TRACKING AND DISTURBANCE REJECTION OF MIMO NONLINEAR SYSTEMS WITH PI CONTROLLER

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Abstract -

We study tracking and disturbance rejection of a class of MIMO nonlinear systems with linear proportional plus integral (PI) compensator. Roughly speaking, we show that if the given nonlinear plant is exponentially stable and has a strictly increasing dc steady-state I/O map, then a simple PI compensator can be used to yield a stable unity-feedback closed-loop system which asymptotically tracks reference inputs that tend to constant vectors and asymptotically rejects disturbances that tend to constant vectors.

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I. Introduction

One of the most important applications of feedback is to achieve servoaction, that is, to obtain a system that tracks a given class of signals and rejects a given class of external disturbances with zero asymptotic error. In the linear time-invariant multi-input multi-output context, this problem has been studied extensively in the literature [See, e.g., Cal. 1; Cal. 2, Chapt. 7; Des. 1; Dav. 1, etc.]. Recently, Morari [Mor. 1] studied the problem of "integral control" of an exponentially stable linear time-invariant multi-input multi-output plant. He obtained a necessary and sufficient condition on the dc gain matrix of the given plant, under which the unity-feedback system consisting of the given plant and an integrator with a sufficiently small coefficient is exp. stable and asymptotically tracks step inputs.

Since any realistic model of a physical system can be linear only as a result of some approximations, it is important to investigate the asymptotic tracking and disturbance rejection of nonlinear systems. Many techniques have been proposed for the design of nonlinear systems [Som. 1, Pec. 1, Sai. 1, Mey. 1, etc.] and specifically for tracking and disturbance rejection [Des. 2, Sol. 1].

In this paper, we study tracking and disturbance rejection for a class of nonlinear MIMO unity-feedback systems; namely, the system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$ consisting of the given nonlinear plant $\mathcal{N}$ and the linear proportional plus integral (PI) compensator $\frac{\varepsilon}{s} I + K$ (see Fig. 2.1 below). The main result of the paper is Theorem 2.1 which shows, roughly speaking, that if the nonlinear plant $\mathcal{N}$ is exp. stable and has a strictly increasing dc steady-state input-output map, then a simple PI compensator $\frac{\varepsilon}{s} I + K$, with $\varepsilon > 0$, K appropriately chosen, can be used to yield a stable unity-feedback closed-loop system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$. Furthermore, the system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$ asymptotically tracks reference inputs which tend to constant vectors and asymptotically rejects disturbances which tend to constant vectors.

The paper is organized as follows: In Section II, some basic notations and definitions are introduced, general assumptions are given and the main result (Theorem 2.1) is precisely stated. Theorem 2.1 is established through Section III and Section IV. Section III studies the special case where the compensator is a pure integrator. Based on the result in Section III and a "loop transformation" technique, Theorem 2.1 is proved in Section IV. Some concluding remarks are given in Section V. Most proofs are given in Appendix A and Appendix B.

II. Problem Formulation and Main Result

II.1. Basic Definitions and Notations

Throughout this paper, $|\cdot|$ denotes the Euclidean norms of $\mathbb{R}^n$ and of $\mathbb{R}^m$. For $g : \mathbb{R}^m \to \mathbb{R}^n$, $Dg(y_0) \in \mathbb{R}^{n\times m}$ denotes the Frechet derivative of g evaluated at y_0. For $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $D_1f(x_0,y_0)$, $(D_2f(x_0,y_0))$, denotes the derivative of f with respect to the first, (second, resp.) variable evaluated at (x_0,y_0). $A \in \mathbb{R}^{n\times n}$ is said to be positive definite, (positive semidefinite), iff $\forall x \in \mathbb{R}^n$, $x \neq \theta_n$, $x^TAx > 0$, (≥ 0, resp.); note that A is not assumed to be symmetric. For $B \in \mathbb{R}^{m\times n}$, $\bar{\sigma}[B]$ ($\underline{\sigma}[B]$) denotes the maximal (minimal, resp.) singular value of B. $f(x) := x^2-1$ means that f(x) is defined to be the given RHS.

II.2. The System ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$

Consider the nonlinear multi-input multi-output unity-feedback system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$ shown in Fig. 2.1, where $\mathcal{N}$ is the given nonlinear plant, u_1 is the reference input, u_2 and d_0 are respectively the plant-input disturbance and plant-output disturbance, $\varepsilon > 0$, and K is a constant gain matrix. The nonlinear plant $\mathcal{N}$ with input e_2, state x, and output η_2 is described by the following equations:

$$\dot{x}(t) = f(x(t), e_2(t)) \tag{2.1a}$$

$$\eta_2(t) = h(x(t)) \tag{2.1b}$$

where $t \geq 0$, $e_2(t) \in \mathbb{R}^m$, $\eta_2(t) \in \mathbb{R}^m$, and $x(t) \in \mathbb{R}^n$. We study the asymptotic tracking and disturbance rejection of ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$.

II.3. General Assumptions

We assume throughout that the nonlinear time-invariant plant $\mathcal{N}$ satisfies the following assumptions:

(N.1) $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are C^1 functions, and $f(\theta_n, \theta_m) = \theta_n$, $h(\theta_n) = \theta_m$. (This together with (N.4) below imply that for every piecewise continuous input $e_2(\cdot)$, for every initial condition (x_0, t_0), Eq. (2.1a) has a unique solution

$$t \mapsto s(t, t_0, x_0, e_2(\cdot)) \tag{2.3}$$

defined on $[t_0, \infty)$.)

(N.2) There exists a C^1 function $g : \mathbb{R}^m \to \mathbb{R}^n$ s.t. $\forall v \in \mathbb{R}^m$,

$$f(\xi,v) = \theta_n \quad \text{iff} \quad g(v) = \xi \quad ; \tag{2.4}$$

(N.3) The map $h \circ g$: $v \mapsto h(g(v))$ is a bijection of $\mathbb{R}^m$ onto $\mathbb{R}^m$;

(N.4) there exists $M > 0$ s.t. $\forall v \in \mathbb{R}^m$, $\forall \xi \in \mathbb{R}^n$,

$$|D_1 f(\xi,v)| < M \quad (2.5a) \quad ; \quad |D_2 f(\xi,v)| < M \quad (2.5b)$$

$$|Dh(\xi)| < M \quad (2.5c) \quad ; \quad |Dg(v)| < M \quad (2.5d)$$

(N.5) There exist $c > 0$, $\alpha > 0$ s.t. $\forall$ constant input $v \in \mathbb{R}^m$, $\forall x_0 \in \mathbb{R}^n$, $\forall t_0 \geq 0$, $\forall t \geq t_o$,

$$|s(t, t_o, x_o, v) - g(v)| \leq c|x_o - g(v)|e^{-\alpha(t-t_o)} \tag{2.6}$$

(N.6) $h \circ g$ is strictly increasing: there exists $\beta > 0$ s.t. $\forall v_1, v_2 \in \mathbb{R}^m$,

$$[hg(v_1) - hg(v_2)]^T [v_1 - v_2] \geq \beta \, |v_1 - v_2|^2 \, . \tag{2.7}$$

Comments

(a) Assumption (N.2) implies that for all constant inputs $v \in \mathbb{R}^m$, the nonlinear plant $\mathcal{N}$ has a unique equilibrium state $x := g(v)$ which by (N.5) is globally uniformly exponentially stable (uniform in the constant input v). Since the read-out map h is continous and satisfies (2.5c), for all constant inputs v, for all initial conditions (x_o, t_o), the corresponding plant output $\eta_2(t)$ tends exponentially to the constant output hg(v). By (N.3), each such steady-state output hg(v) is achieved by one and only one constant plant-input v.

(b) Assumption (N.6) holds [Ort. 1, p. 142] iff $\forall v_1, v_2 \in \mathbb{R}^m$

$$v_1^T D(hg)(v_2) v_1 \geq \beta |v_1|^2 \tag{2.8}$$

(c) Consider the square linear time-invariant plant described by

$$\dot{x} = Ax + Bu \tag{2.9a}$$

$$y = Cx \tag{2.9b}$$

where $u(t)$ and $y(t) \in \mathbb{R}^m$. The linear plant (2.9) satisfies (N.1)-(N.6) if and only if A has all its eigenvalues in $\mathring{\mathbb{C}}_-$ and, with $H(s) := C(sI-A)^{-1}B$, H(0) is positive definite. Morari showed that a pure integrator with a sufficient small coefficient ε can be used fcr the plant (2.9) with the unity-feedback configuration to achieve asymptotic tracking of step inputs if and only if A has all its eigenvalues in $\mathring{\mathbb{C}}_-$ and H(0) has all its eigenvalues in $\mathring{\mathbb{C}}_+$. (The last condition does not imply but is implied by H(0) positive definite)

We assume for ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$ that the reference inputs u_1 and disturbances u_2, d_0 satisfy the assumption

(I.1) $u_1(\cdot)$, $u_2(\cdot)$, $d_0(\cdot) \in C^1$ and $\exists\, \bar{u}_1, \bar{u}_2, \bar{d}_0 \in \mathbb{R}^m$ such that, as $t \to \infty$

$$\begin{cases} u_1(t) \to \bar{u}_1 \\ u_2(t) \to \bar{u}_2 \\ d_0(t) \to \bar{d}_0 \end{cases}, \text{ and } \begin{cases} \dot{u}_1(t) \to \theta_m \\ \dot{u}_2(t) \to \theta_m \\ \dot{d}_0(t) \to \theta_m \end{cases}. \tag{2.10}$$

Note that (I.1) implies that $\exists\, \mu > 0$ such that $\forall t \geq 0$

$$|u_1(t)| < \mu, \ |u_2(t)| < \mu, \text{ and } |d_0(t)| < \mu \ . \tag{2.11}$$

Since $u_1(\cdot)$ and $d_0(\cdot)$ satisfy the same assumption (I.1), the effect of $d_0(\cdot)$ on the closed-loop system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$ can be included in $u_1(\cdot)$. Hence, from now on we assume $d_0(t) \equiv \theta_m$.

