A COMPARATIVE STUDY OF LINEAR AND NONLINEAR MIMO FEEDBACK CONFIGURATIONS

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Abstract

In this paper, we compare several feedback configurations which have appeared in the literature (e.g. unity-feedback, model-reference, etc.). We first consider the linear time-invariant multi-input multi-output case. For each configuration, we specify the stability conditions, the set of all achievable I/O maps and the set of all achievable disturbance-to-output maps, and study the effect of various subsystem perturbations on the system performance. In terms of these considerations, we demonstrate that one of the configurations considered is better than all the others. The results are then extended to the nonlinear multi-input multi-output case.

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I. Introduction

The control system designer must meet various design specifications and to achieve them he has many design configurations to choose from. The standard unity-feedback linear system is the subject of most control textbooks [D'Az. 1, Dor. 1, etc.]. Horowitz discusses briefly a number of different configurations and, in particular the "two-degree of freedom" designs [Hor. 1]. We note also the two-input one-output controller proposed by Astrom [Ast. 1] and developed by Pernebo [Per. 1] and by Desoer and Gustafson [Des. 1] as well as the controller structures used in the model reference adaptive control systems [Lan. 1, Sas. 1].

In this paper, we compare, in a systematic way, several design configurations which have been proposed in the literature. We study first the linear multi-input multi-output case; some of the results are then extended to the nonlinear case.

We adopt the following notations throughout this paper. Let $\mathbb{R}$ ($\mathbb{C}$) denote the field of real (complex, resp.) numbers. Let $\mathbb{R}(s)$($\mathbb{R}_p(s)$,$\mathbb{R}_{p,0}(s)$) denote the set of all rational functions (proper rational functions, strictly proper rational functions, resp.) in s with real coefficients. Let $\mathbb{R}_p(s)^{m\times n}$($\mathbb{R}_{p,0}(s)^{m\times n}$,$\mathbb{C}^{m\times n}$) denote the set of mxn matrices with elements in $\mathbb{R}_p(s)$($\mathbb{R}_{p,0}(s)$, $\mathbb{C}$, resp.). For $P \in \mathbb{R}(s)^{m\times n}$, let $\mathcal{P}[P]$ ($\mathcal{Z}[P]$) denote the list of all poles (all zeros, resp.) of P. For $A \in \mathbb{C}^{m\times n}$, let $\bar{\sigma}[A]$ denote the maximal singular value of A.

For the given linear time-invariant multi-input multi-output plant P(s), any linear output feedback design can be represented as the system ${}^1\Sigma(P,K)$ shown in Fig. 1, where the compensator K has two inputs u_1 and y_2, and one output y_1. Since K is linear, it is uniquely specified by the transfer function from u_1 to y_1 and the transfer function from y_2 to y_1 denoted respectively by $K_{y_1u_1}$ and $K_{y_1y_2}$. More precisely, with

$\pi := K_{y_1u_1}$ and $F := -K_{y_1y_2}$, the system ${}^1\Sigma(P,K)$ and the system shown in Fig. 2 have the same system I/O map $H_{yu} : (u_1,u_2,d_0) \mapsto (y_1,y_2)$. From Fig. 2, H_{yu} can be obtained by inspection:

$$H_{yu} = \left[\begin{array}{c|c|c} (I+FP)^{-1}\pi & -FP(I+FP)^{-1} & -F(I+PF)^{-1} \\ \hline P(I+FP)^{-1}\pi & P(I+FP)^{-1} & (I+PF)^{-1} \end{array}\right] \tag{1.1}$$

The matrix H_{yu} in (1.1) shows that only two submatrices of H_{yu} can be independently specified by a suitable choice of π and F. Therefore, however complicated the structure of the linear compensator K may be, there are only two closed-loop maps that can be independently specified. In most design problems, the two most important maps are $H_{y_2u_1}$ and $H_{y_2d_0}$: $H_{y_2u_1}$ is the map from input u_1 to output y_2 and $H_{y_2d_0}$ is the map from output-disturbance d_0 to output y_2. They specify respectively the servo-performance and regulator-performance of the feedback system ${}^1\Sigma(P,K)$.

In general, the compensator K is implemented as interconnections of several subsystems. Different interconnections of such subsystems result in different <u>feedback configurations</u>. Following Horowitz, [Hor. 1] we say that a feedback configuration is a <u>two-degree of freedom design</u> iff an appropriate choice of the compensation subsystems (i.e. any subsystems that are not the given plant) will change the input-output map $H_{y_2u_1}$ without affecting the disturbance-to-output map $H_{y_2d_0}$, or vice versa. A feedback configuration is said to be a <u>single-degree of freedom design</u> iff it is not a two-degree of freedom design. A transfer function $H(s) \in \mathbb{R}(s)^{m\times n}$ is said to be <u>exp. stable</u> iff a) H(s) is <u>proper</u> and b) all its poles have <u>negative</u> real part. A <u>linear time-invariant</u> feedback configuration is said to be exp. stable[1] iff the system I/O map from any exogenous input to any subsystem input and to any subsystem output is exp. stable.

Throughout Section I-Section IV, we assume that

(A.1) All subsystems which make up the feedback configuration under study are represented by transfer functions $P(s)$, $C(s)$, $\cdots$ etc. with elements in $\mathbb{R}_p(s)$; furthermore none of these subsystems have unstable hidden modes;

(A.2) $P_0(s), P(s) \in \mathbb{R}_{p,0}(s)^{n_o \times n_i}$; $C(s)$, $C_0(s)$, $C_1(s)$, $C_2(s)$, $Q(s)$, $Q_0(s)$ and $Q_1(s) \in \mathbb{R}_p(s)^{n_o \times n_i}$.

We say that the map H is an achievable I/O map (disturbance-to-output map, resp.) of the linear feedback configuration[2] ${}^1\Sigma(P,K)$ iff by some appropriate choice of the compensation subsystems satisfying (A.1), (i) $H_{y_2u_1} = H$, ($H_{y_2d_0} = H$, resp.); (ii) ${}^1\Sigma(P,K)$ is exp. stable.

For each feedback configuration studied in this paper, we obtain stability conditions, the set of all achievable I/O maps and the set of all achievable disturbance-to-output maps; we compute the effects of various subsystem perturbations on the I/O map $H_{y_2u_1}$. Based on these considerations, we demonstrate that the configuration Σ_b (in Section III) is the best among the configurations considered.

The paper is organized as follows: Section II reviews the properties of the unity-feedback configuration ${}^1S(P,C)$; the various two-degree of freedom design configurations are studied in Section III and Section IV. Section V extends the results of Section III to the nonlinear case. Section VI is a brief summary of the paper.

II. Single-degree of freedom design: the unity feedback system ${}^1S(P,C)$

The unity feedback system ${}^1S(P,C)$ shown in Fig. 3 has been studied extensively in the control literature [For. 1, Kai. 1, Oga. 1, Cal. 2, Des. 2, Doy. 1, Vid. 1, Chen. 1]. In this section, we review some of the properties associated with this configuration for the linear time-invariant lumped multi-input multi-output case. Equation (2.1) below shows that ${}^1S(P,C)$ is a single-degree of freedom design.

II.1. The system I/O map

Let P and C satisfy (A.2). For ${}^1S(P,C)$, the system I/O map $H_{yu}:(u_1,u_2,d_0)\mapsto(y_1,y_2)$ is given by

$$H_{yu} = \left[\begin{array}{c|c|c} C(I+PC)^{-1} & -CP(I+CP)^{-1} & -C(I+PC)^{-1} \\ \hline PC(I+PC)^{-1} & P(I+CP)^{-1} & (I+PC)^{-1} \end{array}\right] \tag{2.1}$$

Assumption (A.2) guarantees that all the inverses are well-defined matrices with elements in $\mathbb{R}_p(s)$.

II.2. Stability conditions of ${}^1S(P,C)$

It is easy to check (using the summing node equations) that ${}^1S(P,C)$ is exp. stable iff H_{yu} is exp. stable. Hence, by inspection of (2.1) and the identity $I-M(I+M)^{-1} = (I+M)^{-1}$, we have that

$${}^1S(P,C) \text{ is exp. stable} \Leftrightarrow C(I+PC)^{-1},\ (I+CP)^{-1},\ (I+PC)^{-1} \text{ and } P(I+CP)^{-1} \text{ are exp. stable} \tag{2.3}$$

Note that any one of the four maps in (2.3) can be unstable, while the other three are exp. stable [Des. 2]. It is well-known [Des. 2] that if P is exp. stable, then

$^1S(P,C)$ is exp. stable $\Leftrightarrow C(I+PC)^{-1}$ is exp. stable (2.4)

For the discussions to follow, it is convenient to note that [Des. 3]

a)[3] $$Q := C(I+PC)^{-1} \Leftrightarrow C = Q(I-PQ)^{-1} \qquad (2.5)$$

b) Q is proper (strictly proper) if and only if C is proper (strictly proper, resp.); and (2.5a)

c) with $Q := C(I+PC)^{-1}$,

$$H_{yu} = \left[\begin{array}{c|c|c} Q & -QP & -Q \\ \hline PQ & P(I-QP) & I-PQ \end{array}\right] \qquad (2.1a)$$

The importance of Eq. (2.1a) is that all the I/O properties of $^1S(P,C)$ are specified by P and Q, without requiring any inverse.

II.3. Properties of $^1S(P,C)$

• Dependence of the I/O map and the disturbance-to-output map

Equation (2.1a) shows that the choice of $Q := C(I+PC)^{-1}$ determines simultaneously the I/O map $H_{y_2u_1} = PQ$ and the disturbance-to-output map $H_{y_2d_0} = I-PQ$; clearly we have

$$H_{y_2u_1} + H_{y_2d_0} = I \qquad (2.6)$$

• Achievable I/O and disturbance-to-output maps

Recall that the map H is an achievable I/O map (disturbance-to-output map) of $^1S(P,C)$ iff for some choice of $C \in \mathbb{R}_p(s)^{n_i \times n_o}$, (i) $H_{y_2u_1} = H$, ($H_{y_2d_0} = H$, resp.); (ii) $^1S(P,C)$ is exp. stable. Let $\mathcal{H}_{y_2u_1}$ ($\mathcal{H}_{y_2d_0}$) denote the set of all achievable I/O maps $H_{y_2u_1}$, (the set of all achievable disturbance-to-output maps $H_{y_2d_0}$, resp.). Then clearly,

$$\mathcal{H}_{y_2u_1}(P) = \{PQ | Q := C(I+PC)^{-1}, \text{ where } C \text{ is such that } {}^1S(P,C) \text{ is exp. stable}\} \quad (2.7a)$$

$$\mathcal{H}_{y_2d_0}(P) = \{I-PQ | Q := C(I+PC)^{-1}, \text{ where } C \text{ is such that } {}^1S(P,C) \text{ is exp. stable}\} \quad (2.7b)$$

Let $\mathcal{C}(P)$ be the set of all compensators C that result in ${}^1S(P,C)$ exp. stable. Equations (2.7) show that $\mathcal{C}(P)$ completely characterizes $\mathcal{H}_{y_2u_1}(P)$ and $\mathcal{H}_{y_2d_0}(P)$.

If the plant P is exp. stable, then examination of (2.4), (2.5) and (2.5a) shows that Eqs. (2.7) can be more explicitly written as

$$\mathcal{H}_{y_2u_1}(P) = \{PQ | Q \text{ is exp. stable}\} \quad (2.8a)$$

$$\mathcal{H}_{y_2d_0}(P) = \{I-PQ | Q \text{ is exp. stable}\} \quad (2.8b)$$

If the plant is not exp. stable, then (2.1a) shows that further constraints on Q, in addition to exp. stability, are needed to ensure exp. stability of ${}^1S(P,C)$. There are two-approaches in the literature in characterizing the class of all compensators C which result in an exp. stable ${}^1S(P,C)$ for a given unstable plant P. The first approach is the two-step stabilization scheme proposed by Zames [Zam. 1] and extended by Desoer and Lin [Des. 4]. The second approach uses fractional representations for the plant and the compensator. Youla, Bongiorno and Jabr [You. 1] used polynomial factorization to characterize the class of all stabilizing compensators for a given linear lumped (not necessarily stable) plant. Using more general factorizations Callier and Desoer extended the results to the linear distributed case [Cal. 2]. Further extension into a general algebraic setting was obtained by Desoer, Liu, Murray and Saeks [Des. 5], and by Vidyasagar, Schneider and Francis

[Vid. 1]. For the special case where the unstable plant P contains only one or a few unstable poles, Desoer and Gustafson [Des. 6] obtained $\mathcal{H}_{y_2u_1}(P)$ by explicitly specifying the additional constraints on Q required for stability.

