. By Lemma 4.1 this is equivalent to $V_S^* = (V_{S_1 P_1}^* \oplus \cdots V_{S_n P_n}^*) \leftarrow V_P^*$.

■

An immediate consequence of Theorem 6.2 and the definition of minimal decomposition is the following theorem:

Theorem 6.3. $\{V_P, V_{S_i P_i} \ (i = 1, \ldots n)\}$ is a minimal decomposition of V_S iff $\{V_P^*, V_{S_i P_i}^* \ (i = 1, \ldots n)\}$ is a minimal decomposition of V_S^*.

The next few theorems provide characteristic properties for a minimal decomposition.

Lemma 6.1. Let $\{V_P, V_{S_1 P_1}, \cdots V_{S_n P_n}\}$ be an n-decomposition of V_S. Then $|P_i| \geq r(V_S \cdot S_i) - r(V_S \times S_i)$ $i = 1, \ldots n$.

Proof: From Theorem 6.1 we know that

$$V_S \cdot S_i \subseteq V_{S_i P_i} \cdot S_i \ , \ V_S \times S_i \supseteq V_{S_i P_i} \times S_i \ .$$

Hence

$$r(V_S \cdot S_i) - r(V_S \times S_i) \leq r(V_{S_i P_i} \cdot S_i) - r(V_{S_i P_i} \times S_i) \ .$$

By Lemma 3.2,

$$r(V_{S_i P_i} \cdot S_i) - r(V_{S_i P_i} \times S_i) = r(V_{S_i P_i} \cdot P_i) - r(V_{S_i P_i} \times P_i) \leq |P_i| \ .$$

The Lemma follows.

Lemma 6.2. Let $\{V_P, V_{S_1 P_1}, \cdots V_{S_n P_n}\}$ be an n-decomposition of V_S. Then

$$r(V_S \cdot S_i) - r(V_S \times S_i) \leq r(V_P \cdot P_i) - r(V_P \times P_i) \quad i = 1,...n$$

Proof: Select vectors $\mathbf{f}_{S_i}^1 \cdots \mathbf{f}_{S_i}^k$ which together with a basis of $V_S \times S_i$ form a basis of $V_S \cdot S_i$. Then by the definition of decomposition there exist vectors $\mathbf{f}_{S_i P_i}^r$ ($r = 1,...k$) in $V_{S_i P_i}$ such that

$$\mathbf{f}_{S_i P_i}^r / S_i = \mathbf{f}_{S_i}^r , \ \mathbf{f}_{S_i P_i}^r / P_i \in V_P \cdot P_i$$

Suppose $\{\mathbf{f}_{S_i P_i}^r / P_i \ (r = 1,..k)\}$ does not form a linearly independent set with a basis of $V_P \times P_i$. Then there exists $\mathbf{f}_{S_i P_i} \in V_{S_i P_i}$ such that $\mathbf{f}_{S_i P_i} / P_i$ is a nontrivial linear combination of $\mathbf{f}_{S_i P_i}^r / P_i$ ($r = 1...k$) and belongs to $V_P \times P_i$. This linear combination of $\mathbf{f}_{S_i P_i}^r / S_i$ will however yield a vector which belongs to $V_S \cdot S_i - V_S \times S_i$. Thus the vector $\mathbf{f}_{S_i P_i} \in V_{S_i P_i}$ is such that $\mathbf{f}_{S_i P_i} / S_i \in V_S \cdot S_i - V_S \times S_i$, $\mathbf{f}_{S_i P_i} / P_i \in V_P \times P_i$. Consider a vector $\mathbf{f}_{SP} \in \bigoplus_{i=1}^{n} V_{S_i P_i}$ such that

$$\mathbf{f}_{SP / S_i \cup P_i} = \mathbf{f}_{S_i P_i}$$

$$\mathbf{f}_{SP / (S \cup P) - (S_i \cup P_i)} = 0 \ .$$

Next choose a vector $\mathbf{f}_P \in V^P$ such that $\mathbf{f}_P$ is zero outside P_i and $\mathbf{f}_P / P_i = \mathbf{f}_{S_i P_i} / P_i$. This is possible since $\mathbf{f}_{S_i P_i} / P_i \in V_P \times P_i$. Since $\mathbf{f}_{SP} / P = \mathbf{f}_P$ it follows that $\mathbf{f}_{SP} / S \in V_S$. But then $\mathbf{f}_{SP} / S_i \in V_S \times S_i$. This contradicts the fact that $\{\mathbf{f}_{S_i P_i}^r / S_i\}$ forms a linearly independent set with a basis of $V_S \times S_i$. We con-

clude that $\{\mathbf{f}^r_{S_i P_i}/P_i \ (r = 1,...k)\}$ forms a linearly independent set with a basis of $\mathbf{V}_P \times P_i$ i.e., $r(\mathbf{V}_P{\cdot}P_i) - r(\mathbf{V}_P{\times}P_i) \geq r(\mathbf{V}_S{\cdot}S_i) - r(\mathbf{V}_S{\times}S_i)$.

∎

We later present an algorithm for the construction of a minimal decomposition of a vector space $\mathbf{V}_S$ that is the coboundary space of a graph and has $|P| = \Sigma(r(\mathbf{V}_S{\cdot}S_i) - r(\mathbf{V}_S{\times}S_i))$. A simple algorithm of the same kind can be given even for a general vector space $\mathbf{V}_S$ [6]. We therefore assert

Lemma 6.3. $\{\mathbf{V}_P, \mathbf{V}_{S_1 P_1},\cdots\mathbf{V}_{S_n P_n}\}$ is a minimal n-decomposition of $\mathbf{V}_S$ iff $|P_i| = r(\mathbf{V}_S{\cdot}S_i) - r(\mathbf{V}_S{\times}S_i)$. (i = 1,...n).

Theorem 6.4. Let $\{\mathbf{V}_P, \mathbf{V}_{S_1 P_1},\cdots\mathbf{V}_{S_n P_n}\}$. be an n-decomposition of $\mathbf{V}_S$. Then the following statements are equivalent:

(a) It is minimal

(b) $|P_i| = r(\mathbf{V}_S{\cdot}S_i) - r(\mathbf{V}_S{\times}S_i)$ (i = 1,...n)

(c) P_i has no circuits or bonds in $M(\mathbf{V}_{S_i P_i})$ (i=1,...n) or $M(\mathbf{V}_P)$.

Proof: By Lemma 6.3 we know that (a) and (b) are equivalent. We will now show that (b) and (c) are equivalent. Let $|P_i| = r(\mathbf{V}_S{\cdot}S_i) - r(\mathbf{V}_S{\times}S_i)$ (i = 1,...n). Then for all i, by Theorem 6.1, $|P_i| \leq r(\mathbf{V}_{S_i P_i}{\cdot}S_i) - r(\mathbf{V}_{S_i P_i}{\times}S_i)$. Hence by Lemma 3.2, $|P_i| \leq r(\mathbf{V}_{S_i P_i}{\cdot}P_i) - r(\mathbf{V}_{S_i P_i}{\times}P_i)$. By Lemma 6.2, $|P_i| \leq r(\mathbf{V}_P{\cdot}P_i) - r(\mathbf{V}_P{\times}P_i)$. It follows that $|P_i| = r(\mathbf{V}_{S_i P_i}{\cdot}P_i) = r(\mathbf{V}_p{\cdot}P_i)$ and $0 = r(\mathbf{V}_{S_i P_i}{\times}P_i) = r(\mathbf{V}_P{\times}P_i)$ for all i. Hence by Theorem P7, $M(\mathbf{V}_{S_i P_i}{\cdot}P_i)$ and $M(\mathbf{V}_P{\cdot}P_i)$ do not have circuits and $M(\mathbf{V}_{S_i P_i}{\times}P_i)$ and $M(\mathbf{V}_P{\times}P_i)$ do not have bonds. Hence P_i has no circuits or bonds in $M(\mathbf{V}_{S_i P_i})$, (i = 1,...n) or $M(\mathbf{V}_P)$ by Theorem P6. Conversely suppose P_i has no circuits or bonds in $M(\mathbf{V}_{S_i P_i})$, or $M(\mathbf{V}_P)$. Then for all i

$$|P_i| = \tau(V_{S_iP_i} \cdot P_i) = \tau(V_P \cdot P_i) \ ,$$
$$0 = \tau(V_{S_iP_i} \times P_i) = \tau(V_P \times P_i) \ .$$