For the system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$ shown in Fig. 2.1, let $\eta(t) := \varepsilon\int_0^t e_1(t')dt'$ and choose $(x(t), \eta(t))$ as state variable. We <u>assume</u> that for all $\varepsilon > 0$ and $K \in \mathbb{R}^{m\times m}$, the system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$ is <u>reachable</u>: more precisely, for all states (x_o, η_o), $(x_1, \eta_1) \in \mathbb{R}^n \times \mathbb{R}^m$, there exists inputs $u_1, u_2 \in C^1$, with compact support, say [0,T], which take $(x(0), \eta(0)) = (x_o, \eta_o)$ to $(x(T), \eta(T)) = (x_1, \eta_1)$.

II.4. The Main Result

Roughly speaking, we shall establish the following: Given the nonlinear plant $\mathcal{N}$, which satisfies the smoothness, stability and dc steady-state assumptions (N.1)-(N.6), if the constant gain matrix K is chosen positive semidefinite, and if $|K|$ is small enough, then the PI compensator $\frac{\varepsilon}{s} I + K$, with $\varepsilon > 0$ small enough, yields stability and asymptotic tracking of the closed-loop system ${}^{1}S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$ for all initial conditions and for all inputs and disturbances satisfying (I.1). More precisely, we shall prove the following theorem.

Theorem 2.1.

Consider the nonlinear feedback system ${}^{1}S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$ where $\mathcal{N}$ satisfies (N.1)-(N.6). U.t.c., if

(i) $K \in \mathbb{R}^{mxn}$ is positive semidefinite; and

(ii) $|K|$ is small enough, then there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}]$, for all initial conditions $(x(0), \eta(0)) \in \mathbb{R}^n \times \mathbb{R}^m$, for all $u_1(\cdot)$, $u_2(\cdot)$, and $d_0(\cdot)$ satisfying (I.1), the corresponding $e_1(\cdot)$, $e_2(\cdot)$, $x(\cdot)$, and $y_2(\cdot)$ are bounded and $e_1(t) \to \theta_m$ as $t \to \infty$.

Comment. We establish Theorem 2.1 through the next two sections. In Section III, we study the tracking and disturbance rejection problem for the system ${}^{1}S(\mathcal{N}, \frac{\varepsilon}{s} I)$ (i.e., ${}^{1}S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$ with $K = \theta_{mxm}$) with a slightly more general class of reference inputs and disturbances. Based on this result (Theorem 3.1), and the "loop transformation" technique, Theorem 2.1 is proved in Section IV.

III. Integral Control - The System ${}^{1}S(\mathcal{N}, \frac{\varepsilon}{s} I)$

In this section, we study the tracking and disturbance rejection problem for the system ${}^{1}S(\mathcal{N}, \frac{\varepsilon}{s} I)$ shown in Fig. 2.1, with $K = \theta_{mxm}$.

For convenience, let C denote the class of all continuous functions. Let the assumptions (N.1)-(N.6) still hold but the class of reference inputs and disturbances considered in this section be slightly more general than those satisfying (I.1): for the system ${}^{1}S(\mathcal{N}, \frac{\varepsilon}{s} I)$, we assume that $u_1(\cdot)$, $u_2(\cdot)$, and $d_0(\cdot)$ satisfy

(I.2) $u_1(\cdot), d_0(\cdot) \in C$, $u_2(\cdot) \in C^1$ and $\exists\, \bar{u}_1, \bar{u}_2, \bar{d}_0 \in \mathbb{R}^m$ such that, as $t \to \infty$,

$$\begin{cases} u_1(t) \to \bar{u}_1 \\ u_2(t) \to \bar{u}_2 \\ d_0(t) \to \bar{d}_0 \end{cases} , \quad \text{and} \quad \dot{u}_2(t) \to \theta_m \ .$$

We shall prove the following tracking and disturbance rejection result.

<u>Theorem 3.1</u>.

Given that the nonlinear plant $\mathcal{N}$ satisfies (N.1)-(N.6), consider the system ${}^{1}S(\mathcal{N}, \frac{\varepsilon}{s} I)$ shown in Fig. 2.1 with $K = \theta_{mxm}$. U.t.c., $\exists\, \varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$, for all $(x(0), e_2(0)) \in \mathbb{R}^n \times \mathbb{R}^m$, for all $u_1(\cdot)$, $u_2(\cdot)$ and $d_0(\cdot)$ satisfying (I.2), the corresponding $e_1(\cdot)$, $e_2(\cdot)$, $x(\cdot)$ and $y_2(\cdot)$ are bounded; furthermore, $e_1(t) \to \theta_m$ as $t \to \infty$.

<u>Comments</u>

(a) Roughly speaking, the theorem says that given that the nonlinear plant $\mathcal{N}$ satisfies (N.1)-(N.6), a simple integrator $\frac{\varepsilon}{s} I$, with $\varepsilon > 0$ small enough, yields stability and asymptotic tracking of the closed-loop system ${}^{1}S(\mathcal{N}, \frac{\varepsilon}{s} I)$, for all inputs and disturbances satisfying (I.2).

(b) Theorem 3.1 is a special case of Theorem 2.1 except that the class of reference inputs $u_1(\cdot)$ and plant-output disturbance $d_0(\cdot)$ is slightly more general. (Compare I.1 with I.2).

(c) From the stability and tracking point of view, the effect of $u_2(\cdot)$ on the closed-loop system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I)$ is equivalent to the effect of adding the derivative $\dot{u}_2(\cdot)$ to the reference input $u_1(\cdot)$. Since $\dot{u}_2(\cdot) \in C$, and $\dot{u}_2(t) \to \theta_m$, we may assume, without loss of generality that $u_2(t) \equiv \theta_n$ and $d_0(t) \equiv \theta_m$.

(d) In the following analysis, we first examine the case where the reference input u_1 is constant; by a change of variables and a change of time scale, the problem is converted into a singular perturbation framework.

Consider the nonlinear feedback system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I)$, with $u_2(t) \equiv d_0(t) \equiv \theta_m$, $u_1 \in C$ and $u_1(t) \to \bar{u}_1$ as $t \to \infty$; refer to Fig. 2.1, with $K = \theta_{mxm}$, write the equations describing ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I)$:

$$\dot{x}(t) = f(x(t), e_2(t)) \tag{3.1a}$$

$$\dot{e}_2(t) = \varepsilon(u_1(t) - h(x(t))) \tag{3.1b}$$

$$y_2(t) = h(x(t)) \tag{3.2}$$

with $\varepsilon > 0$, $t \geq 0$, $x(0) = x_0$, and $e_2(0) = e_{20}$. Let us shift the origin to the equilibrium point: First define $\bar{e}_2$ by

$$\bar{u}_1 = hg(\bar{e}_2) \tag{3.3a}$$

and then define the new variables $\tilde{u}_1(t)$, $\tilde{e}_2(t)$ and $\tilde{x}(t)$ by

$$\tilde{u}_1(t) := u_1(t) - \bar{u}_1 \tag{3.3b}$$

$$\tilde{e}_2(t) := e_2(t) - \bar{e}_2 \tag{3.3c}$$

$$\tilde{x}(t) := x(t) - g(e_2(t)) \tag{3.3d}$$

Using $\tilde{x}(t)$ and $\tilde{e}_2(t)$ as state variables, Eqs. (3.1) become

$$\dot{\tilde{x}} = f(g(\bar{e}_2+\tilde{e}_2)+\tilde{x},\ \bar{e}_2+\tilde{e}_2) - \varepsilon\cdot Dg(\bar{e}_2+\tilde{e}_2)(hg(\bar{e}_2) - h(g(\bar{e}_2+\tilde{e}_2)+x))$$

$$- \varepsilon\cdot Dg(\bar{e}_2+\tilde{e}_2)\cdot\tilde{u}_1 \tag{3.5a}$$

$$\dot{\tilde{e}}_2 = \varepsilon[hg(\bar{e}_2) - h(g(\bar{e}_2+\tilde{e}_2) + \tilde{x})] + \varepsilon\cdot\tilde{u}_1 \tag{3.5b}$$

with $t \geq 0$, $\tilde{x}(0) = x_o - g(e_{20})$ and $\tilde{e}_2(0) = e_{20} - \bar{e}_2$. Assumptions (N.1) and (N.4) imply that for all initial conditions and all piecewise continuous $u_1(\cdot)$ Eqs. (3.1) have a unique solution on $\mathbb{R}_+$, consequently, by (3.3), the solution $\tilde{x}(\cdot)$ and $\tilde{e}_2(\cdot)$ of (3.5) are uniquely defined on $\mathbb{R}_+$.