• Plant perturbation

In practice, the given plant P is usually not known exactly, therefore the design must be based on a certain nominal value of the plant, say P_0. Plant variation also contributes to make P different from P_0. By plant perturbation, we mean the difference between the actual plant P and the nominal P_0. For ${}^1S(P_0,C)$ with the given nominal plant P_0, the plant perturbation $P_0 \leftarrow P_0 + \Delta P := P$ entails

$$\Delta H_{y_2u_1} := H_{y_2u_1} - H^o_{y_2u_1} = (I+PC)^{-1}\ \Delta PC(I+P_0C)^{-1} \tag{2.9}$$

where $H^0_{y_2u_1}$ is the nominal input-output map. Standard derivation of (2.9) can be found in [Cru. 1, Cal. 1].

• Remark

Equation (2.6) of the ${}^1S(P,C)$ configuration constrains the design, hence a compromise between servo performance and regulation (desensitization) is necessary. For example, suppose the design objectives are

(i) $\bar{\sigma}[H_{y_2d_0}(j\omega)] = \bar{\sigma}[I-PQ(j\omega)] << 1$ for all $\omega \in [0,\omega_d]$; and

(ii) $\bar{\sigma}[H_{y_2u_1}(j\omega)] = \bar{\sigma}[PQ(j\omega)] << 1$ for all $\omega \in (\omega_o,\infty)$, with $\omega_o < \omega_d$.

It is clear that there are conflicting requirements over the frequency interval (ω_o,ω_d) : objective (i) requires that the product PQ be close to the identity matrix over (ω_o,ω_d), while objective (ii) requires that PQ be close to the zero matrix over (ω_o,ω_d).

III. Two-degree of freedom design-group 1: the four configurations Σ_a, Σ_b, Σ_c, and Σ_d

In this section, we study the four feedback configurations Σ_a, Σ_b, Σ_c, and Σ_d shown in Fig. 4. It is assumed that C_0 and Q_0 are related by $Q_0 = C_0(I+P_0C_0)^{-1}$ or equivalently $C_0 = Q_0(I-P_0Q_0)^{-1}$. As shown in Fig. 4, each of these four configurations falls into the scheme of Fig. 1: K is the two-input, namely u_1 and y_2, one-output, namely y_1, compensator. In these four cases, $K_{y_1u_1} = (I+C_1P_0)Q_0$ and $K_{y_1y_2} = -C_1$. Therefore Σ_a, Σ_b, Σ_c, and Σ_d have the same system I/O map $H_{yu} : (u_1,u_2,d_0) \mapsto (y_1,y_2)$. Equation (3.1) below shows that each of the four configurations is a two-degree of freedom design: indeed, for $P = P_0$, $H_{y_2u_1} = P_0Q_0$, $H_{y_2d_0} = (I+P_0C_1)^{-1}$.

Σ_a has a model reference structure: P is the given plant, P_0 is the nominal plant model, Q_0 is the precompensator, and C_1 is the "comparator." Note that if the plant is nominal (i.e. $P = P_0$) and if there is no disturbance (i.e. $n_1 = d_0 = u_2 \equiv 0$), then there is no feedback in this configuration. Σ_a has been called conditional feedback in [Hor. 1, p. 246] for the single-input single-output case.

Σ_b has also a model reference structure. The important difference between Σ_a and Σ_b is the following: in Σ_b, the map $H_{\xi_1u_1} : u_1 \mapsto \xi_1$ is the result of a closed-loop configuration, whereas in Σ_a, $H_{\xi_1u_1}$ is the result of an open-loop configuration. The structure of Σ_b has been used by Meyer et al. in the design of flight control systems [Mey. 1]. For the configuration Σ_b, it is easy to see that $H_{\xi_1u_1} = Q_0$.

Σ_c consists of the given plant P, the precompensator $(I+C_1P_0)Q_0$, and the feedback compensator C_1. We assume that the compensator $(I+C_1P_0)Q_0$ is built as one transfer function. Zames used the structure of Σ_c in the study of effects of plant uncertainty [Zam. 1, p. 316].

Σ_d is obtained from Σ_c by introducing the transfer function pair π and π^{-1} as shown in Fig. 4d. We assume that the precompensator $\pi^{-1}(I+C_1P_0)Q_0$ and the feedback compensator $\pi^{-1}C_1$ a) are each built as a single transfer function, and b) have all their elements in $\mathbb{R}_p(s)$.

When the given plant is nominal i.e., $P = P_0$, we call the resulting nominal feedback configuration and denote it by Σ_a^o, Σ_b^o, Σ_c^o, and Σ_d^o respectively. We use H_{yu}^o, $H_{y_2u_1}^o$, and $H_{y_2d_0}^o$ to denote respectively the system I/O map, the input-output map and the disturbance-to-output map of Σ_a^o, Σ_b^o, Σ_c^o, and Σ_d^o.

III.1. The system I/O map

The system I/O map $H_{yu} : (u_1,u_2,u_0) \mapsto (y_1,y_2)$ of Σ_a, Σ_b, Σ_c, and Σ_d is given by (3.1), for the nominal system, and by (3.2) for the case where $P \neq P_0$,

$$H_{yu}^o = \left[\begin{array}{c|c|c} Q_0 & -C_1P_0(I+C_1P_0)^{-1} & -C_1(I+P_0C_1)^{-1} \\ \hline P_0Q_0 & P_0(I+C_1P_0)^{-1} & (I+P_0C_1)^{-1} \end{array}\right] \tag{3.1}$$

When $P \neq P_0$, (see derivation in Appendix)

$$H_{yu}^o = \left[\begin{array}{c|c|c} (I+C_1P)^{-1}(I+C_1P_0)Q_0 & -C_1P(I+C_1P)^{-1} & -C_1(I+PC_1)^{-1} \\ \hline P(I+C_1P)^{-1}(I+C_1P_0)Q_0 & P(I+C_1P)^{-1} & (I+PC_1)^{-1} \end{array}\right] \tag{3.2}$$

III.2. Stability conditions of Σ_a^o, Σ_b^o, Σ_c^o, and Σ_d^o

Recall the definition of the exp. stability of a linear feedback configuration. It can be easily checked (using the summing node equations) that Σ_a^o is exp. stable iff the map $H^o : (u_1,u_2,d_0,n_1) \mapsto (y_1,y_2,y_0,\xi_1)$ of

Σ_a^o is exp. stable. Hence, by (i) Eq. (3.1), (ii) inspection of the configuration Σ_a in Fig. 4a, and (iii) that the composition of exp. stable maps is exp. stable, we conclude that

$$\Sigma_a^o \text{ is exp. stable } \Leftrightarrow P_0,\ Q_0, \text{ and } {}^1S(P_0,C_1) \text{ are exp. stable} \tag{3.3}$$

Similarly, we have

$$\Sigma_b^o \text{ is exp. stable } \Leftrightarrow {}^1S(P_0,C_0) \text{ and } {}^1S(P_0,C_1) \text{ are exp. stable} \tag{3.4}$$

$$\Sigma_c^o \text{ is exp. stable } \Leftrightarrow (I+C_1P_0)Q_0 \text{ and } {}^1S(P_0,C_1) \text{ are exp. stable} \tag{3.5}$$

With the system ${}^1S(P,\pi,\pi^{-1}C_1)$ defined in Fig. 5,

$$\Sigma_d^o \text{ is exp. stable } \Leftrightarrow \pi^{-1}(I+C_1P_0)Q_0 \text{ and } {}^1S(P_0,\pi,\pi^{-1}C_1) \text{ are stable} \tag{3.6}$$

The following fact relates the exp. stability of Σ_c^o and the exp. stability of Σ_d^o.

Fact 3.1: If π and π^{-1} are exp. stable, then

$$\Sigma_c^o \text{ is exp. stable } \Leftrightarrow \Sigma_d^o \text{ is exp. stable.} \tag{3.7}$$

Proof: (see Appendix)

Remarks

a) Σ_a is the only configuration that requires the nominal plant P_0 be stable, because there is no feedback around the model P_0.

b) The stability conditions (3.3)-(3.6) are robust in the following sense: suppose that in the configuration Σ_a^o, Σ_b^o, Σ_c^o and Σ_d^o, we impose arbitrary but small (in the graph topology)[4] perturbations on all subsystems,[5] then [Vid. 1, Chen 2] each of the resulting perturbed systems Σ_a, Σ_b, Σ_c and Σ_d is also exp. stable.

III.3. Properties of Σ_a^o, Σ_b^o, Σ_c^o, and Σ_d^o

• Nominal design

The four nominal configurations Σ_a^o, Σ_b^o, Σ_c^o, and Σ_d^o have the same system I/O map H_{yu}^o; furthermore Q_o specifies the nominal I/O map

$$H^o_{y_2u_1} = P_0Q_0 \; ; \tag{3.8}$$

C_1 specifies the nominal disturbance-to-output map

$$H^o_{y_2d_0} = (I+P_0C_1)^{-1} \tag{3.9}$$

Any achievable I/O map (disturbance-to-output map) must have the form specified by (3.8) for some Q_0 ((3.9) for some C_1, resp.) where Q_0, (C_1, resp.) satisfies (A.2) and each configuration satisfies the stability requirements.

• Achievable I/O maps

We denote the set of all achievable I/O maps for Σ_a^o, Σ_b^o, Σ_c^o, and Σ_d^o by $\mathcal{H}^a_{y_2u_1}$, $\mathcal{H}^b_{y_2u_1}$, $\mathcal{H}^c_{y_2u_1}$, and $\mathcal{H}^d_{y_2u_1}$, respectively.

(a) For Σ_a^o:

(i) If P_0 is not exp. stable, then (3.3) shows that the configuration Σ_a^o is unstable for all Q_0 and C_1 satisfying (A.2).

(ii) If P_0 is exp. stable, then

$$\mathcal{H}^a_{y_2u_1}(P_0) = \{P_0Q_0 \mid Q_0 \text{ is exp. stable}\} \tag{3.11a}$$

Proof of (ii):

Let $\mathcal{H} := \{P_0Q_0 \mid Q_0$ is exp. stable$\}$. It is clear from (3.8) and (3.3) that every achievable I/O map of Σ_a^o is of the form P_0Q_0 for some exp. stable Q_0. Hence, $\mathcal{H}^a_{y_2u_1}(P_0) \subset \mathcal{H}$. To show $\mathcal{H} \subset \mathcal{H}^a_{y_2u_1}(P_0)$, we note that 1) for any $H \in \mathcal{H}$, there exists an exp. stable Q_0 such that $H^o_{y_2u_1} = P_0Q_0 = H$, 2) from (3.3), given that P_0 and Q_0 are exp. stable, Σ^o_a is exp. stable iff ${}^1S(P_0,C_1)$ is exp. stable, 3) there are many C_1's such that ${}^1S(P_0,C_1)$ is exp. stable; for example $C_1 = 0$. 1), 2), and 3) together show that $H \in \mathcal{H}$ implies $H \in \mathcal{H}^a_{y_2u_1}(P_0)$, hence $\mathcal{H} \subset \mathcal{H}^a_{y_2u_1}(P_0)$. This proves the assertion.

(b) For Σ^o_b:

Since $P_0 \in \mathbb{R}_{p,0}(s)^{n_o \times n_i}$, there exists $C_1 \in \mathbb{R}_p(s)^{n_i \times n_o}$ such that ${}^1S(P_0,C_1)$ is exp. stable [Bra. 1, You. 1]. By using similar arguments as those in the proof of (3.11a), it is easily shown that

$$\mathcal{H}_{y_2u_1}(P_0) = \left\{ P_0Q_0 \,\middle|\, \begin{array}{l} Q_0 = C_0(I+P_0C_0)^{-1} \text{ where } C_0 \text{ is such that } {}^1S(P_0,C_0) \\ \text{is exp. stable} \end{array} \right\} \tag{3.11b}$$

(c) For Σ^o_c: By Eq. (3.8) and the stability condition (3.5), we see that

$$\mathcal{H}^c_{y_2u_1}(P_0) = \left\{ P_0Q_0 \,\middle|\, \begin{array}{l} Q_0 \text{ is such that } (I+C_1P_0)Q_0 \text{ is exp. stable for} \\ \text{some } C_1 \text{ which yields } {}^1S(P_0,C_1) \text{ exp. stable} \end{array} \right\} \tag{3.11c}$$

Note that the Q_0's in (3.11c) are necessarily exp. stable, because $\mathcal{Z}[I+C_1P_0] = \mathcal{P}[(I+C_1P_0)^{-1}] \subset \mathring{\mathbb{C}}_-$.