So

$$|P_i| = \tau(V_{S_iP_i} \cdot P_i) - \tau(V_{S_iP_i} \times P_i)$$
$$= \tau(V_{S_iP_i} \cdot S_i) - \tau(V_{S_iP_i} \times S_i)$$

by Lemma 3.2. Suppose $|P_j| > \tau(V_S \cdot S_j) - \tau(V_S \times S_j)$ for some j. Hence $\tau(V_{S_jP_j} \cdot S_j) - \tau(V_{S_jP_j} \times S_j) > \dot{\tau}(V_S \cdot S_j) - \tau(V_S \times S_j)$. Then by Theorem 6.1 there exists

$$f_{S_j} \in (V_{S_jP_j} \cdot S_j - V_S \cdot S_j) \quad \text{or} \quad g_{S_j} \in (V_S \times S_j - V_{S_jP_j} \times S_j) \ .$$

Assume the former. Then there exists a vector $f_{S_jP_j} \in V_{S_jP_j}$ such that $f_{S_jP_j}/S_j = f_{S_j}$. Since $V_P \cdot P_j$ has full rank there must exist a vector $f_P \in V_P$ such that $f_{P/P_j} = f_{S_jP_j}/P_j$. Since $V_{S_iP_i} \cdot P_i$ has full rank for all i it follows that there exist vectors $f_{S_iP_i}$ for all i such that $f_{S_iP_i}/P_i = f_{P/P_i}$. Then by the definition of n-decomposition the vector $f_S = \sum_{i=1}^{n} f_{S_iP_i}/S_i$ belongs to V_S. This

contradicts our assumption. Next suppose there exists $g_{S_j} \in V_S \times S_j - V_{S_jP_j} \times S_j$. Then there exists a vector $g_S \in V_S$ such that $g_S/S_j = g_{S_j}$ and $g_S/(S-S_j) = 0$. By the definition of decomposition there exist vectors $g_{S_iP_i} \in V_{S_iP_i}$ and $g_P \in V_P$ such that

$$g_{S_iP_i}/S_i = g_S/S_i \quad (i = 1,...n)$$

and

$$g_{P/P_i} = g_{S_iP_i}/P_i \quad (i = 1,...n) \ .$$

But then

$$g_{S_iP_i}/S_i = 0 \quad i \neq j \ .$$

Hence $g_{S_i P_i / P_i} \in V_{S_i P_i} \times P_i, i \neq j$. But $V_{S_i P_i} \times P_i$ has zero rank for all i. Hence $g_{P / P_i} = 0$ for $i \neq j$. Hence $g_{P / P_j} \in V_P \times P_j$. But $V_P \times P_j$ also has zero rank. Hence $g_{P / P_j} = 0$ and hence $g_{S_j P_j / P_j} = 0$. This contradicts the fact that

$$g_{S_j P_j / S_j}(= g_{S_j}) \in V_S \times S_j - V_{S_j P_j} \times S_j \quad .$$

∎

We now present an algorithm for minimal n-decomposition of a space V_S that is graphic. It seems difficult (if not impossible) to give an algorithm that makes both $V_{S_i P_i}$ (i = 1...n) and V_P graphic. Justification for this algorithm is given in Appendix I.

III. Algorithm for a minimal n-decomposition of a coboundary space Let $\widehat{G}$ be a graph on S. Let $\widehat{A}$ be its incidence matrix. Let S be partitioned into S_1, S_2 ... S_n. Let $V_{cob}(\widehat{G}) = V_S$.

Step 1. Construct $\widehat{G} \times S_i$ (i = 1,2,...n).

Step 2. Construct trees $\widehat{t}_i$ for $\widehat{G} \times S_i$ (i = 1,2,...n).

Step 3. Construct $\widehat{G} \times (S - \bigcup_{i=1}^{n} \widehat{t}_i)$. Add the edges $\bigcup_{i=1}^{n} \widehat{t}_i$ back again as self loops. Let the resulting graph on S be G. Let **A** be its reduced incidence matrix.

Step 4. Select trees t_i for $G \cdot S_i$. Let A_{t_i} be the submatrix of A corresponding to the columns t_i.

Step 5. Construct the matrix A_P as shown below.

$$A_P = [\overset{P_1}{A_{t_1}} \vdots \cdots \overset{P_n}{A_{t_n}}]$$

V_P is the space generated by the rows of this matrix.

Step 6. Let A_{iS_i} be the reduced incidence matrix of $G \cdot S_i$ (i = 1,...n). Let A_{it_i} be the submatrix of A_{iS_i} corresponding to columns t_i. Let $\widehat{A}_{ii}$ be the reduced incidence matrix of $\widehat{G} \times S_i$ (i = 1,...n) The generator matrix of $V_{S_i P_i}$ (i = 1,...n) is shown below

$$A_{S_i P_i} = \begin{matrix} S_i & P_i \\ \begin{bmatrix} \widehat{A}_{ii} & 0 \\ A_{iS_i} & A_{it_i} \end{bmatrix} \end{matrix}$$

∎

Remark: Observe that A_P is the reduced incidence matrix of a graph, while the spaces $V_{S_i P_i}$ need not be coboundary spaces of graphs. An algorithm that makes $V_{S_i P_i}$ (i = 1,...n) graphic (but V_P non graphic) is given in [6].

6.3. Applications to network theory

The idea of decomposition is particularly relevant when the network can be naturally partitioned into different types of elements such as resistors, inductors, capacitors or linear, nonlinear or faulty and good elements etc. Solutions that can exist in such networks may be classified as of two kinds: "Trapped" and "interactive" solutions. Trapped solutions "lie" entirely within one type of element. A solution that is not trapped will be called interactive. By examining sections of such solutions corresponding to one part of the network, one can get an idea of what is happening to another part of the network. Trapped solutions can yield no such information. Interactive solutions can be studied more conveniently by working with a network defined on the coupler space of the decomposition. The new reduced network will be minimal when the decomposition is minimal.

Definition 6.2. Let $N = (S, V_S, D_S)$ be a generalized network. Let S be partitioned into $S_1, \cdots S_n$. Let these sets appear decoupled in the device characteris-

tic. We say that $(\mathbf{v}_S,\mathbf{i}_S)$ is a *trapped solution* with respect to the partition $(S_1,\cdots S_n)$ iff $\mathbf{v}_S \in \sum_{i=1}^{n} \mathbf{V}_S \times S_i$ and $\mathbf{i}_S \in \sum_{i=1}^{n} \mathbf{V}_S^{\bullet} \times S_i$.

Observe that in the above definition if $\mathbf{V}_S$ is the coboundary space of a graph G, a solution $(\mathbf{v}_S,\mathbf{i}_S)$ is a trapped solution iff $\mathbf{v}_S$ can exist in $\bigoplus_{i=1}^{n} G \times S_i$ and $\mathbf{i}_S$ can exist in $\bigoplus_{i=1}^{n} G \cdot S_i$. One may imagine the voltages lying trapped within cutsets of G lying entirely in a single S_i and currents as trapped within circuits of G lying entirely in a single S_i. Such solutions cannot be observed at the ports of the component multiports and hence will not be reflected onto the network defined on the coupler space of the decomposition.

Multiport decomposition can be used in theoretical network analysis in the following manner.

Let $\mathbf{N} = (S,\mathbf{V}_S,\mathbf{D}_S)$ be a generalized network and let S be partitioned into $(S_1,S_2\cdots S_n)$. Let $S_1, S_2 \cdots S_n$ appear decoupled in the device characteristic. Let $(\mathbf{V}_P,\mathbf{V}_{S_1 P_1},\cdots \mathbf{V}_{S_n P_n})$ be an n-decomposition of $\mathbf{V}_S$. Define networks $\mathbf{N}_{iP_i} = (S_i \cup P_i,\mathbf{V}_{S_i P_i},\mathbf{D}_{S_i P_i})$. (i = 1,...n). $\mathbf{D}_{S_i P_i}$ imposes no constraint at all on P_i and precisely the same constraints on S_i as $\mathbf{D}_S$. Solving the network $\mathbf{N}$ is then equivalent to finding solutions $(\mathbf{v}_{S_i P_i},\mathbf{i}_{S_i P_i})$ (i = 1,...n) of $\mathbf{N}_i$ such that $\sum_{i=1}^{n} \mathbf{v}_{S_i P_i}/P_i \in \mathbf{V}_P$ and $\sum_{i=1}^{n} \mathbf{i}_{S_i P_i}/P_i \in \mathbf{V}_P^{\bullet}$.

In other words, in order to solve $\mathbf{N}$, we solve the "multiports" $\mathbf{N}_{iP_i}$ in terms of the port variables, in the process obtaining the "port behavior" of $\mathbf{N}_{iP_i}$ (i.e., the set of all $\mathbf{v}_{P_i}$, $\mathbf{i}_{P_i}$ that can coexist on P_i in the multiport $\mathbf{N}_{iP_i}$). We then define a new coupler network $\mathbf{N}_P$ on P with coboundary space $\mathbf{V}_P$ (the "port connection diagram") and device characteristic on P_i as the port behavior of the multiports $\mathbf{N}_{iP_i}$. Solving $\mathbf{N}_P$ gives the possible port vectors. Since in each $\mathbf{N}_{iP_i}$ the

other variables have been determined with respect to the port variables this completes the solution of **N**. Observe that trapped solutions will coexist with zero vectors on the portedges which means that the trapped solutions of the network correspond to the zero solution of $\mathbf{N}_P$. The nonzero solutions of $\mathbf{N}_P$ will correspond to the interactive solutions of **N**. We will illustrate these ideas for an RLMC network.

