CIRCUITS, k-PORTS, HIDDEN MODES AND STABILITY OF INTERCONNECTED k-PORTS

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Title page

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Abstract

This paper considers exclusively lumped k-ports and circuits which contain linear time-invariant elements, independent sources and controlled sources. The k-ports are represented by their hybrid matrices. Tableau equations of circuits are used as a special case of polynomial matrix description. The hidden modes of a circuit are determined by inspection from its tableau equations in the Hermite row form. The same form is used also to determine the exponential stability of the circuit and that of the k-port. Finally, necessary and sufficient conditions for exponential stability of interconnected k-ports are given; the hidden modes of the interconnection are studied. The paper is self-contained.

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0. INTRODUCTION

This paper investigates the dynamics of k-ports obtained from lumped, linear, time-invariant circuits, of circuits resulting from k-ports driven by independent sources and of interconnections of two k-ports. In all these cases, the circuits may include R, L, C's, ideal transformers, independent and controlled sources. As in [Bel. 1] we describe our circuits by polynomial equations; more specifically we use tableau equations.

Belevitch was the first to systematically use polynomial equations to derive properties of circuits. Later Rosenbrock [Ros. 1] applied similar methods to control problems. More recently, Callier and Civalleri [Cal. 2] used the Polynomial Matrix Description (PMD) to state conditions for complete controllability and observability of n-ports and circuits based on the Hermite Normal Form and the Smith Canonical Form.

In this paper we use tableau equations to study the relationship between a k-port and a circuit obtained by driving the k-port by independent sources. The k-ports are represented by appropriate hybrid matrices [see e.g., Chu. 1, Chu. 2, Chu. 3]. In Section 6 we determine the hidden modes of the circuit by inspection from its tableau equations and relate the hidden modes of each individual circuit to those of the interconnected circuit. The tableau equations bookkeep the behavior of all branch voltages and all branch currents; therefore the stability results account for the exponential stability of the circuit as a whole instead of that of a chosen set of output variables.

Roughly speaking, the stability results are as follows: the exponential stability of the <u>k-port</u> is necessary for, but does not guarantee, the stability of the <u>circuit</u> driven by independent sources at the ports of the k-port. The interconnected circuit is not exponentially stable if the individual circuits in the interconnection have unstable hidden modes.

The first three sections give the construction of the k-port and the circuit, the formulation of tableau equations and the stability results based on them. Section 4 relates the tableau equations to the PMD and is followed by an example. The concepts of hidden modes and exponential stability of the interconnection of two k-ports are treated in sections 7 and 8.

Notation:

$\mathbb{R}$ ($\mathbb{C}$) field of real (complex) numbers; $\mathbb{C}_+ := \{s \in \mathbb{C} : \mathrm{Re}(s) \geq 0\}$, equivalently, the closed right-half of the complex plane; $\mathring{\mathbb{C}}_- := \{s \in \mathbb{C} : \mathrm{Re}(s) < 0\}$, equivalently, the open left-half of the complex plane; $\mathbb{R}[p]$ euclidean ring of polynomials in p with real coefficients; $\mathbb{R}(p)$ field of rational functions in p with real coefficients; A^{mxn} set of mxn arrays of elements belonging to the set A (e.g., $\mathbb{R}^{mxn}$, $\mathbb{R}[p]^{mxn}$, $\mathbb{R}(p)^{mxn}, \cdots$); rk($\underset{\sim}{A}$) the rank of matrix A; $\mathcal{Z}[f]$ the list of zeros of the function f; $\mathcal{P}[\underset{\sim}{H}]$ the list of poles of the matrix function $\underset{\sim}{H}$; $\uplus$ as in ($A \uplus B$) the concatenation of the lists A and B; $\underset{\sim}{1}_k$ the kxk identity matrix.

1. GENERATING THE k-PORT K FROM THE GIVEN CIRCUIT $\mathcal{N}$

The given circuit $\mathcal{N}$ is an arbitrary interconnection of lumped, linear, time-invariant circuit elements including independent sources. It has a connected graph of b branches and n_t nodes.

Assumption 1.1. The circuit $\mathcal{N}$ is uniquely solvable.

The uncommitted circuit $\mathcal{K}$ is obtained from $\mathcal{N}$ as follows: k 1-ports are generated by soldering-iron entries to some nodes of $\mathcal{N}$ and by pliers entries to some branches of $\mathcal{N}$. A branch, whose nature is not yet specified, is connected to each port. These branches are called the uncommitted port-branches [Chu. 1, p. 260] and they are assigned the voltage and current reference directions of the port where they are connected (see Fig. 1.1). The resulting circuit is called $\mathcal{K}$: it has k port-branches, b internal branches (the same as the branches of $\mathcal{N}$), and $n := n_k - 1$ nodes excluding datum, where $n_k \geq n_t$.

In the circuit $\mathcal{K}$, let $\underset{\sim}{i}$ and $\underset{\sim}{v}$ denote the b-vectors of internal branch currents and of voltages. The superscript p distinguishes the port-branch variables from the internal branch variables: $\underset{\sim}{i}$ and $\underset{\sim}{v}$ have associated reference directions whereas $\underset{\sim}{i}^p$ and $\underset{\sim}{v}^p$ have non-associated reference directions as far as the port-branches are concerned (see Fig. 1.1).

Let us now 1) remove all the uncommitted port-branches from the circuit $\mathcal{K}$ and 2) put all its internal branches and internal nodes inside a black-box; the result is a k-port called K (see Fig. 1.2). Note that the port-variables $\underset{\sim}{i}^p$ and $\underset{\sim}{v}^p$ are the only measurable variables of the k-port K.

Def. 1.2 [Bel. 1, p. 66] An n-port is said to be well-defined iff there is at least one way of choosing n independent sources to terminate its n ports such that the port-variables of the circuit thus formed are uniquely solvable for all values of the independent sources. Equivalently, an n-port is well-defined iff it has at least one hybrid representation.

Fact 1.3 Assumption 1.1 implies that the k-port K is well-defined.

Proof: One way of choosing the k independent sources is to connect independent current (voltage) sources to the ports of K that were created by soldering-iron entries (pliers-entries, resp.) on the circuit $\mathcal{N}$. The circuit thus obtained is zero-input equivalent to $\mathcal{N}$, which is uniquely solvable by assumption 1.1. ¤

2. THE HYBRID REPRESENTATION FOR THE k-PORT K

The Circuit $\mathcal{K}_h$

Let k_1 ($k_2 := k-k_1$) ports of the k-port K be driven by independent voltage (current) sources and call these ports the "voltage-ports" ("current-ports", resp.).

Assumption 2.1: The hybrid representation corresponding to this partitioning of the k-ports of K exists. At least one such hybrid representation is guaranteed to exist by Fact 1.3.

The circuit obtained by driving the k-port K by k_1 independent voltage sources and k_2 independent current-sources is called $\mathcal{K}_h$ (see Fig. 2.1). $\mathcal{K}_h$ has the same digraph as the uncommitted circuit $\mathcal{K}$: the uncommitted port-branches of the circuit $\mathcal{K}$ are now specified as k_1 voltage-port branches and k_2 current-port branches in the circuit $\mathcal{K}_h$.

Partition the port-branch variables $\underset{\sim}{i}^p$ and $\underset{\sim}{v}^p$ of the circuit $\mathcal{K}_h$ as follows:

$$\underset{\sim}{v}^p := \begin{bmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{v}_2^p \end{bmatrix} \quad \text{where } \underset{\sim}{v}_1^p := [\underset{\sim}{v}_{s_1}\ \underset{\sim}{v}_{s_2} \cdots \underset{\sim}{v}_{s_{k_1}}]^T =: \underset{\sim}{v}_s$$

$$\underset{\sim}{i}^p := \begin{bmatrix} \underset{\sim}{i}_1^p \\ \underset{\sim}{i}_2^p \end{bmatrix} \quad \text{where } \underset{\sim}{i}_2^p := [\underset{\sim}{i}_{s_1}\ \underset{\sim}{i}_{s_2} \cdots \underset{\sim}{i}_{s_{k_2}}]^T =: \underset{\sim}{i}_s \quad .$$

Therefore $\underset{\sim}{v}_1^p$ and $\underset{\sim}{i}_1^p$ ($\underset{\sim}{i}_2^p$ and $\underset{\sim}{v}_2^p$) are the port-branch variables of the voltage-ports (current-ports, resp.). The tableau equations for the circuit $\mathcal{K}_h$ can be written using the node voltages ($\underset{\sim}{e}$), the internal branch voltages and currents ($\underset{\sim}{v}$ and $\underset{\sim}{i}$), and the port-branch variables ($\underset{\sim}{v}^p$ and $\underset{\sim}{i}^p$).

Tableau Equations for the Circuit $\mathcal{K}_h$

For tableau equations we use the form

$$\underset{\sim}{T}(p)\ \underset{\sim}{w}(t) = \underset{\sim}{u}(t)$$

where $\frac{d}{dt}$ is denoted by p. For the hybrid circuit $\mathcal{K}_h$

$$\begin{array}{c|c:c:c:c:c|}
 & n & b & b & k & k \\ \hline
n & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{A}_b & \underset{\sim}{0} & \underset{\sim}{A}_p \\ \hdashline
b & -\underset{\sim}{A}_b^T & \underset{\sim}{1}_b & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \hdashline
k & -\underset{\sim}{A}_p^T & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1}_k & \underset{\sim}{0} \\ \hdashline
b & \underset{\sim}{0} & \underset{\sim}{M}(p) & \underset{\sim}{N}(p) & \underset{\sim}{0} & \underset{\sim}{0} \\ \hdashline
k & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \begin{array}{cc} \underset{\sim}{1}_{k_1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{array} & \begin{array}{cc} \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1}_{k_2} \end{array} \\ \hline
\end{array}
\left[\begin{array}{c}
\underset{\sim}{e}(t) \\ \hdashline
\underset{\sim}{v}(t) \\ \hdashline
\underset{\sim}{i}(t) \\ \hdashline
-\underset{\sim}{v}_1^p(t) \\
-\underset{\sim}{v}_2^p(t) \\ \hdashline
\underset{\sim}{i}_1^p(t) \\
\underset{\sim}{i}_2^p(t)
\end{array}\right]
\begin{array}{c} n \\ b \\ b \\ k \\ \\ k \\ \\ \end{array}
=
\left[\begin{array}{c}
\underset{\sim}{0} \\ \hdashline
\underset{\sim}{0} \\ \hdashline
\underset{\sim}{0} \\ \hdashline
\underset{\sim}{u}_s \\ \hdashline
-\underset{\sim}{v}_s \\
\underset{\sim}{i}_s
\end{array}\right]
\begin{array}{c} n \\ b \\ k \\ b \\ k_1 \\ k_2 \end{array}
\qquad (2.1)$$

$\underset{\sim}{T}(p) \in \mathbb{R}[p]^{(2b+n+2k)x(2b+n+2k)}$ is the tableau matrix, $\underset{\sim}{A} = [\underset{\sim}{A}_b \vdots \underset{\sim}{A}_p] \in \mathbb{R}^{nx(b+k)}$ is the reduced incidence matrix of $\mathcal{K}_h$ where $\underset{\sim}{A}_b$ ($\underset{\sim}{A}_p$) corresponds to the internal branches (port-branches, resp.). (For an example, see (5.1) below.)

In the tableau equations (2.1), the first n are KCL equations, the next b+k are KVL equations, and the equations $\underset{\sim}{M}(p)\ \underset{\sim}{v}(t) + \underset{\sim}{N}(p)\ \underset{\sim}{i}(t) = \underset{\sim}{u}_s$, $\underset{\sim}{M}, \underset{\sim}{N} \in \mathbb{R}[p]^{bxb}$, correspond to the b internal branch equations with $\underset{\sim}{u}_s$ representing the <u>internal independent sources</u>. The last k are the branch equations of the independent sources connected at the ports. Note the minus sign in front of the port voltages due to nonassociated reference directions.

The Tableau Matrix T(p)

The circuit $\mathcal{K}_h$, and in fact any uniquely solvable circuit obtained by connecting independent sources to the ports of the k-port K, has the same digraph and internal branches as the uncommitted circuit $\mathcal{K}$. Therefore the first 2b+n+k tableau equations of the circuit $\mathcal{K}_h$ are identical to those of any other well-defined circuit obtained from the k-port K. Their respective tableau matrices differ only in the last k rows.

Fact 2.2. Let assumptions 1.1 and 2.1 hold. Then

a) the tableau matrix $\underset{\sim}{T}(p)$ is nonsingular;

b) the first 2b+n+k rows of $\underset{\sim}{T}(p)$ are linearly independent in the module $(\mathbb{R}[p]^{2b+n+2k}, \mathbb{R}[p])$. [Sig. 1, Chap. 6].

Proof: a) The circuit $\mathcal{K}_h$ is uniquely solvable (assumption 2.1) and thus its tableau equations have a unique solution; equivalently, the polynomial det $\underset{\sim}{T}(p) \not\equiv 0$.

b) Let $\mathcal{K}_{h_p}$ be the particular circuit obtained by connecting independent current (voltage) sources to the ports of the k-port K that were created by soldering-iron entries (pliers-entries, resp.). Then the circuit $\mathcal{K}_{h_p}$ is uniquely solvable by assumption 1.1 since it is zero-input equivalent to the given circuit $\mathcal{N}$. Let $\underset{\sim}{T}_p(p)$ be the tableau matrix for the circuit $\mathcal{K}_{h_p}$ which differs only in the last k rows from the tableau matrix $\underset{\sim}{T}(p)$ of the circuit $\mathcal{K}_h$. Since $\mathcal{K}_{h_p}$ is uniquely solvable, $\underset{\sim}{T}_p(p)$ is nonsingular with its 2b+n+2k rows (or any subset of its rows) linearly independent in the module $(\mathbb{R}[p]^{2b+n+2k}, \mathbb{R}[p])$. Then the first 2b+n+k rows of $\underset{\sim}{T}(p)$, which are identical to those of $\underset{\sim}{T}_p(p)$, are linearly independent in the same module. ¤

Let $\tilde{\underset{\sim}{R}}(p) \in \mathbb{R}[p]^{(2b+n+k)\times(2b+n+2k)}$ denote the rectangular matrix that corresponds to the first 2b+n+k rows of $\underset{\sim}{T}(p)$. Then $\tilde{\underset{\sim}{R}}(p)$ is full row-rank by fact 2.2.