(d) For Σ^o_d with π and π^{-1} exp. stable: Since in this case Σ^o_d is exp. stable iff Σ^o_c is exp stable (Fact 3.1), and Σ^o_d and Σ^o_c have the same input-output map $H_{y_2u_1}$, we have

$$\mathcal{H}^d_{y_2u_1}(P_0) = \mathcal{H}^c_{y_2u_1}(P_0) \tag{3.11d}$$

• <u>Achievable disturbance-to-output maps</u>

We denote the set of all achievable disturbance-to-output maps for Σ_a^o, Σ_b^o, Σ_c^o, and Σ_d^o by $\mathcal{H}^a_{y_2d_0}$, $\mathcal{H}^b_{y_2d_0}$, $\mathcal{H}^c_{y_2d_0}$, and $\mathcal{H}^d_{y_2d_0}$, respectively.

(a) For Σ_a^o:

(i) If <u>P_0 is not exp. stable,</u> then no stable design is possible.

(ii) If <u>P_0 is exp. stable,</u> then (3.3) and (3.9) imply that

$$\mathcal{H}^a_{y_2d_0}(P_0) = \{(I+P_0C_1)^{-1} | C_1 \text{ is such that } {}^1S(P_0,C_1) \text{ is exp. stable}\} \quad (3.13a)$$

Alternatively, if we set $Q_1 := C_1(I+P_0C_1)^{-1}$ -- hence ${}^1S(P_0,C_1)$ is exp. stable iff Q_1 is exp. stable -- then,

$$\mathcal{H}^a_{y_2d_0}(P_o) = \{I-P_0Q_1 | Q_1 \text{ is exp. stable}\}$$

(b) For Σ_b^o: The stability condition (3.4) and Eq. (3.9) show that

$$\mathcal{H}^b_{y_2d_0}(P_0) = \left\{ (I+P_0C_1)^{-1} \middle| \begin{array}{l} C_1 \text{ is such that } {}^1S(P_0,C_1) \text{ is exp. stable; and} \\ {}^1S(P_0,C_0) \text{ is exp. stable for some } C_0 \end{array} \right\}$$

Since $P_0 \in \mathbb{R}_{p,0}(s)^{n_o \times n_i}$, there always exists $C_0 \in \mathbb{R}_p(s)^{n_i \times n_o}$ such that ${}^1S(P_0,C_0)$ is exp. stable. Therefore, the above expression simplies to

$$\mathcal{H}^b_{y_2d_0}(P_o) = \{(I+P_0C_1)^{-1} | C_1 \text{ is such that } {}^1S(P_0,C_1) \text{ is exp. stable}\} \quad (3.13b)$$

(c) For Σ_c^o: The stability condition (3.5) and Eq. (3.9) show that

$$\mathcal{H}^c_{y_2d_0}(P_0) = \left\{ (I+P_0C_1)^{-1} \middle| \begin{array}{l} C_1 \text{ is such that } {}^1S(P_0,C_1) \text{ is exp. stable and} \\ \text{such that } (I+C_1P_0)Q_0 \text{ is exp. stable for} \\ \text{some } Q_0 \end{array} \right\}$$

(3.13c)

(d) For Σ_d^o with both π and π^{-1} exp. stable: Since in this case, Σ_d^o is exp. stable iff Σ_c^o is exp. stable (Fact 3.1), we have that

$$\mathcal{H}^d_{y_2d_0}(P_0) = \left\{ (I+P_0C_1)^{-1} \middle| \begin{array}{l} C_1 \text{ is such that } {}^1S(P_0,C_1) \text{ is exp. stable} \\ \text{and such that } (I+C_1P_0)Q_0 \text{ is exp. stable for} \\ \text{some } Q_0 \end{array} \right\} \quad (3.13d)$$

$$= \mathcal{H}^c_{y_2d_0}(P_0)$$

Remarks

(i) For the configurations Σ_a^o and Σ_b^o, we can simultaneously achieve any $H^o_{y_2u_1} \in \mathcal{H}^a_{y_2u_1}(P_0)$, ($\mathcal{H}^b_{y_2u_1}(P_0)$, resp.) and any $H^o_{y_2d_0} \in \mathcal{H}^a_{y_2d_0}(P_0)$, ($\mathcal{H}^b_{y_2d_0}(P_0)$, resp.) i.e., the choices of $H^o_{y_2u_1}$ and $H^o_{y_2d_0}$ (hence the choices of Q_0 and C_1) are independent. For the configurations Σ_c^o and Σ_d^o, the choices of $H^o_{y_2u_1}$ and $H^o_{y_2d_0}$ are constrained: indeed, Q_0 and C_1 must be chosen so that the transfer function $[(I+C_1P_0)Q_0]$ is exp. stable.

(ii) Although Σ_c^o and Σ_d^o have the same achievable I/O maps and achievable disturbance-to-output maps, Σ_d^o offers more flexibility in implementation: for example, π may be used to adjust the signal level at the summing node.

• Plant perturbation

For Σ_a^o, Σ_b^o, Σ_c^o, and Σ_d^o, the plant perturbation $P_0 \leftarrow P_0 + \Delta P := P$ entails

$$\Delta H_{y_2u_1} := H_{y_2u_1} - H^o_{y_2u_1} = (I+PC_1)^{-1}\Delta P Q_0 \quad (3.17)$$

(3.17) follows by the same calculation for (2.9).

• Model perturbation (for Σ_a^o and Σ_b^o)

By model perturbation we mean any inaccuracy and variation in the model P_0 (of Σ_a^o and Σ_b^o).

(a) For Σ_a^o, the I/O map $H^o_{y_2u_1}$ is <u>sensitive</u> to perturbations in the model P_0. Indeed, the perturbation $P_0 \leftarrow P_0 + \Delta P_m$ <u>in the model</u> implies that[6] (see Appendix)

$$\Delta^m H^a_{y_2u_1} := {}^m H^a_{y_2u_1} - H^o_{y_2u_1} = P_0 C_1 (I+P_0C_1)^{-1} \Delta P_m Q_0 \tag{3.18}$$

$$\approx \Delta P_m Q_0 \qquad \text{over } \Omega_d$$

where Ω_d is the frequency band of interest for disturbance rejection. Note that an arbitrary small but unstable ΔP_m will in general cause system instability.

(b) For Σ_b^o, the I/O map $H^o_{y_2u_1}$ is relatively insensitive to perturbations in the model P_0, compared to Σ_a^o. Indeed, the perturbation $P_0 \leftarrow P_0 + \Delta P_m := P$, <u>in the model</u> implies that (see Appendix)

$$\Delta^m H^b_{y_2u_1} := {}^m H^b_{y_2u_1} - H^o_{y_2u_1} = [(I+P_0C_0)^{-1} - (I+P_0C_1)^{-1}]\Delta P_m C_0 (I+PC_0)^{-1} \tag{3.19}$$

$$= [(I+P_0C_0)^{-1} - (I+P_0C_1)^{-1}](I+\Delta P_m Q_0)^{-1} \Delta P_m Q_0 \tag{3.19a}$$

Note that if the perturbation is small, more precisely, if $\bar{\sigma}[\Delta P_m(j\omega)]\ \bar{\sigma}[Q_0(j\omega)] << 1$ for all $\omega \in [0, \infty)$, then (3.19a) shows that

$$\Delta^m H^b_{y_2u_1} \approx [(I+P_0C_0)^{-1} - (I+P_0C_1)^{-1}]\Delta P_m Q_0 \tag{3.19b}$$

- <u>Perturbation in precompensator</u>

For Σ_a, Σ_c and Σ_d, the precompensator is not under feedback hence the I/O map $H^o_{y_2u_1}$ is sensitive to perturbations in the precompensator: namely, Q_0 in Σ_a, $(I+C_1P_0)Q_0$ in Σ_c, and $\pi^{-1}(I+C_1P_0)Q_0$ in Σ_d.

• Conclusions

a) Σ_b is better than Σ_a:

1) Σ_a is more sensitive to changes in the model P_0 (see (3.18) and (3.19)).

2) Σ_b can accommodate unstable P_0's.

b) Σ_b is better than Σ_a, Σ_c, and Σ_d:

Σ_a, Σ_c, and Σ_d are sensitive to changes in the precompensator. (In Σ_b, the precompensator is realized as a feedback configuration, hence Σ_b is less sensitive, if well designed).

c) Σ_b is better than Σ_c and Σ_d:

In Σ_b, the choices of the I/O map $H^o_{y_2u_1}$ and the disturbance-to-output map $H^o_{y_2d_0}$ are independent, whereas in Σ_c and Σ_d, the choices are constrained.

IV. Two-degree of freedom design-group 2: the configurations Σ_e and Σ_f

The configuration Σ_e has the same model reference structure as that of Σ_a except that the output of the comparator C_2 in Σ_e is feedback to the input of Q_0, rather than as in Σ_a, to the plant input. For the single-input single-output case, Σ_e has been called model feedback by Horowitz [Hor. 1, p. 246].

The structure of Σ_f has been considered by Cruz, et al. [Cru. 1] among others. Note that for the special case when $C_2 = I$, the configuration Σ_f reduces to the unity-feedback configuration ${}^1S(P,C_0)$, with $C_0 := Q_0(I-P_0Q_0)^{-1}$.

We use Σ^o_e and Σ^o_f to denote the nominal feedback configurations, and H^o_{yu} to denote the nominal I/O map.

IV.1. The system I/O map

For the nominal system Σ^o_e (i.e., when $P = P_o$), the system I/O map $H^o_e : (u_1,u_2,d_0,n_1) \mapsto (y_1,y_2,y_0,e_1)$ is given by

$$H_e^o = \left[\begin{array}{c|c|c|c} Q_0 & -Q_0C_2P_0 & -Q_0C_2 & Q_0C_2P_0 \\ \hline P_0Q_0 & P_0(I-Q_0C_2P_0) & I-P_0Q_0C_2 & P_0Q_0C_2P_0 \\ \hline P_0Q_0 & -P_0Q_0C_2P_0 & -P_0Q_0C_2 & P_0 \\ \hline I & -C_2P_0 & -C_2 & C_2P_0 \end{array}\right] \tag{4.1}$$

For the nominal system Σ_f^o (i.e., when $P = P_0$), the system I/O map $H_f^o : (u_1,u_2,d_0) \mapsto (y_1,y_2,e_1')$ is given by

$$H_f^o = \left[\begin{array}{c|c|c} Q_0 & -Q_0C_2P_0 & -Q_0C_2 \\ \hline P_0Q_0 & P_0(I-Q_0C_2P_0) & I-P_0Q_0C_2 \\ \hline I-C_2P_0Q_0 & -C_2(I-P_0Q_0C_2)P_0 & -C_2(I-P_0Q_0C_2) \end{array}\right] \tag{4.2}$$

When $P \neq P_0$, Σ_e and Σ_f have the same system I/O map $H_{yu} : (u_1,u_2,d_0) \mapsto (y_1,y_2)$: indeed, with $\Delta P := P-P_o$, (see Appendix),

$$H_{yu} = \left[\begin{array}{c|c|c} Q_0(I+C_2\Delta PQ_0)^{-1} & -(I+Q_0C_2\Delta P)^{-1}Q_0C_2P & -Q_0C_2(I+\Delta PQ_0C_2)^{-1} \\ \hline PQ_0(I+C_2\Delta PQ_0)^{-1} & (I-P_0Q_0C_2)(I+\Delta PQ_0C_2)^{-1}P & (I-P_0Q_0C_2)(I+\Delta PQ_0C_2)^{-1} \end{array}\right] \tag{4.3}$$

IV.2. Stability conditions of Σ_e^o and Σ_f^o

It can be checked (using the summing node equations) that

$$\left.\begin{array}{l}\text{the nominal configuration } \Sigma_e^o \ (\Sigma_f^o) \text{ is exp. stable} \\ \text{iff the system I/O map } H_e^o, \ (H_f^o, \text{ resp.}), \text{ is exp. stable.}\end{array}\right\} \tag{4.4}$$

Hence, by inspection of (4.1), we have that

$$\Sigma_e^o \text{ is exp. stable } \Leftrightarrow P_0, Q_0 \text{ and } C_2 \text{ are exp. stable} \tag{4.5}$$

To test the exp. stability of Σ_f^o, we have to check all the submatrices in (4.2). However, in the special case where P_0 is exp. stable, Q_0 and C_2 are exp. stable implies that Σ_f^o is exp stable.