We study first the case where the reference input $u_1(t)$ is constant, i.e., $u_1(t) \equiv \bar{u}_1$ or equivalently $\tilde{u}_1(t) \equiv \theta_m$; and derive a stability and tracking result.

Set $\tilde{u}_1(t) \equiv \theta_n$, Eqs. (3.5) become

$$\dot{\tilde{x}} = f(g(\bar{e}_2+\tilde{e}_2)+\tilde{x},\ \bar{e}_2+\tilde{e}_2) - \varepsilon\cdot Dg(\bar{e}_2+\tilde{e}_2)(hg(\bar{e}_2) - h(g(\bar{e}_2+\tilde{e}_2)+\tilde{x})) \tag{3.6a}$$

$$\dot{\tilde{e}}_2 = \varepsilon[hg(\bar{e}_2) - h(g(\bar{e}_2+\tilde{e}_2)+\tilde{x})] \tag{3.6b}$$

It can be easily checked that (N.2), (N.3) and (3.3b) imply that $(\tilde{x}(t),\ \tilde{e}_2(t)) \equiv (\theta_n,\ \theta_m)$ is the <u>unique</u> equilibrium point of Eqs. (3.6). Since h and g are continuous, $(\tilde{x}(t),\ \tilde{e}_2(t)) \to (\theta_n,\ \theta_m)$ as $t \to \infty$ implies that $y_2(t) \to \bar{u}_1$ i.e., $e_1(t) \to \theta_m$ as $t \to \infty$.

Let $\tau := \varepsilon t$ be the new "time scale," τ is "slow time" compared to t. Define the variables $z(\cdot)$ and $w(\cdot)$ by

$$z(\tau) := \tilde{x}(t) \quad , \quad w(\tau) := \tilde{e}_2(t) \tag{3.7}$$

Using τ, z, and w as new variables (hence $\dot{z}$ denotes $\frac{dz}{d\tau}$, etc.), rewrite Eqs. (3.6) as

$$\varepsilon\dot{z} = f(g(\bar{e}_2+w)+z,\ \bar{e}_2+w) - \varepsilon Dg(\bar{e}_2+w)[hg(\bar{e}_2) - h(g(\bar{e}_2+w)+z)] \tag{3.8a}$$

$$\dot{w} = hg(\bar{e}_2) - h(g(\bar{e}_2+w) + z) \tag{3.8b}$$

with $z(0) = \tilde{x}(0)$, $w(0) = \tilde{e}_2(0)$, where we have used that $\bar{u}_1 = hg(\bar{e}_2)$. Clearly, $(z,w) = (\theta_n,\theta_m)$ is the equilibrium point of Eqs. (3.8). Since $\varepsilon > 0$, $(z(\tau), w(\tau)) \to (\theta_n, \theta_m)$ as $\tau \to \infty$ iff $(\tilde{x}(t), \tilde{e}_2(t)) \to (\theta_n, \theta_m)$ as $t \to \infty$. Note that Eqs. (3.8) are in the form of standard singular perturbation [Tih. 1, Bar. 1, Hab. 1, Kok. 1]: for $\varepsilon \ll 1$, z is the "fast" variable and w is the slow variable. Note that Eqs. (3.6) and Eqs. (3.8) describe the same system but in a different time scale.

If in Eqs. (3.6), we let $\varepsilon = 0$ and hence $\tilde{e}_2+\bar{e}_2 = e_{20}$, we have

$$\begin{aligned} \dot{\tilde{x}} &= f(g(e_{20}) + \tilde{x}, e_{20}) \\ \tilde{e}_2 &\equiv \tilde{e}_2(0) = e_{20} - \bar{e}_2 \end{aligned} \tag{3.9}$$

The system described by (3.10) with e_{20} treated as a fixed parameter is called the "<u>boundary-layer system</u>" [Tih. 1, Bar. 1, Hab. 1, Kok. 1] of the singularly perturbed system (3.8). Note that the boundary-layer system is defined in the "fast" time scale t.

Let $\phi(\cdot, t_o, \tilde{x}_o, v)$ denote the solution of (3.9) with $e_{20} = v$ starting with the initial condition $(\tilde{x}_o, t_o)$. (3.10a)

It can be easily checked that Assumption (N.5) implies that, there exist $\alpha > 0$ and $c > 0$ -- see (2.6) -- such that

$$\begin{aligned} &\forall\ (\tilde{x}_o, v) \in \mathbb{R}^n \times \mathbb{R}^m,\ \ \forall t_o \geq 0,\ \ \forall t \geq t_o\ , \\ &\quad |\phi(t, t_o, \tilde{x}_o, v)| \leq c|\tilde{x}_o|\ e^{-\alpha(t-t_o)} \end{aligned} \tag{3.11}$$

If in Eqs. (3.8a) we set $\varepsilon = 0$, then Eqs. (3.8) reduce to

$$\begin{aligned} z &= \theta_n \\ \dot{w} &= hg(\bar{e}_2) - hg(\bar{e}_2+w) \end{aligned} \tag{3.12}$$

The system (3.12) is the "reduced system" of the singularly perturbed system (3.8) [Bar. 1, Hab. 1, Kok. 1]. Note that θ_m is the equilibrium point of (3.12).

Let $\psi(\cdot, \tau_0, w_0)$ be the solution of (3.12) starting with the initial condition $w(\tau_0) = w_0$. (3.13)

Assumption (N.6) implies that the equilibrium point θ_m of the reduced system (3.12) is globally uniformly exp. stable: More precisely, we have the following

Fact: There is a $\beta > 0$ such that the solution of (3.12) satisfies that $\forall w_0 \in \mathbb{R}^m$, $\forall \tau_0 \geq 0$, $\forall \tau \geq \tau_0$,

$$|\psi(\tau, \tau_0, w_0)| \leq |w_0|\, e^{-\beta(\tau-\tau_0)} \tag{3.14}$$

Proof: Let $P : \mathbb{R}^m \to \mathbb{R}_+$ be defined by $P(w) = |w|^2$. The derivative $\dot{P}_{(3.12)}$ along the trajectory of (3.12) is given by

$$\dot{P}_{(3.12)}(w) = 2\dot{w}^T w = 2[hg(\bar{e}_2) - hg(\bar{e}_2+w)]^T w$$

$$\leq -2\beta|w|^2 \tag{3.15}$$

where we have used Assumption (N.6). From (3.15) and the definition of P, we have that $\forall w_0 \in \mathbb{R}^m$, $\forall \tau_0 \geq 0$, $\forall \tau \geq \tau_0$,

$$\frac{d}{d\tau}\left(|\psi(\tau, \tau_0, w_0)|^2\right) \leq -2\beta|\psi(\tau, \tau_0, w_0)|^2$$

Hence, $\forall w_0 \in \mathbb{R}^m$, $\forall \tau_0 \geq 0$, $\forall \tau \geq \tau_0$,

$$|\psi(\tau, \tau_0, w_0)| \leq |w_0|\, e^{-\beta(\tau-\tau_0)}$$

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Lemma 3.2. If the nonlinear plant $\mathcal{N}$ satisfies the Assumptions (N.1)-(N.6), then $\exists\, \varepsilon^* > 0$ such that $\forall \varepsilon \in (0,\varepsilon^*]$, the equilibrium point $(z(\tau), w(\tau)) \equiv (\theta_n, \theta_m)$ of Eqs. (3.8) is globally uniformly exp. stable.

Comments

(a) Lemma 3.2 implies that $\forall \varepsilon \in (0,\varepsilon^*]$, the equilibrium point $(\tilde{x}(t), \tilde{e}_2(t)) \equiv (\theta_n, \theta_m)$ of Eqs. (3.6) is globally uniformly exp. stable, i.e., for ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I)$, for constant input $u_1(\tau) = \bar{u}_1$, the equilibrium point $(g(\bar{e}_2), \bar{e}_2)$ is globally uniformly exp. stable

(b) Since $u_1(t) \equiv \bar{u}_1$ i.e., $\tilde{u}_1(t) \equiv \theta$, from (a), $\forall\, (\tilde{x}(0), \tilde{e}_2(0)) \in \mathbb{R}^n \times \mathbb{R}^m$ (or equivalently for any $(x(0), e_2(0)) \in \mathbb{R}^n \times \mathbb{R}^m$), $\tilde{x}(t) \to \theta_n$, $\tilde{e}_2(t) \to \theta_m$ as $t \to \infty$. Since h and g are continuous, $e_2(t) \to \bar{e}_2$, $x(t) \to g(\bar{e}_2)$, $y_2(t) \to \bar{u}_1$, and $e_1(t) \to \theta_m$ as $t \to \infty$ (see (3.3)). In particular, $e_1(\cdot)$, $e_2(\cdot)$, $x(\cdot)$, $y_2(\cdot)$ are bounded on $\mathbb{R}_+$.