The Hermite Row Form of $\tilde{\underset{\sim}{R}}(p)$

By elementary column operations (in the ring $\mathbb{R}[p]$) performed on the rectangular matrix $\tilde{\underset{\sim}{R}}(p)$, let us now

1) make a change of variables from $-\underset{\sim}{v}^p$ to $\underset{\sim}{v}^p$, and

2) reorder $\begin{bmatrix} \underset{\sim}{v}^p \\ \hdashline \underset{\sim}{i}^p \end{bmatrix} = \begin{bmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{v}_2^p \\ \hdashline \underset{\sim}{i}_1^p \\ \underset{\sim}{i}_2^p \end{bmatrix}$ as $\begin{bmatrix} \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \\ \hdashline \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{bmatrix}$ in the first 2b+n+k tableau equations

in (2.1).

With all <u>internal</u> independent sources of the circuit $\mathcal{K}_h$ turned-off, these equations read:

$$\tilde{\underset{\sim}{R}}(p)\begin{bmatrix} \underset{\sim}{x} \\ \hdashline \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \\ \hdashline \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{bmatrix} = \underset{\sim}{0} \quad \text{where} \quad \underset{\sim}{x} := \begin{bmatrix} \underset{\sim}{e} \\ \underset{\sim}{v} \\ \underset{\sim}{i} \end{bmatrix} \tag{2.2}$$

<u>Fact 2.3</u>: Let assumptions 1.1 and 2.1 hold. Then the polynomial matrix $\tilde{\underset{\sim}{R}}(p)$ defined in eqn. (2.2) can be put in the following <u>Hermite row-form</u> by elementary row operations in the ring $\mathbb{R}[p]$:

$$\begin{array}{cc} \begin{array}{r} 2b+n \\ k \end{array} & \left[\begin{array}{c|c|c} \underset{\sim}{U} & \underset{\sim}{W} & \underset{\sim}{\tilde{W}} \\ \hline \underset{\sim}{0} & \underset{\sim}{A} & -\underset{\sim}{B} \end{array}\right] \\ & \begin{array}{ccc} 2b+n & k & k \end{array} \end{array} \left[\begin{array}{c} \underset{\sim}{x} \\ \hline \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \\ \hline \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{array}\right] = \underset{\sim}{0} \qquad (2.3)$$

where $\underset{\sim}{U}(p) \in \mathbb{R}[p]^{(2b+n)\times(2b+n)}$ is an upper-triangular nonsingular matrix with nonzero monic polynomials in p of degree at most 1 on the main diagonal, $\underset{\sim}{A}(p)$, $\underset{\sim}{B}(p) \in \mathbb{R}[p]^{k\times k}$. (The matrix on the left of equation (2.3) is called $\underset{\sim}{\hat{R}}(p) \in \mathbb{R}[p]^{\nu\times(\nu+k)}$ where $\nu := 2b+n+k$.)

<u>Comment</u>: From the Hermite row form, all entries of $\underset{\sim}{U}(p)$ above the main diagonal are (possibly zero) constants.

<u>Proof</u>: The square tableau matrix $\underset{\sim}{T}(p)$ for the circuit $\mathcal{K}_h$ in equation (2.1) has polynomial entries of degree at most 1 since $\frac{d}{dt} := p$. The last k equations in (2.1) specify the nature of the port-branches and hence, the last k rows of $\underset{\sim}{T}(p)$ are all zeros except in the last 2k columns. $\underset{\sim}{T}(p)$ can be reduced to an upper-triangular Hermite row form by elementary row operations in the ring $\mathbb{R}[p]$ without using the last k rows to bring zeros below the main diagonal in the first 2b+n rows. Since $\underset{\sim}{T}(p)$ is nonsingular, its Hermite row form has nonzero diagonal entries. By elementary row operations performed on the first 2b+n+k rows of $\underset{\sim}{T}(p)$ we obtain $\underset{\sim}{\hat{R}}(p)$, the hermite row form of the submatrix $\underset{\sim}{\tilde{R}}(p)$ which corresponds to the first 2b+n+k rows of $\underset{\sim}{T}(p)$. ¤

<u>The Hybrid Matrix $\underset{\sim}{H}$ for the k-port K</u>

The last k equations in (2.3), namely,

$$\underset{\sim}{A}(p)\begin{bmatrix} \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{bmatrix} - \underset{\sim}{B}(p)\begin{bmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{bmatrix} = \underset{\sim}{0} \tag{2.4}$$

are the constraints imposed on the port-variables by the k-port K.

In the circuit $\mathcal{K}_h$, $\begin{bmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{bmatrix} = \begin{bmatrix} \underset{\sim}{v}_s \\ \underset{\sim}{i}_s \end{bmatrix}$ are the "independent" port-variables. Since the choice of independent sources corresponds to a uniquely solvable circuit $\mathcal{K}_h$, the hybrid representation is well-defined and the polynomial matrix $\underset{\sim}{A} \in \mathbb{R}[p]^{k\times k}$ is non-singular. Then from equation (2.3), the dependent port-variables $\begin{bmatrix} \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{bmatrix}$ are represented in terms of the independent port-variables:

$$\begin{bmatrix} \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{bmatrix} = \underset{\sim}{A}^{-1}(p)\ \underset{\sim}{B}(p)\begin{bmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{bmatrix} = \underset{\sim}{H}(p)\begin{bmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{bmatrix} \tag{2.5}$$

where $\underset{\sim}{H} := \underset{\sim}{A}^{-1}\underset{\sim}{B} \in \mathbb{R}(p)^{k\times k}$ is the <u>hybrid-matrix</u> for the k-port K.

3. STABILITY OF THE CIRCUIT $\mathcal{K}_h$ AND OF THE k-PORT K

If the hybrid matrix $\underset{\sim}{H}$ of some k-port is not proper (i.e., has a pole at ∞), then some bounded inputs produce unbounded outputs even if $\underset{\sim}{H}$ is analytic in $\mathbb{C}_+$. To wit, in the one-port case, with a first order pole at ∞, the bounded output due to $\sin(\omega_0 t^2)$ includes the term $2\omega_0\, t\cos(\omega_0 t^2)$ which is <u>not</u> bounded on $\mathbb{R}_+$. Consequently, our definition for the exponential stability of a k-port will be a variant on the "bounded-input bounded-output" stability definition.

The stability of the *k-port* K, which is a black-box with access only to its ports, does not guarantee that the *circuit* $\mathcal{K}_h$ is also exponentially stable. The internal behavior of the *circuit* $\mathcal{K}_h$ is not available for measurement at the k ports of K unless all modes of the *circuit* $\mathcal{K}_h$ are both controllable by the given inputs at the ports *and* observable at the output port-variables.

Let $\begin{bmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{bmatrix} = \begin{bmatrix} \underset{\sim}{v}_s \\ \underset{\sim}{i}_s \end{bmatrix}$ be the input to the circuit $\mathcal{K}_h$ and let $\begin{bmatrix} \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{bmatrix}$ be the output. Then the hybrid matrix $\underset{\sim}{H} \in \mathbb{R}(p)^{kxk}$ of the k-port K is the network function of the circuit $\mathcal{K}_h$ from the given inputs to the outputs.

Def. 3.1. *The k-port K* with the hybrid matrix $\underset{\sim}{H}$ is said to be *exponentially stable* iff bounded inputs with bounded support (say on [0,T]) create zero-state responses which go to zero exponentially as t approaches ∞.

Def. 3.2. *The network function* $\underset{\sim}{H} \in \mathbb{R}(p)^{kxk}$ is said to be *exponentially stable* iff $\mathcal{P}[\underset{\sim}{H}] \subset \mathring{\mathbb{C}}_-$.

Comment: In most control applications the following definition is adopted: the transfer function (network function) $\underset{\sim}{H}$ is exponentially stable iff $\underset{\sim}{H}$ is *proper* and $\mathcal{P}[\underset{\sim}{H}] \subset \mathbb{C}_-$. (For input/output properties of exponentially stable transfer functions so defined, see [Cal. 1, p. 127].)

Def. 3.3. $\lambda \in \mathbb{C}$ is called a *natural frequency of the circuit* $\mathcal{K}_h$ iff, for some initial condition, the zero-input response of the circuit $\mathcal{K}_h$ is of the form

$$\begin{bmatrix} \underset{\sim}{x} \\ \underset{\sim}{v}^p \\ \underset{\sim}{i}^p \end{bmatrix} = \underset{\sim}{\alpha} e^{\lambda t} \quad \text{where} \quad \underset{\sim}{\alpha} \in \mathbb{C}^{(2b+n+2k)} \quad .$$

Any zero-input response $t \mapsto \underset{\sim}{\alpha} e^{\lambda t}$ is called a mode of the circuit $\mathcal{K}_h$ associated with the natural frequency λ.

It can be shown that λ is a natural frequency of the circuit $\mathcal{K}_h$ iff $\det \underset{\sim}{T}(\lambda) = 0$ where $\underset{\sim}{T}(p)$ is the tableau matrix of $\mathcal{K}_h$.

Remark: Associated with a given natural frequency, say λ, there may be several modes:

$$\underset{\sim}{\alpha}_1 e^{\lambda t} \ , \ \underset{\sim}{\alpha}_2 e^{\lambda t} \ , \cdots, \underset{\sim}{\alpha}_k e^{\lambda t}$$

where the vectors $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \cdots, \underset{\sim}{\alpha}_k \in \mathbb{C}^{(2b+n+2k)}$ are linearly independent members of the nullspace of $\underset{\sim}{T}(\lambda)$.

Def. 3.4. The circuit $\mathcal{K}_h$ is said to be exponentially stable iff, for all initial conditions, the zero-input response (i.e., all branch voltages and all branch currents) goes to zero exponentially as $t \to \infty$. Equivalently, all natural frequencies of the circuit $\mathcal{K}_h$ have negative real parts, ($\sigma(\mathcal{K}_h) \subset \mathring{\mathbb{C}}_-$, where $\sigma(\mathcal{K}_h)$ is the list of natural frequencies of the circuit $\mathcal{K}_h$.)

Analysis and Stability Theorems

With $\begin{bmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{bmatrix}$ as the input to the circuit $\mathcal{K}_h$, rewrite eqn. (2.3) in the form:

$$\begin{matrix} 2b+n \\ \\ k \end{matrix} \underset{\begin{matrix} 2b+n & \quad k \end{matrix}}{\left[\begin{array}{c|c} \underset{\sim}{U} & \underset{\sim}{W} \\ \hline \underset{\sim}{0} & \underset{\sim}{A} \end{array}\right]} \left[\begin{array}{c} \underset{\sim}{x} \\ \hline \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{array}\right] = \left[\begin{array}{c} \underset{\sim}{\tilde{W}} \\ \hline \underset{\sim}{B} \end{array}\right] \begin{bmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{bmatrix} \tag{3.1}$$

The matrix on the left of eqn. (3.1) is called $\hat{\underset{\sim}{T}}(p) \in \mathbb{R}[p]^{\nu \times \nu}$.

<u>Comment</u>: We refer to $\hat{\underset{\sim}{T}}(p)$ as the tableau matrix of the circuit $\mathcal{K}_h$ although it is obtained by elementary operations from the original tableau matrix of equations (2.1). Since the zeros of det $\underset{\sim}{T}(p)$ are the natural frequencies of the circuit $\mathcal{K}_h$, it is important to note that det $\hat{\underset{\sim}{T}}(p)$ is equal to det $\underset{\sim}{T}(p)$ modulo a nonzero constant and $\mathcal{Z}[\det \hat{\underset{\sim}{T}}] = \mathcal{Z}[\det \underset{\sim}{T}]$.

Let $\hat{\underset{\sim}{L}}(\cdot) \in \mathbb{R}[p]^{k \times k}$ be any g.c.l.d. (greatest common left divisor) of the polynomial matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ of eqn. (3.1) [Cal. 1, p. 24, Kal. 1, p. 376, Ros. 1, p. 70]. Equivalently, there exists polynomial matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$ such that

$$\underset{\sim}{A} = \hat{\underset{\sim}{L}}\bar{\underset{\sim}{A}} \ , \quad \underset{\sim}{B} = \hat{\underset{\sim}{L}}\bar{\underset{\sim}{B}} \tag{3.2}$$

and the pair $(\bar{\underset{\sim}{A}},\bar{\underset{\sim}{B}})$is left-coprime. Then the hybrid matrix $\underset{\sim}{H} = \underset{\sim}{A}^{-1}\underset{\sim}{B}$ of the k-port K can also be expressed as

$$\underset{\sim}{H} = \bar{\underset{\sim}{A}}^{-1}\bar{\underset{\sim}{B}} \tag{3.3}$$

where $\mathcal{P}[\underset{\sim}{H}] = \mathcal{Z}[\det \bar{\underset{\sim}{A}}]$.

With these notations in mind, we have the

<u>Theorem 3.5</u>. The k-port K specified by the hybrid matrix $\underset{\sim}{H}$ is exponentially stable iff the hybrid matrix (network function) $\underset{\sim}{H}$ is exponentially stable.