IV.3. Properties of Σ_e^o and Σ_f^o

• Nominal design

For Σ_e^o and Σ_f^o, Q_0 specifies the nominal I/O map

$$H_{y_2u_1}^o = P_0Q_0 ; \tag{4.6}$$

C_2 and Q_0 together specifies the nominal disturbance-to-output map

$$H_{y_2d_0}^o = I - P_0Q_0C_2 = I - H_{y_2u_1}^o \ C_2 \tag{4.7}$$

In the following, we specify the set of all achievable I/O maps and the set of all achievable disturbance-to-output map for Σ_e^o and Σ_f^o.

• Achievable I/O maps

(a) For Σ_e^o:

(i) If P_0 is not exp. stable, then Σ_e^o is not exp. stable for any choice of Q_0 and C_2 satisfying (A.2).

(ii) If P_o is exp. stable, then (4.5) and (4.6) together show that

$$\mathcal{H}^{e}_{y_2u_1}(P_0) = \{P_0Q_0 | Q_0 \text{ is exp. stable}\} \tag{4.8}$$

(b) For Σ_f^o: By the stability condition (4.4) and Eq. (4.6), we have that

$$\mathcal{H}^{f}_{y_2u_1}(P_0) = \{P_0Q_0 | Q_0 \text{ is such that } \exists\, C_2 \text{ which yields } H_f^o \text{ exp. stable}\} \tag{4.10a}$$

For the special case where <u>P_0 is exp. stable,</u> it can be easily checked that

$$\mathcal{H}^{f}_{y_2u_1}(P_0) = \{P_0Q_0 | Q_0 \text{ is exp. stable}\} \tag{4.10b}$$

• <u>Achievable disturbance-to-output maps</u>

(a) For Σ_e^o: If P_0 is exp. stable, then (4.5) and (4.7) together show

$$\begin{aligned}\mathcal{H}^{e}_{y_2u_1}(P_0) &= \{I-P_0Q_0C_2 | Q_0 \text{ and } C_2 \text{ are exp. stable}\} \\ &= \{I-P_0Q | Q \text{ is exp. stable}\}\end{aligned} \tag{4.11}$$

(b) For Σ_f^o: By the stability condition (4.4) and Eq. (4.7), we have that

$$\mathcal{H}^{f}_{y_2d_0}(P_0) = \{I-P_0Q_0C_2 | Q_0 \text{ and } C_2 \text{ are such that } H_f^o \text{ is exp. stable}\} \tag{4.12a}$$

For the special case where P_0 is exp. stable, we have that Q_0 and C_2 are exp. stable implies that Σ_f^o is exp. stable, hence,

$$\mathcal{H}^{f}_{y_2d_0}(P_0) \supset \{I-P_0Q_0C_2 | Q_0 \text{ and } C_2 \text{ are exp. stable}\}.$$

Therefore,

$$\mathcal{H}^{f}_{y_2d_0}(P_0) \supset \{I-P_0Q | Q \text{ is exp. stable}\}. \tag{4.12b}$$

However, the stability condition (4.4) and Eq. (4.2) show that Σ_f^o is exp. stable implies that the product Q_0C_2 is exp. stable, hence

$$\mathcal{H}^{f}_{y_2d_0}(P_0) \subset \{I-P_0Q_0C_2 | Q_0C_2 \text{ is exp. stable}\} \tag{4.12c}$$

$$= \{I-P_0Q | Q \text{ is exp. stable}\}$$

We conlcude from (4.12b) and (4.12c) that if P_0 is exp. stable, then

$$\mathcal{H}^{f}_{y_2d_0}(P) = \{I-P_0Q | Q \text{ is exp. stable}\} \tag{4.12d}$$

• Plant perturbation

For Σ_e^o and Σ_f^o, the plant perturbation $P_0 \leftarrow P_0+\Delta P$ entails

$$\Delta H_{y_2u_1} := H_{y_2u_1} - H^o_{y_2u_1} = \underbrace{(I-P_0Q_0C_2)(I+\Delta PQ_0C_2)^{-1}}_{H_{y_2d_0}}\Delta PQ_0 \tag{4.13}$$

(see Appendix for the derivation of (4.13))

• Model perturbation (for Σ_e)

For Σ_e^o, the I/O map $H^o_{y_2u_1}$ is sensitive to perturbations in the model P_0. Indeed, the model perturbation $P_0 \leftarrow P_0+P_m$ implies that $\Delta^m H^e_{y_2u_1}$, the corresponding change in $H^o_{y_2u_1}$, is given by (see Appendix)

$$\Delta^m H^e_{y_2u_1} = P_0Q_0C_2\Delta P_mQ_0(I-C_2\Delta P_mQ_0)^{-1} \tag{4.14}$$

$$= P_0Q_0C_2[I-\Delta P_mQ_0C_2)^{-1}\Delta P_mQ_0 \tag{4.15}$$

If $\bar{\sigma}[\Delta P_m(j\omega)]\,\bar{\sigma}[Q_0C_2(j\omega)] << 1$ for all $\omega \in \Omega_d$ and if $\|H_{y_2d_0}(j\omega)\| << 1$ for all $\omega \in \Omega_d$, then, from (4.15),

$$\Delta^m H^e_{y_2u_1} \sim \Delta P_mQ_0 \quad \text{over} \quad \Omega_d \tag{4.16}$$

where Ω_d is the frequency band of interest for disturbance rejection.

• Perturbation in the compensators Q_0 and C_2 in Σ_e^0

By inspection of Σ_e in Fig. 6(a), it is clear that when the plant is nominal ($P=P_0$), there is no feedback in Σ_e^o i.e., Σ_e^o is an open-loop system. If the compensators Q_0 and C_2 undergo the perturbations $Q_0 \leftarrow Q_0+\Delta Q_0$ and $C_2 \leftarrow C_2+\Delta C_2$, then the resulting I/O map $H^c_{y_2u_1}$ and the resulting disturbance-to-output map $H^c_{y_2d_0}$ are given by

$$H^c_{y_2u_1} = P_0(Q_0+\Delta Q_0), \qquad \text{and}$$

$$H^c_{y_2d_0} = I - P_0(Q_0+\Delta Q_0)(C_2+\Delta C_2)$$

• Conclusion

Σ_b and Σ_f are better than Σ_e: indeed,

1) Σ_e requires that P_0 be exp. stable;

2) Σ_e is sensitive to changes in the model P_0; and

3) Σ_e is sensitive to changes in the compensation subsystems Q_0 and C_2.

• Generalization

So far, in studying feedback configurations, we restrict ourselves to the continuous linear time-invariant lumped systems. However, it should be noted that in deriving stability conditions and various properties of each configuration, the only necessary restrictions are linearity and time-invariance. Hence, all the results developed in the present section and Section II and III can be easily generalized to the discrete linear time-invariant case and to the continuous linear time-invariant distributed case.

V. Configurations Σ_a, Σ_b, Σ_c, and Σ_d: the nonlinear case

In Section III, we compare the four configurations Σ_a, Σ_b, Σ_c, and Σ_d for the linear case. We specify the set of all achievable I/O maps and the set of all achievable disturbance-to-output maps, and study the effects of various subsystem perturbations on the I/O map $H_{y_2u_1}$. In this section, we do the same comparison for this four configurations in the nonlinear context. We shall see that, under suitable assumptions, most of the results in Section III still hold for the nonlinear case.

We use an input-output description of the nonlinear system. Let $(\mathcal{L},\|\cdot\|)$ be a normed space of "time functions": $\mathcal{J} \to \mathcal{V}$ where $\mathcal{J} \subset \mathbb{R}_+$ ($\mathcal{J} = \mathbb{R}_+$ ($\mathbb{N}$, resp.) for the continuous-time case (discrete-time case, resp.)), $\mathcal{V}$ is a normed space and $\|\cdot\|$ is the chosen norm in $\mathcal{L}$. Let $\mathcal{L}_e$ be the corresponding extended space [Wil. 1], [Des. 7], [Vid. 2]. A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class K (denoted by $\phi \in K$) iff ϕ is continuous and increasing. ϕ is said to belong to class K_0 iff $\phi \in K$ and $\phi(0) = 0$. A nonlinear causal map $H : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$ is said to be $\mathcal{S}$-stable iff $\exists\ \phi \in K$ s.t. $\forall x \in \mathcal{L}_e^{n_i}$, $\forall T \in \mathcal{J}$,

$$\|Hx\|_T \leq \phi(\|x\|_T) \ .$$

H is said to be incrementally $\mathcal{S}$-stable (incr. $\mathcal{S}$-stable) iff (i) H is $\mathcal{S}$-stable, (ii) $\exists\ \tilde{\phi} \in K_0$ s.t. $\forall x, x' \in \mathcal{L}_e^{n_i}$, $\forall T \in \mathcal{J}$,

$$\|Hx-Hx'\|_T \leq \tilde{\phi}(\|x-x'\|_T) \ .$$

Note that if $\phi : x \to \gamma x$, γ constant ($\tilde{\phi} : x \to \tilde{\gamma}x$, $\tilde{\gamma}$ constant), then we have finite-gain stability, (finite incremental gain stability, resp.). It can be easily checked that the sum and the composition of $\mathcal{S}$-stable maps, (incr. $\mathcal{S}$-stable maps) are $\mathcal{S}$-stable, (incre. $\mathcal{S}$-stable, resp.).

We make the following assumptions throughout this section:

(N.1) P_0, $P : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$ and $Q_0 : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ are nonlinear causal maps;

(N.2) $C_1 : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_i}$ is linear and causal;

(N.3) $\pi^{-1}, \pi : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_i}$ are linear and causal;

(N.4) for each configuration, both the nominal and perturbed system are well-posed i.e., the relation from the exogenous inputs into each subsystem variable (i.e., input or output) is a well-defined nonlinear causal map between the corresponding extended spaces;

(N.5) The nonlinear maps C_0 and Q_0 are related by

$$C_0 = Q_0(I-P_0Q_0)^{-1} \text{ or equivalently } Q_0 = C_0(I+P_0C_0)^{-1} .$$

We say that a well-posed feedback configuration is $\mathcal{S}$-stable iff the map from the exogenous inputs to any subsystem variable (i.e., input or output) is $\mathcal{S}$-stable. The map $H : \mathcal{L}_e^{n_o} \to \mathcal{L}_e^{n_o}$ is said to be an achievable I/O map (achievable disturbance-to-output map, resp.) of the nonlinear feedback configuration Σ_a (Σ_b, Σ_c, Σ_d, resp.) iff by some appropriate choice of the compensation sybsystems satisfying (N.1)-(N.5),
(i) $H_{y_2u_1} = H$, ($H_{y_2d_0} = H$, resp.); (ii) Σ_a, (Σ_b, Σ_c, Σ_d, resp.) is $\mathcal{S}$-stable.

It is crucial to note that although the formulas belows have the same form as those in the linear case, they have here a completely different meaning: for example in the previous sections PC meant the product of the transfer function P with the transfer function C, in the nonlinear case PC means the composition of the function P with the function C: e.g., when we write PCe, we mean P(C(e)) or equivalently $P \circ C(e)$.