(c) The lemma is proved by showing that if $\varepsilon > 0$ is small enough, then there exists a Lyapunov function for the system (3.8), which leads to the global uniform exp. stability of the equilibrium point. Similar results on the stability of nonlinear singularly perturbed system are also available in [Bar. 1], [Hab. 1], [Cho. 1].

Proof: See Appendix A.

Proof of Theorem 3.1: Let

$$\hat{x}(t) := x(t) - g(\bar{e}_2) = \tilde{x}(t) + g(\bar{e}_2+\tilde{e}_2(t)) - g(\bar{e}_2) \tag{3.17}$$

Use† $(\hat{x}, \tilde{e}_2)$ as state variable, rewrite Eqs. (3.1) as

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† $\bar{e}_2$, $\tilde{u}_1(\cdot)$, and $\tilde{e}_2(\cdot)$ are defined in (3.3).

$$\dot{\hat{x}} = f(g(\bar{e}_2) + \hat{x}, \bar{e}_2 + \tilde{e}_2) \qquad (3.18a)$$

$$\dot{\tilde{e}}_2 = \varepsilon[hg(\bar{e}_2) - h(g(\bar{e}_2) + \hat{x})] + \varepsilon \tilde{u}_1 \quad . \qquad (3.18b)$$

Clearly, with $\tilde{u}_1(t) \equiv \theta_m$, (θ_n, θ_m) is the equilibrium point of (3.18). Now Lemma 3.2 implies that, with $\tilde{u}_1(t) \equiv \theta_m$, $\exists\, \varepsilon^* > 0$ s.t. $\forall \varepsilon \in (0,\varepsilon^*]$, for all $(\tilde{x}(0), \tilde{e}_2(0))$, as $t \to \infty$, $(\tilde{x}(t), \tilde{e}_2(t)) \to (\theta_n, \theta_m)$ exponentially, and since $|\hat{x}(t) - \tilde{x}(t)| = |g(\bar{e}_2 + \tilde{e}_2(t)) - g(\bar{e}_2)| \leq M|\tilde{e}_2(t)|$, we have that, with $\tilde{u}_1(t) \equiv \theta_m$, the equilibrium point (θ_n, θ_m) of (3.18) is globally uniformly exponentially stable. By (2.5), the RHS of (3.18) as a function of $(\hat{x}, \tilde{e}_2, \tilde{u}_1)$ is globally Lipschitz continuous. A result of Vidyasagar and Vannelli [Vid. 1, Cor. 1.1] states that if $(\hat{x}(0), \tilde{e}_2(0)) = (\theta_n, \theta_m)$, then $\forall \varepsilon \in (0,\varepsilon^*]$, for all $\tilde{u}_1(\cdot) \in C$ with $\tilde{u}_1(t) \to \theta_m$ as $t \to \infty$, the corresponding $(\hat{x}(t), \tilde{e}_2(t)) \to (\theta_n, \theta_m)$ as $t \to \infty$. Therefore, for the system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I)$, for every $u_1(\cdot) \in C$ with $u_1(t) \to \bar{u}_1$ as $t \to \infty$, for all† $u_2(\cdot)$ and $d_o(\cdot)$ satisfying (I.2), if $(x(0), e_2(0)) = (g(\bar{e}_2), \bar{e}_2)$, (where $\bar{e}_2 := (hg)^{-1}(\bar{u}_1)$), then the corresponding $e_1(\cdot)$, $e_2(\cdot)$, $x(\cdot)$ and $y_2(\cdot)$ are bounded and $e_1(t) \to \theta_m$ (i.e., $y_2(t) \to \bar{u}_1$) as $t \to \infty$. Since the system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I)$ is time-invariant and is assumed to be reachable,†† we have that for the system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I)$, for all $\varepsilon \in (0,\varepsilon^*]$, for all $(x(0), e_2(0)) \in \mathbb{R}^n \times \mathbb{R}^m$, for all $u_1(\cdot)$, $u_2(\cdot)$ and $d_o(\cdot)$ satisfying (I.2), the corresponding $e_1(\cdot)$, $e_2(\cdot)$, $x(\cdot)$ and $y_2(\cdot)$ are bounded and $e_1(t) \to \theta_m$ as $t \to \infty$. ¤

IV. Asymptotic Tracking and Disturbance Rejection of ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$

In this section, we prove Theorem 2.1 based on Theorem 3.1 and the "loop transformation" technique. We first prove that the assumptions

†In the derivation, w.l.o.g., we assumed $u_2(t) = d_o(t) \equiv \theta_m$

††See section II.3.

(N.1)-(N.6) are "invariant" under sufficient small positive semidefinite feedback K (Lemma 4.1 below). The proof proceeds by examining the relation between the (proportional plus integral control) system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$ and the (integral control plus local constant output feedback) system ${}^1S(\tilde{\mathcal{N}}, \frac{\varepsilon}{s} I)$, shown in Fig. 4.2 below.

For the given nonlinear plant $\mathcal{N}$ described by (2.1) and satisfying (N.1)-(N.6) apply a constant output feedback K as in Fig. 4.1, and call the resulting closed-loop system $\tilde{\mathcal{N}}$: it has input $\hat{e}_2$, state x, and output y_2; it is described by

$$\dot{x} = f(x, \hat{e}_2 - Kh(x)) =: \tilde{f}(x, \hat{e}_2) \tag{4.1a}$$

$$y_2 = h(x) \tag{4.1b}$$

For any $v \in \mathbb{R}^m$, let $\tilde{s}(\cdot, t_0, x_0, v)$ be the solution of

$$\dot{x} = \tilde{f}(x,v), \quad t \geq 0 \tag{4.2}$$

corresponding to the initial condition (x_0,t_0). The following lemma shows that if K is <u>positive semidefinite</u> and if $|K|$ is <u>small enough</u>, then the resulting nonlinear plant $\tilde{\mathcal{N}}$ preserves the qualitative properties (N.1) to (N.6) of $\mathcal{N}$. More precisely, we have

<u>Lemma 4.1</u>. Let $\mathcal{N}$ satisfy (N.1)-(N.6). If K is positive semidefinite and if $|K|$ is small enough, then $\tilde{\mathcal{N}}$ satisfies the following assumptions ($\tilde{N}$.1)-($\tilde{N}$.6):

($\tilde{N}$.1) $\tilde{f}, h \in C^1$, $\tilde{f}(\theta_n, \theta_m) = \theta_n$, and $h(\theta_n) = \theta_m$;

($\tilde{N}$.2) $\forall v \in \mathbb{R}^m$, $\tilde{f}(\xi,v) = \theta_n$ iff $\tilde{g}(v) := g(I+Khg)^{-1}(v) = \xi$; furthermore $\tilde{g} := g(I+Khg)^{-1} \in C^1$;

($\tilde{N}$.3) the map $v \mapsto h\tilde{g}(v)$ is a bijection of $\mathbb{R}^m$ onto $\mathbb{R}^m$;

($\tilde{N}$.4) there exists $\tilde{M} > 0$, s.t. $\forall v_0 \in \mathbb{R}^m$, $\forall \xi \in \mathbb{R}^n$,

$$|D_1\tilde{f}(\xi, v_0)| < \tilde{M} \quad ; \quad |D_2\tilde{f}(\xi, v_0)| < \tilde{M}$$

$$|Dh(\xi)| < \tilde{M} \quad ; \quad |D\tilde{g}(v_0)| < \tilde{M}$$

(N.5) there exists $\tilde{c} > 0$, $\tilde{\alpha} > 0$ s.t. $\forall v \in \mathbb{R}^m$, $\forall t_0 \geq 0$, $\forall x_0 \in \mathbb{R}^n$, $\forall t \geq t_0$,

$$|\tilde{s}(t, t_0, x_0, v) - \tilde{g}(v)| \leq \tilde{c}|x_0 - \tilde{g}(v)|\, e^{-\tilde{\alpha}(t-t_0)} \quad ;$$

($\tilde{N}$.6) there exists $\tilde{\beta} > 0$ s.t. $\forall v_1, v_2 \in \mathbb{R}^m$,

$$[h\tilde{g}(v_1) - h\tilde{g}(v_2)]^T[v_1 - v_2] \geq \tilde{\beta}|v_1 - v_2|^2 .$$

Comment: $h\tilde{g}$ is the dc steady-state I/O map of the nonlinear system $\tilde{\mathcal{N}}$ shown in Fig. 4.1.

Proof of Lemma 4.1: See Appendix B.

An immediate consequence of Lemma 4.1 is the following.