<u>Proof</u>: The <u>k-port</u> K is a black-box which allows access only to the port-variables $\underset{\sim}{v}^p$, $\underset{\sim}{i}^p$ of the circuit $\mathcal{K}_h$; thus, the modes of the <u>k-port</u> K are those modes of the <u>circuit</u> $\mathcal{K}_h$ which are available for measurement at the k ports. Therefore the k-port K is characterized by the network function $\underset{\sim}{H}$ from the given inputs at the ports to the output port-variables. Using

Laplace transforms it is easily shown that for bounded inputs *with bounded support* (say on [0,T]), the resulting outputs go to zero exponentially as t → ∞ iff this network function $\underset{\sim}{H}$ is exponentially stable; equivalently, iff $\mathcal{P}[\underset{\sim}{H}] = \mathcal{Z}[\det \bar{\underset{\sim}{A}}] \subset \mathring{\mathbb{C}}_-$. ¤

Theorem 3.6. Let assumptions 1.1 and 2.1 hold. Then the *circuit* $\mathcal{K}_h$ with tableau equations (3.1) is exponentially stable iff the characteristic polynomial $\chi(p) := \det \hat{\underset{\sim}{T}}(p)$ has no zeros in $\mathbb{C}_+$; equivalently a) $\det \underset{\sim}{U}(p)$ has no zeros in $\mathbb{C}_+$ (i.e., all diagonal entries of $\underset{\sim}{U}(p)$ are strictly Hurwitz), *and* b) given *any* g.c.l.d. $\hat{\underset{\sim}{L}}$ of $(\underset{\sim}{A}, \underset{\sim}{B})$, $\det \hat{\underset{\sim}{L}}(p)$ has no zeros in $\mathbb{C}_+$ *and* c) the *k-port* K is exponentially stable (equivalently, $\mathcal{P}[\underset{\sim}{H}] \subset \mathring{\mathbb{C}}_-$).

Proof: $\sigma(\mathcal{K}_h)$, the list of all natural frequencies of the circuit $\mathcal{K}_h$, is given by $\sigma(\mathcal{K}_h) = \mathcal{Z}[\chi] = \mathcal{Z}[\det \hat{\underset{\sim}{T}}]$. With $\underset{\sim}{A} = \hat{L}\bar{\underset{\sim}{A}}$, and from eqn. (3.1) we obtain

$$\chi(p) = \det \hat{\underset{\sim}{T}}(p) = \det \underset{\sim}{U}(p) \ \det \underset{\sim}{A}(p) = \det \underset{\sim}{U}(p) \det \hat{\underset{\sim}{L}}(p) \det \bar{\underset{\sim}{A}}(p). \quad (3.4)$$

Hence,

$$\mathcal{Z}[\chi] = \mathcal{Z}[\det \underset{\sim}{U}] \uplus \mathcal{Z}[\det \hat{\underset{\sim}{L}}] \uplus \mathcal{Z}[\det \bar{\underset{\sim}{A}}] \quad (3.5)$$

Furthermore,

$$\mathcal{P}[\underset{\sim}{H}] = \mathcal{P}[\bar{\underset{\sim}{A}}^{-1}\bar{\underset{\sim}{B}}] = \mathcal{Z}[\det \bar{\underset{\sim}{A}}] \quad (3.6)$$

since $\bar{A}$, $\bar{B}$ are left coprime. The conclusion follows from (3.5) and (3.6). ¤

Def. 3.7. A mode of the *circuit* $\mathcal{K}_h$ is said to be a *hidden mode* iff it is not a mode of the *k-port* K; equivalently, it is not controllable by the inputs $\begin{bmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{bmatrix}$ and/or not observable at the output port-variables $\begin{bmatrix} \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{bmatrix}$.

<u>Comment</u>: We shall see later that 1) the list of "hidden modes" of the circuit $\mathcal{K}_h$ is $\mathcal{Z}[\det \underset{\sim}{U}] \uplus \mathcal{Z}[\det \hat{\underset{\sim}{L}}]$. If $z \in \mathbb{C}$ is in this list and $\det \overline{\underset{\sim}{A}}(z) \neq 0$, then no input with bounded support, (say on $[0,T]$), can create a response at the output port-variables which contains the term e^{zt} for all $t > T$.

2) If there are no hidden modes, or if all the hidden modes are <u>exponentially stable</u>, then the exponential stability of the <u>k-port</u> K is equivalent to the exponential stability of the <u>circuit</u> $\mathcal{K}_h$.

4. THE POLYNOMIAL MATRIX DESCRIPTION (PMD) OF THE CIRCUIT $\mathcal{K}_h$

Let us rewrite equation (2.3) for the circuit $\mathcal{K}_h$ in the following form:

$$\underbrace{\left[\begin{array}{c:c} \underset{\sim}{U} & \underset{\sim}{W} \\ \hdashline \underset{\sim}{0} & \underset{\sim}{A} \end{array}\right]}_{\underset{\sim}{D}(p)} \underbrace{\left[\begin{array}{c} \underset{\sim}{x} \\ \hdashline \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{array}\right]}_{\underset{\sim}{\xi}(t)} = \underbrace{\left[\begin{array}{c} \tilde{\underset{\sim}{W}} \\ \hdashline \underset{\sim}{B} \end{array}\right]}_{\underset{\sim}{N}_\ell(p)} \underbrace{\left[\begin{array}{c} \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{array}\right]}_{\underset{\sim}{u}(t)} \tag{4.1}$$

$$\underbrace{[\,\underset{\sim}{0} \,\vdots\, \underset{\sim}{1}_k]}_{\underset{\sim}{N}_r(p)} \underbrace{\left[\begin{array}{c} \underset{\sim}{x} \\ \hdashline \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{array}\right]}_{\underset{\sim}{\xi}(t)} = \underbrace{\left[\begin{array}{c} \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{array}\right]}_{\underset{\sim}{y}(t)} \tag{4.2}$$

Equations (4.1)-(4.2) define the Polynomial Matrix Description (PMD) $\mathcal{D} = [\underset{\sim}{D}, \underset{\sim}{N}_\ell, \underset{\sim}{N}_r, \underset{\sim}{0}]$ of the circuit $\mathcal{K}_h$ [Cal. 1, sec. 3.2, Kai. 1, sec. sec. 6.2.3, Ros. 1 sec. 2.2]. In equations (4.1)-(4.2),

a) $\underset{\sim}{D}(p) \in \mathbb{R}[p]^{\nu \times \nu}$, $\underset{\sim}{N}_\ell(p) \in \mathbb{R}[p]^{\nu \times k}$, $\underset{\sim}{N}_r(p) \in \underline{\mathbb{R}}[p]^{k \times \nu}$;

b) $\underset{\sim}{D}(\cdot)$ is nonsingular;

c) $\underset{\sim}{u}(\cdot) : \mathbb{R}_+ \to \mathbb{R}^k$, $\underset{\sim}{\xi}(\cdot) : \mathbb{R}_+ \to \mathbb{R}^\nu$, $\underset{\sim}{y}(\cdot) : \mathbb{R}_+ \to \mathbb{R}^k$ are called the input, pseudo-state, and output of the PMD;

d) if we want to avoid δ-functions, $\underset{\sim}{u}(\cdot)$ must be piecewise sufficiently differentiable [Cal. 1, p. 93].

Call $\underset{\sim}{H}(s) \in \mathbb{R}(s)^{k\times k}$ the transfer function of the PMD. Then $\underset{\sim}{H}(s) = \underset{\sim}{N}_r(s)\ \underset{\sim}{D}^{-1}(s)\ \underset{\sim}{N}_\ell(s)$ and the Laplace transform of the zero-state response is $\hat{\underset{\sim}{y}}(s) = \underset{\sim}{H}(s)\ \hat{\underset{\sim}{u}}(s)$.

Decoupling zeros of the PMD: [Cal. 1, sec. 3.2, Ros. 1, p. 64]

Let $\tilde{\underset{\sim}{L}}(\cdot) \in \mathbb{R}[p]^{\nu\times\nu}$ be any g.c.l.d. of $(\underset{\sim}{D},\underset{\sim}{N}_\ell)$. Equivalently, there are polynomial matrices $\tilde{\underset{\sim}{D}}$ and $\tilde{\underset{\sim}{N}}_\ell$ such that

$$\underset{\sim}{D} = \tilde{\underset{\sim}{L}}\,\tilde{\underset{\sim}{D}} \quad \text{and} \quad \underset{\sim}{N}_\ell = \tilde{\underset{\sim}{L}}\,\tilde{\underset{\sim}{N}}_\ell \tag{4.3}$$

and $(\tilde{\underset{\sim}{D}},\tilde{\underset{\sim}{N}}_\ell)$ is left-coprime.

Def. 4.1. A point $z_i \in \mathbb{C}$ is called an input-decoupling zero (i-d zero) of the PMD $\mathcal{D}$ described by (4.1)-(4.2) iff given any g.c.l.d. $\tilde{\underset{\sim}{L}}(\cdot)$ of $(\underset{\sim}{D},\underset{\sim}{N}_\ell)$, det $\tilde{\underset{\sim}{L}}(z_i) = 0$ [Cal. 1, p. 101].

Rank Test 4.2 [Cal. 1, p. 101, ex. 37, Ros. 1, chapter 2]. $z_i \in \mathbb{C}$ is an i-d zero of the PMD $\mathcal{D}$ described by (4.1)-(4.2) iff $\text{rk}[\underset{\sim}{D}(z_i) \vdots \underset{\sim}{N}_\ell(z_i)] < \nu$.

It can be shown that every i-d zero of the PMD $\mathcal{D}$ is associated with an uncontrollable mode of the circuit $\mathcal{H}_h$.

Let $R(\cdot) \in \mathbb{R}[p]^{\nu\times\nu}$ be any greatest common right divisor (g.c.r.d) of $(\underset{\sim}{N}_r,\underset{\sim}{D})$. Equivalently, there are polynomial matrices $\bar{\underset{\sim}{N}}_r$, $\hat{\underset{\sim}{D}}$ such that

$$\underset{\sim}{D} = \hat{\underset{\sim}{D}}\,\underset{\sim}{R} \qquad \underset{\sim}{N}_r = \bar{\underset{\sim}{N}}_r\,\underset{\sim}{R} \tag{4.4}$$

and the pair $(\bar{\underset{\sim}{N}}_r,\hat{\underset{\sim}{D}})$ is right-coprime.

Def. 4.3. A point $z_0 \in \mathbb{C}$ is called an output-decoupling zero (o-d zero) of the PMD $\mathcal{D}$ described by (4.1)-(4.2) iff given any g.c.r.d. $\underset{\sim}{R}(\cdot)$ of $(\underset{\sim}{N}_r, \underset{\sim}{D})$, $\det \underset{\sim}{R}(z_0) = 0$ [Cal. 1, p. 104; Ros. 1, p. 65].

Rank Test 4.4 [Cal. 1, p. 104, ex 26, Ros. 1, chapter 2]. $z \in \mathbb{C}$ is an o-d zero of the PMD $\mathcal{D}$ described by (4.1)-(4.2) iff $\operatorname{rk} \begin{bmatrix} \underset{\sim}{D}(z_0) \\ \hdashline \underset{\sim}{N}_r(z_0) \end{bmatrix} < \nu$

Fact 4.5. $z_0 \in \mathbb{C}$ is an o-d zero of the PMD $\mathcal{D}$ described by (4.1)-(4.2) iff $\det \underset{\sim}{U}(z_0) = 0$.

Proof: Using the rank test 4.4, z_0 is an o-d zero iff

$$\operatorname{rk} \begin{bmatrix} \underset{\sim}{U} & \underset{\sim}{W} \\ \underset{\sim}{0} & \underset{\sim}{A} \\ \hdashline \underset{\sim}{0} & \underset{\sim}{1}_k \end{bmatrix} (z_0) < \nu \Leftrightarrow \operatorname{rk} \underset{\sim}{U}(z_0) < \nu$$ and the conclusion follows. ¤

Comment: Fact 4.5 implies that $\underset{\sim}{R}(\cdot)$ in eqn. (4.4) and $\underset{\sim}{U}(\cdot)$ differ by a unimodular factor.

It can be shown that every o-d zero of the PMD $\mathcal{D}$ is associated with an unobservable mode of the circuit $\mathcal{K}_h$.

Let $\underset{\sim}{L}(\cdot)$ be any g.c.l.d. of $(\hat{\underset{\sim}{D}}, \underset{\sim}{N}_\ell)$. Equivalently there are polynomial matrices $\overline{\underset{\sim}{D}}$ and $\overline{\underset{\sim}{N}}_\ell$ such that

$$\hat{\underset{\sim}{D}} = \underset{\sim}{L}\, \overline{\underset{\sim}{D}}, \quad \underset{\sim}{N}_\ell = \underset{\sim}{L}\, \overline{\underset{\sim}{N}}_\ell \tag{4.5}$$

and the pair $(\overline{\underset{\sim}{D}}, \overline{\underset{\sim}{N}}_\ell)$ is left-coprime. Then we obtain the network function as

$$\underset{\sim}{H} = \underset{\sim}{N}_r \underset{\sim}{D}^{-1} \underset{\sim}{N}_\ell = \overline{\underset{\sim}{N}}_r \overline{\underset{\sim}{D}}^{-1} \overline{\underset{\sim}{N}}_\ell \tag{4.6}$$

where $(\overline{\underset{\sim}{N}}_r, \overline{\underset{\sim}{D}})$ is right-coprime and $(\overline{\underset{\sim}{D}}, \overline{\underset{\sim}{N}}_\ell)$ is left-coprime; consequently $\mathcal{P}[\underset{\sim}{H}] = \mathcal{Z}[\det \overline{\underset{\sim}{D}}]$.

Comment: Consider $\tilde{\underset{\sim}{L}}(\cdot)$, $\underset{\sim}{L}(\cdot)$ and $\hat{\underset{\sim}{L}}(\cdot)$ defined in (4.3), (4.5) and (3.2) respectively. By def. 4.1, the complete list of the i-d zeros of the PMD $\mathcal{D}$ is given by $\mathcal{Z}[\det \tilde{\underset{\sim}{L}}(\cdot)]$ whereas the list $\mathcal{Z}[\det \underset{\sim}{L}]$ gives all of the i-d zeros that are not o-d zeros since $(\bar{\underset{\sim}{N}}_{\ell},\hat{D})$ is right-coprime. (See proposition 6.1 for determining those i-d zeros that are also o-d zeros.) From the rank test 4.2 and eqn. (4.1) it follows that if det $\hat{L}(\underset{\sim}{z}_i) = 0$ then z_i is an i-d zero of the PMD $\mathcal{D}$ and that if z_i is an i-d zero then $(\det \underset{\sim}{U}(z_i) \cdot \det \hat{\underset{\sim}{L}}(z_i)) = 0$. Therefore $\underset{\sim}{L}(\cdot)$ and $\hat{\underset{\sim}{L}}(\cdot)$ differ only by a unimodular factor and the i-d zeros that are not o-d zeros are given by $\mathcal{Z}[\det \hat{\underset{\sim}{L}}]$ as well.