V.1. The system I/O map

With the assumption that C_1 and π are linear, it can be easily verified that the partial system I/O maps of the four configurations are given by the Eqs. (5.1) and (5.2) below; each entry of (5.1) and (5.2) is composition of nonlinear causal maps. By assumption (N.4), all the inverses in (5.1) and (5.2) are well-defined causal maps. Let for $k = 1,2$, $F_k : (u_1, u_2, d_0) \mapsto y_k$; so F_1 and F_2 specify the closed-loop map. We denote the partial maps by the same notation as in Section III: for example in terms of partial maps, we have $H_{y_1u_1} := F_1(u_1,0,0)$ and $H_{y_2d_0} := F_2(0,0,d_2)$. When $P = P_0$, the partial maps relating (u_1,u_2,d_0) to (y_1,y_2) are given by

$$\left[\begin{array}{c|c|c} H^o_{y_1u_1} & H^o_{y_1u_2} & H^o_{y_1d_0} \\ \hline H^o_{y_2u_1} & H^o_{y_2u_2} & H^o_{y_2d_0} \end{array}\right] = \left[\begin{array}{c|c|c} Q_0 & -C_1P_0(I+C_1P_0)^{-1} & -C_1[I-P_0(-C_1)]^{-1} \\ \hline P_0Q_0 & P_0(I+C_1P_0)^{-1} & [I-P_0(-C_1)]^{-1} \end{array}\right] \tag{5.1}$$

When $P \neq P_0$,

$$\left[\begin{array}{c|c|c} H_{y_1u_1} & H_{y_1u_2} & H_{y_1d_0} \\ \hline H_{y_2u_1} & H_{y_2u_2} & H_{y_2d_0} \end{array}\right] = \left[\begin{array}{c|c|c} (I+C_1P)^{-1}(I+C_1P_0)Q_0 & -C_1P(I+C_1P)^{-1} & -C_1[I-P(-C_1)]^{-1} \\ \hline P(I+C_1P)^{-1}(I+C_1P_0)Q_0 & P(I+C_1P)^{-1} & [I-P(-C_1)]^{-1} \end{array}\right] \tag{5.2}$$

In the following all the symbols Σ_a^o, $\mathcal{H}^a_{y_2u_1}(P_0)$, ${}^1S(P_0,C_0)$, etc. have the same meaning as in Section III except that they are associated with the nonlinear configurations Σ_a, Σ_b, ... etc.

V.2. Stability conditions of the nominal nonlinear feedback configurations Σ_a^o, Σ_b^o, Σ_e^o and Σ_d^o

Unlike the linear case, each partial map of (5.1) being $\mathcal{S}$-stable does not imply that the nominal nonlinear feedback configurations are

$\mathcal{S}$-stable. The following stability conditions can be obtained by (i) that the composition of $\mathcal{S}$-stable maps are $\mathcal{S}$-stable, and (ii) inspection[7] of the block diagrams of the nonlinear configurations in Fig. 4.

(a) Σ_a^o is $\mathcal{S}$-stable $\Leftrightarrow$ Q_0, P_0 and ${}^1S(C_1,P_0)$ are $\mathcal{S}$-stable (5.3a)

(b) Σ_b^o is $\mathcal{S}$-stable $\Leftrightarrow$ ${}^1S(P_0,C_0)$ and ${}^1S(C_1,P_0)$ are $\mathcal{S}$-stable (5.3b)

(c) Σ_c^o is $\mathcal{S}$-stable $\Leftrightarrow$ $(I+C_1P_0)Q_0$ and ${}^1S(C_1,P_0)$ are $\mathcal{S}$-stable (5.3c)

(d) Σ_d^o is $\mathcal{S}$-stable $\Leftrightarrow$ $\pi^{-1}(I+C_1P_0)Q_0$ and ${}^1S(P_0,\pi,\pi^{-1}C_1)$ are $\mathcal{S}$-stable (5.3d)

<u>Fact 5.1</u>. If π and π^{-1} are linear and incr. $\mathcal{S}$-stable, then

$$\Sigma_d^o \text{ is } \mathcal{S}\text{-stable} \Leftrightarrow \Sigma_c^o \text{ is } \mathcal{S}\text{-stable} \tag{5.3e}$$

Proof: (See Appendix).

V.3. <u>Properties of nonlinear configurations Σ_a^o, Σ_b^o, Σ_c^o, and Σ_d^o</u>

• <u>Nominal design</u>

As in the linear case, for the nonlinear feedback configurations Σ_a^o, Σ_b^o, Σ_c^o, and Σ_d^o, Q_0 specifies the <u>nominal</u> I/O map

$$H^o_{y_2u_1} = P_0Q_0 \ ; \tag{5.4}$$

C_1 specifies the <u>nominal</u> disturbance-to-output map

$$H^o_{y_2d_0} = [I - P_0(-C_1)]^{-1} \tag{5.5}$$

<u>Remark</u>: With C_1 linear, $\forall x \in \mathcal{L}_e^{n_o}$,

$$[I-P_0(-C_1)](x) = x-P_0C_1(-x) = -(-x) - P_0C_1(-x) = -(I+P_0C_1)(-x) \ .$$

In the following, we specify the set of all achievable I/O maps and the set of all achievable disturbance-to-output maps for each configuration. We assume that there exists a linear C_1 such that $^1S(C_1,P_0)$ is $\mathcal{S}$-stable, and that there exists a C_0 such that $^1S(P_0,C_0)$ is $\mathcal{S}$-stable.

• Achievable I/O maps

(a) For Σ_a^o:

(i) If P_0 is not $\mathcal{S}$-stable, (5.3a) shows that the configuration Σ_a is not $\mathcal{S}$-stable.

(ii) If P_0 is incr. $\mathcal{S}$-stable, then [Des. 8]

$$\mathcal{H}^a_{y_2u_1}(P_0) = \{P_0Q_0 \mid Q_0 \text{ is } \mathcal{S}\text{-stable}\} \tag{5.6}$$

(b) For Σ_b^o:

$$\mathcal{H}^b_{y_2u_1}(P_0) = \left\{ P_0Q_0 \;\middle|\; \begin{array}{l} Q_0 = C_0(I+P_0C_0)^{-1} \text{ where } C_0 \text{ is such that} \\ \quad {}^1S(P_0,C_0) \text{ is } \mathcal{S}\text{-stable} \end{array} \right\} \tag{5.7}$$

(c) For Σ_c^o:

$$\mathcal{H}^c_{y_2u_1}(P_0) = \left\{ P_0Q_0 \;\middle|\; \begin{array}{l} Q_0 \text{ is such that } (I+C_1P_0)Q_0 \text{ is } \mathcal{S}\text{-stable for} \\ \text{some } C_1 \text{ which yields } {}^1S(C_1,P_0)\ \mathcal{S}\text{-stable} \end{array} \right\} \tag{5.8}$$

(d) For Σ_d^o with π and π^{-1} linear and incr. $\mathcal{S}$-stable:

$$\mathcal{H}^d_{y_2u_1}(P_0) = \left\{ P_0Q_0 \;\middle|\; \begin{array}{l} Q_0 \text{ is such that } (I+C_1P_0)Q_0 \text{ is } \mathcal{S}\text{-stable for} \\ \text{some } C_1 \text{ which yields } {}^1S(C_1,P_0)\ \mathcal{S}\text{-stable} \end{array} \right\} \tag{5.9}$$

$$= \mathcal{H}^c_{y_2u_1}(P_0)$$

Remark: For Σ_c^o and Σ_d^o, we did not need that C_1 be linear.

• Achievable disturbance-to-output maps

(a) For Σ_a^o:

If P_0 is incr. $\mathcal{S}$-stable, then

$$\mathcal{H}^a_{y_2d_0}(P_0) = \left\{ [I-P_0(-C_1)]^{-1} \,\middle|\, \begin{array}{l} C_1 \text{ is linear and is such that} \\ {}^1S(C_1,P_0) \text{ is } \mathcal{S}\text{-stable} \end{array} \right\} \quad (5.10)$$

(b) For Σ_b^o:

$$\mathcal{H}^b_{y_2d_0}(P_0) = \left\{ [I-P_0(-C_1)]^{-1} \,\middle|\, \begin{array}{l} C_1 \text{ is linear and is such that} \\ {}^1S(C_1,P_0) \text{ is } \mathcal{S}\text{-stable} \end{array} \right\} \quad (5.11)$$

(c) For Σ_c^o:

$$\mathcal{H}^c_{y_2d_0}(P_0) = \left\{ [I-P_0(-C_1)]^{-1} \,\middle|\, \begin{array}{l} C_1 \text{ is such that } {}^1S(C_1,P_0) \text{ is } \mathcal{S}\text{-stable} \\ \text{and that } (I+C_1P_0)Q_0 \text{ is } \mathcal{S}\text{-stable for} \\ \text{some } Q_0. \end{array} \right\} \quad (5.12)$$

(d) For Σ_d^o with π and π^{-1} linear and incr. $\mathcal{S}$-stable:

$$\mathcal{H}^d_{y_2d_0}(P_0) = \left\{ [I-P_0(-C_1)]^{-1} \,\middle|\, \begin{array}{l} C_1 \text{ is such that } {}^1S(C_1,P_0) \text{ is } \mathcal{S}\text{-stable} \\ \text{and that } (I+C_1P_0)Q_0 \text{ is } \mathcal{S}\text{-stable} \\ \text{for some } Q_0 \end{array} \right\} \quad (5.13)$$

$$= \mathcal{H}^c_{y_2d_0}(P_0)$$

Remarks

(i) From (5.6), (5.7), (5.10), and (5.11), it is clear that for the configurations Σ_a^o and Σ_b^o, we can simultaneously achieve any $H^o_{y_2u_1} \in \mathcal{H}^a_{y_2u_1}(P_0)$, ($\mathcal{H}^b_{y_2u_1}(P_0)$, resp.) and any $H^o_{y_2d_0} \in \mathcal{H}^a_{y_2d_0}(P_0)$,

($\mathcal{H}^b_{y_2d_0}(P_0)$, resp.) i.e., the choices of $H^o_{y_2u_1}$ and $H^o_{y_2d_0}$, (hence the choices of Q_0 and C_1) are independent. For the configurations Σ^o_c and Σ^o_d, the choices of $H^o_{y_2u_1}$ and $H^o_{y_2d_0}$ are constrained: indeed, Q_0 and C_1 must be chosen so that the map $[(I+C_1P_0)Q_0]$, ($[\pi^{-1}(I+C_1P_0)Q_0]$, resp.) is $\mathcal{S}$-stable.

(ii) As in the linear case, Σ^o_d offers more flexibility in implementation than Σ^o_c does.

• Plant perturbation

For each of the four configurations Σ^o_a, Σ^o_b, Σ^o_c, and Σ^o_d, where C_1 is assumed linear, the plant perturbation $P_0 \leftarrow P_0+\Delta P := P$ has the same effect on the I/O map $H^o_{y_2u_1}$. More precisely, let $\Delta H_{y_2u_1} := H_{y_2u_1} - H^o_{y_2u_1}$. Then for any input $u_1 \in \mathcal{L}_e^{n_o}$,

$$\Delta H_{y_2u_1}(u_1) = \int_0^1 [I+D(P)C_1]^{-1} d\alpha \cdot \Delta P Q_0(u_1) \tag{5.14}$$

where D(P) is the Frechet derivative of P, (see [Die. 1], [Des. 9]), and is evaluated at $(I+C_1P)^{-1}[(I+C_1P_0)Q_0(u_1) + \alpha C_1 \Delta P Q_0(u_1)]$ with $\alpha \in [0,1]$. See Appendix for derivation of (5.14).

Remark: Equation (5.14) tells us that if the linear compensator C_1 is chosen so that along the trajectory, defined in (5.14), where D(P) is evaluated, all the linear maps $D(P)C_1$ has "large gain," then, for Σ^o_a, Σ^o_b, $\cdots$ Σ^o_d, the output y_2 (corresponding to the fixed input u_1) is very insensitive to perturbations in the nominal plant P_0 (in comparison with the equivalent open-loop system).

• Model perturbation

(i) For Σ^o_a, let $\Delta^m H^a_{y_2u_1}$ be the change in the I/O map $H^o_{y_2u_1}$ caused by the model perturbation $P_0 \leftarrow P_0+\Delta P_m$, then $\forall u_1 \in \mathcal{L}_e^{n_o}$,

$$\Delta^m H^a_{y_2u_1}(u_1) = \int_0^1 [I-(I+D(P_0)C_1)^{-1}]d\alpha\cdot\Delta P_m Q_0(u_1) \qquad (5.15)$$

where $D(P_0)$ is evaluated at $(I+C_1P_0)^{-1}[(I+C_1P_0)Q_0(u_1)+\alpha C_1\Delta P_m Q_0(u_1)]$ with $\alpha\in[0,1]$. See Appendix.