Lemma 4.2: Given that the nonlinear plant $\mathcal{N}$ satisfies (N.1)-(N.6), consider the system ${}^1S(\tilde{\mathcal{N}}, \frac{\varepsilon}{s} I)$ shown in Fig. 4.2, where $\tilde{\mathcal{N}}$ is defined in Eqs. (4.1). U.t.c. if K is positive semidefinite and if $|K|$ is small enough, then $\exists\ \bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0,\bar{\varepsilon}]$, for all $(x(0),\hat{e}_2(0)) \in \mathbb{R}^n \times \mathbb{R}^m$, for all $\hat{u}_1(\cdot)$, $\hat{u}_2(\cdot)$ and $d_0(\cdot)$ satisfying (I.2), the corresponding $\hat{e}_1(\cdot)$, $\hat{e}_2(\cdot)$, $x(\cdot)$ and $y_2(\cdot)$ (see Fig. 4.2) are bounded and $\hat{e}_1(t) \to \theta_m$ as $t \to \infty$.

Proof: From Lemma 4.1, $\tilde{N}$ satisfies ($\tilde{N}$.1)-($\tilde{N}$.6). The proof can be constructed using exactly the same techniques as those in the proof of Theorem 3.1. ¤

Consider the system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$, (see Fig. 2.1) with $d_o(t) \equiv \theta_m$, and write the equations†

$$\dot{x} = f(x,e_2) \tag{4.5a}$$

$$e_2 = u_2 + \eta + K(u_1 - h(x)) \tag{4.5b}$$

$$\dot{\eta} = \varepsilon(u_1 - h(x)) \tag{4.5c}$$

$$y_2 = h(x) \tag{4.5d}$$

Let $\hat{e}_2 := e_2 + Kh(x)$, rewrite the equations (4.5) as

$$\dot{x} = f(x, \hat{e}_2 - Kh(x)) = \tilde{f}(x,\hat{e}_2) \tag{4.6a}$$

$$\dot{\hat{e}}_2 = -\varepsilon h(x) + \varepsilon(u_1 + \frac{\dot{u}_2}{\varepsilon} + \frac{K\dot{u}_1}{\varepsilon}) \tag{4.6b}$$

$$y_2 = h(x) \tag{4.6c}$$

Note that Eqs. (4.6) describe the system ${}^1S(\tilde{\mathcal{N}}, \frac{\varepsilon}{s} I)$ with reference input $\hat{u}_1 := u_1 + \frac{1}{\varepsilon}\dot{u}_2 + \frac{1}{\varepsilon} K \dot{u}_1$, and disturbances $\hat{u}_2(t) \equiv d_o(t) \equiv \theta_m$. So, use Lemma 4.2, to obtain the following

<u>Proof of Theorem 2.1</u>: Since $u_1(\cdot)$ and $u_2(\cdot)$ satisfy (I.1), $\hat{u}_1(\cdot) \in C$ and $\hat{u}_1(t) \to \bar{u}_1$ as $t \to \infty$. By Lemma 4.2, $\hat{e}_1(\cdot) := u_1(\cdot) + (\dot{u}_2(\cdot) + K\dot{u}_1(\cdot))/\varepsilon - y_2(\cdot)$, $\hat{e}_2(\cdot) = e_2(\cdot) + Ky_2(\cdot)$, $x(\cdot)$ and $y_2(\cdot)$ are bounded and $\hat{e}_1(t) \to \theta_m$ as $t \to \infty$. Therefore, for the system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$, $e_1(\cdot) = u_1(\cdot) - y_2(\cdot)$, $e_2(\cdot)$, $x(\cdot)$ and $y_2(\cdot)$ are bounded. Since $\dot{u}_1(t) \to \theta_m$, $\dot{u}_2(t) \to \theta_m$ and $\hat{e}_1(t) \to \theta_m$ as $t \to \infty$, and since

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†Recall that $\eta := \varepsilon\int_0^t e_1(t')dt'$.

$e_1 = \hat{e}_1 - (\dot{u}_2 + K\dot{u}_1)/\varepsilon$, $e_1(t) \to \theta_m$ as $t \to \infty$. ⌑

V. Concluding Remarks

In this paper, we study tracking and disturbance rejection of a class of nonlinear time-invariant systems with proportional plus integral (PI) compensator. The basic assumptions on the plant -- exponential stability and monotonicity of the dc steady-state input-output map -- are satisfied by typical chemical process (with appropriate choices of inputs and outputs). We have shown that, for a large class of nonlinear plants, a simple PI controller (which is commonly used in process control) can be used to achieve closed-loop stability, asymptotic tracking for inputs which tend to constant vector and asymptotic disturbance rejection for disturbances which tend to constant vectors.

Appendix A

Proof of Lemma 3.2

We first construct Lyapunov functions for the boundary-layer system (3.10) and for the reduced system (3.12), then we show that if $\varepsilon > 0$ is small enough, then the sum of these two functions is a Lyapunov function of the system (3.8). We proceed with the following lemma.

Lemma A.1: With $\phi(\cdot,\cdot,\cdot,\cdot)$ and $\psi(\cdot,\cdot,\cdot)$ defined in (3.10a) and (3.13), $\exists\, \alpha_1 > 0$ such that $\forall \tau \geq 0$, $\forall z \in \mathbb{R}^n$, $\forall w \in \mathbb{R}^m$,

$$|D_3\phi(\tau,0,z,w)| \leq \alpha_1 \cdot e^{\alpha_1 \tau} \tag{A.1}$$

$$|D_4\phi(\tau,0,z,w)| \leq \alpha_1\, e^{\alpha_1 \tau} \tag{A.2}$$

$$|D_3\psi(\tau,0,w)| \leq \alpha_1\, e^{\alpha_1 \tau} \tag{A.3}$$

Proof:

From (3.10), we have

$$\phi(\tau,0,z,w) = z + \int_0^\tau f(g(w) + \phi(\tau',0,z,w),\, g(w))\, d\tau' \tag{A.4}$$

Hence

$$D_3(\tau,0,z,w) = I + \int_0^\tau D_1 f(g(w) + \phi(\tau',0,z,w),\, g(w)) \circ D_3\phi(\tau',0,z,w)\, d\tau' \tag{A.5}$$

By taking norms on both sides of (A.5) and using (2.5) we obtain,

$$|D_3(\tau,0,z,w)| \leq 1 + \int_0^\tau M \cdot |D_3(\tau',0,z,w)|\, d\tau \tag{A.6}$$

Hence, $\forall\, (\tau,z,w) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m$,

$$|D_3(\tau,0,z,w)| \leq e^{M\tau} \tag{A.7}$$

Ineq. (A.2) and (A.3) can be similarly established. ⌑

Let γ be a positive integer and $\gamma \geq \max(\frac{\alpha_1+2\alpha}{2\alpha}, \frac{\alpha_1+2\beta}{2\beta})$, where α_1, α, β are defined in Lemma A.1, (2.6), and (2.7) respectively. Following [Bar. 1], let

$$U(w) := \int_0^\infty |\psi(\tau,0,w)|^{2\gamma} \, d\tau \, . \tag{A.8}$$

Lemma A.2 below summarizes the properties of $U(\cdot)$.

Lemma A.2: The function $U : \mathbb{R}^m \to \mathbb{R}_+$ defined in (A.8) satisfies the following properties:

(a) $$\forall w \in \mathbb{R}^m, \ \frac{1}{2\gamma\beta} |w|^{2\gamma} \geq U(w) \geq \frac{1}{2M^2} \frac{|w|^{2\gamma}}{4^\gamma} \tag{A.9}$$

(b) $\dot{U}_{(3.12)}(w)$, the derivative of U along the trajectory of (3.12) satisfies

$$\dot{U}_{(3.12)}(w) = -|w|^{2\gamma}, \quad \forall w \in \mathbb{R}^m \tag{A.10}$$

(c) $\exists \, \alpha_2 > 0$, s.t. $\forall w \in \mathbb{R}^m$,

$$|DU(w)| \leq \alpha_2 |w|^{2\gamma-1} \tag{A.11}$$

Proof:

$$U(w) = \int_0^\infty |\psi(\tau',0,w)|^{2\gamma} d\tau' \leq \int_0^\infty |w|^{2\gamma} \, e^{-2\alpha\beta\tau'} d\tau' = \frac{1}{2\gamma\beta} |w|^{2\gamma} \tag{A.12}$$

Since by (3.12) $\psi(\tau,0,w) = w + \int_0^\tau [hg(\bar{e}_2) - hg(\bar{e}_2+\psi(\tau',0,w))] \, d\tau'$, we obtain using (2.5) and (3.14)

$$|\psi(\tau,0,w)| \geq |w| - \int_0^\tau \max_\eta |D(hg)(\eta)| \cdot |\psi(\tau',0,w)| d\tau'$$

$$\geq |w| - \int_0^\tau M^2 |\psi(\tau',0,w)| d\tau' \geq |w| - \int_0^\tau M^2 |w| d\tau'$$

Therefore, for $0 < \tau \leq \frac{1}{2M^2}$,

$$|\psi(\tau,0,w)| \geq \frac{|w|}{2}$$

Consequently,

$$U(w) \geq \int_0^{\frac{1}{2M^2}} |\psi(\tau',0,w)|^{2\gamma} d\tau' \geq \frac{1}{2M^2} \frac{|w|^{2\gamma}}{4^\gamma} \tag{A.13}$$

Assertion (a) follows from (A.12) and (A.13).