Let $\mathbf{N} = (S, \mathbf{V}_S, \mathbf{D}_S)$ be a generalized linear RLMC network. Let S be partitioned S_R, S_L, S_C corresponding to resistors, inductors and capacitors. $\mathbf{D}_S$ is equivalent to the constraints

$$\mathbf{v}_R^T = (\mathbf{R}) \, \mathbf{i}_R^T$$

$$\mathbf{v}_L^T = (\mathbf{L}) \, \mathbf{i}_L^T$$

$$\mathbf{i}_C^T = (\mathbf{C}) \, \dot{\mathbf{v}}_C^T$$

where **R**, **L**, **C** are positive definite matrices and subscripts R, L, C have been used in place of S_R, S_L, S_C. Let $\mathbf{V}_S$ have the minimal decomposition $\{\mathbf{V}_P, \mathbf{V}_{S_R P_R}, \mathbf{V}_{S_L P_L}, \mathbf{V}_{S_C P_C}\}$. Let $\mathbf{V}_{S_R P_R}^{\bullet}$, $\mathbf{V}_{S_L P_L}^{\bullet}$, $\mathbf{V}_{S_C P_C}$ have generating matrices:

$$\begin{bmatrix} \mathbf{B}_{1R} & \vdots & 0 \\ \mathbf{B}_{2R} & \vdots & \mathbf{U} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{B}_{1L} & \vdots & 0 \\ \mathbf{B}_{2L} & \vdots & \mathbf{U} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{Q}_{1C} & \vdots & 0 \\ \mathbf{Q}_{2C} & \vdots & \mathbf{U} \end{bmatrix}$$

Observe that the matrices $\begin{bmatrix} \mathbf{B}_{1R} \\ \mathbf{B}_{2R} \end{bmatrix}, \begin{bmatrix} \mathbf{B}_{1L} \\ \mathbf{B}_{2L} \end{bmatrix}, \begin{bmatrix} \mathbf{Q}_{1C} \\ \mathbf{Q}_{2C} \end{bmatrix}$ have linearly independent rows. This follows from the fact that for minimal decomposition $\mathbf{V}_{S_i P_i} \times P_i$ has zero rank. We know that the solution of the original network is equivalent to the solution of the following equations (6.1), (6.2), (6.3), and (6.4)

$$\left.\begin{array}{l} \begin{bmatrix} \mathbf{B}_{1R} & \vdots & 0 \\ \mathbf{B}_{2R} & \vdots & \mathbf{U} \end{bmatrix} \begin{matrix} \mathbf{v}_R^T \\ \mathbf{v}_{P_R}^T \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\[4pt] \begin{bmatrix} \mathbf{B}_{1R}^T & \mathbf{B}_{2R}^T \\ 0 & \mathbf{U} \end{bmatrix} \begin{matrix} \mathbf{i}_R^T \\ \mathbf{i}_{P_R}^T \end{matrix} = \begin{pmatrix} \mathbf{i}_R^T \\ \mathbf{i}_{P_R}^T \end{pmatrix} \\[6pt] (R)\mathbf{i}_R^T = \mathbf{v}_R^T \end{array}\right\}$$

Equations for the
resistive multiport
$$\mathbf{N}_{RP_R} = \left\{ S_R \cup P_R,\ \mathbf{V}_{S_R P_R},\ D_{S_R P_R} \right\} \tag{6.1}$$

$$\left.\begin{array}{l} \begin{bmatrix} \mathbf{B}_{1L} & \vdots & 0 \\ \mathbf{B}_{2L} & \vdots & \mathbf{U} \end{bmatrix} \begin{matrix} \mathbf{i}_L^T \\ \mathbf{v}_{P_L}^T \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\[4pt] \begin{bmatrix} \mathbf{B}_{1L}^T & \vdots & \mathbf{B}_{2L}^T \\ 0 & \vdots & \mathbf{U} \end{bmatrix} \begin{matrix} \mathbf{i}_L^T \\ \mathbf{i}_{P_L}^T \end{matrix} = \begin{pmatrix} \mathbf{i}_L^T \\ \mathbf{i}_{P_L}^T \end{pmatrix} \\[6pt] (L)\mathbf{i}_L^T = \mathbf{v}_L^T \end{array}\right\}$$

Equations for the
inductive multiport
$$\mathbf{N}_{LP_L} = \left\{ S_L \cup P_L,\ \mathbf{V}_{S_L P_L},\ D_{S_L P_L} \right\} \tag{6.2}$$

$$\left.\begin{array}{l} \begin{bmatrix} \mathbf{Q}_{1C} & \vdots & 0 \\ \mathbf{Q}_{2C} & \vdots & \mathbf{U} \end{bmatrix} \begin{matrix} \mathbf{i}_C^T \\ \mathbf{i}_{P_C}^T \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\[4pt] \begin{bmatrix} \mathbf{Q}_{1C}^T & \vdots & \mathbf{Q}_{2C}^T \\ 0 & \vdots & \mathbf{U} \end{bmatrix} \begin{matrix} \mathbf{v}_{tC}^T \\ \mathbf{v}_{P_C}^T \end{matrix} = \begin{pmatrix} \mathbf{v}_C^T \\ \mathbf{v}_{P_C}^T \end{pmatrix} \\[6pt] (C)\dot{\mathbf{v}}_C^T = \mathbf{i}_C^T \end{array}\right\}$$

Equations for the
capacitive multiport
$$\mathbf{N}_{CP_C} = \left\{ S_C \cup P_C,\ \mathbf{V}_{S_C P_C},\ D_{S_C P_C} \right\} \tag{6.3}$$

$$\left.\begin{array}{l} [\mathbf{Q}_{P_R}^C \vdots \mathbf{Q}_{P_L}^C \vdots \mathbf{Q}_{P_C}^C]\, \begin{matrix} \mathbf{i}_{P_R}^T \\ \mathbf{i}_{P_L}^T \\ \mathbf{i}_{P_C}^T \end{matrix} = 0 \\[10pt] \begin{bmatrix} (\mathbf{Q}_{P_R}^C)^T \\ (\mathbf{Q}_{P_L}^C)^T \\ (\mathbf{Q}_{P_C}^C)^T \end{bmatrix} (\mathbf{v}_t^c)^T = \begin{bmatrix} \mathbf{v}_{P_R}^T \\ \mathbf{v}_{P_L}^T \\ \mathbf{v}_{P_C}^T \end{bmatrix} \end{array}\right\}$$

Coupler KCL
and KVL Equations
$$\tag{6.4}$$

We now solve the resistive, inductive and capacitive multiports with respect to the port variables. In the process we get relations D_{P_R} between i_{P_R} and $\mathbf{v}_{P_R}$, D_{P_L} between i_{P_L} and $\mathbf{v}_{P_L}$ and D_{P_C} between i_{P_C} and $\mathbf{v}_{P_C}$. In this case from the resistive, inductive and capacitive multiport equations we get,

$$V_{P_R}^T = -\left[\begin{array}{cc} 0 & U \end{array}\right]\left\{\begin{pmatrix} B_{1R} \\ B_{2R} \end{pmatrix}(R)\begin{bmatrix} B_{1R}^T & B_{2R}^T \end{bmatrix}^{-1}\begin{pmatrix} 0 \\ U \end{pmatrix}\right\}^{-1}(i_{P_R}^T) \equiv D_{P_R}$$

$$V_{P_L}^T = -\left[\begin{array}{cc} 0 & U \end{array}\right]\begin{pmatrix} B_{1L} \\ B_{2L} \end{pmatrix}(L)\begin{bmatrix} B_{1L}^T & B_{2L}^T \end{bmatrix}^{-1}\begin{pmatrix} 0 \\ U \end{pmatrix}^{-1}(i_{P_L}^T) \equiv D_{P_L} \qquad - (6.5)$$

$$i_{P_C}^T = -\left[\begin{array}{cc} 0 & U \end{array}\right]\begin{pmatrix} Q_{1C} \\ Q_{2C} \end{pmatrix}(C)\begin{bmatrix} Q_{1C}^T & Q_{2C}^T \end{bmatrix}^{-1}\begin{pmatrix} 0 \\ U \end{pmatrix}^{-1}(V_{P_C}^T) \equiv D_{P_C}$$

The coefficient matrices on the right are all negative definite.
we next use D_{P_R}, D_{P_L}, D_{P_C} as the device characteristic for the
coupler network N with V as its coboundary space. Solving the
coupler network we get possible values of i_{P_R}, V_{P_R}, i_{P_L}, V_{P_L}, i_{P_C}, V_{P_C}.
Since in each of the multiports we already know the rest of the
variables with respect to the port variables this completes the
solution of the network.