(ii) For Σ_b^o, if we assume that both C_0 and C_1 are linear, then it can be checked that $\forall u_1 \in \mathcal{L}_e^{n_o}$, $\Delta^m H^b_{y_2u_1}(u_1) = P_0(I+C_1P_0)^{-1}[(C_1-C_0)$ $\cdot\int_0^1 (I+D(P)C_0)^{-1}d\alpha\ \Delta P_m C_0(I+P_0C_0)^{-1}(u_1)]$ where $P := P_0+\Delta P_m$, and $D(P)$ is evaluated at

$$C_0(I+PC_0)^{-1}(u_1+\alpha\Delta P_m C_0(I+P_0C_0)^{-1}(u_1)) \text{ for } \alpha\in[0,1].$$

• Perturbation in the precompensators

For Σ_a, Σ_c and Σ_d, the I/O map $H^o_{y_2u_1}$ is sensitive to changes in the precompensators, namely Q_0 in Σ_a, $(I+C_1P_0)Q_0$ in Σ_c and $\pi^{-1}(I+C_1P_0)Q_0$ in Σ_d, since they are outside the feedback loop.

• Conclusions. For nonlinear P_0, P, linear C_1 and π,

(i) Σ_b is better than Σ_a in that Σ_b can accommodate unstable plants.

(ii) Σ_b is better than Σ_a, Σ_c and Σ_d: the latter are sensitive to changes in the precompensator. (In Σ_b, the precompensator is realized as a feedback configuration, hence is less sensitive if well-designed).

(iii) Σ_b is better than Σ_c and Σ_d: In Σ_b, the choices of the I/O map $H^o_{y_2u_1}$ and the disturbance-to-output map $H^o_{y_2d_0}$ are independent, whereas in Σ_c and Σ_d the choices are constrained.

Conclusions

In this paper, we study several feedback configurations which have appeared in the control literature. We start with the definitions of two-degree of freedom design and of achievable I/O and disturbance-to-output map. In section II, we show the basic limitation of linear unity feedback configuration ${}^1S(P,C)$, namely the dependence of the I/O and disturbance-to-output map. We study the four two-degree of freedom design configurations Σ_a, Σ_b, Σ_c and Σ_d in section III, in terms of their achievable I/O maps and disturbance-to-output maps and their sensitivity to subsystem perturbations, we demonstrate that Σ_b is better than Σ_a, Σ_c and Σ_d. In section IV, the two-degree of freedom design configurations Σ_e and Σ_f are studied and compared to Σ_b. In our discussion, we have restricted ourselves to the linear time-invariant lumped case, however the same results hold for the linear time-invariant distributed and the linear time-invariant discrete-time cases. Finally, we study Σ_a, Σ_b, Σ_c and Σ_d in the nonlinear context, it is seen that some of the linear properties are also held for the nonlinear case.

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Appendix

Derivation of (3.2):

(i) For Σ_a and Σ_b:

(a) Let $u_2 = d_0 = n_1 = 0$, then we obtain successively

$$y_1 = Q_0u_1 - C_1[Py_1 - P_0Q_0u_1]$$

$$(I+C_1P)y_1 = (I+C_1P_0)Q_0u_1$$

$$y_1 = (I+C_1P)^{-1}(I+C_1P_0)Q_0u_1 \tag{A.1}$$

$$y_2 = P(I+C_1P)^{-1}(I+C_1P_0)Q_0u_1 \tag{A.2}$$

From (A.1) and (A.2), we have

$$H_{y_2u_1} = P(I+C_1P)^{-1}(I+C_1P_0)Q_0$$

$$H_{y_1u_1} = (I+C_1P)^{-1}(I+C_1P_0)Q_0.$$

(b) Let $u_1 = d_0 = n_1 = 0$, then $\xi_1 = y_0 = 0$. Thus by inspection,

$$H_{y_2u_2} = P(I+C_1P)^{-1}$$

$$H_{y_1u_2} = -C_1P(I+C_1P)^{-1}$$

(c) Let $n_1 = u_1 = u_2 = 0$, then $\xi_1 = y_0 = 0$. Again by inspection

$$H_{y_2d_0} = (I+PC_1)^{-1}$$

$$H_{y_1d_0} = -C_1(I+PC_1)^{-1}$$

(ii) For Σ_c and Σ_d, Eq. (3.2) can be easily verified by inspection.

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Proof of Fact 3.1: It can be seen from Fig. 5 that the system ${}^1S(P_0,\pi,\pi^{-1}C_1)$ is exp. stable iff the map $H_{eu}:(u_1,u_2d_0)\to(e_1,e_2,y_2)$ is exp. stable. By simple calculation, we have

$$H_{eu}=\left[\begin{array}{c|c|c} \pi^{-1}(I+C_1P_0)^{-1}\pi & -\pi^{-1}C_1P_0(I+C_1P_0)^{-1} & -\pi^{-1}C_1(I+P_0C_1)^{-1} \\ \hline (I+C_1P_0)^{-1}\pi & (I+C_1P_0)^{-1} & -C_1(I+P_0C_1)^{-1} \\ \hline P_0(I+C_1P_0)^{-1}\pi & P_0(I+C_1P_0)^{-1} & (I+P_0C_1)^{-1} \end{array}\right] \qquad (A.3)$$

By assumption, π and π^{-1} are exp. stable, hence (i) $(I+C_1P_0)Q_0$ is exp. stable $\Leftrightarrow$ $\pi^{-1}(I+C_1P_0)Q_0$ is exp. stable; and from (A.3), (ii) ${}^1S(P_0,C_1)$ is exp. stable $\Leftrightarrow$ ${}^1S(P_0,\pi,\pi^{-1}C_1)$ is exp. stable. Therefore, (3.7) follows from (i), (ii), (3.5) and (3.6). ¤

Derivation of (3.18): By computation, we have that the corresponding perturbed I/O map ${}^mH^a_{y_2u_1} = P_0(I+C_1P_0)^{-1}(I+C_1P)Q_0$. Therefore,

$$\begin{aligned}\Delta{}^mH^a_{y_2u_1} &:= {}^mH^a_{y_2u_1} - H^o_{y_2u_1} = P_0(I+C_1P_0)^{-1}(I+C_1P)Q_0 - P_0Q_0 \\ &= P_0(I+C_1P_0)^{-1}(I+C_1P_0+C_1\Delta P_m)Q_0 - P_0Q_0 \\ &= P_0Q_0 + P_0(I+C_1P_0)^{-1}C_1\Delta P_mQ_0 - P_0Q_0 \\ &= P_0C_1(I+P_0C_1)^{-1}\Delta P_mQ_0\end{aligned}$$

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Derivation of (3.19): By computation, we have that the corresponding perturbed I/O map ${}^mH^b_{y_2u_1} = [P_0(I+C_1P_0)^{-1} + P_0C_1(I+P_0C_1)^{-1}P]C_0(I+PC_0)^{-1}$. Therefore,

$$\Delta^m H^b_{y_2u_1} := {}^m H^b_{y_2u_1} - H^o_{y_2u_1}$$

$$= [P_0(I+C_1P_0)^{-1} + P_0C_1(I+P_0C_1)^{-1}P]C_0(I+PC_0)^{-1} - P_0C_0(I+P_0C_0)^{-1}$$

$$= (I+P_0C_1)^{-1}P_0C_0(I+PC_0)^{-1} + [I-(I+P_0C_1)^{-1}]PC_0(I+PC_0)^{-1} - P_0C_0(I+P_0C_0)^{-1}$$

$$= (I+P_0C_1)^{-1}P_0C_0(I+PC_0)^{-1} - (I+P_0C_1)^{-1}PC_0(I+PC_0)^{-1} + PC_0(I+PC_0)^{-1}$$
$$-P_0C_0(I+P_0C_0)^{-1}$$

$$= -(I+P_0C_1)^{-1}\Delta P_m C_0(I+PC_0)^{-1} + (I+PC_0)^{-1}\Delta P_m C_0(I+P_0C_0)^{-1}$$

$$= -(I+P_0C_1)^{-1}\Delta P_m C_0(I+PC_0)^{-1} + (I+P_0C_0)^{-1}[(I+P_0C_0)(I+PC_0)^{-1}]$$
$$\cdot\ \Delta P_m C_0(I+P_0C_0)^{-1}$$

$$= -(I+P_0C_1)^{-1}\Delta P_m C_0(I+PC_0)^{-1} + (I+P_0C_0)^{-1}[I-\Delta P_m C_0(I+PC_0)^{-1}]$$
$$\cdot\ \Delta P_m C_0(I+P_0C_0)^{-1}$$

$$= -(I+P_0C_1)^{-1}\Delta P_m C_0(I+PC_0)^{-1} + (I+P_0C_0)^{-1}\Delta P_m C_0[I-(I+PC_0)^{-1}\Delta P_m C_0](I+P_0C_0)^{-1}$$

$$= -(I+P_0C_1)^{-1}\Delta P_m C_0(I+PC_0)^{-1} + (I+P_0C_0)^{-1}\Delta P_m C_0(I+PC_0)^{-1}\underbrace{(I+P_0C_0)(I+P_0C_0)^{-1}}_{I}$$

$$= [(I+P_0C_0)^{-1}-(I+P_0C_1)^{-1}]\Delta P_m C_0(I+PC_0)^{-1} \qquad ¤$$

<u>Derivation of (4.3)</u>:

(i) For Σ_e:

(a) Let $n_1 = u_2 = d_0 = 0$, and $u_1 \neq 0$, then we obtain successively

$$e_1 = u_1 - C_2(P-P_0)Q_0e_1$$

$$[I+C_2(P-P_0)Q_0]e_1 = u_1$$

$$e_1 = (I+C_2\Delta PQ_0)^{-1}u_1$$

$$y_1 = Q_0(I+C_2\Delta PQ_0)^{-1}u_1 \qquad \text{(A.6)}$$

$$y_2 = PQ(I+C_2\Delta PQ_0)^{-1}u_1 \tag{A.7}$$

From (A.6) and (A.7), we have

$$H_{y_1u_1} = Q_0(I+C_2\Delta PQ_0)^{-1}$$

$$H_{y_2u_1} = PQ_0(I+C_2\Delta PQ_0)^{-1}$$

(b) Let $u_1 = n_1 = d_0 = 0$, and $u_2 \neq 0$, then

$$y_1 = -Q_0C_2e_d = -Q_0C_2(P(y_1+u_2) - P_0y_1) = -Q_0C_2(\Delta Py_1+Pu_2)$$

Hence,

$$y_1 = -(I+Q_0C_2\Delta P)^{-1}Q_0C_2Pu_2; \text{ thus}$$

$$H_{y_1u_2} = -(I+Q_0C_2\Delta P)^{-1}Q_0C_2P$$

Since $y_2 = P(y_1+u_2)$,

$$\begin{aligned} H_{y_2u_2} &= P(H_{y_1u_2}+I) \\ &= P[-(I+Q_0C_2\Delta P)^{-1}Q_0C_2P+I] \\ &= [I-PQ_0C_2(I+\Delta PQ_0C_2)^{-1}]P \\ &= (I+\Delta PQ_0C_2-PQ_0C_2)(I+\Delta PQ_0C_2)^{-1}P \\ &= (I-P_0Q_0C_2)(I+\Delta PQ_0C_2)^{-1}P \;. \end{aligned}$$

(c) Let $u_1 = u_2 = n_1 = 0$, and $d_0 \neq 0$, then

$$y_1 = -Q_0C_2e_d = -Q_0C_2(Py_1+d_0-P_0y_1) = -Q_0C_2(\Delta Py_1+d_0)$$

Hence,

$$y_1 = -(I+Q_0C_2\Delta P)^{-1}Q_0C_2d_0; \text{ thus}$$

$$H_{y_1d_0} = -(I+Q_0C_2\Delta P)^{-1}Q_0C_2 = -Q_0C_2(I+\Delta PQ_0C_2)^{-1}$$

Since $y_2 = Py_1 + d_0$

$$H_{y_2d_0} = I + PH_{y_1d_0} = I - PQ_0C_2(I+\Delta PQ_0C_2)^{-1}$$
$$= (I+\Delta PQ_0C_2-PQ_0C_2)(I+\Delta PQ_0C_2)^{-1}$$
$$= (I-P_0Q_0C_2)(I+\Delta PQ_0C_2)^{-1}$$