Taking limits as $\Delta t \searrow 0$, we obtain successively

$$\begin{aligned}
\dot{U}_{(3.12)}(w) &= \lim[U(\psi(\Delta t,0,w)) - U(w)]/\Delta t \\
&= \lim[\int_0^\infty |\psi(\tau,0,\psi(\Delta t,0,w))|^{2\gamma} d\tau - \int_0^\infty |\psi(\tau,0,w)|^{2\gamma} d\tau]/\Delta t \\
&= \lim[\int_0^\infty |\psi(\tau+\Delta t,0,w)|^{2\gamma} d\tau - \int_0^\infty |\psi(\tau,0,w)|^{2\gamma} d\tau]/\Delta t \\
&= \lim[-\int_0^{\Delta t} |\psi(\tau,0,w)|^{2\gamma} d\tau]/\Delta t \\
&= -|w|^{2\gamma}
\end{aligned}$$

since $\psi(0,0,w) = w$. This proves (b).

From (A.8),

$$DU(w) = \int_0^\infty \gamma|\psi(\tau,0,w)|^{2(\gamma-1)} \cdot 2 \cdot \psi(\tau,0,w)^T \cdot D_3\psi(\tau,0,w)\, d\tau \tag{A.14}$$

Therefore, by (3.14) and (A.3),

$$\begin{aligned}
|DU(w)| &\leq \int_0^\infty 2\gamma|w|^{2(\gamma-1)} e^{-2(\gamma-1)\beta\tau} \cdot |w| e^{-\beta\tau} \cdot |D_3\psi(\tau,0,w)| d\tau \\
&\leq \int_0^\infty 2\gamma|w|^{2\gamma-1} \cdot e^{-(2\gamma-1)\beta\tau} \cdot \alpha_1 e^{\alpha_1\tau} d\tau
\end{aligned} \tag{A.15}$$

Since $\gamma > \frac{2\beta+\alpha_1}{2\beta}$, the integral in (A.15) is finite and

$$|DU(w)| \leq \frac{2\gamma\alpha_1}{(2\gamma-1)\beta-\alpha_1} |w|^{2\gamma-1} =: \alpha_2|w|^{2\gamma-1} .$$

¤

Similarly referring to (3.10a) and (3.11), we define $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ by

$$V(z,w) := \int_0^\infty |\phi(\tau,0,z,w)|^{2\gamma} d\tau \tag{A.16}$$

The following lemma can be proved using the same techniques as in the proof of Lemma A.2.

Lemma A.3: The function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ defined by (A.16) satisfies the following conditions:

(a) $\forall\ (z,w) \in \mathbb{R}^n \times \mathbb{R}^m,\ \frac{c^{2\gamma}}{2\gamma\alpha} |z|^{2\gamma} \geq V(z,w) \geq \frac{1}{2Mc} \frac{|z|^{2\gamma}}{4^\gamma}$ (A.17)

where $M > 0$, $\alpha > 0$ and $c > 0$ are given in (2.5) and (2.6);

(b) $\dot{V}_{(3.10)}$, the derivative of V along the trajectory of (3.10) satisfies that $\forall\ (z,w) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\dot{V}_{(3.10)}(z,w) = -|z|^{2\gamma} \tag{A.18}$$

(c) $\exists\, \alpha_3 > 0$ such that $\forall\ (z,w) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$|D_1 V(z,w)| \leq \alpha_3 |z|^{2\gamma-1} \tag{A.19}$$

$$|D_2 V(z,w)| \leq \alpha_3 |z|^{2\gamma-1} \tag{A.20}$$

Now consider the function $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ defined by

$$L(z,w) = V(z,w) + U(w) \tag{A.21}$$

The derivative of L along the trajectory of (3.8) is given by

$$\dot{L}_{(3,8)}(z,w) = D_1V(z,w)\,\frac{1}{\varepsilon}\,\{f(g(\bar{e}_2+w)+z,\bar{e}_2+w) - \varepsilon Dg(\bar{e}_2+w)$$

$$\cdot[hg(\bar{e}_2) - h(g(\bar{e}_2+w)+z)]\}$$

$$+ [D_2V(z,w) + DU(w)][hg(\bar{e}_2) - h(g(\bar{e}_2+w)+z)] \qquad (A.22)$$

By adding and substracting $DU(w)[hg(\bar{e}_2) - hg(\bar{e}_2+w)]$ to (A.22), and by using (A.10), (A.18), (2.5c) and (2.5d), we have

$$\dot{L}_{(3.8)}(z,w) = -\frac{1}{\varepsilon}\,|z|^{2\gamma} - D_1V(z,w)\;Dg(\bar{e}_2+w)\;[hg(\bar{e}_2) - h(g(\bar{e}_2+w)+z)]$$

$$+ D_2V(z,w)[hg(\bar{e}_2) - h(g(\bar{e}_2+w)+z)]$$

$$+ DU(w)[hg(\bar{e}_2) - hg(\bar{e}_2+w) + hg(\bar{e}_2) - h(g(\bar{e}_2+w)+z)$$

$$- hg(\bar{e}_2) + hg(\bar{e}_2+w)]$$

$$\leq -\frac{1}{\varepsilon}\,|z|^{2\gamma} + |D_1V(z,w)|\cdot M\cdot M(|z|+M|w|)$$

$$+ |D_2V(z,w)|\cdot M\cdot(|z|+M|w|) - |w|^{2\gamma} + |DU(w)|\cdot M\;|z| \qquad (A.23)$$

Using (A.11), (A.19) and (A.20) in (A.23), we have

$$\dot{L}_{(3.8)}(z,w) = -\frac{1}{\varepsilon}\,|z|^{2\gamma} + \alpha_3M^2|z|^{2\gamma} + \alpha_3M^3|z|^{2\gamma-1}|w| + \alpha_3M|z|^{2\gamma} + \alpha_3M^2|z|^{2\gamma-1}|w|$$

$$- |w|^{2\gamma} + \alpha_2M|w|^{2\gamma-1}|z| \quad .$$

Hence for some constant β_1, β_2, $\beta_3 > 0$,

$$\dot{L}_{(3.8)}(z,w) \leq (\beta_1 - \tfrac{1}{\varepsilon})|z|^{2\gamma} + \beta_2|z|^{2\gamma-1}|w| + \beta_3|z|\,|w|^{2\gamma-1} - |w|^{2\gamma}$$

$$= [(\beta_1 - \tfrac{1}{2\varepsilon})|z|^{2\gamma} + \beta_2|z|^{2\gamma-1}|w| + \beta_2|z|\,|w|^{2\gamma-1} - \tfrac{1}{2}\,|w|^{2\gamma}]$$

$$- \frac{1}{2\varepsilon}\,|z|^{2\gamma} - \frac{1}{2}\,|w|^{2\gamma} \qquad (A.24)$$

Now, there exists[†] $\varepsilon^* > 0$ s.t. for all $\varepsilon \in (0,\varepsilon^*]$, for all $(z,w) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$(\beta_1 - \frac{1}{2\varepsilon})|z|^{2\gamma} + \beta_2|z|^{2\gamma-1}|w| + \beta_3|z||w|^{2\gamma-1} - \frac{1}{2}|w|^{2\gamma} \leq 0 \tag{A.25}$$

Consequently, for all $\varepsilon \in (0,\varepsilon^*]$,

$$\dot{L}_{(3.8)}(z,w) \leq -\frac{1}{2\varepsilon}|z|^{2\gamma} - \frac{1}{2}|w|^{2\gamma} \tag{A.26}$$

We have shown that (i) for some constant a_1, $a_2 > 0$,

$$a_1(|z|^{2\gamma} + |w|^{2\gamma}) \leq L(z,w) \leq a_2(|z|^{2\gamma} + |w|^{2\gamma}) \tag{A.27}$$

and (ii) $\exists\, 1 > \varepsilon^* > 0$ s.t. for all $\varepsilon \in (0,\varepsilon^*]$

$$\dot{L}_{(3.8)}(z,w) \leq -\frac{1}{2}(|z|^{2\gamma} + |w|^{2\gamma}) \tag{A.28}$$

$$\leq -\frac{1}{2a_2}L(z,w) \tag{A.29}$$

Let $(\tilde{\phi}(\cdot, t_0, z,w), \tilde{\psi}(\cdot, t_0, z,w))$ be the solution of (3.8) corresponding to the initial condition $(z(t_0), w(t_0)) = (z_0,w_0)$. By using (A.27) and (A.29) we obtain that $\forall\varepsilon \in (0,\varepsilon^*]$,

$$|\tilde{\phi}(t,t_0,z_0,w_0)| + |\tilde{\psi}(t,t_0,z_0,w_0)| \leq 2\cdot(a_2/a_1)^{1/2\gamma}(|w_0|+|z_0|)\exp[-(t-t_0)/(4a_2\gamma)]$$

This shows that $\forall\varepsilon \in (0,\varepsilon^*]$, the equilibrium point (θ_n,θ_m) of (3.8) is globally uniformly exp. stable.