Using this technique it is possible to show that the trapped
solutions correspond to zero eigen values and the state equations
of the reduced network have the same Jordan canonical form as the
state equations of the original network except for the zero
eigenvalues. We outline proofs for these facts in the Appendix II.
Example 6.2

In this example we illustrate the idea of the decomposition
of an RLMC network. Fig 6.2(a) shows an RLMC network N.
The set S of edges of the graph $\hat{G}$ (Fig 6.2(b)) of N has been
partitioned into $S_L = \{L_1, L_2, L_3\}$, $S_R = \{R_1, R_2, R_3\}$,
$S_C = \{C_1, C_2, C_3\}$.

We now proceed to decompose $\hat{G}$ through Algorithm III. Constructing $\hat{G} \times S_L$, $\hat{G} \times S_R$, $\hat{G} \times S_C$, We find that we can choose $\hat{t}_L = \{L_3\}$, $\hat{t}_R = \{R_3\}$, $\hat{t}_C = \{C_3\}$. Graph G is shown in Fig 6.2(c). $G \cdot S_R$, $G \cdot S_L$, $G \cdot S_C$ have trees $t_R = \{R_1, R_2\}$, $t_L = \{L_1, L_2\}$, $t_C = \{C_1, C_2\}$. Fig 6.2(d) shows a minimal decomposition of graph $\hat{G}$. The graphs $G_{S_R F_R}$, $G_{S_L F_L}$, $G_{S_C F_C}$ have coboundary spaces (constructed according to Algorithm III) $V_{S_R F_R}$, $V_{S_L F_L}$, $V_{C_C F_C}$, V_P respectively. In this case both the component spaces and the coupler of the decomposition of the original coboundary space have turned out to be graphic. (This may not be possible in general). The network N may be thought of as being obtained by the 'connection', of the resistive, inductive and capacitive multiports on graphs $\hat{G}_{S_R F_R}$, $\hat{G}_{S_L F_L}$, $\hat{G}_{S_C F_C}$ according to the 'port connection diagram' $\hat{G}_P$. This will correspond to physical connection if the ports of the multiports are connected to 1 : 1 transformers and the secondaries are connected according to $\hat{G}_P$. Next by using equation 6.5 or simply by inspection of the multiports we can obtain D_{P_R}, D_{P_L}, D_{P_C} for N_P as

$$
\begin{array}{l} V_{P_{R_1}} \\ V_{P_{R_2}} \end{array} = - \begin{bmatrix} R_1 + R_3 & R_3 \\ R_3 & R_2 + R_3 \end{bmatrix} \begin{array}{l} i_{P_{R_1}} \\ i_{P_{R_2}} \end{array}
$$

$$
\begin{array}{l} V_{P_{L_1}} \\ V_{P_{L_2}} \end{array} = - \begin{bmatrix} L_1 + L_3 & L_3 \\ L_3 & L_2 + L_3 \end{bmatrix} \begin{array}{l} \dot{i}_{P_{L_1}} \\ \dot{i}_{P_{L_2}} \end{array}
$$

$$\begin{bmatrix} i_{?_1} \\ i_{?_2} \end{bmatrix} = - \begin{pmatrix} \dfrac{C_1(C_2 + C_3)}{C_1 + C_2 + C_3} & \dfrac{-C_1 C_2}{C_1 + C_2 + C_3} \\[2ex] \dfrac{-C_1 C_2}{C_1 + C_2 + C_3} & \dfrac{C_2(C_1 + C_3)}{C_1 + C_2 + C_3} \end{pmatrix} \begin{matrix} \dot{V}_{P_{C_1}} \\[2ex] \dot{V}_{P_{C_2}} \end{matrix}$$

N_p would in general be an accurate representation of all the interactive solutions of N ('modulo' trapped solutions). In this (RLMC) case it has the same Jordon block as N except for zero eigen values.

Conclusion

In this paper we have introduced and made systematic use of the generalized minor operation on vector spaces. We have shown that it arises naturally in the context of connection of ideal transformers. We have defined the notion of the minimal extension of two or more vector spaces and used it to describe a method of network analysis where one could use any desired network topology, at a certain cost, to solve a given network. We have defined the notion of decomposition of a vector space to formalize the intuitive idea of decomposition of a network into multiports and a port connection diagram. Using the example of an RLMC network we have shown that some of the properties of the original network could be more conveniently studied by examining a reduced network defined suitably on the port connection diagram. While all the concepts introduced in this paper have been for arbitrary vector spaces, in order to show their relevance to network theory, we

have presented algorithms, wherever necessary, which are particularly appropriate to coboundary spaces of graphs.

References

1. G. Kron, *Tensor Analysis of Networks*, John Wiley and sons, Inc., New York, 1939.

2. G. Kron, "A Set of Principles to Interconnect the Solutions of Physical Systems," *Journal of Applied Physics*, Vol. 24, No. 8, pp. 965-980, 1953.

3. K. Hoffman and R. Kunze, *Linear Algebra*, Prentice Hall, Englewood Cliffs, N.J., 1961.

4. W. T. Tutte, "Lectures on Matroids," *J. Res. Natl. Bur. Stand.*, Sect. B, Vol 69B, pp. 1-47, 1965.

5. V. Belevitch, *Classical Network Theory*, Holden-Day, San Francisco, Cambridge, London, Amsterdam, 1968.

6. H. Narayanan, "On the Decomposition of Vector Spaces," Research report, E. E. Dept., I. I. T. Bombay, April 1983.

APPENDIX I

A.I. Justification for Algorithm III. Observe that the rows of the matrix $\tilde{A}$ shown below generate the coboundary space of $\hat{G}$.

$$\tilde{A} = \begin{bmatrix} \hat{A}_{11} & & \\ & \ddots & \\ & & \hat{A}_{nn} \\ A_{S_1} & \cdots & A_{S_n} \end{bmatrix}$$

where A_{S_i} is the submatrix of A corresponding to columns S_i.

This is because (a) The rows of A belong to the coboundary space of $\hat{G}$ (They are obtained by taking suitable vectors of $V_S \times (S - \cup \hat{t}_i)$ and adding zero columns corresponding to edges in $\cup \hat{t}_i$).

(b) The rows of $\begin{bmatrix} \overset{S_1}{0} & \cdots & \overset{S_i}{\hat{A}_{ii}} & \cdots & \overset{S_n}{0} \end{bmatrix}$ belong to the coboundary space of $\hat{G}$ since the rows of $\hat{A}_{ii}$ belong to $(V_{cob}(\hat{G}) \times S_i)$.

(c) The rank of $\tilde{A}$ = rank of $\hat{G}$. $\Big[$ The columns $\hat{t}_i$ of $\hat{A}_{ii}$ are linearly independent, where as these columns are zero columns in the matrix A. Hence

$$\text{rank of } \tilde{A} = \sum_{i=1}^{n} r(\hat{G} \times S_i) + r(\hat{G} \times (S - \overset{n}{\underset{i=1}{\cup}} \hat{t}_i)) \ .$$
$$= |\overset{n}{\underset{i=1}{\cup}} \hat{t}_i| + r(\hat{G} \times (S - \overset{n}{\underset{i=1}{\cup}} \hat{t}_i))$$

Since $\overset{n}{\underset{i=1}{\cup}} \hat{t}_i$ contains no circuits in $\hat{G}$

$$\text{rank of } \tilde{A} = r(\hat{G} \cdot (\overset{n}{\underset{i=1}{\cup}} \hat{t}_i)) + r(\hat{G} \times (S - \overset{n}{\underset{i=1}{\cup}} \hat{t}_i))$$
$$= r(\hat{G}) \ . \Big]$$

A_{S_i} can be expressed as $k_i(A_{iS_i})$ since A_{iS_i} is a reduced incidence matrix of $G \cdot S_i$. For the spaces $\overset{n}{\underset{i=1}{\oplus}} V_{S_i P_i}$ and V_P we have the generator matrices $\hat{A}_\oplus$, A_P

shown below

$$
\hat{A}_\oplus =
\begin{array}{c}
\\
\hat{A}_\oplus =
\end{array}
$$

$$
\begin{array}{ccccccc}
 & S_1 & \cdots & S_i & \cdots & S_n & \quad P_1 \cdots P_i \cdots P_n
\end{array}
$$

$$
\hat{A}_\oplus = \left[
\begin{array}{ccc|ccc}
\hat{A}_{11} & & 0 & 0 & 0 & 0 \\
A_{1S_1} & & & A_{1t_1} & & 0 \\
\cdot & \hat{A}_{ii} & & & & \cdot \\
0 & 0 & A_{iS_i} & & A_{it_i} & 0 \\
0 & & \hat{A}_{nn} & & & 0 \\
0 & & 0 & A_{nS_n} & 0 & 0 & A_{nt_n}
\end{array}
\right]
$$

$$
A_P = [\ \cdots \ \overset{P_1 \quad P_i \quad P_n}{k_i(A_{it_i})} \ \cdots \ ]
$$