Stability of the PMD $\mathcal{D}$

Let the PMD $\mathcal{D}$ described by (4.1)-(4.2) have the network function $\underset{\sim}{H} = \bar{\underset{\sim}{N}}_r \bar{\underset{\sim}{D}}^{-1} \bar{\underset{\sim}{N}}_{\ell}$ and let (4.3)-(4.6) hold. U.t.c. we have the

Theorem 4.6. The circuit $\mathcal{N}_h$ which has the PMD $\mathcal{D}$ described by (4.1)-(4.2) is exponentially stable iff the characteristic polynomial $\chi(p) := \det \underset{\sim}{D}(p)$ has all its zeros in $\mathring{\mathbb{C}}_-$; equivalently,

a) the PMD $\mathcal{D}$ has no i-d or o-d zeros in $\mathbb{C}_+$

and

b) the network function $\underset{\sim}{H}$ is exponentially stable (equivalently, $\mathcal{P}[\underset{\sim}{H}] \subset \mathring{\mathbb{C}}_-$).

Comment: For exponential stability of the PMD, most control applications would require in addition that the PMD $\mathcal{D}$ is well-formed (equivalently, $\underset{\sim}{D}^{-1}$, $\underset{\sim}{N}_r \underset{\sim}{D}^{-1}$, $\underset{\sim}{D}^{-1}\underset{\sim}{N}_{\ell}$, $\underset{\sim}{H}$ are proper) [Cal. 1, p. 128].

Proof: Use the same reasoning as in the proof of theorem 3.6 with $\chi(p) = \det \underset{\sim}{D}(p) = \det \underset{\sim}{L}(p) \cdot \det \bar{\underset{\sim}{D}}(p) \cdot \det \underset{\sim}{R}(p)$ and $\mathcal{Z}[\chi] = \mathcal{Z}[\det \underset{\sim}{L}]$

$\uplus\ \mathcal{Z}[\det \underset{\sim}{R}] \uplus \mathcal{Z}[\det \overline{\underset{\sim}{D}}]$. The list of decoupling zeros is $\mathcal{Z}[\det \underset{\sim}{L}]$ $\uplus\ \mathcal{Z}[\det \underset{\sim}{R}]$, and $\mathcal{P}[\underset{\sim}{H}] = \mathcal{Z}[\det \overline{\underset{\sim}{D}}]$. ¤

<u>Comment</u>: A mode of the circuit $\mathcal{K}_h$ is a <u>hidden mode</u> iff it is associated with an i-d zero or an o-d zero of the PMD $\mathcal{D}$. Therefore the list of the natural frequenices of the circuit $\mathcal{K}_h$ associated with the hidden modes of $\mathcal{K}_h$ is $\mathcal{Z}[\det \underset{\sim}{R}] \uplus \mathcal{Z}[\det \underset{\sim}{L}] = \mathcal{Z}[\det \underset{\sim}{U}] \uplus \mathcal{Z}[\det \hat{\underset{\sim}{L}}]$.

5. EXAMPLE

Consider the linear time-invariant active circuit $\mathcal{N}$ shown in Fig. 5.1.

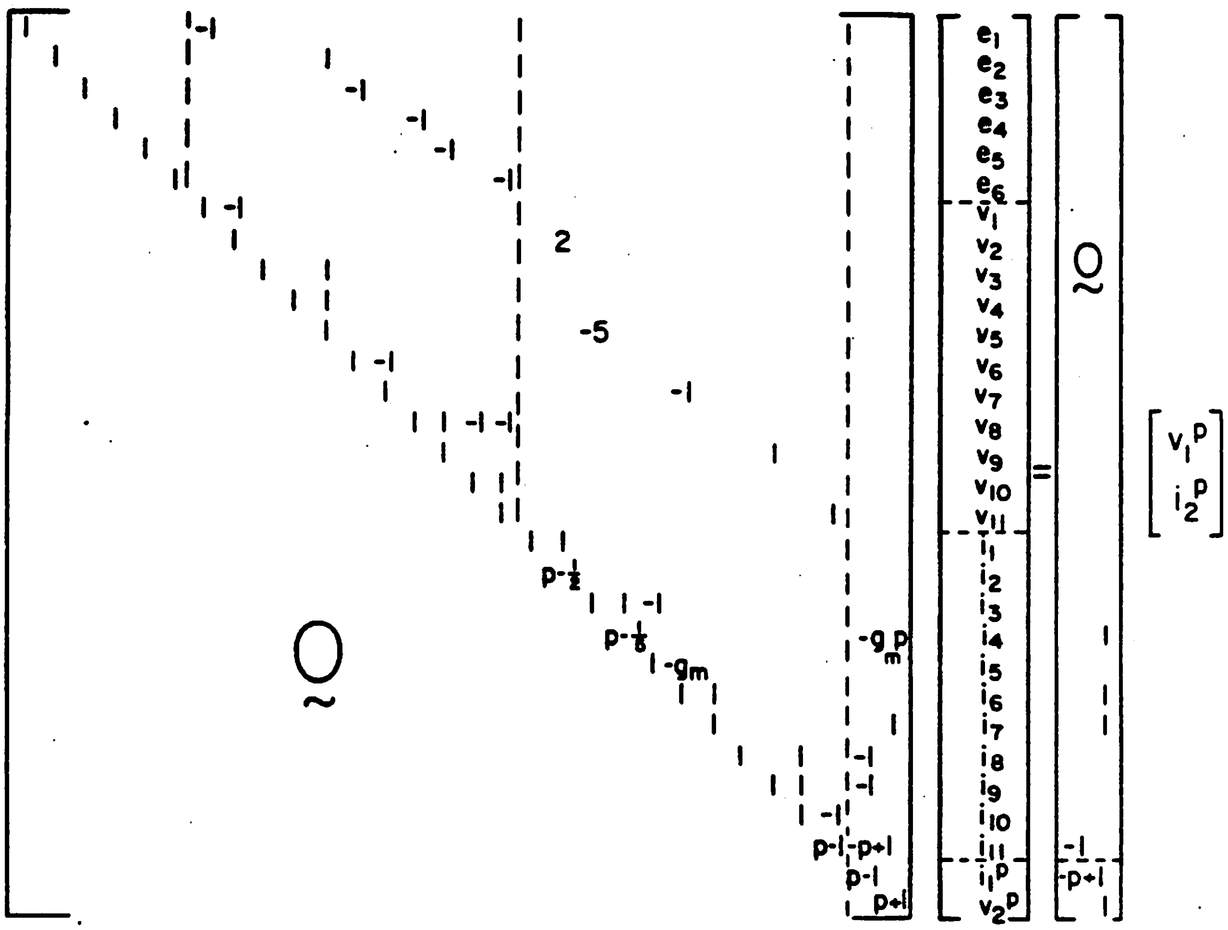
Generate two 1-ports by soldering port-branches as shown in Fig. 5.2 to obtain the uncommitted circuit $\mathcal{K}$.

The circuit $\mathcal{K}$ has k=2 port-branches, b=11 internal branches and n=6 nodes, and the <u>k-port</u> K of section 1 reduces here to a 2-port (see Fig. 5.3).

Designate port I (port II) as a voltage-port (current-port, resp.). Then the <u>circuit</u> $\mathcal{K}_h$ corresponds to the uncommitted circuit $\mathcal{K}$ where port-branch 1 (port-branch 2) is an independent voltage (current, resp.) source. We write the tableau equations as in eqn. (1.1) (see eqn. (5.1)).

$$
\left[\begin{array}{c|c|c|c|c}
\underset{\sim}{0} & \underset{\sim}{0} & \begin{array}{ccccccccccc} 1&1&&&&&&&&& \\ &&1&1&-1&&&&&& \\ &&&&&1&1&&&& \\ &&&&&&&1&&1& \\ &&&&&&&-1&1&& \\ &&&&&&&&&-1&1 \end{array} & \underset{\sim}{0} & \begin{array}{cc} & \\ & \\ &-1 \\ -1& \\ & \\ & \end{array} \\
\hline
\begin{array}{cccccc} -1&&&&& \\ -1&&&&& \\ &-1&&&& \\ &-1&&&& \\ &1&&&& \\ &&-1&&& \\ &&-1&&& \\ &&&-1&1& \\ &&&&-1& \\ &&&-1&&1 \\ &&&&&-1 \end{array} & \begin{array}{ccccccccccc} 1&&&&&&&&&& \\ &1&&&&&&&&& \\ &&1&&&&&&&& \\ &&&1&&&&&&& \\ &&&&1&&&&&& \\ &&&&&1&&&&& \\ &&&&&&1&&&& \\ &&&&&&&1&&& \\ &&&&&&&&1&& \\ &&&&&&&&&1& \\ &&&&&&&&&&1 \end{array} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\
\hline
\begin{array}{cccccc} &&&1&& \\ &&1&&& \end{array} & \underset{\sim}{0} & \underset{\sim}{0} & \begin{array}{cc} 1& \\ &1 \end{array} & \underset{\sim}{0} \\
\hline
\underset{\sim}{0} & \begin{array}{ccccccccccc} p&&&&&&&&&& \\ &1&&&&&&&&& \\ &&1&&&&&&&& \\ &&&p&&&&&&& \\ &&&&0&g_m&&&&& \\ &&&&&1&&&&& \\ &&&&&&p&&&& \\ &&&&&&&1&&& \\ &&&&&&&&1&& \\ &&&&&&&&&p& \\ &&&&&&&&&&1 \end{array} & \begin{array}{ccccccccccc} -1&&&&&&&&&& \\ &2&&&&&&&&& \\ &&5&&&&&&&& \\ &&&-1&&&&&&& \\ &&&&-1&&&&&& \\ &&&&&-1&&&&& \\ &&&&&&-1&&&& \\ &&&&&&&-p&&& \\ &&&&&&&&1&& \\ &&&&&&&&&-1& \\ &&&&&&&&&&1 \end{array} & \underset{\sim}{0} & \underset{\sim}{0} \\
\hline
\underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \begin{array}{cc} 1&0 \\ 0&0 \end{array} & \begin{array}{cc} 0&0 \\ 0&1 \end{array}
\end{array}\right]
\left[\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ \hline v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ \hline i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \\ i_9 \\ i_{10} \\ i_{11} \\ \hline -v_1p \\ -v_2p \\ i_1p \\ v_2p \end{array}\right]
=
\left[\begin{array}{c} \underset{\sim}{0} \\ \hline \underset{\sim}{0} \\ \hline \underset{\sim}{0} \\ \hline -v_s \\ i_s \end{array}\right]
\tag{5.1}
$$

Here $\nu := 2b + n + k = 30$ and the tableau matrix $\underset{\sim}{T}(p) \in \mathbb{R}[p]^{32\text{x}32}$. Next we put the first ν rows of $\underset{\sim}{T}(p)$ in the Hermite row form described in section 2 and obtain the tableau equations for the circuit $\mathcal{N}_h$ in the form of eqn. (4.1) (see eqn. 5.2).



(5.2)

The last 2 equations in (5.2) give the 2-port equations of the 2-port K:

$$\underset{\underset{\sim}{A}}{\boxed{\begin{matrix} p-1 & 0 \\ 0 & p+1 \end{matrix}}} \begin{bmatrix} i_1^p \\ v_2^p \end{bmatrix} - \underset{\underset{\sim}{B}}{\boxed{\begin{matrix} -p+1 & 0 \\ 0 & 1 \end{matrix}}} \begin{bmatrix} v_1^p \\ i_2^p \end{bmatrix} = 0 \qquad (5.3)$$

From (5.3) we obtain the hybrid matrix $\underset{\sim}{H} \in \mathbb{R}(p)^{2\times 2}$:

$$\underset{\sim}{H} = \underset{\sim}{A}^{-1}\underset{\sim}{B} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{p+1} \end{bmatrix} . \tag{5.4}$$

The characteristic polynomial $\chi(p)$ of the <u>circuit</u> $\mathcal{K}_h$ is obtained from the matrix $R(p) \in \mathbb{R}[p]^{\nu \times \nu}$ in the left of equation (5.2) as:

$$\chi(p) = \det \underset{\sim}{R}(p) = \det \underset{\sim}{U}(p) \det \underset{\sim}{A}(p) = (p-\tfrac{1}{2})(p-\tfrac{1}{5})(p-1)(p-1)(p+1). \tag{5.5}$$

Thus the list of natural frequencies for the circuit $\mathcal{K}_h$ is $\sigma(\mathcal{K}_h) = (\frac{1}{2}, \frac{1}{5}, 1, 1, -1)$. Since $\sigma(\mathcal{K}_h)$ is <u>not</u> a subset of $\mathring{\mathbb{C}}_-$, the <u>circuit</u> $\mathcal{K}_h$ is <u>not exponentially stable</u> but the <u>k-port</u> K is exponentially stable since $\wp[\underset{\sim}{H}] = (-1) \subset \mathring{\mathbb{C}}_-$. Here, $\lambda_1 = 1/2$, $\lambda_2 = 1/5$, $\lambda_3 = 1$, $\lambda_4 = 1$ are the natural frequencies of the circuit $\mathcal{K}_h$ that correspond to hidden modes. In fact the polynomial matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are not left-coprime, i.e.,

$$\underset{\sim}{A} = \begin{bmatrix} p-1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & p+1 \end{bmatrix} := \hat{\underset{\sim}{L}}\, \overline{\underset{\sim}{A}}$$

$$\underset{\sim}{B} = \begin{bmatrix} p-1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} := \hat{\underset{\sim}{L}}\, \overline{\underset{\sim}{B}}$$

as in eqn. (3.2), and since $\det \hat{\underset{\sim}{L}} = p-1$, the g.c.l.d $\hat{\underset{\sim}{L}}$ is not unimodular. Thus $\lambda_4 = 1$ is an i-d zero of the PMD for the circuit $\mathcal{K}_h$. Since $\det \underset{\sim}{U} = (p-\frac{1}{2})(p-\frac{1}{5})(p-1)$, $\underset{\sim}{U}$ is not unimodular and $\lambda_1 = 1/2$, $\lambda_2 = 1/5$, $\lambda_3 = 1$ are the o-d zeros of the PMD for the circuit $\mathcal{K}_h$. Note that using the rank test 4.2 we see that $\lambda_1 = 1/2$ is an i-d zero in addition to being an o-d zero. The only controllable and observable mode of the circuit is $\lambda_5 = -1$ since $\det \overline{\underset{\sim}{A}} = p+1$.