[†] Let $y := \frac{|w|}{|z|}$, the left hand side of (A.25) can be written as $(\beta_1 - \frac{1}{2\varepsilon} + \beta_2 y + \beta_3 y^{2\gamma-1} - \frac{1}{2}y^{2\gamma})|z|^{2\gamma}$ and the polynomial $\beta_2 y + \beta_3 y^{2\gamma-1} - \frac{1}{2}y^{2\gamma}$ has a global maximum in $\mathbb{R}_+$.

Appendix B

Proof of Lemma 4.1

(i) From (4.1a), the function $\tilde{f} \in C^1$ as the composition of two C^1 functions. Also $\tilde{f}(\theta_n, \theta_m) = f(\theta_n, \theta_m - Kh(\theta_n)) = \theta_n$. Thus ($\tilde{N}$.1) is satisfied.

(ii) Claim: If $|K| < 1/M^2$, then $\tilde{\mathcal{N}}$ satisfies ($\tilde{N}$.2) and ($\tilde{N}$.3).

From (2.5), we have $\forall v \in \mathbb{R}^m$, $|hg(v)| = |hg(v) - hg(\theta_m)| < M^2|v|^2$ and hence $|(I+Khg)(v)| > |v| - |K|\; M^2|v| = (1-|K|M^2)|v|$. Since $|K|\cdot M^2 < 1$, $(I+Khg)$ is a <u>proper</u> C^1 function. Now by (2.5), $\forall v \in \mathbb{R}^m$,

$$|[I + D(Khg)\,(v)]| > (1 - |K|M^2) \quad , \tag{B.1}$$

hence the matrix $I + D(Khg)(v)$ is nonsingular, $\forall v \in \mathbb{R}^m$. By the <u>global</u> inverse function theorem [Pal. 1, Wu. 1], the map $(I + Khg)^{-1}$ is a C^1 bijection of $\mathbb{R}^m$ onto $\mathbb{R}^m$. Finally, by (N.3), and since $h, g \in C^1$, the map $v \to h\tilde{g}(v) = hg(I + Khg)^{-1}(v)$ is a C^1 bijection of $\mathbb{R}^m$ onto $\mathbb{R}^m$. Thus ($\tilde{N}$.3) is established.

Consider

$$\tilde{f}(\xi,v) = f(\xi,v-Kh(\xi)) = \theta_n \; , \tag{B.2}$$

$g : \mathbb{R}^m \to \mathbb{R}^n$ defined in (N.2) is a C^1 injection by (N.2) and (N.3), hence (B.2) holds if and only if

$$\xi = g(v-Kh(\xi)) \;\Leftrightarrow\; g^{-1}(\xi) = v - Kh(\xi) =: v_1$$

$$\Leftrightarrow\; v - Khg(v_1) = v_1 \;\Leftrightarrow\; (I + Khg)^{-1}(v) = v_1$$

$$\Leftrightarrow\; \xi = g(I + Khg)^{-1}(v) = \tilde{g}(v) \quad . \tag{B.3}$$

†A continuous map is said to be proper iff the inverse image of every compact set is compact.

Note that $\tilde{g}$ is a C^1 map.

(iii) Claim: If $|K| < 1/M^2$, then $\tilde{\mathcal{N}}$ satisfies ($\tilde{N}$.4).

Since $|K| < 1/M^2$, by (ii), $(I+Khg)^{-1}$ is well-defined. Now $\forall\ (v_1,v_0,\xi) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$, we obtain using standard differential calculus [Die. 1, pp. 148-154]:

$$|D_1\tilde{f}(\xi,v_o)| = |D_1 f(\xi,v_o-Kh(\xi)) + D_2 f(\xi,v_o-Kh(\xi)) \cdot (-KDh(\xi))|$$

$$\leq |D_1 f(\xi,v_o-Kh(\xi))| + |D_2 f(\xi,v_o-Kh(\xi))||KDh(\xi)|$$

$$< M + M^2 \cdot |K| \qquad \text{by (2.5a-c)}$$

$$|D_2\tilde{f}(\xi,v_o)| = |D_2 f(\xi,v_o-Kh(\xi)) \cdot I| < M \qquad \text{by (2.5b)}$$

We have shown in (ii) that, with $\phi(\cdot) := (I+Khg)(\cdot)$, $\forall v \in \mathbb{R}^m$,

$$1 - |K|M^2 < |D\phi(v)| < 1 + |K|M^2 \tag{B.4}$$

hence

$$\forall v \in \mathbb{R}^m,\ |D\phi^{-1}(\phi(v))| < 1/(1-|K|M^2) \tag{B.5}$$

i.e.,

$$\forall w \in \mathbb{R}^m,\ |D\phi^{-1}(w)| < 1/(1-|K|M^2)\ , \tag{B.6}$$

since ϕ is a bijection of $\mathbb{R}^m$ onto $\mathbb{R}^m$. Consequently, $\forall v \in \mathbb{R}^m$,

$$|D\tilde{g}(v)| = |D(g \circ \phi^{-1})(v)| \leq |Dg(\phi^{-1}(v))| \cdot |D\phi^{-1}(v)| < M/(1-|K|M^2) \tag{B.7}$$

(iv) Claim: If $|K|$ is small enough, then $\tilde{\mathcal{N}}$ satisfies ($\tilde{N}$.5).

Consider the system (4.2) with $v \in \mathbb{R}^m$, let $v_1 := (I+Khg)^{-1}v$. Define $\tilde{x} := x - g(v_1) = x - g(I+Khg)^{-1}(v)$, and write Eq. (4.2) as

$$\dot{\tilde{x}} = f(g(v_1) + \tilde{x},\ v - Kh(g(v_1) + \tilde{x}))$$

$$= f[g(v_1) + \tilde{x},\ (I + Khg)v_1 - Kh(g(v_1) + \tilde{x})]$$

$$\dot{\tilde{x}} = f(g(v_1) + \tilde{x},\ v_1) + \Delta f(\tilde{x}, v_1) \tag{B.8}$$

where $\Delta f(\tilde{x},v_1) := f[g(v_1) + \tilde{x},\ (I + Khg)v_1 - Kh(g(v_1) + \tilde{x})] - f(g(v_1) + \tilde{x}, v_1)$

By (2.5c),

$$|\Delta f(\tilde{x},v_1)| \leq M|K||hg(v_1) - h(g(v_1) + \tilde{x})|$$

$$\leq M^2|K||\tilde{x}| \quad . \tag{B.9}$$

We know by (N.5) that $\forall v_1 \in \mathbb{R}^m$ the equilibrium point $\tilde{x} \equiv \theta_n$ of the system

$$\dot{\tilde{x}} = f(g(v_1) + \tilde{x},\ v_1) \tag{B.10}$$

is globally uniformly exp. stable uniform in the constant input v_1, i.e., $\exists\, c > 0,\ \alpha > 0$ s.t. $\forall \tilde{x}(0) \in \mathbb{R}^n$, $\forall v_1 \in \mathbb{R}^m$, $\forall t \geq 0$

$$|\tilde{x}(t)| \leq c|\tilde{x}(0)|\, e^{-\alpha t}$$

By a converse stability theorem [Hah. 1, p. 273], $\exists$ Lyapunov function $\tilde{V} : \mathbb{R}^n \to \mathbb{R}_+$ satisfying

1) $\exists\, \gamma_2 > \gamma_1 > 0$ s.t. $\forall \tilde{x} \in \mathbb{R}^n$, $\gamma_1 |\tilde{x}|^2 \leq \tilde{V}(\tilde{x}) \leq \gamma_2|\tilde{x}|^2$; (B.11a)

2) $\dot{\tilde{V}}_{(B.10)}(\tilde{x}) \leq -\frac{1}{2}|\tilde{x}|^2$, $\forall \tilde{x} \in \mathbb{R}^n$; (B.11b)

3) $\exists\, \lambda > 0$ s.t. $\forall \tilde{x} \in \mathbb{R}^n$, $|D\tilde{V}(x)| \leq \lambda|\tilde{x}|$. (B.11c)

Now for the system (B.8), use the same Lyapunov function $\tilde{V}$, then

$$\dot{\tilde{V}}_{(B.8)}(\tilde{x}) = [D\tilde{V}(\tilde{x})][f(g(v_1)+\tilde{x}, v_1) + \Delta f(\tilde{x}, v_1)]$$

$$\leq -\frac{1}{2}|\tilde{x}|^2 + |D\tilde{V}(\tilde{x})| \cdot |\Delta f(\tilde{x}, v_1)| \qquad \text{by (B.11b)}$$

$$\leq -\frac{1}{2}|\tilde{x}|^2 + \lambda \cdot |K| \cdot M^2 |\tilde{x}|^2 \qquad \text{by (B.9) and (B.11c)}$$

$$= (\lambda \cdot |K| \cdot M^2 - \tfrac{1}{2})|\tilde{x}|^2$$

So if $|K| < \frac{1}{2\lambda M^2}$, then $\dot{\tilde{V}}_{(B.8)}(\tilde{x}) \leq -c_1|\tilde{x}|^2$ for some $c_1 > 0$. Consequently, if $|K| < \frac{1}{2\lambda M^2}$, the equilibrium point $\tilde{x} = \theta_n$ of (4.2) is globally uniformly exp. stable [Kra. 1, p. 60], thus $\tilde{\mathcal{N}}$ satisfies ($\tilde{N}$.5).