Consider the submatrix $A_\oplus$ of $\hat{A}_\oplus$ as shown below:

$$
\begin{array}{cccccc}
 & S_1 \cdots S_i \cdots S_n & \quad P_1 \cdots P_i \cdots P_n
\end{array}
$$

$$
A_\oplus = \left[
\begin{array}{ccc|ccc}
A_{1S_1} & & & A_{1t_1} & & \\
 & A_{iS_i} & & & A_{it_i} & \\
 & & A_{nS_n} & & & A_{nt_n}
\end{array}
\right]
$$

If we premultiply this matrix by the matrix

$$
[\ \overset{P_1}{k_1} : \ \cdots \ \overset{P_i}{k_i} \ \cdots \ \overset{P_n}{: k_n} \ ]
$$

we get the matrix

$$
[\ \overset{S_1}{A_{S_1}} \cdots \overset{S_i}{A_{S_i}} \cdots \overset{S_n}{A_{S_n}} : \overset{P_1}{A_{t_1}} \cdots \overset{P_i}{A_{t_i}} \cdots \overset{P_n}{A_{t_n}} \ ]
$$

observe that A_P appears as the submatrix corresponding to columns P. From the definition of g-minors it follows that $V_S \subseteq ((\overset{n}{\underset{i=1}{\oplus}} V_{S_i P_i} \leftarrow V_P)$. We will now show the reverse inequality. Let f_S be a vector in $(\overset{n}{\underset{i=1}{\oplus}} V_{S_i P_i}) \leftarrow V_P$. Then f_S can be written as $f_S^1 + f_S^2$ where $f_S \in \overset{n}{\underset{i=1}{\oplus}} (V_S \times S_i)$ and f_S^2 is the restriction of a

vector f^2_{SP} which is linearly dependent on the rows of $A_{\oplus}$ and further $f^2_{SP/P} \in V_P$. f^1_S clearly belongs to V_S. We will show that f^2_S also does. We have $f^2_{SP} = (\lambda)(A_{\oplus})$. Hence $f^2_{SP/P} = (\lambda)(A_{\oplus P})$. We know that $f^2_{SP/P}$ is linearly dependent on the rows of

$$
\begin{array}{c}
\begin{array}{ccc} P_1 & \cdots & P_n \end{array} \\
[A_{t_1} \cdots A_{t_n}]
\end{array}
= (k_1 \cdots k_n)
\begin{array}{c}
\begin{array}{ccc} P_1 & \cdot\cdot & P_i & \cdot & P_n \end{array} \\
\begin{bmatrix} A_{1t_1} & & \\ & A_{it_i} & \\ & & A_{nt_n} \end{bmatrix}
\end{array} .
$$

The matrix postmultiplying $(k_1 .. k_n)$ is the matrix $A_{\oplus P}$. Since this is nonsingular λ is linearly dependent on the rows of $(k_1 \cdots k_n)$. Hence $f^2_{SP/S}$ is spanned by the rows of $\begin{array}{c}\begin{array}{ccc} S_1 & & S_n \end{array} \\ [A_{S_1} \cdots A_{S_n}]\end{array}$. Hence $f^2_S \in V_S$.

ENDS

APPENDIX II

In the following discussion we refer to state equations of a generalized RLMC network. By this we mean a set of equations of the form

$$\dot{\mathbf{x}}(t)^T = \mathbf{A}(\mathbf{x}(t))^T$$

such that $\mathbf{V}_S^T(t)$, $\mathbf{i}_S^T(t)$ can be expressed as $(\mathbf{k}_v)\mathbf{x}^T(t)$ and $(\mathbf{k}_i)\mathbf{x}^T(t)$. Just as in the case of ordinary RLMC networks we may choose the state variables to be a maximal linearly independent set of capacitor voltages and a maximal linearly independent set of inductor currents. We refer below to independent solutions of the network as well as to independent solutions of the state equations of the network. The latter requires no explanation. Solutions $(\mathbf{V}_S^1\mathbf{i}_S^1) \dots (\mathbf{v}_S^k, \mathbf{i}_S^k)$ of the network are linearly independent iff there exist no nontrivial $\lambda_1, \dots, \lambda_n$ such that

$$(\lambda_1 \mathbf{v}_s^1, \lambda^1 \mathbf{i}_s^1) + \cdots + (\lambda_k \mathbf{v}_s^k, \lambda_k \mathbf{i}_s^k) = (0,0) \ .$$

Let **N** be a generalized RLMC network. Let S_R, S_L, S_C, P_R, P_L, P_C, $\mathbf{N}_P$ be defined as in subsection 6.3. We define a generalized negative RLMC network as one where **R**, **L**, **C** matrices are negative definite. We reemphasize that we are dealing with minimal decompositions.

Theorem A1. The number of independent state variables of $\mathbf{N} = r(\mathbf{V}_S \cdot S_C) + r(\mathbf{V}_S^{\bullet} \cdot S_L)$.

Proof The number of independent state variables is the same as the number of independent initial conditions. Capacitor voltages and inductor currents can be chosen as state variables. The possible capacitor initial voltage vectors and the possilbe inductor initial current vectors form respectively the spaces $\mathbf{V}_S \cdot S_C$

and $\mathbf{V}_S^{\bullet} \cdot S_L$. Hence the number of independent state variables of $\mathbf{N} = \tau(\mathbf{V}_S \cdot S_C) + \tau(\mathbf{V}_S^{\bullet} \cdot S_L)$.

<div align="right">Q.E.D.</div>

Theorem A2. Let $\mathbf{N}$ be a generalized RLMC or negative RLMC network.

(a) A solution of $\mathbf{N}$ is a trapped solution iff it is a constant solution of $\mathbf{N}$.

(b) The number of independent constant solutions of $\mathbf{N}$ is equal to the number of independent constant solutions of the state equations of $\mathbf{N}$.

(c) The number of independent trapped solutions of $\mathbf{N}$ is the number of its zero eigenvalues and equals $\tau(\mathbf{V}_S \times S_C) + \tau(\mathbf{V}_S^{\bullet} \times S_L)$.

Proof We will prove the theorem for the case where $\mathbf{N}$ is a generalized RLMC network. The negative RLMC network case is essentially the same and so is omitted.

(a) Let $(\mathbf{v}_S^1(t), \mathbf{i}_S^1(t))$ be a trapped solution of $\mathbf{N}$. Then, by definition

$$\mathbf{v}_R^1(t) \in \mathbf{V}_S \times S_R \ ,$$

$$\mathbf{i}_R^1(t) \in \mathbf{V}_S^{\bullet} \times S_R \ ,$$

$$(\mathbf{R})(\mathbf{i}_R^1(t))^T = (\mathbf{v}_R^1(t))^T$$

Since $\mathbf{V}_S^{\bullet} \times S_R \subseteq (\mathbf{V}_S \times S_R)^{\bullet}$, we have $\langle \mathbf{v}_R^1(t), \mathbf{i}_R^1(t) \rangle = 0$ i.e., $(\mathbf{i}_R^1(t))(\mathbf{R})(\mathbf{i}_R^1(t))^T = 0$. Since $\mathbf{R}$ is positive definite we conclude that $\mathbf{i}_R^1(t)$ and therefore $\mathbf{v}_R^1(t)$ is a zero vector. Next we have