6. PHYSICAL INTERPRETATION OF THE MODES OF THE CIRCUIT $\mathcal{K}_h$ AND OF THE k-PORT K

Consider the PMD equations (4.1)-(4.2) for the circuit $\mathcal{K}_h$. We now determine the hidden modes of the circuit from eqn. (4.1) by inspection and give a physical interpretation for each natural frequency.

The uncontrollable and unobservable hidden modes of the circuit $\mathcal{K}_h$

Proposition 6.1: Consider eqn. (4.1). If the ith diagonal entry of the upper triangular matrix $\underset{\sim}{U}$ is the monic first degree polynomial $(p-\lambda_1)$ and if the ith rows of the matrices $\underset{\sim}{W}(\lambda_1)$ and $\underset{\sim}{\tilde{W}}(\lambda_1)$ are both zero, then associated with the natural frequency λ_1, there is an uncontrollable and unobservable mode.

Proof: By assumption, $\det \underset{\sim}{U}(\lambda_1) = 0 \Rightarrow \underset{\sim}{\hat{I}}(\lambda_1) = 0$ and λ_1 is a natural frequency of the circuit $\mathcal{K}_h$. By the rank test 4.4, λ_1 is an o-d zero since $\underset{\sim}{U}$ drops rank at λ_1. By the rank test 4.2, λ_1 is an i-d zero since the ith rows of $\underset{\sim}{U}(\lambda_1)$, $\underset{\sim}{W}(\lambda_1)$ and $\underset{\sim}{\tilde{W}}(\lambda_1)$ are zero. Therefore λ_1 is both an o-d zero and an i-d zero and the conclusion follows. □

In the example of section 5, $\lambda_1 = 1/2$ corresponds to an uncontrollable and unobservable mode since row 19 of equation (5.2) is:

$$(p-1/2)\, i_2 = 0 \ .$$

The controllable but unobservable hidden modes of the circuit $\mathcal{K}_h$

Proposition 6.2: Consider eqn. (4.1). If the jth diagonal entry of the upper triangular matrix $\underset{\sim}{U}$ is $(p-\lambda_2)$ and if $\text{rk}[\underset{\sim}{D}(\lambda_2) \vdots \underset{\sim}{N}_\ell(\lambda_2) = \nu$, then associated with the natural frequency λ_2 there is an unobservable mode that is controllable.

Proof: By assumption, det $\underset{\sim}{U}(\lambda_2) = 0$; hence λ_2 is a natural frequency of the circuit $\mathcal{K}_h$. By the rank test 4.4, λ_2 is an o-d zero, but since $\text{rk}[\underset{\sim}{D}(\lambda_2) \vdots \underset{\sim}{N}_\ell(\lambda_2)] = \nu$, λ_2 is not an i-d zero and the conclusion follows.

¤

In the example of section 5, $\lambda_2 = 1/5$ and $\lambda_3 = 1$ correspond to unobservable but controllable modes. From rows 21 and 30 of equation (5.2)

$$(p-\tfrac{1}{5})i_4 - g_m p v_2^p = 0 \quad \text{and} \quad (p+1)v_2^p = i_2^p = i_s$$

$$\Rightarrow \quad i_4 = \frac{1}{p-1/5}\,\frac{g_m p}{(p+1)}\, i_s$$

From row 28, $(p-1)i_{11} + (1-p)i_1^p = -v_1^p = -v_s$. Note that $\lambda_4 = 1 = \lambda_3$ is a multiple natural frequency: λ_4 is an i-d zero but λ_3 is an o-d zero.

The observable but uncontrollable hidden modes of the circuit $\mathcal{K}_h$

Proposition 6.3: Consider eqn. (4.1). Let $\det A(\lambda_4) = 0$. Then associated with the natural frequency λ_4 there is an uncontrollable mode that is observable iff $\text{rk}[\underset{\sim}{A}(\lambda_4) \vdots \underset{\sim}{B}(\lambda_4)] < k$.

Proof: The assumption implies that det $\hat{\underset{\sim}{T}}(\lambda_4) = 0$ and that λ_4 is a natural frequency of the circuit $\mathcal{K}_h$. Since $Z[\det \underset{\sim}{\hat{L}}]$ is the complete list of the i-d zeros that are not o-d zeros, λ_4 is associated with an uncontrollable but observable mode iff $\det[\hat{\underset{\sim}{L}}(\lambda_4)] = 0$, where $\hat{\underset{\sim}{L}}$ is as in eqn. (3.2). But $\det[\hat{\underset{\sim}{L}}(\lambda_4)] = 0 \Leftrightarrow \text{rk}[\underset{\sim}{A}(\lambda_4) \vdots \underset{\sim}{B}(\lambda_4)] < k$ and the conclusion follows.

¤

Comment: If, in addition, det $\underset{\sim}{U}(\lambda_4) = 0$, λ_4 is a repeated natural frequency and associated with λ_4 there is another mode that is unobservable.

In the example of section 5, $\lambda_4 = 1$ corresponds to an uncontrollable but observable mode. From row 29 of equation (5.2)

$$(p-1)i_1^p = -(p-1)v_1^p \quad .$$

(The first rows of both $\underset{\sim}{A}$ and $\underset{\sim}{B}$ only have $(p-1)$).

The observable and controllable modes of the circuit. $\mathcal{K}_h$

Proposition 6.4: Consider eqns. (4.1) and (3.4)-(3.5). If $\det \bar{\underset{\sim}{A}}(\lambda_5) = 0$ then associated with the natural frequency λ_5 there is a mode that is both observable and controllable. Hence, this mode is a mode of the k-port K.

Proof: By assumption $\lambda_5 \in \sigma(\mathcal{K}_h) = \mathcal{Z}[\chi]$. By the rank tests,

$\text{rk}[\underset{\sim}{D}(\lambda_5) \vdots \underset{\sim}{N}_\ell(\lambda_5)] = \nu$ and $\text{rk}\begin{bmatrix} \underset{\sim}{D}(\lambda_5) \\ \hdashline \underset{\sim}{N}_r(\lambda_5) \end{bmatrix} = \nu$. Therefore λ_5 is not a decoupling zero. ¤

Comment: If λ_5 is a repeated natural frequency, then associated with λ_5, there may also be another mode that is hidden if $\det \underset{\sim}{U}(\lambda_5) = 0$ and/or if $\det \hat{\underset{\sim}{L}}(\lambda_5) = 0$.

In the example of section 5, $\lambda_5 = -1$ corresponds to a controllable and observable mode: from row 30 of equation (5.2)

$$(p+1)v_2^p = i_2^p = i_s \quad .$$

Comment: The impulse response of the k-port K contains a term of the form $p(t)e^{\lambda t}$ for some λ (with $p(t) \in \mathbb{R}[t]$) $\Leftrightarrow$ $\lambda \in \mathcal{P}[\underset{\sim}{H}]$. The zero-input response of the circuit $\mathcal{K}_h$ may contain terms of the form $p'(t)\, e^{\lambda' t}$ such terms are created by initial conditions inside the k-port K that cannot be set up by appropriate port excitations.

7. INTERCONNECTION OF TWO k-PORTS

Construction

For $\alpha = 1,2$, let the linear, time-invariant circuits $\mathcal{N}_\alpha$ be uniquely solvable and have no independent sources. Let assumption 1.1 hold for the circuits $\mathcal{N}_\alpha$, and let $\mathcal{N}_\alpha$ have b_α branches and n_{t_α} nodes.

The uncommitted circuits $\mathcal{K}_\alpha$ with b_α internal branches, n_α nodes, and k port-branches are obtained from the given circuits $\mathcal{N}_\alpha$ as $\mathcal{K}$ was obtained from $\mathcal{N}$ in section 1. Continue the procedure of section 1 to obtain the k-ports K_α from the circuits $\mathcal{K}_\alpha$. From Fact 1.3, the k-ports K_1 and K_2 are well-defined.

Next partition the k ports such that k_1 ports of the k-port K_1 (K_2) are voltage-ports (current-ports) and the rest are current-ports (voltage-ports, resp.) and *assume* that the corresponding hybrid representations exist. Therefore the circuits $\mathcal{K}_{h_1}$ and $\mathcal{K}_{h_2}$ obtained similarly as the circuit $\mathcal{K}_h$ are uniquely solvable. Observe that if we use a hybrid representation for the k-port K_1 with the *k_1 voltage-ports* and the *$k_2 := k-k_1$ current-ports* chosen above, then for the k-port K_2 we use the (dual) hybrid representation with the *k_1 current-ports* and the *k_2 voltage-ports* chosen above.

Connect the k_1 *voltage-ports* (k_2 current-ports) of the k-port K_1 to the k_1 *current-ports* (k_2 voltage-ports, resp.) of the k-port K_2. Call the resulting circuit *the interconnected circuit $\mathcal{K}_i$* (see Fig. 7.1).

Let $\underset{\sim}{x}_1$ ($\underset{\sim}{x}_2$) denote the $2b_1+n_1$ ($2b_2+n_2$) internal-branch variables and

$\begin{pmatrix}\underset{\sim}{v}^p\\ \underset{\sim}{i}^p\end{pmatrix}_1 \left(\begin{pmatrix}\underset{\sim}{v}^p\\ \underset{\sim}{i}^p\end{pmatrix}_2\right)$ denote the 2k port-branch variables of the circuit $\mathcal{K}_{h_1}$ ($\mathcal{K}_{h_2}$, resp.). Then $\begin{pmatrix}\underset{\sim}{x}_1\\ \underset{\sim}{x}_2\end{pmatrix}$ represents the internal-branch variables and $\left(\begin{pmatrix}\underset{\sim}{v}^p\\ \underset{\sim}{i}^p\end{pmatrix}_1^T, \begin{pmatrix}\underset{\sim}{v}^p\\ \underset{\sim}{i}^p\end{pmatrix}_2^T\right)^T$ are the driving-port variables of the circuit $\mathcal{K}_i$. The independent-source drive of the interconnection is $\underset{\sim}{u}_1 := \begin{pmatrix}\underset{\sim}{v}_{s_1}\\ \underset{\sim}{i}_{s_2}\end{pmatrix}$ and $\underset{\sim}{u}_2 := \begin{pmatrix}-\underset{\sim}{i}_{s_1}\\ -\underset{\sim}{v}_{s_2}\end{pmatrix}$. We call $\begin{pmatrix}\underset{\sim}{u}_1\\ \underset{\sim}{u}_2\end{pmatrix}$ the driving-point input of the interconnected circuit $\mathcal{K}_i$.

8. TABLEAU EQUATIONS, PMD, AND STABILITY OF THE CIRCUIT $\mathcal{K}_i$

For each of the circuits $\mathcal{K}_{h_1}$ and $\mathcal{K}_{h_2}$, we write tableau equations as in eqn. (2.1) and put each of the tableau equations into the Hermite row-form (2.3). These tableau equations lead us to the PMD's for the circuits $\mathcal{K}_{h_1}$ and $\mathcal{K}_{h_2}$ similar to the PMD $\mathscr{D}$ of eqn. (4.1)-(4.2).

The PMD for the circuit $\mathcal{K}_{h_1}$

From the tableau equations for the circuit $\mathcal{K}_{h_1}$, we obtain its PMD $\mathscr{D}_1 = [\underset{\sim}{D}_1, \underset{\sim}{N}_{\ell_1}, \underset{\sim}{N}_{r_1}, \underset{\sim}{0}]$

$$\begin{array}{cc} & \begin{array}{cc} 2b_1+n_1 & \quad k_1 \end{array} \\ \begin{array}{c} 2b_1+n_1 \\ \\ k \end{array} & \underbrace{\left[\begin{array}{c:c} \underset{\sim}{U}_1 & \underset{\sim}{W}_1 \\ \hdashline \underset{\sim}{0} & \underset{\sim}{A}_1 \end{array}\right]}_{\underset{\sim}{D}_1(p)} \end{array} \underbrace{\left[\begin{array}{c} \underset{\sim}{x}_1 \\ \hdashline \begin{pmatrix}\underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p\end{pmatrix}_1 \end{array}\right]}_{\underset{\sim}{\xi}_1(t)} = \begin{array}{c} k \\ \underbrace{\left[\begin{array}{c} \tilde{\underset{\sim}{W}}_1 \\ \hdashline \underset{\sim}{B}_1 \end{array}\right]}_{\underset{\sim}{N}_\ell(p)} \end{array} \underbrace{\left[\begin{pmatrix}\underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p\end{pmatrix}\right]}_{\underset{\sim}{e}_1(t)} \qquad (8.1)$$

$$\underbrace{k\,[\,\mathbf{0} \;\vdots\; \mathbf{1}_k\,]}_{\mathbf{N}_{r_1}(p)} \underbrace{\left[\begin{array}{c} \mathbf{x}_1 \\ \hline \begin{pmatrix}\mathbf{i}_1^p\\ \mathbf{v}_2^p\end{pmatrix}_1\end{array}\right]}_{\boldsymbol{\xi}_1(t)} = \underbrace{\left[\begin{pmatrix}\mathbf{i}_1^p\\ \mathbf{v}_2^p\end{pmatrix}_2\right]}_{\mathbf{y}_1(t)} \qquad 8.2)$$

where $\mathbf{D}_1(p) \in \mathbb{R}[p]^{\nu_1 x \nu_1}$, $\mathbf{N}_{\ell_1}(p) \in \mathbb{R}[p]^{\nu_1 xk}$, $\mathbf{N}_{r_1}(p) \in \mathbb{R}[p]^{kx\nu_1}$, $\nu_1 := 2b_1 + n_1 + k$. Since the circuit $\mathcal{K}_{h_1}$ is uniquely solvable, $\det \mathbf{D}_1(p) = \det \mathbf{U}_1(p) \det \mathbf{A}_1(p) \not\equiv 0$.