(ii) For Σ_f, (4.3) can be easily verified by inspection of Fig. 6(b) and simple computations. ¤

<u>Derivation of (4.13)</u>: By definition and the system I/O map (4.3),

$$\Delta H_{y_2u_1} := H_{y_2u_1} - H^o_{y_2u_1} = PQ_0(I+C_2\Delta PQ_0)^{-1} - P_0Q_0$$
$$= [PQ_0 - P_0Q_0(I+C_2\Delta PQ_0)](I+C_2\Delta PQ_0)^{-1}$$
$$= (I-P_0Q_0C_2)\Delta PQ_0(I+C_2\Delta PQ_0)^{-1}$$
$$= (I-P_0Q_0C_2)(I+\Delta PQ_0C_2)^{-1}\Delta PQ_0$$ ¤

<u>Derivation of (4.14)</u>: By simple computations, we have the corresponding perturbed I/O map ${}^mH^e_{y_2u_1} = P_0Q_0(I-C_2\Delta P_mQ_0)^{-1}$. Hence,

$$\Delta^m H^e_{y_2u_1} := {}^mH^e_{y_2u_1} - H^o_{y_2u_1}$$
$$= P_0Q_0(I-C_2\Delta P_mQ_0)^{-1} - P_0Q_0$$
$$= P_0Q_0[(I-C_2\Delta P_mQ_0)^{-1} - I]$$
$$= P_0Q_0[I - (I - C_2\Delta P_mQ_0)](I-C_2\Delta P_mQ_0)^{-1}$$
$$= P_0Q_0C_2\Delta PQ_0(I-C_2\Delta P_mQ_0)^{-1}$$ ¤

Proof of Fact 5.1:

Since π and π^{-1} are incr.$\mathcal{S}$-stable, $(I+C_1P_0)Q_0$ is $\mathcal{S}$-stable iff $\pi^{-1}(I+C_1P_0)Q_0$ is $\mathcal{S}$-stable. Thus from (5.3c) and (5.3d), to show (5.3e) we only have to show that ${}^1S(C_0,P_0)$ is $\mathcal{S}$-stable iff ${}^1S(P_0,\pi,\pi^{-1}C_1)$ is $\mathcal{S}$-stable.

Since π is linear, it can be easily seen, from Fig. 5, that ${}^1S(P_0,\pi,\pi^{-1}C_1)$ is $\mathcal{S}$-stable implies that ${}^1S(C_1,P_0)$ is $\mathcal{S}$-stable. The proof is complete if we show that ${}^1S(C_1,P_0)$ is $\mathcal{S}$-stable implies that ${}^1S(P_0,\pi,\pi^{-1}C_1)$ is $\mathcal{S}$-stable; we prove this next.

Consider the system ${}^1S(P_0,\pi,\pi^{-1}C_1)$ shown in Fig. 5 with input (u_1,u_2,d_0); write the equations determining e_2 and y_2:

$$e_2 = \pi(u_1-\pi^{-1}C_1y_2) + u_2 \qquad \text{(A.8)}$$

$$y_2 = d_0 + P_0e_2 \qquad \text{(A.9)}$$

Let $\tilde{u}_2 := \pi(u_1-\pi^{-1}C_1y_2) - \pi(-\pi^{-1}C_1y_2)$, and (A.10)

$$\tilde{d}_0 := d_0; \qquad \text{(A.11)}$$

and then rewrite (A.8) and (A.9) as

$$e_2 = \tilde{u}_2 + u_2 + \pi(-\pi^{-1}C_1y_2) = \tilde{u}_2 + u_2 - C_1y_2 \qquad \text{(A.12)}$$

$$y_2 = \tilde{d}_0 + P_0e_2 \qquad \text{(A.13)}$$

where in (A.12) we have used the linearity of π.

Note that (A.12) and (A.13) describe the system ${}^1S(C_1,P_0)$ with input $(\tilde{u}_2+u_2,\tilde{d}_0)$. Since by assumption ${}^1S(C_1,P_0)$ is $\mathcal{S}$-stable, for the system ${}^1S(P_0,\pi,\pi^{-1}C_1)$, the map $H : (d_0,\tilde{u}_2+u_2) \mapsto (e_2,y_2)$ is $\mathcal{S}$-stable. Since π is incre.$\mathcal{S}$-stable, it can be easily shown that the map $\psi : (u_1,u_2,d_0) \mapsto (d_0,\tilde{u}_2+u_2)$ is $\mathcal{S}$-stable. Therefore, for ${}^1S(P_0,\pi,\pi^{-1}C_1)$, the composite map $H\psi: (u_1,u_2,d_0) \mapsto (e_2,y_2)$ is $\mathcal{S}$-stable. Since $y_1 = e_2 - u_2$ and $e_1 = \pi^{-1}y_1$, the map $(u_1,u_2,d_0) \mapsto (e_1,y_1)$ is also $\mathcal{S}$-stable, consequently, the system ${}^1S(P_0,\pi,\pi^{-1}C_1)$ is $\mathcal{S}$-stable. ¤

<u>Derivation of (5.14)</u>: By definition of $\Delta H_{y_2u_1}$ and Eq. (5.2),

$$\begin{aligned}\Delta H_{y_2u_1} &= P(I+C_1P)^{-1}(I+C_1P_0)Q_0 - P_0Q_0 \\ &= P(I+C_1P)^{-1}(I+C_1P_0)Q_0 - PQ_0 + PQ_0 - P_0Q_0 \\ &= P(I+C_1P)^{-1}(I+C_1P_0)Q_0 - P(I+C_1P)^{-1}(I+C_1P)Q_0 + \Delta PQ_0\end{aligned}$$

For $u_1 \in \mathcal{L}_e^{n_o}$, let $\eta_1 := (I+C_1P_0)Q_0(u_1)$, $\Delta\eta_1 := C_1\Delta PQ_0(u_1)$, then

$$\Delta H_{y_2u_1}(u_1) = P(I+C_1P)^{-1}(\eta_1) - P(I+C_1P)^{-1}(\eta_1+\Delta\eta_1) + \Delta PQ_0(u_1)$$

Using Taylor's formula, [Die. 1, Theorem 8.14.3],

$$\begin{aligned}\Delta H_{y_2u_1}(u_1) &= [-\int_0^1 D[P(I+C_1P)^{-1}](\eta_1+\alpha\Delta\eta_1)\cdot\Delta\eta_1 d\alpha + \Delta PQ_0(u_1)] \\ &= [-\int_0^1 D(P)\ [I+C_1D(P)]^{-1}\Delta\eta_1,\ d\alpha + \Delta PQ_0(u_1)]\end{aligned}$$

where in both instances $D(P)$ is evaluated at $(I+C_1P)^{-1}(\eta_1+\alpha\Delta\eta_1)$. Note that in the last step we only used the chain rule, the inverse function rule and the linearity of C_1 [Die. 1, Theorems 8.2.1, 8.2.3]. Now, since C_1 is linear

$$\begin{aligned}\Delta H_{y_2u_1}(u_1) &= [-\int_0^1 D(P)(I+C_1D(P))^{-1}C_1\Delta PQ_0(u_1)d\alpha + \Delta PQ_0(u_1)] \\ &= \int_0^1 [I - D(P)C_1(I+D(P)C_1)^{-1}]\Delta PQ_0(u_1)d\alpha \\ &= \int_0^1 (I+D(P)C_1)^{-1}d\alpha\cdot\Delta PQ_0(u_1)\ .\end{aligned}$$

¤

<u>Derivation of (5.15)</u>: By definition,

$$\Delta^m H^a_{y_2u_1} = H^m_{y_2u_1} - H^o_{y_2u_1}$$

$$= P_0(I+C_1P_0)^{-1}(I+C_1P)Q_0 - P_0Q_0$$

$$= P_0(I+C_1P_0)^{-1}(I+C_1P)Q_0 - P_0(I+C_1P_0)^{-1}(I+C_1P_0)Q_0 \ .$$

For $u_1 \in \mathcal{L}_e^{n_o}$, let $\eta_1 := (I+C_1P_0)Q_0(u_1)$

$$\Delta\eta_1 := C_1\Delta P_m Q_0(u_1) \ , \text{ then}$$

$$\Delta^m H^a_{y_2u_1}(u_1) = P_0(I+C_1P_0)^{-1}(\eta_1+\Delta\eta_1) - P_0(I+C_1P_0)^{-1}(\eta_1)$$

By using Taylor's expansion,

$$\Delta^m H^a_{y_2u_1}(u_1) = \int_0^1 D[P_0(I+C_1P_0)^{-1}](\eta_1+\alpha\Delta\eta_1)\cdot\Delta\eta_1 d\alpha$$

$$= \int_0^1 D(P_0)(I+C_1D(P_0))^{-1}C_1\Delta P_m Q_0(u_1)d\alpha$$

where $D(P_0)$ is evaluated at $(I+C_1P_0)^{-1}(\eta_1+\alpha\Delta\eta_1)$. Now since C_1 is linear,

$$\Delta^m H^a_{y_2u_1}(u_1) = \int_0^1 D(P_0)C_1(I+D(P_0)C_1)^{-1}d\alpha\,\Delta P_m Q_0(u_1)$$

$$= \int_0^1 [I-(I+D(P_0)C_1)^{-1}]d\alpha\ \Delta P_m Q_0(u_1) \qquad ¤$$

List of Figure Captions

Fig. 1. The system ${}^{1}\Sigma(P,K)$.

Fig. 2. ${}^{1}\Sigma(P,K)$ with the controller K replaced by the two subsystems π and F.

Fig. 3. Single degree of freedom design: ${}^{1}S(P,C)$ which takes (u_1,u_2,d_0) into (y_1,y_2).

Fig. 4. Two-degree of freedom designs-group 1: feedback configurations Σ_a, Σ_b, Σ_c, and Σ_d. It is assumed that $Q_0 = C_0(I+P_0C_0)^{-1}$

(a) Σ_a

(b) Σ_b

(c) Σ_c

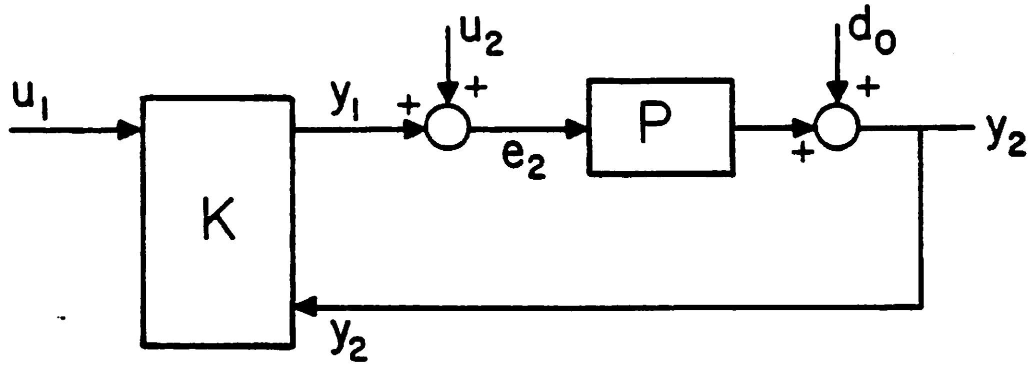
(d) Σ_d .