$$\mathbf{v}_L^1(t) \in \mathbf{V}_S \times S_L \ ,$$

$$\mathbf{i}_L^1(t) \in \mathbf{V}_S^{\bullet} \times S_L \ .$$

<div align="center">-5-</div>

$$(\mathbf{L})(\dot{\mathbf{i}}_L^1(t))^T = \mathbf{v}_L^1(t)$$

As in the previous case we have

$$\langle \mathbf{v}_L^1(t), \mathbf{i}_L^1(t) \rangle = 0 \quad .$$

Hence, since (L) is positive definite we have

$$\frac{d}{dt}(\mathbf{i}_L^1(t)(\mathbf{L})(\mathbf{i}_L^1(t))^T) = 0 \quad .$$

This means that $(\mathbf{i}_L^1(t)(\mathbf{L})(\mathbf{i}_L^1(t))^T)$ is constant. Since (L) is positive definite it can be factorized as $(\mathbf{k}\,\mathbf{k}^T)$ where $\mathbf{k}$ is a constant matrix with linearly independent rows. Hence $(\mathbf{i}_L^1(t)\mathbf{k})(\mathbf{i}_L^1(t)\mathbf{k})^T$ is constant. Hence $(\mathbf{i}_L^1(t)\mathbf{k})$ is a constant vector and hence $\mathbf{i}_L^1(t)$ is a constant vector. Hence $\mathbf{v}_L^1(t)$ is a zero vector. The capacitor case can be handled similarly and we can show that $\mathbf{v}_C^1(t)$ is a constant vector and $\mathbf{i}_C^1(t)$ is a zero vector. Thus we see that trapped vectors are constant vectors. Next suppose $(\mathbf{v}_S^1(t), \mathbf{i}_S^1(t))$ is a constant solution. From the device characteristic it is clear that we must have $\dot{\mathbf{v}}_C(t) = 0$, $\dot{\mathbf{i}}_L(t) = 0$. It follows that $\langle \mathbf{v}_L^1(t), \mathbf{i}_L^1(t) \rangle = 0$ and $\langle \mathbf{v}_C^1(t), \mathbf{i}_C^1(t) \rangle = 0$. Since $\langle \mathbf{v}_S^1(t), \mathbf{i}_S^1(t) \rangle = 0$ it follows that $\langle \mathbf{v}_R^1(t), \mathbf{i}_R^1(t) \rangle = 0$. Since $\mathbf{R}$ is positive definite it follows $\mathbf{v}_R^1(t) = \mathbf{i}_R^1(t) = 0$. Hence $\mathbf{i}_L^1(t) \in V_S^* \times S_L$ and $\mathbf{v}_C^1(t) \in V_S \times S_C$. Hence

$$\mathbf{v}_S^1(t) \in V_S \times S_R \ \oplus \ V_S \times S_L \ \oplus \ V_S \times S_C \quad .$$

$$\mathbf{i}_S^1(t) \in V_S^* \times S_R \ \oplus \ V_S^* \times S_L \ \oplus \ V_S^* \times S_C \quad .$$

Hence $(\mathbf{v}_S^1(t), \mathbf{i}_S^1(t))$ is a trapped solution.

(b), (c). We thus see that $(\mathbf{v}_S^1(t), \mathbf{i}_S^1(t))$ is a constant solution of N iff $\mathbf{v}_C^1(t)$, $\mathbf{i}_L^1(t)$ are constant vectors, $\mathbf{v}_C^1(t) \in V_S \times S_C$, $\mathbf{i}_L^1(t) \in V_S^* \times S_L$ and $\mathbf{v}_R^1(t)$, $\mathbf{v}_L^1(t)$, $\mathbf{i}_R^1(t)$, $\mathbf{i}_C^1(t)$ are zero vectors. The number of independent constant solutions of N is therefore the same as the number of independent constant solutions of the

state equations of **N**. The latter equals the number of zero eigenvalues of **N**. While the former is equal to the number of independent trapped solutions which equals $\tau(\mathbf{V}_S \times S_C) + \tau(\mathbf{V}_S^{\bullet} \times S_L)$.

<div align="right">Q.E.D.</div>

From Theorems 6.4 and A2 we have

Theorem A3 (a) Number of independent state variables of $\mathbf{N}_P = |P_L| + |P_C|$.

(b) $\mathbf{N}_P$ has no zero eigenvalue.

Proof Since $\{\mathbf{V}_P, \mathbf{V}_{S_R P_R}, \mathbf{V}_{S_L P_L}, \mathbf{V}_{S_C P_C}\}$ is a minimal decomposition of $\mathbf{V}_S$ we have by Theorem 6.4 and Lemma 6.2

$$\begin{aligned}
|P_L| &= \tau(\mathbf{V}_S \cdot S_L) - \tau(\mathbf{V}_S \times S_L) \\
&= \tau(\mathbf{V}_P \cdot P_L) - \tau(\mathbf{V}_P \times P_L) \\
|P_C| &= \tau(\mathbf{V}_S \cdot S_C) - \tau(\mathbf{V}_S \times S_C) \\
&= \tau(\mathbf{V}_P \cdot P_C) - \tau(\mathbf{V}_P \times P_C) \ .
\end{aligned}$$

Hence we have

$$\begin{aligned}
\tau(\mathbf{V}_P \cdot P_L) &= |P_L| \ , \ \tau(\mathbf{V}_P \times P_L) = 0 \\
\tau(\mathbf{V}_P \cdot P_C) &= |P_C| \ , \ \tau(\mathbf{V}_P \times P_C) = 0 \ .
\end{aligned}$$

In particular

$$\tau(\mathbf{V}_P \cdot P_C) = |P_C|$$

and

$$\tau(\mathbf{V}_P^{\bullet} \cdot P_L) = |P_L| - \tau(\mathbf{V}_P \times P_L) = |P_L|$$

by Theorems P1 and P4. Hence by Theorem A1 the number of independent state variables of $\mathbf{N}_P = |P_L| + |P_C|$. The number of nonzero eigenvalues of

$\mathbf{N}_P = \tau(\mathbf{V}_S \cdot S_{C)} + \tau(\mathbf{V}_S^{\bullet} \cdot S_L) - \tau(\mathbf{V}_S \times S_C) - \tau(\mathbf{V}_S^{\bullet} \times S_L)$ (by Theorems A1 and A2)

$= \tau(\mathbf{V}_S \cdot S_C) - \tau(\mathbf{V}_S \times S_C) + |S_L| - \tau(\mathbf{V}_S \times S_L) - |S_L| + \tau(\mathbf{V}_S \cdot S_L)$

$= |P_C| + |P_L|$ by Theorem 6.4. Hence N_P has no zero eigenvalues.

<div align="right">Q.E.D.</div>

Theorem A4   The state equations of $\mathbf{N}$ and $\mathbf{N}_P$ have the same Jordan canonical form except for zero eigenvalues.

Proof   Since the decomposition $\{\mathbf{V}_P, \mathbf{V}_{S_R P_R}, \mathbf{V}_{S_L P_L}, \mathbf{V}_{S_C P_C}\}$ of $\mathbf{V}_S$ is minimal by Theorem 6.4, $M(\mathbf{V}_{S_L P_L})$, $M(\mathbf{V}_{S_C P_C})$ have no circuits or bonds in P_L, P_C respectively. Hence in generator matrices of $\mathbf{V}_{S_L P_L}$, $\mathbf{V}_{S_L P_L}^{\bullet}$ the columns corresponding to P_L are linearly independent and in generator matrices of $\mathbf{V}_{S_C P_C}^{\bullet}$, $\mathbf{V}_{S_C P_C}$ the columns corresponding to P_C are linearly independent. Hence we can write $\mathbf{i}_{P_L}^T = (\mathbf{k}_L)\mathbf{i}_L^T$ and $\mathbf{v}_{P_C}^T = \mathbf{k}_C \mathbf{v}_C^T$, where $\mathbf{k}_L$, $\mathbf{k}_C$ have linearly independent rows. Let the state equations of $\mathbf{N}$ and $\mathbf{N}_p$ be respectively

$$\dot{\mathbf{x}}^T = \mathbf{A}\mathbf{x}^T \tag{A.1}$$

$$\dot{\widetilde{\mathbf{x}}}^T = \widetilde{\mathbf{A}}\widetilde{\mathbf{x}}^T . \tag{A.2}$$

Let $\mathbf{X}$ and $\widetilde{\mathbf{X}}$ be the corresponding state spaces. We then have a map from $\mathbf{X}$ onto $\widetilde{X}$ defined through $\widetilde{\mathbf{x}}^T = (\mathbf{k})\mathbf{x}^T$, where $\mathbf{k}$ is the direct sum of $\mathbf{k}_L$, $\mathbf{k}_C$. We know that equation (A.2) has no nonzero constant solutions since $\mathbf{N}_P$ has no zero eigenvalues by Theorem A3. Hence the space of constant solutions of equations (A.1) maps to the zero vector under $(\mathbf{k})$. Next $|P_L| = \tau(\mathbf{V}_S \cdot S_L) - \tau(\mathbf{V}_S \times S_L)$ $|P_C| = \tau(\mathbf{V}_S \cdot S_C) - \tau(\mathbf{V}_S \times S_C)$ by Theorem 6.4. Noting that $\tau(\mathbf{V}_S \cdot S_L) - \tau(\mathbf{V}_S \times S_L) = \tau(\mathbf{V}_S^{\bullet} \cdot S_L) - \tau(\mathbf{V}_S^{\bullet} \times S_L)$, we have by Theorems A1, A2, and A3, $\dim(\widetilde{X}) = |P_L| + |P_C| = \dim \mathbf{X} - \dim$ (constant solutions of (A.1)). Now there exists a transformation $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ has the form

$$\begin{pmatrix} 0 & \vdots & \\ \cdots & \vdots & \cdots \\ & \vdots & \mathbf{A}_z \end{pmatrix}$$ where $\mathbf{A}_z$ has no zero eigenvalues. Let $\mathbf{z}^T = \mathbf{T}^{-1}\mathbf{x}^T$. Then $\dot{\mathbf{z}}^T = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{z}^T$. We could write this as

$$\begin{pmatrix} \dot{\mathbf{z}}_1^T \\ \cdots \\ \dot{\mathbf{z}}_2^T \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \vdots & \\ \cdots & \vdots & \cdots \\ & \vdots & \mathbf{A}_z \end{pmatrix} \begin{pmatrix} \mathbf{z}_1^T \\ \cdots \\ \mathbf{z}_2^T \end{pmatrix} \tag{A.3}$$