From $\mathbf{A}_1 \begin{pmatrix}\mathbf{i}_1^p\\ \mathbf{v}_2^p\end{pmatrix}_1 = \mathbf{B}_1 \begin{pmatrix}\mathbf{v}_1^p\\ \mathbf{i}_2^p\end{pmatrix}_1$ we obtain the hybrid matrix $\mathbf{H}_1 := \mathbf{A}_1^{-1}\mathbf{B}_1 \in \mathbb{R}(p)^{kxk}$ where $\mathbf{A}_1$ is nonsingular since the hybrid representation exists by assumption. The transfer function (network function) from

$\begin{pmatrix}\mathbf{v}_1^p\\ \mathbf{i}_2^p\end{pmatrix}_1$ to $\begin{pmatrix}\mathbf{i}_1^p\\ \mathbf{v}_2^p\end{pmatrix}_1$ is

$$\mathbf{H}_1 = \mathbf{N}_{r_1}\mathbf{D}_1^{-1}\mathbf{N}_{\ell_1} = \mathbf{A}_1^{-1}\mathbf{B}_1 \; . \qquad (8.3)$$

The PMD for the circuit $\mathcal{K}_{h_2}$

Similarly for $\mathcal{K}_{h_2}$, we have the PMD $\mathcal{P}_2 = [\mathbf{D}_2, \mathbf{N}_{\ell_2}, \mathbf{N}_{r_2}, \mathbf{0}]$

$$\underbrace{\begin{array}{c|cc} & 2b_2+n_2 & k \\ \hline 2b_2+n_2 & \underset{\sim}{U}_2 & \underset{\sim}{W}_2 \\ k & \underset{\sim}{0} & \underset{\sim}{B}_2 \end{array}}_{\underset{\sim}{D}_2(p)} \underbrace{\left[\begin{array}{c} \underset{\sim}{x}_2 \\ \hline \begin{pmatrix} \underset{\sim}{v}_1^p \\ -\underset{\sim}{i}_2^p \end{pmatrix}_2 \end{array}\right]}_{\underset{\sim}{\xi}_2(p)} = \underbrace{\left[\begin{array}{c} k \\ \underset{\sim}{\tilde{W}}_2 \\ \hline \underset{\sim}{A}_2 \end{array}\right]}_{\underset{\sim}{N}_{\ell_2}(p)} \left[\begin{pmatrix} -\underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{pmatrix}_2\right] \tag{8.4}$$

$$k \ \underbrace{\left[\begin{array}{c|c} \underset{\sim}{0} & \underset{\sim}{1}_k \end{array}\right]}_{\underset{\sim}{N}_{r_2}(p)} \underbrace{\left[\begin{array}{c} \underset{\sim}{x}_2 \\ \hline \begin{pmatrix} \underset{\sim}{v}_1^p \\ -\underset{\sim}{i}_2^p \end{pmatrix}_2 \end{array}\right]}_{\xi_2(t)} = \underbrace{\left[\begin{pmatrix} \underset{\sim}{v}_1^p \\ -\underset{\sim}{i}_2^p \end{pmatrix}_2\right]}_{\underset{\sim}{y}_2(t)} \tag{8.5}$$

where $\underset{\sim}{D}_2(p) \in \mathbb{R}[p]^{\nu_2 \times \nu_2}$, $\underset{\sim}{N}_{\ell_2}(p) \in \mathbb{R}[p]^{\nu_2 \times k}$, $\underset{\sim}{N}_{r_2}(p) \in \mathbb{R}[p]^{k \times \nu_2}$, $\nu_2 := 2b_2+n_2+k$. Since the circuit $\mathcal{K}_{h_2}$ is uniquely solvable, $\det \underset{\sim}{D}_2(p) = \det \underset{\sim}{U}_2(p) \det \underset{\sim}{B}_2(p) \not\equiv 0$. As for the k-port K_1 we have $\underset{\sim}{H}_2 := \underset{\sim}{B}_2^{-1}\underset{\sim}{A}_2$, and

$$\underset{\sim}{H}_2 = \underset{\sim}{N}_{r_2}\underset{\sim}{D}_2^{-1}\underset{\sim}{N}_{\ell_2} = \underset{\sim}{B}_2^{-1}\underset{\sim}{A}_2 \ . \tag{8.6}$$

The PMD of the circuit $\mathcal{K}_i$

First we concatenate the PMD equations (8.1)-(8.2) and (8.4)-(8.5), and reorder variables to obtain:

$$\underbrace{\left[\begin{array}{c|c|c|c} \underset{\sim}{U}_1 & \underset{\sim}{0} & \underset{\sim}{W}_1 & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & \underset{\sim}{U}_2 & \underset{\sim}{0} & \underset{\sim}{W}_2 \\ \hline \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{A}_1 & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{B}_2 \end{array}\right]}_{\underset{\sim}{D}_i(p)} \underbrace{\left[\begin{array}{c} \underset{\sim}{x}_1 \\ \hline \underset{\sim}{x}_2 \\ \hline \begin{pmatrix} \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{pmatrix}_1 \\ \hline \begin{pmatrix} v_1^p \\ -i_2^p \end{pmatrix}_2 \end{array}\right]}_{\underset{\sim}{\xi}(t)} = \underbrace{\left[\begin{array}{c|c} \underset{\sim}{W}_1 & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & \underset{\sim}{W}_2 \\ \hline \underset{\sim}{B}_1 & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & \underset{\sim}{A}_2 \end{array}\right]}_{\underset{\sim}{N}_{\ell_i}(p)} \underbrace{\left[\begin{array}{c} \begin{pmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{pmatrix}_1 \\ \hline \begin{pmatrix} -\underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{pmatrix}_2 \end{array}\right]}_{\underset{\sim}{e}(t)} \tag{8.7}$$

$$\underbrace{\left[\begin{array}{c|c|c|c} \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1}_k & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1}_k \end{array}\right]}_{\underset{\sim}{N}_{r_i}(p)} \underset{\sim}{\xi}(t) = \underbrace{\left[\begin{array}{c} \begin{pmatrix} \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{pmatrix}_1 \\ \hline \begin{pmatrix} \underset{\sim}{v}_1^p \\ -\underset{\sim}{i}_2^p \end{pmatrix}_2 \end{array}\right]}_{\underset{\sim}{y}(t)} \tag{8.8}$$

where $\det \underset{\sim}{D}_i(p) = \det \underset{\sim}{U}_1(p) \det \underset{\sim}{A}_1(p) \det \underset{\sim}{U}_2(p) \det \underset{\sim}{B}_2(p) = \det \underset{\sim}{D}_1(p) \cdot \det \underset{\sim}{D}_2(p) \not\equiv 0$. Using KVL and KCL, we obtain the connection equations from Fig. 7.1.

$$\begin{gathered} \begin{pmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{pmatrix}_1 = \begin{pmatrix} \underset{\sim}{v}_1^p \\ -\underset{\sim}{i}_2^p \end{pmatrix}_2 + \begin{pmatrix} \underset{\sim}{v}_{s_1} \\ \underset{\sim}{i}_{s_2} \end{pmatrix} \\ \begin{pmatrix} -\underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{pmatrix} = \begin{pmatrix} \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{pmatrix}_1 + \begin{pmatrix} -\underset{\sim}{i}_{s_1} \\ -\underset{\sim}{v}_{s_2} \end{pmatrix} \end{gathered} \tag{8.9}$$

Eliminating $\begin{pmatrix} \underset{\sim}{v}_1^p \\ \underset{\sim}{i}_2^p \end{pmatrix}_1$ and $\begin{pmatrix} -\underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{pmatrix}_2$ from (8.7)-(8.8) by using (8.9), we get the PMD $\mathcal{D}_i = [\underset{\sim}{D}_g, \underset{\sim}{N}_{\ell_i}, \underset{\sim}{N}_{r_i}, \underset{\sim}{0}]$ of the circuit $\mathcal{K}_i$:

$$
\begin{array}{c} 2b_1+n_1 \\ 2b_2+n_2 \\ k \\ k \end{array}
\underbrace{\left[\begin{array}{c|c|c|c}
\underset{\sim}{U}_1 & \underset{\sim}{0} & \underset{\sim}{W}_1 & -\underset{\sim}{W}_1 \\ \hline
\underset{\sim}{0} & \underset{\sim}{U}_2 & -\underset{\sim}{W}_2 & \underset{\sim}{W}_2 \\ \hline
 & & \underset{\sim}{A}_1 & -\underset{\sim}{B}_1 \\ \hline
\multicolumn{2}{c|}{\underset{\sim}{\bigcirc}} & -\underset{\sim}{A}_2 & \underset{\sim}{B}_2
\end{array}\right]}_{\underset{\sim}{D}_g(p)}
\underbrace{\left[\begin{array}{c}
\underset{\sim}{x}_1 \\ \hline
\underset{\sim}{x}_2 \\ \hline
\begin{pmatrix} \underset{\sim}{i}_1^p \\ \underset{\sim}{v}_2^p \end{pmatrix}_1 \\ \hline
\begin{pmatrix} \underset{\sim}{v}_1^p \\ -\underset{\sim}{i}_2^p \end{pmatrix}_2
\end{array}\right]}_{\underset{\sim}{\xi}(t)}
=
\underbrace{\begin{array}{c}
\begin{array}{cc} k & k \end{array} \\
\left[\begin{array}{c|c}
\underset{\sim}{\tilde{W}}_1 & \underset{\sim}{0} \\ \hline
\underset{\sim}{0} & \underset{\sim}{\tilde{W}}_2 \\ \hline
\underset{\sim}{B}_1 & \underset{\sim}{0} \\ \hline
\underset{\sim}{0} & \underset{\sim}{A}_2
\end{array}\right]
\end{array}}_{\underset{\sim}{N}_{\ell_i}(p)}
\underbrace{\left[\begin{array}{c}
\begin{pmatrix} \underset{\sim}{v}_{s_1} \\ \underset{\sim}{i}_{s_2} \end{pmatrix} \\
\begin{pmatrix} -\underset{\sim}{i}_{s_1} \\ -\underset{\sim}{v}_{s_2} \end{pmatrix}
\end{array}\right]}_{\underset{\sim}{u}(t)}
\qquad (8.10)
$$

$$
\begin{array}{c} k \\ k \end{array}
\underbrace{\left[\begin{array}{c|c} \underset{\sim}{0} & \underset{\sim}{1}_{2k} \end{array}\right]}_{\underset{\sim}{N}_{r_i}(p)}
\underset{\sim}{\xi}(t) = \underset{\sim}{y}(t) \qquad (8.11)
$$

where $\underset{\sim}{N}_{\ell_i}(p) \in \mathbb{R}[p]^{(\nu_1+\nu_2)xk}$, $\underset{\sim}{N}_{r_i}(p) \in \mathbb{R}[p]^{kx(\nu_1+\nu_2)}$, $\nu_1+\nu_2 = (2b_1+n_1+k) + (2b_2+n_2+k)$.

Equations (8.10) are the tableau equations of the circuit $\mathcal{K}_i$ modulo elementary operations. The characteristic polynomial of the circuit $\mathcal{K}_i$ is $\chi_i(p) = \det \underset{\sim}{D}_g(p)$.

Fact 8.1: (well-posedness condition) Let the circuits $\mathcal{K}_{h_1}$ and $\mathcal{K}_{h_2}$ be uniquely solvable (equivalently, $\det \underset{\sim}{D}_1(\cdot) \not\equiv 0$, $\det \underset{\sim}{D}_2(\cdot) \not\equiv 0$.)

U.t.c., the circuit $\mathcal{K}_i$ is uniquely solvable (equivalently, det $D_g(\cdot) \not\equiv 0$) $\Leftrightarrow$ det $(I-H_1H_2)$ = det $(I-H_2H_1) \not\equiv 0$.