Fig. 5. The system ${}^{1}S(P,\pi,\pi^{-1}C_1)$

Fig. 6. Two-degree of freedom designs-group 2: feedback configurations Σ_e and Σ_f

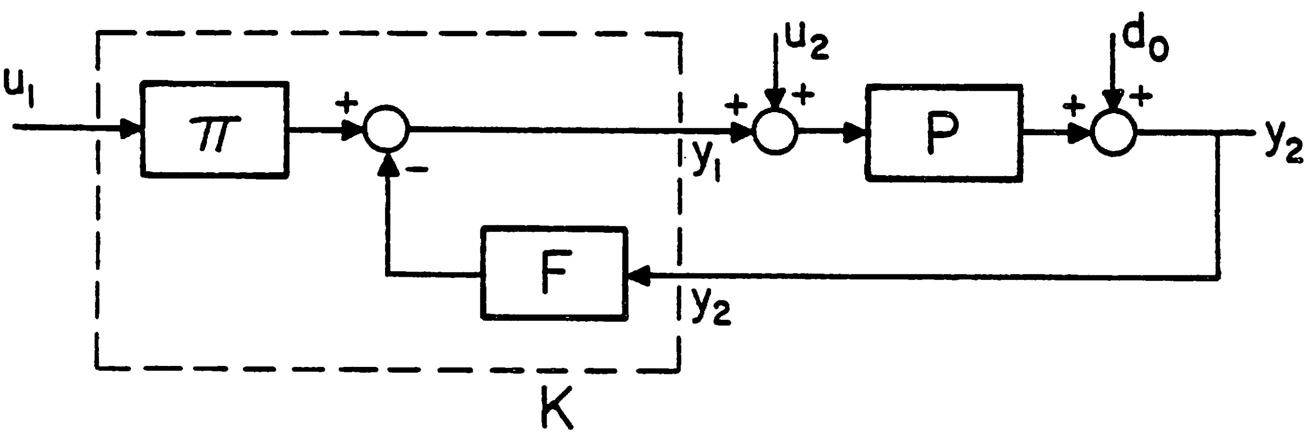
(a) Σ_e

(b) Σ_f .



$^{1}\Sigma(P,K)$

Fig. 1



$^{1}\Sigma(P,K)$

Fig. 2

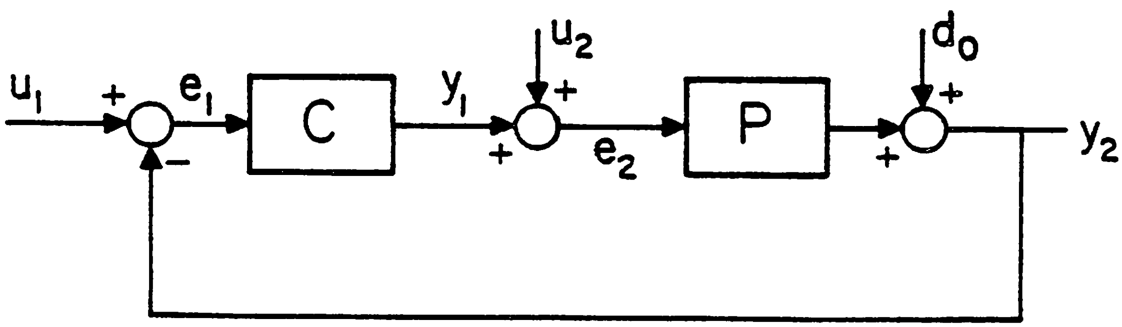


Fig. 3

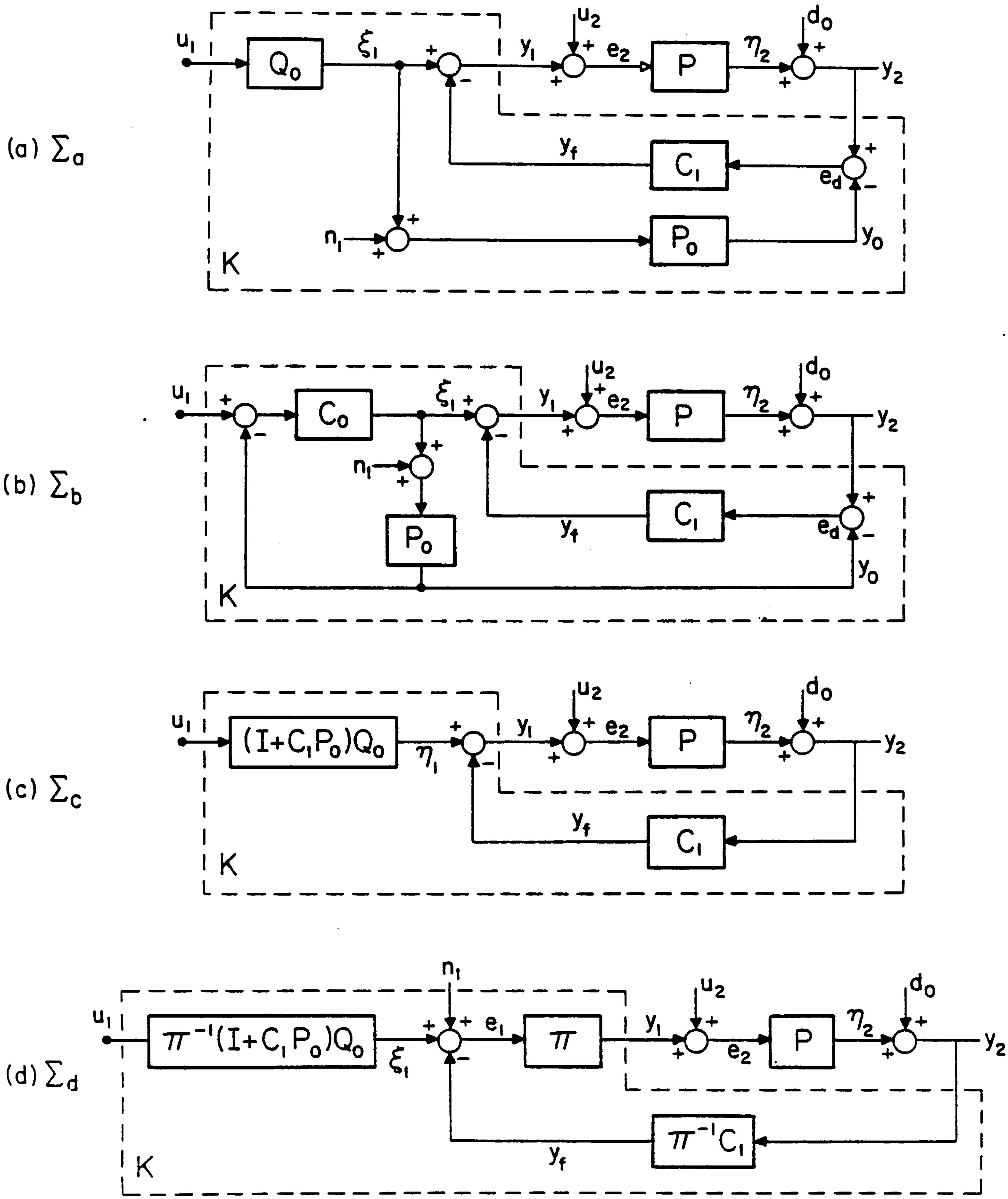
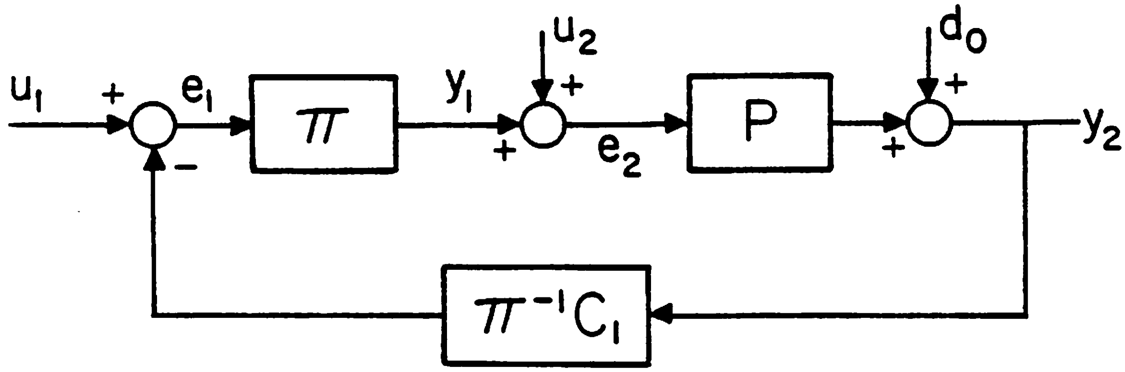
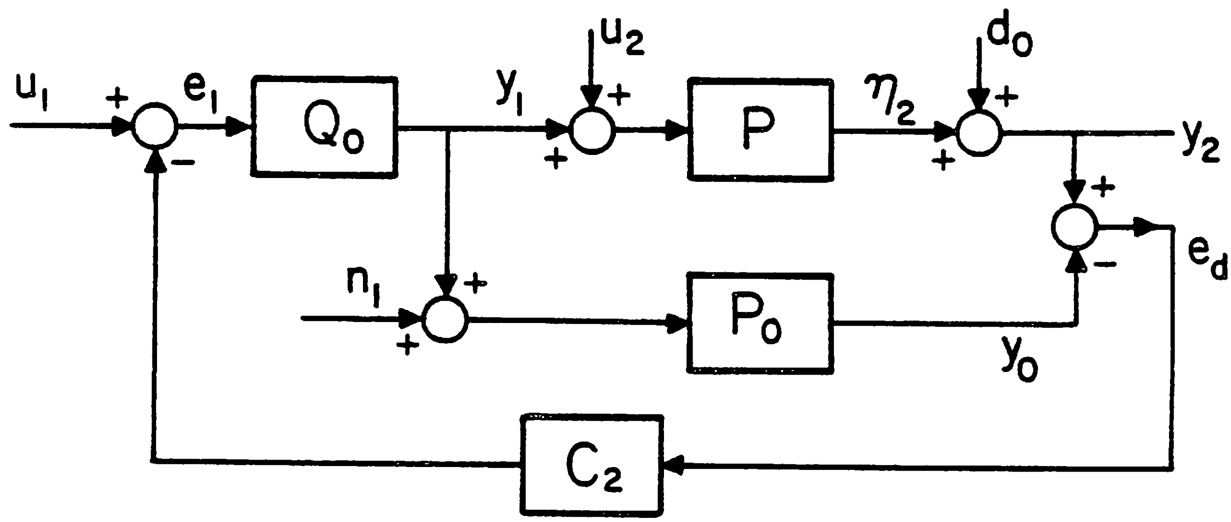

(a) Σa
u1
Q0
ξ1
y1
u2
e2
P
η2
d0
y2
yf
C1
ed
n1
P0
y0
K
(b) Σb
C0
(c) Σc
(I+C1P0)Q0
η1
(d) Σd
π-1(I+C1P0)Q0
e1
π
π-1C1

Fig. 4

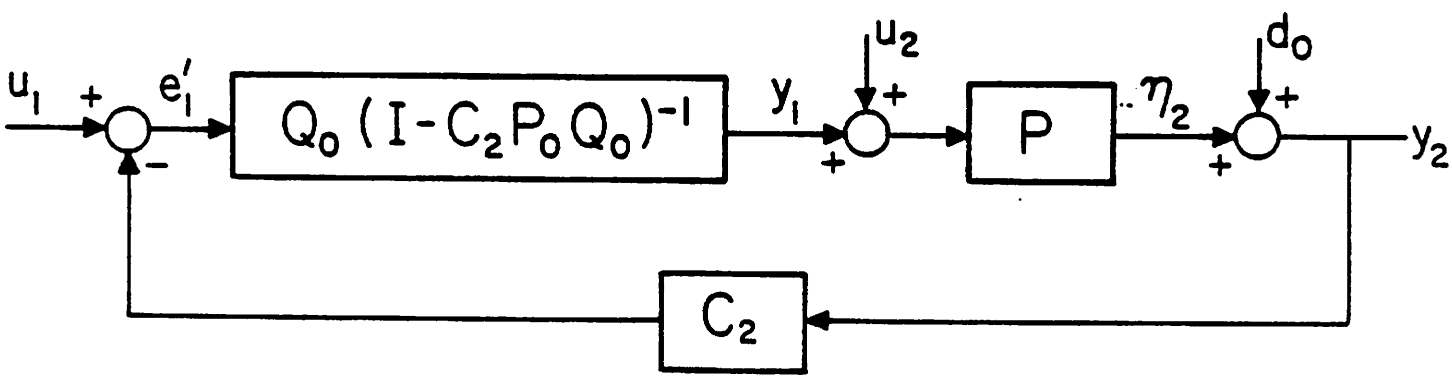


$^1S(P, \pi, \pi^{-1}C_1)$

Fig. 5



(a) Configuration Σ_e



(b) Configuration Σ_f

Fig. 6