The transformation $\mathbf{T}^{-1}$ maps all constant solutions of equation (A.1) to vectors of the form $\begin{pmatrix} \mathbf{z}_1^T \\ 0 \end{pmatrix}$ while $\mathbf{T}$ maps all solutions of the form $\begin{pmatrix} \mathbf{z}_1 \\ 0 \end{pmatrix}$ of (A.3) to constant solutions of (A.1).

Now

$$\widetilde{\mathbf{x}} = \mathbf{k}\mathbf{T}\mathbf{z}^T$$

Since equation (A.2) has no constant solutions and $\dim(\widetilde{X}) = \dim X - \dim$ (constant solutions of (A.1)) it follows that $(\mathbf{k}\mathbf{T}) = (0 \vdots \mathbf{M})$ where M is nonsingular and

$$\widetilde{\mathbf{x}}^T = \mathbf{M}\mathbf{z}_2^T$$

Hence

$$\dot{\widetilde{\mathbf{x}}}^T = (\mathbf{M}(\mathbf{A}_z)\mathbf{M}^{-1})\widetilde{\mathbf{x}}^T \ .$$

Now $\mathbf{A}_z$ and $(\mathbf{M}\mathbf{A}_z\mathbf{M}^{-1})$ have the same Jordan canonical form and $\mathbf{A}_z$ has the same canonical form as $\mathbf{A}$ except for the zero eigenvalues. The theorem follows.

Q.E.D.

Figure Captions

Fig. 4.1   An ideal transformer

Fig. 4.2 $I_S \longleftarrow I_P$
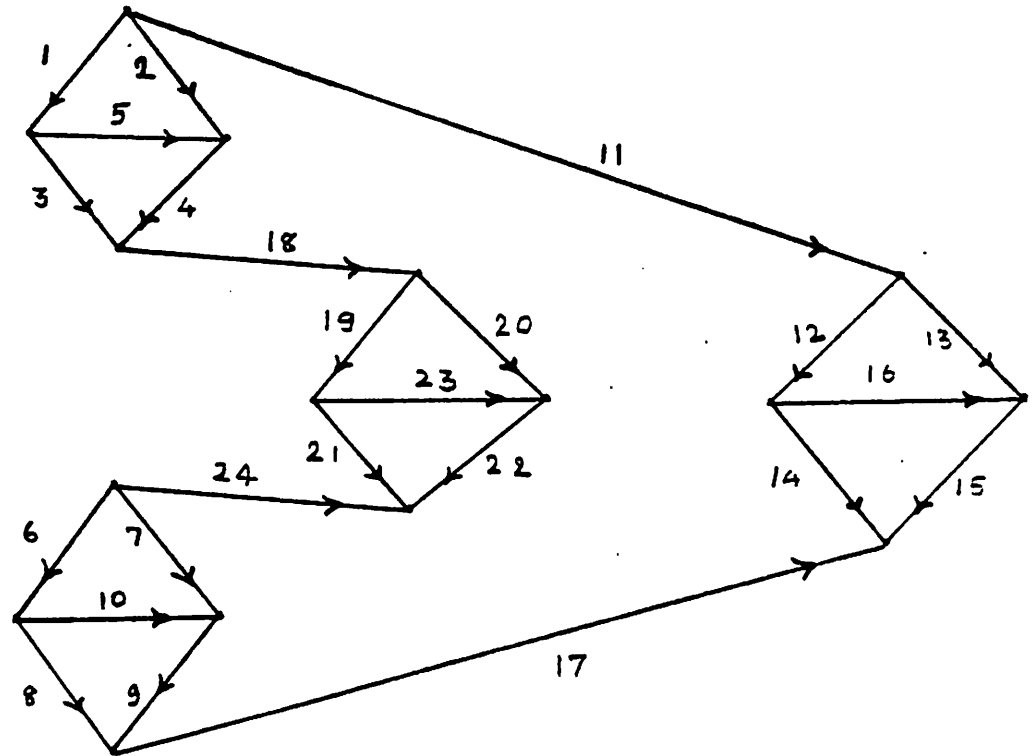
Fig. 5.1(a)   Graph G
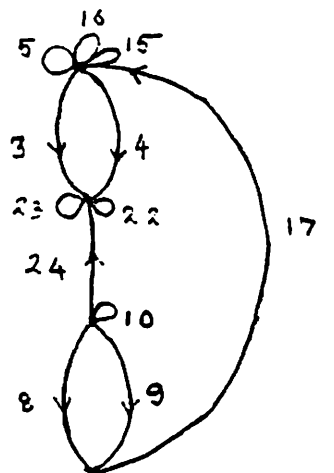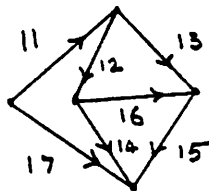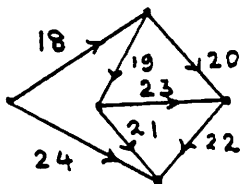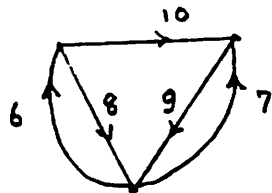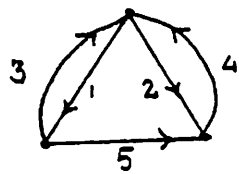
Fig. 5.1(b) $\overset{4}{\underset{i=1}{\oplus}}$ $(G \times S_i)$ Fig. 5.1(c) $G \times (S - \overset{4}{\underset{i=1}{\cup}} t_i)$
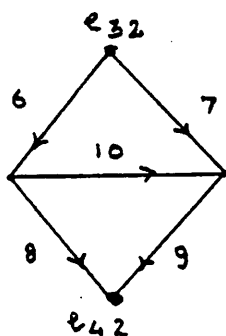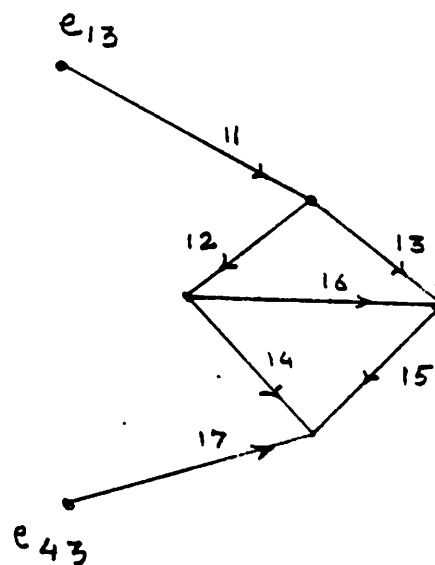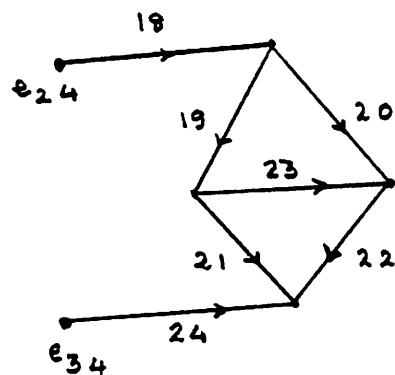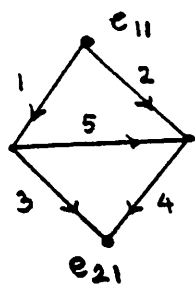
Fig. 5.2(a) $\bigoplus\limits_{i=1}^{4}$ (G. S_i)
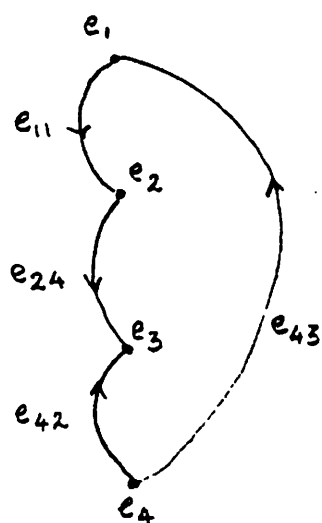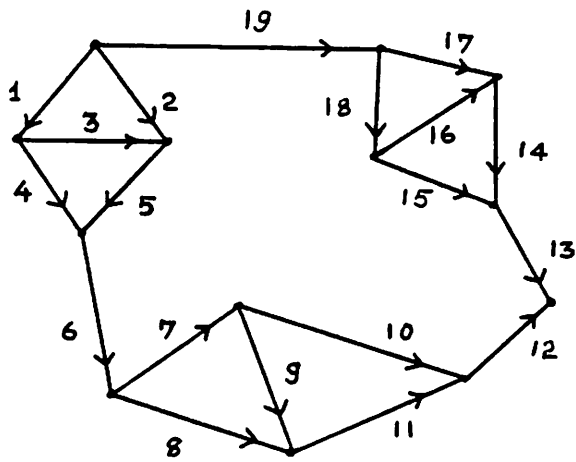