Proof: From (8.10) we obtain

$$\det D_g(p) = \det U_1(p) \det U_2(p) \det \begin{bmatrix} A_1(p) & -B_1(p) \\ -A_2(p) & B_2(p) \end{bmatrix} .$$

By elementary column operations,

$$\det D_g(p) = \det U_1(p) \det U_2(p) \det A_1(p) \det B_2(p) \det(I-B_2^{-1}A_2A_1^{-1}B_1) \tag{8.12}$$

$$= \det D_1(p) \det D_2(p) \det(I-H_2H_1) \tag{8.13}$$

and the conclusion follows. ¤

Let $\hat{L}_1(\cdot) \in \mathbb{R}[p]^{k\times k}$ be any g.c.l.d. of (A_1,B_1) and $\hat{L}_2(\cdot) \in \mathbb{R}[p]^{k\times k}$ be any g.c.l.d. of (B_2,A_2). Equivalently $\exists$ polynomial matrices $\bar{A}_1$, $\bar{B}_1$ and $\bar{B}_2$, $\bar{A}_2$ such that

$$\begin{aligned} &A_1 = \hat{L}_1\bar{A}_1 \quad B_1 = \hat{L}\,\bar{B}_1 \quad (\bar{A}_1,\bar{B}_1) \text{ is left-coprime} \\ &B_2 = \hat{L}_2B_2 \quad A_2 = \hat{L}_2\bar{A}_2 \quad (\bar{B}_2,\bar{A}_2) \text{ is left-coprime} \end{aligned} \tag{8.14}$$

Then the hybrid matrices H_1 and H_2 for the <u>k-ports</u> K_1 and K_2 defined by (8.2) and (8.6) resp. become

$$H_1 = \bar{A}_1^{-1}\bar{B}_1 \quad H_2 = \bar{B}_2^{-1}\bar{A}_2 \ . \tag{8.15}$$

The network function H_{yu} from the driving-point input u to the output y is given by

$$\utilde{H}_{yu} := \utilde{N}_{r_i} \utilde{D}_g^{-1} \utilde{N}_{\ell_i} = \begin{bmatrix} \utilde{H}_1(\utilde{I}-\utilde{H}_2\utilde{H}_1)^{-1} & \utilde{H}_1\utilde{H}_2(\utilde{I}-\utilde{H}_1\utilde{H}_2)^{-1} \\ \utilde{H}_2\utilde{H}_1(\utilde{I}-\utilde{H}_2\utilde{H}_1)^{-1} & \utilde{H}_2(\utilde{I}-\utilde{H}_1\utilde{H}_2)^{-1} \end{bmatrix} . \tag{8.16}$$

From (8.14)-(8.15) we have:

$$\mathcal{P}[\utilde{H}_1] = \mathcal{Z}[\det \utilde{\bar{A}}_1], \; \mathcal{P}[\utilde{H}_2] = \mathcal{Z}[\det \utilde{\bar{B}}_2]$$

and an easy calculation shows that

$$\mathcal{P}[\utilde{H}_{yu}] = \mathcal{Z}[\det \utilde{\bar{A}}_1 \det \utilde{\bar{B}}_2 \det(\utilde{I}-\utilde{H}_2\utilde{H}_1] \tag{8.17}$$

From (8.12)-(8.14) we obtain the characteristic polynomial $\chi_i(p)$ of the circuit $\mathcal{K}_i$:

$$\chi_i(p) = \det \utilde{D}_g(p) = \det \utilde{U}_1(p) \det \utilde{U}_2(p) \det \utilde{\hat{L}}_1(p) \det \utilde{\hat{L}}_2(p) \; . \\ [\det \utilde{\bar{A}}_1(p) \det \utilde{\bar{B}}_2(p) \det (\utilde{I}-\utilde{H}_2\utilde{H}_1)] \tag{8.18}$$

Let $\sigma(\mathcal{K}_i)$ denote the list of all natural frequencies of the <u>circuit</u> $\mathcal{K}_i$. Then from (8.17)-(8.18),

$$\sigma(\mathcal{K}_i) = \mathcal{Z}[\chi_i] = \mathcal{Z}[\det \utilde{U}_1] \uplus \mathcal{Z}[\det \utilde{U}_2] \uplus \mathcal{Z}[\det \utilde{\hat{L}}_1] \uplus \mathcal{Z}[\det \utilde{\hat{L}}_2] \\ \uplus \mathcal{P}[\utilde{H}_{yu}] \; . \tag{8.19}$$

<u>Stability of the circuit $\mathcal{K}_i$ and physical interpretation of the modes</u>

Equation (8.19) will be used to give a physical interpretation of the natural frequencies of the circuit $\mathcal{K}_i$. First we show that the circuit $\mathcal{K}_i$ inherits of all hidden modes of the circuits $\mathcal{K}_{h_1}$ and $\mathcal{K}_{h_2}$ and <u>no other</u> hidden modes result from the interconnection. From equations

(8.1)-(8.2) ((8.4)-(8.5), resp.):

a) z_1 (z_2) is an o-d zero of the PMD for the circuit $\mathcal{K}_{h_1}$ ($\mathcal{K}_{h_2}$) iff $\det \underset{\sim}{U}_1(z_1) = 0$ ($\det \underset{\sim}{U}_2(z_2) = 0$, resp.),

b) s_1 (s_2) is an i-d zero of the PMD for the circuit $\mathcal{K}_{h_1}$ ($\mathcal{K}_{h_2}$) iff $\mathrm{rk}[\underset{\sim}{D}_1(s_1) \vdots \underset{\sim}{N}_{\ell_1}(s_1)] < \nu_1$ ($\mathrm{rk}[\underset{\sim}{D}_2(s_2) \vdots \underset{\sim}{N}_{\ell_2}(s_2)] < \nu_2$, resp.). Then we have the

Theorem 8.2. Consider the PMD's for the circuits $\mathcal{K}_{h_1}$, $\mathcal{K}_{h_2}$, and $\mathcal{K}_i$ defined by equations (8.1)-(8.2), (8.4)-(8.5), and (8.10)-(8.11), resp.

i) The list of all o-d zeros of the PMD for the circuit $\mathcal{K}_i$ is the concatenated list of the o-d zeros of the PMD's for the circuits $\mathcal{K}_{h_1}$ and $\mathcal{K}_{h_2}$.

ii) The list of all i-d zeros of the PMD for the circuit $\mathcal{K}_i$ is the concatenated list of the i-d zeros of the PMD's for the circuits $\mathcal{K}_{h_1}$ and $\mathcal{K}_{h_2}$.

iii) The complete list of the decoupling zeros of the PMD for the circuit $\mathcal{K}_i$ is the list

$$\mathcal{Z}[\det \underset{\sim}{U}_1] \uplus \mathcal{Z}[\det \underset{\sim}{U}_2] \uplus \mathcal{Z}[\det \hat{\underset{\sim}{L}}_1] \uplus \mathcal{Z}[\det \hat{\underset{\sim}{L}}_2] \ . \tag{8.20}$$

iv) The circuit $\mathcal{K}_i$ is exponentially stable iff the circuits $\mathcal{K}_{h_1}$ and $\mathcal{K}_{h_2}$ have no unstable hidden modes and $\mathcal{P}[\underset{\sim}{H}_{yu}] \subset \mathring{\mathbb{C}}_-$.

Proof: i) $z_o \in \mathbb{C}$ is an o-d zero of the PMD for $\mathcal{K}_i$, (see (8.10)-(8.11)),

$$\Leftrightarrow \ \mathrm{rk}\begin{bmatrix} \underset{\sim}{D}_g(z_o) \\ \hdashline \underset{\sim}{N}_{r_i}(z_o) \end{bmatrix} < \nu_1 + \nu_2 \ \Leftrightarrow \ \det \underset{\sim}{U}_1(z_o) \det \underset{\sim}{U}_2(z_o) = 0$$

$\Leftrightarrow$ z_o is an o-d zero of $\mathcal{K}_{h_1}$ and/or of $\mathcal{K}_{h_2}$ and the conclusion follows.

ii) $z_i \in \mathbb{C}$ is an i-d zero of the PMD for $\mathcal{K}_i$.

$$\Leftrightarrow \mathrm{rk}[\underset{\sim}{D}_g(z_i) \vdots N_{\ell_i}(z_i)] < \nu_1+\nu_2$$

$$\Leftrightarrow \mathrm{rk}\left[\begin{array}{cccc|cc} \underset{\sim}{U}_1 & \underset{\sim}{0} & \underset{\sim}{W}_1 & -\underset{\sim}{\tilde{W}}_1 & \underset{\sim}{\tilde{W}}_1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{U}_2 & -\underset{\sim}{\tilde{W}}_2 & \underset{\sim}{W}_2 & \underset{\sim}{0} & \underset{\sim}{\tilde{W}}_2 \\ & & \underset{\sim}{A}_1 & -\underset{\sim}{B}_1 & \underset{\sim}{B}_1 & \underset{\sim}{0} \\ \multicolumn{2}{c}{\underset{\sim}{\bigcirc}} & -\underset{\sim}{A}_2 & \underset{\sim}{B}_2 & \underset{\sim}{0} & \underset{\sim}{A}_2 \end{array}\right](z_i) < \nu_1+\nu_2$$

and by elementary column operations,

$$\Leftrightarrow \mathrm{rk}\left[\begin{array}{cccc|cc} \underset{\sim}{U}_1 & \underset{\sim}{0} & \underset{\sim}{W}_1 & \underset{\sim}{0} & \underset{\sim}{\tilde{W}}_1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{U}_2 & \underset{\sim}{0} & \underset{\sim}{W}_2 & \underset{\sim}{0} & \underset{\sim}{\tilde{W}}_2 \\ & & \underset{\sim}{A}_1 & \underset{\sim}{0} & \underset{\sim}{B}_1 & \underset{\sim}{0} \\ \multicolumn{2}{c}{\underset{\sim}{\bigcirc}} & \underset{\sim}{0} & \underset{\sim}{B}_2 & \underset{\sim}{0} & \underset{\sim}{A}_2 \end{array}\right](z_i) < \nu_1+\nu_2$$

by column and row exchanges,

$$\Leftrightarrow \mathrm{rk}\left[\begin{array}{cc|c|cc|c} \underset{\sim}{U}_1 & \underset{\sim}{W}_1 & \underset{\sim}{\tilde{W}}_1 & & \underset{\sim}{\bigcirc} & \\ \underset{\sim}{0} & \underset{\sim}{A}_1 & \underset{\sim}{B}_1 & & & \\ \hline & \underset{\sim}{\bigcirc} & & \underset{\sim}{U}_2 & \underset{\sim}{W}_2 & \underset{\sim}{\tilde{W}}_2 \\ & & & \underset{\sim}{0} & \underset{\sim}{B}_2 & \underset{\sim}{A}_2 \end{array}\right](z_i) < \nu_1+\nu_2$$

$$\Leftrightarrow \mathrm{rk}\left[\begin{array}{cc|c|c} \underset{\sim}{D}_1 & \underset{\sim}{N}_{\ell_1} & \multicolumn{2}{c}{\underset{\sim}{\bigcirc}} \\ \hline \multicolumn{2}{c|}{\underset{\sim}{\bigcirc}} & \underset{\sim}{D}_2 & \underset{\sim}{N}_{\ell_2} \end{array}\right](z_i) < \nu_1+\nu_2$$

$\Leftrightarrow$ z_i is an i-d zero of $\mathcal{K}_{h_1}$ and/or of $\mathcal{K}_{h_2}$ and the conclusion follows.

iii) From i), the list of all o-d zeros of the PMD for $\mathcal{K}_i$ is given by

$$\mathcal{Z}[\det \underset{\sim}{U}_1] \uplus \mathcal{Z}[\det \underset{\sim}{U}_2] \; . \qquad (8.21)$$

Let $\hat{\underset{\sim}{L}}_1(\cdot)$ and $\hat{\underset{\sim}{L}}_2(\cdot)$ be as in eqn. (8.14). Then z_1 (z_2) is an i-d zero that is not an o-d zero of the PMD for the circuit $\mathcal{K}_{h_1}$ $(\mathcal{K}_{h_2})$ iff det $\hat{\underset{\sim}{L}}_1(z_1) = 0$ (det $\hat{\underset{\sim}{L}}_2(z_2) = 0$, resp.). From ii), the list of all i-d zeros that are not o-d zeros of the PMD for $\mathcal{K}_i$ is given by

$$\mathcal{Z}[\det \hat{\underset{\sim}{L}}_1] \uplus \mathcal{Z}[\det \hat{\underset{\sim}{L}}_2] \quad . \tag{8.22}$$

The concatenation of the lists from (8.21)-(8.22) gives the complete list of the decoupling zeros of the PMD for $\mathcal{K}_i$ and the conclusion follows.

iv) The circuit $\mathcal{K}_i$ has a hidden mode associated with each decoupling zero of the circuit $\mathcal{K}_{h_1}$ and of the circuit $\mathcal{K}_{h_2}$. The list of all natural frequencies $\sigma(\mathcal{K}_i)$, given by (8.19), is the concatenation of $\mathcal{P}[\underset{\sim}{H}_{yu}]$ and the decoupling zeros listed in (8.20). Associated with each decoupling zero $z_d \in \mathbb{C}_+$ there is an unstable hidden mode of the circuit $\mathcal{K}_i$. Since the circuit $\mathcal{K}_i$ is exponentially stable iff $\sigma(\mathcal{K}_i) \subset \mathring{\mathbb{C}}_-$, the conclusion (iv) follows. ¤

The physical interpretation of the natural frequencies of the circuit $\mathcal{K}_i$ listed in (8.20) is as follows: Associated with each natural frequency from the list in (8.21), the circuit $\mathcal{K}_i$ has an unobservable hidden mode. Some of these modes may also be uncontrollable as well as being unobservable. (To determine the o-d zeros that are also i-d zeros for each of the circuits $\mathcal{K}_{h_1}$ and $\mathcal{K}_{h_2}$, see section 6). Associated with each natural frequency from the list in (8.22), the circuit $\mathcal{K}_i$ has an uncontrollable (but observable) hidden mode. Associated with each natural frequency, say λ_{co}, from the list in (8.17), the circuit $\mathcal{K}_i$ has a controllable and observable mode and the impulse response at the driving point of $\mathcal{K}_i$ includes an exponential term of the form $p(t)e^{\lambda_{co}t}$ (where $p(t)$ is a polynomial).

Remark:

In this paper we choose to write tableau equations as a general method of circuit analysis although all of the discussion above is also valid for Modified Node Analysis (MNA), using as circuit variables the node voltages, the additional branch currents, and the port variables $\underset{\sim}{v}^p$ and $\underset{\sim}{i}^p$.

9. CONCLUSION

This paper investigates the dynamics of lumped, linear time-invariant k-ports, and of circuits obtained from them, by using tableau equations. In a polynomial matrix description framework, the concepts of modes, hidden modes, uncontrollable and unobservable modes are explained and conditions are obtained for the exponential stability of k-ports (theorem 3.5), of circuits (theorem 3.6), and of interconnected k-ports (theorem 8.2). It is shown that the interconnection of two k-ports inherits all hidden modes of each of the individual circuits in the interconnection and that the presence of any unstable hidden modes causes the circuits to be exponentially unstable even though the network functions from their inputs to their outputs are exponentially stable.

References

[Bel. 1] V. Belevitch, Classical Network Theory, San Francisco: Holden-Day 1968.

[Cal. 1] F. M. Callier and C. A. Desoer, Multivariable Feedback Systems New York-Heidelberg-Berlin: Springer-Verlag 1982.

[Cal. 2] F. M. Callier and P. P. Civalleri, "On the Controllability and Observability of Linear Lumped Time-Invariant N-Ports: Hermite Normal Form versus Smith Canonical Form," Alta Frequenza, vol. XLVII-N.1.

[Car. 1] H. J. Carlin and A. B. Giordano, Network Theory, Englewood Cliffs, NJ: Prentice-Hall 1964.

[Chu. 1] L. O. Chua and P-M. Lin, Computer-Aided Analysis and Design of Electronic Circuits, Englewood Cliffs, NJ: Prentice-Hall 1975.

[Chu. 2] L. O. Chua and Y-F. Lam, "Dimension of N-Ports," IEEE Trans. on Circuits and Systems, vol. CAS-21, no. 3, May 1974.

[Chu. 3] L. O. Chua and B. J. Leon, "Unified N-Port System Theory," US Government Research and Development Reports, vol. 41, January 1966.

[Kai. 1] T. Kailath, Linear Systems, Englewood Cliffs, NJ: Prentice-Hall 1980.

[Ros. 1] H. H. Rosenbrock, State Space and Multivariable Theory, J. Wiley & Sons, New York 1970.

[Sig. 1] L. E. Sigler, Algebra, UTM, New York-Heidelberg-Berlin: Springer-Verlag 1976.

[Vla. 1] J. Vlach and K. Singhal, Computer Methods for Circuit Analysis and Design, Van Nostrand Reinhold: New York 1983.

Figure Captions

Fig. 1.1. The uncommitted circuit $\mathcal{K}$ with b internal branches and k port-branches.

Fig. 1.2. The k-port K.

Fig. 2.1. The circuit $\mathcal{K}_h$ with b internal branches, k_1 voltage-port branches and k_2 current-port branches.

Fig. 5.1. The given circuit $\mathcal{N}$ for the example.

Fig. 5.2. The uncommitted circuit $\mathcal{K}$ for the example.

Fig. 5.3. The k-port K for the example.

Fig. 7.1. The interconnected circuit $\mathcal{K}_i$.

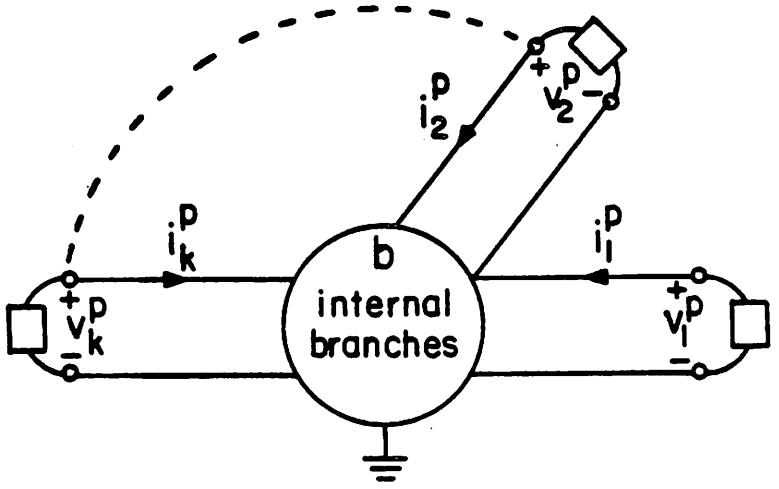


Fig. 1.1

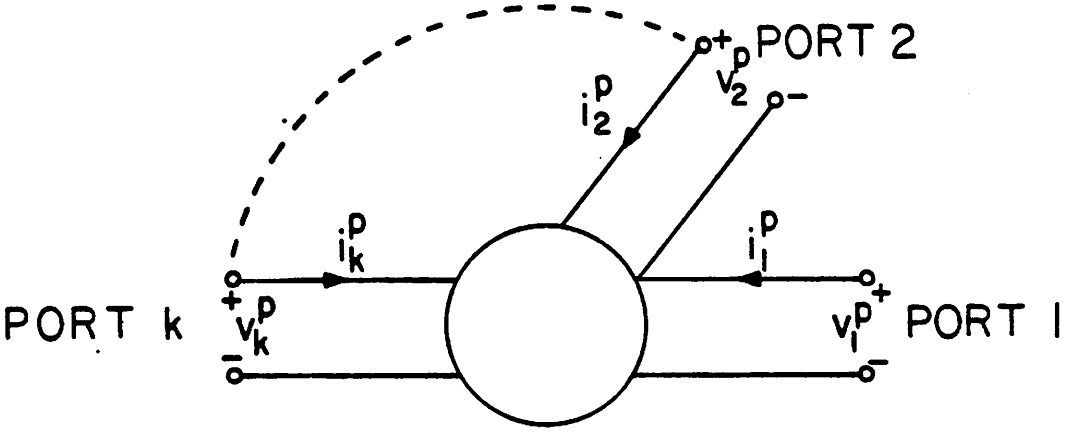


Fig. 1.2

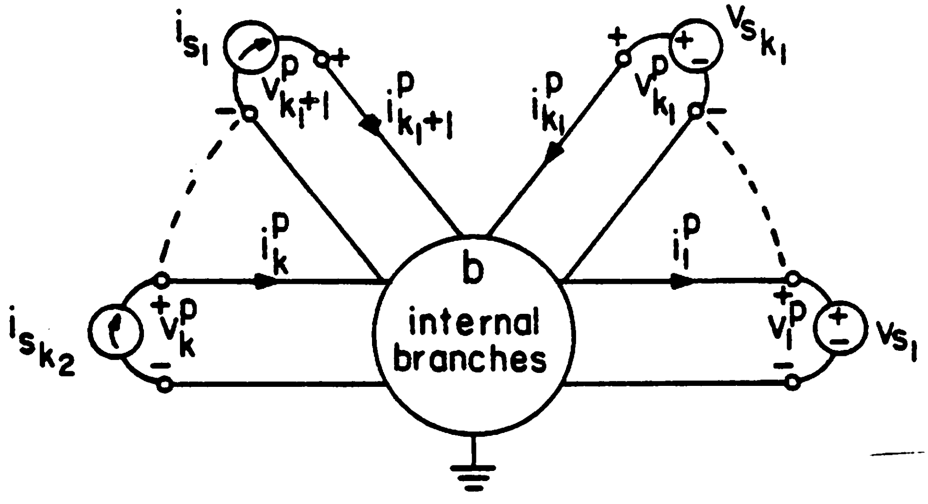


Fig. 2.1

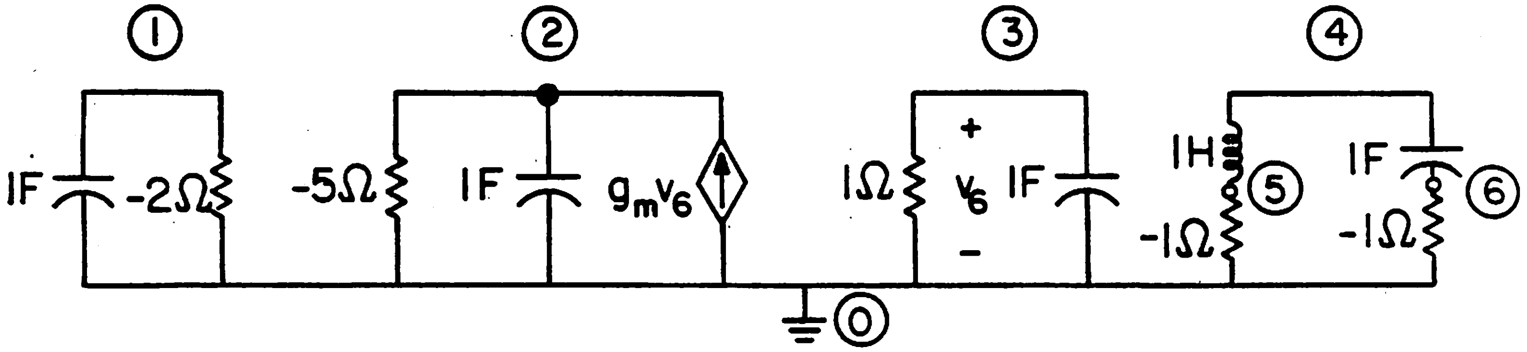

①
②
③
④
1F
-2Ω
-5Ω
1F
$g_m v_6$
1Ω
+
v_6
−
1F
1H
⑤
-1Ω
1F
⑥
-1Ω
⓪

Fig. 5.1

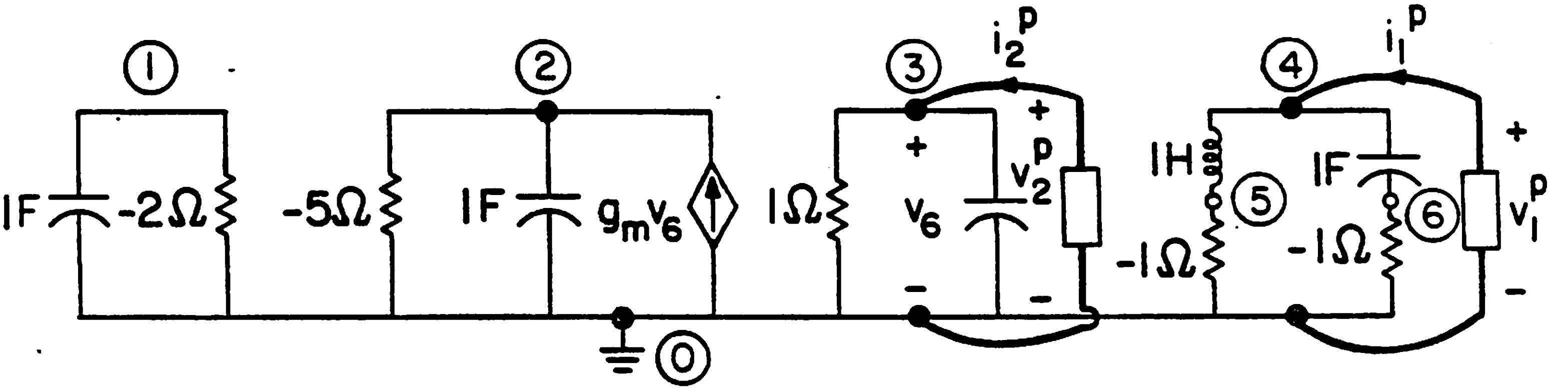
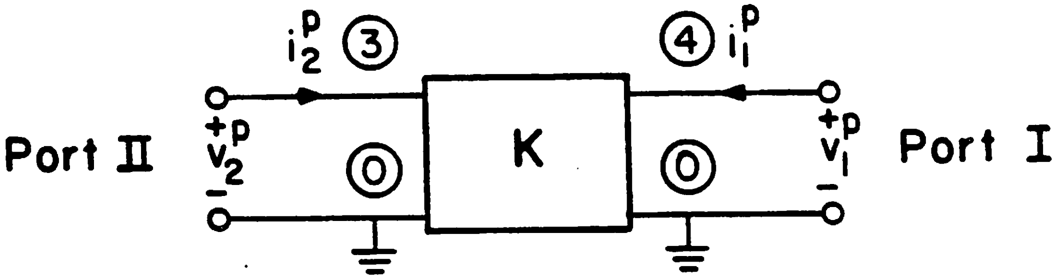
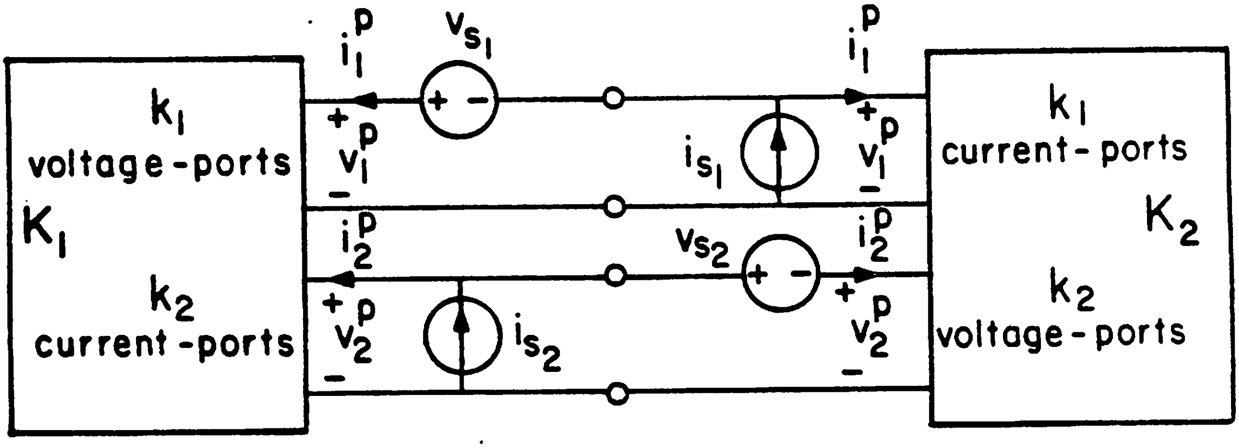

①
②
③
④
i_2^p
i_1^p
1F
-2Ω
-5Ω
1F
$g_m v_6$
1Ω
+
v_6
−
+
v_2^p
−
1H
⑤
-1Ω
1F
⑥
-1Ω
+
v_1^p
−
⓪

Fig. 5.2



i_2^p
③
④
i_1^p
Port II
$+$
v_2^p
$-$
⓪
K
⓪
$+$
v_1^p
$-$
Port I

Fig. 5.3



k_1
voltage-ports
K_1
k_2
current-ports
i_1^p
v_{s_1}
$+$
v_1^p
$-$
i_2^p
$+$
v_2^p
$-$
i_{s_2}
i_1^p
i_{s_1}
$+$
v_1^p
$-$
v_{s_2}
i_2^p
$+$
v_2^p
$-$
k_1
current-ports
K_2
k_2
voltage-ports

Fig. 7.1