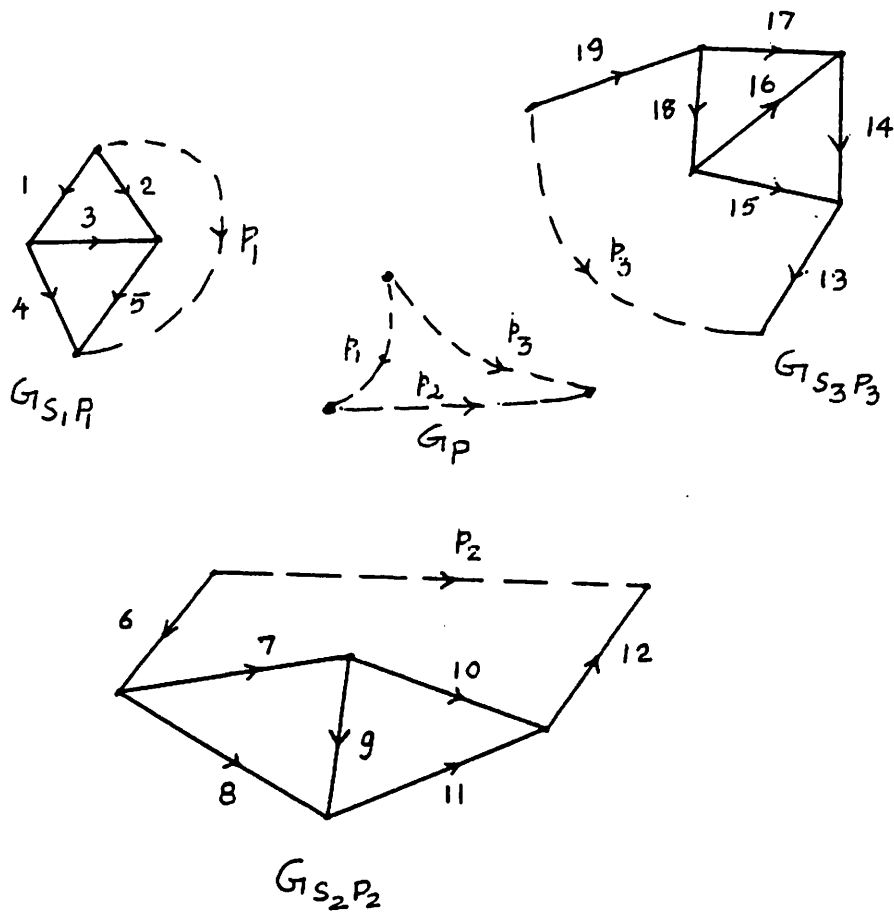Fig. 5.2(b) The graph G_b

Fig. 6.1(a)   Graph G

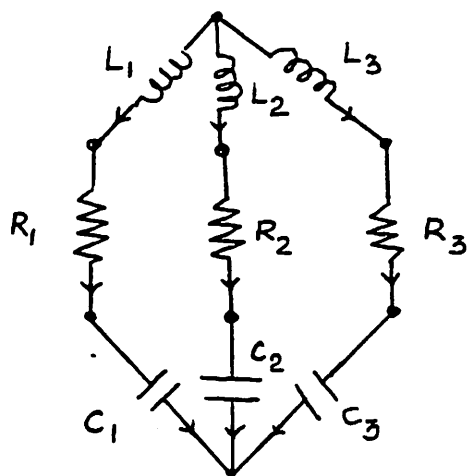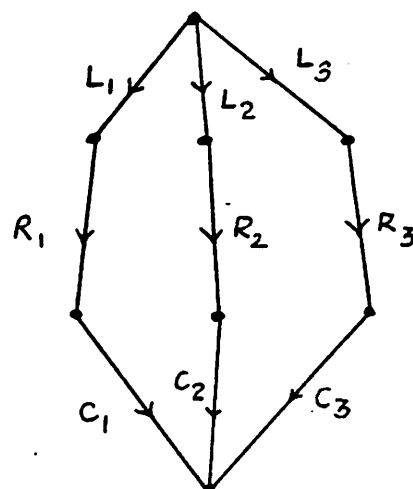Fig. 6.1(b)   The decomposition of G

Fig. 6.2(a)   The RLMC network N

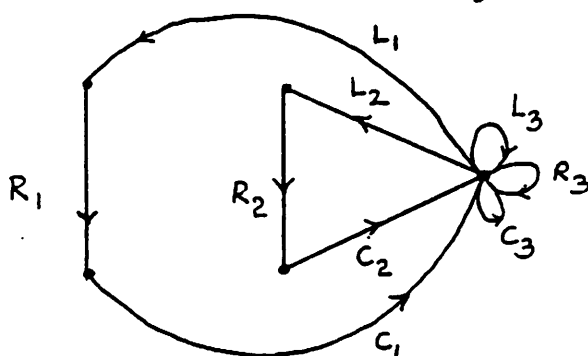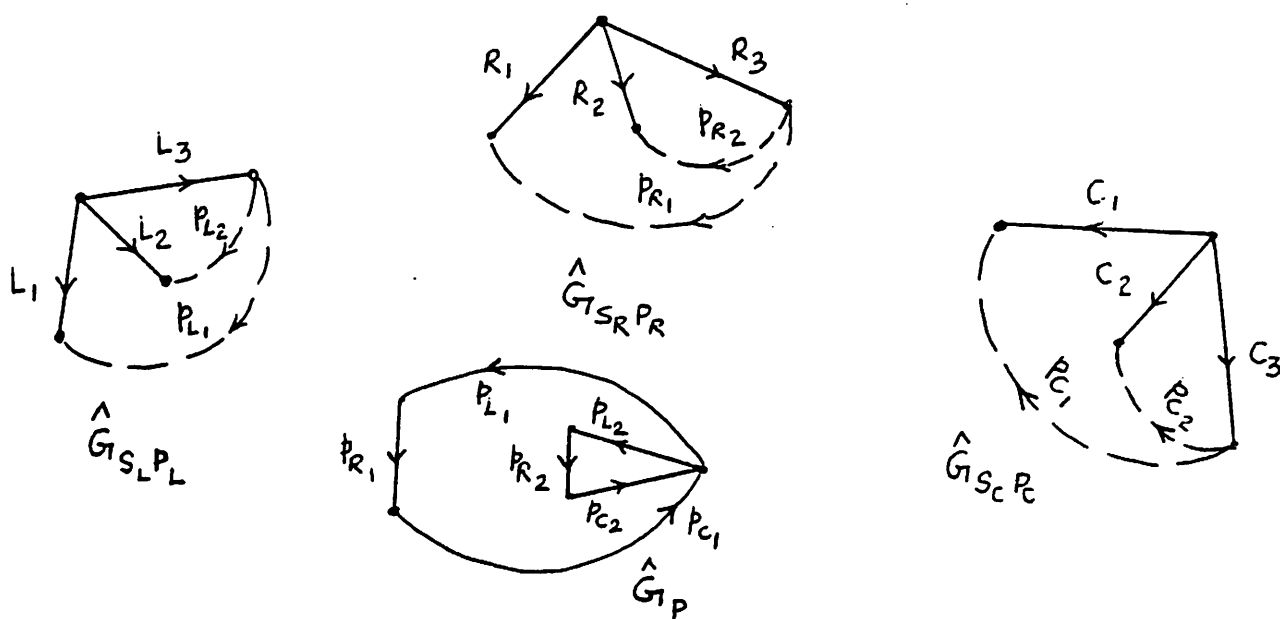

Fig. 6.2(b)   The graph $\hat{G}$ of N

Fig. 6.2(c)   The graph G

Fig. 6.2(d)   A minimal decomposition of G.