(v) Claim: If K is positive semidefinite and $|K| < \frac{1}{M^2}$ then $\exists\, \tilde{\beta} > 0$ s.t. $\forall v, v_0 \in \mathbb{R}^m$, $v^T D(h\tilde{g})(v_0)v \geq \tilde{\beta}|v|^2$.

Let $A(v_1) := D(hg)(v_1)$, then for $v_1 := (I + Khg)^{-1}v_0$,

$$D(h\tilde{g})(v_0) = A(v_1)(I + KA(v_1))^{-1} = (A(v_1)^{-1} + K)^{-1}. \qquad \text{(B.12)}$$

Also $hg : \mathbb{R}^m \to \mathbb{R}^m$ is a C^1 bijection which by (2.7) and (2.5) satisfies $\forall v_1 \in \mathbb{R}^m$,

$$M^2 > |D(hg)(v_1)| = |A(v_1)| \geq \beta > 0 \quad \text{and} \quad \underline{\sigma}[A(v_1)] \geq \beta \qquad \text{(B.13)}$$

Let $v_3 := A(v_1)^{-1}v$, then

$$v^T A(v_1)^{-1} v = v_3^T A(v_1)^T v_3 = v_3^T A(v_1) v_3 \geq \beta |v_3|^2 \qquad \text{(B.14)}$$

Now $|v_3| = |A(v_1)^{-1}v| \geq \underline{\sigma}[A(v_1)^{-1}] \cdot |v|$

$$= |v| \cdot 1/\bar{\sigma}[A(v_1)] > |v|/M^2 \qquad \text{by (2.5)}$$

So $$v^T A(v_1)^{-1} v > (\beta/M^4)|v|^2 \qquad \text{(B.15)}$$

Then $\forall v, v_0 \in \mathbb{R}^m$ with v_1 defined above,

$$
\begin{aligned}
v^T D(h\tilde{g})(v_o)v &= v^T(A(v_1)^{-1} + K)^{-1}v && \text{by (B.12)} \\
&= v_2(A(v_1)^{-1} + K)^T v_2 && \text{with } v_2 := (A(v_1)^{-1} + K)^{-1}v \\
&\geq v_2^T A(v_1)^{-1} v_2 && \text{since K is positive semidefine} \\
&> (\beta/M^4)|v_2|^2 && \text{by (B.15)} \\
&> \frac{\beta}{M^4}\left(\frac{1}{\beta^{-1}+M^{-2}}\right)^2 |v|^2 = \frac{\beta^3}{(M^2+\beta)^2}|v|^2 && \text{by (B.13)}
\end{aligned}
$$

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References

[Bar. 1] J. F. Barman, "Well-posedness of feedback systems and singular perturbations," Ph.D. Thesis, University of California, Berkeley, 1973.

[Cal. 1] F. M. Callier and C. A. Desoer, Multivariable Feedback Systems, Springer-Verlag: New York-Heidelbergy-Berlin, 1982.

[Cal. 2] F. M. Callier and C. A. Desoer, "Stabilization, tracking and disturbance rejection in multivariable convolution systems," Annales de la Société Scientifique de Bruxelles, T. 94. I, 1980, pp. 7-51.

[Cho. 1] J. H. Chow, "Asymptotic stability of a class of nonlinear singularly perturbed systems," J. Franklin Inst., vol. 305, no.5, 1978, pp. 275-281.

[Dav. 1] E. J. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," IEEE Trans. Automat. Contr., vol. AC-21, Feb. 1976, pp. 25-34.

[Des. 1] C. A. Desoer and Y. T. Wang, "Linear time-invariant robust servomechanism problem: a self contained exposition," in Advances in Control and Dynamical Systems, vol. 16, C. T. Leondes (Ed.), Academic Press, New York, 1980, pp. 81-129.

[Des. 2] C. A. Desoer and Y. T. Wang, "The robust nonlinear servomechanism problem," Int. J. Control, vol. 20, no. 5 1979, pp. 803-828.

[Die. 1] J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1969.

[Hab. 1] P. Habets, "Stabilite asymptotique pour des problèmes de perturbations singulières," Centro Internazionale Matematico Estivo, Edizioni Cremonese Roma, 1974.

[Hah. 1] W. Hahn, Stability of Motion, Springer-Verglag, New York, 1967.

[Kok. 1] P. V. Kokotovic, R. E. O'Malley Jr, and P. Sannuti, "Singular perturbation and order reduction in control theory-an overview," Automatica, vol. 12, pp. 123-132.

[Kra. 1] N. N. Krasovskii, Stability of Motion, Stanford University Press, Stanford, California, 1963.

[Mey. 1] G. Meyer, "The design of exact nonlinear model follower," Proceeding of the JACC, FA-3A, 1980.

[Ort. 1] J. M. Ortega and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, 1970.

[Mor. 1] M. Morari, "Robust stability of systems with integral control," Proc. 22nd IEEE Conf. on Decision and Control, San Antonio TX, Dec. 14-16, 1983, pp. 865-869.

[Pal. 1] R. S. Palais, "Natural operations on differential forms," Trans. Amer. Math. Soc., vol. 92, 1959, pp. 125-141.

[Pec. 1] J. L. Peczkowski and S. A. Stopher, "Nonlinear Multivariable synetheses with transfer functions," Proc. 1980 Joint American Control Conference, WA8-D.

[Sai. 1] M. K. Sain and J. L. Peczkowski, "Nonlinear multivariable design by total synthesis," Proc. 1982 Joint American Control Conference, pp. 252-260.

[Sol. 1] A. Solomon and E. J. Davison, "Design of controllers to solve the robust servomechanism problem for a class of nonlinear systems," Proc. 22nd IEEE Conf. on Decision and Control, San Antonio, TX, Dec. 14-16, 1983, pp. 335-341.

[Som. 1] R. Sommer, "Control design for multivariable nonlinear time-varying systems," Int. J. Control, vol. 31-5, 1980, pp. 883-891.

[Tih. 1] A. N. Tihonov, A. B. Vasileva, and V. M. Volosov, Chap. IV, §5 in Mathematics Applied to Physics, Springer-Verlag, New York and UNESCO, Paris, 1970.

[Vid. 1] M. Vidyasagar and A. Vannelli, "New relationship between input-output stability and Lyapunov stability," IEEE Trans. on Automat. Contr. vol. AC-27, no. 2, April 1982, pp. 481-483.

[Wu. 1] F. F. Wu and C. A. Desoer, "Global inverse function theorem," IEEE Trans. CAS, March 1972, pp. 199-201.

Figure Captions

Fig. 2.1. The system ${}^1S(\mathcal{N}, \frac{\varepsilon}{s} I + K)$, where $\mathcal{N}$ is described by (2.1).

Fig. 4.1. The nonlinear plant $\tilde{\mathcal{N}}$.

Fig. 4.2. The system ${}^1S(\tilde{\mathcal{N}}, \frac{\varepsilon}{s} I)$.

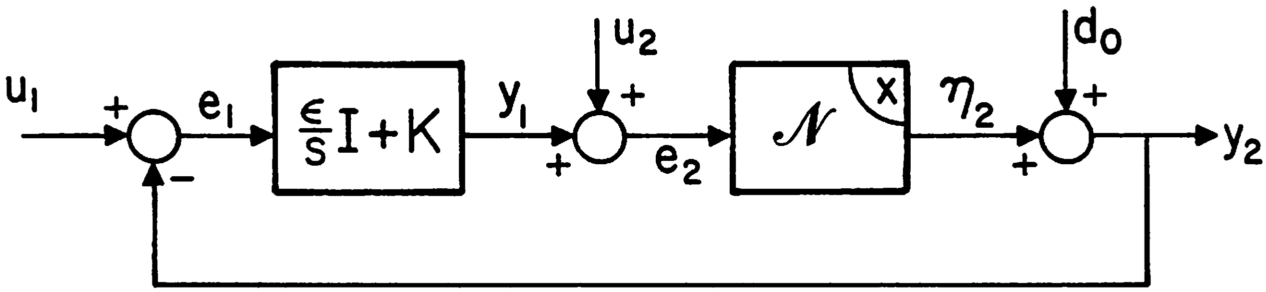

Fig. 2.1

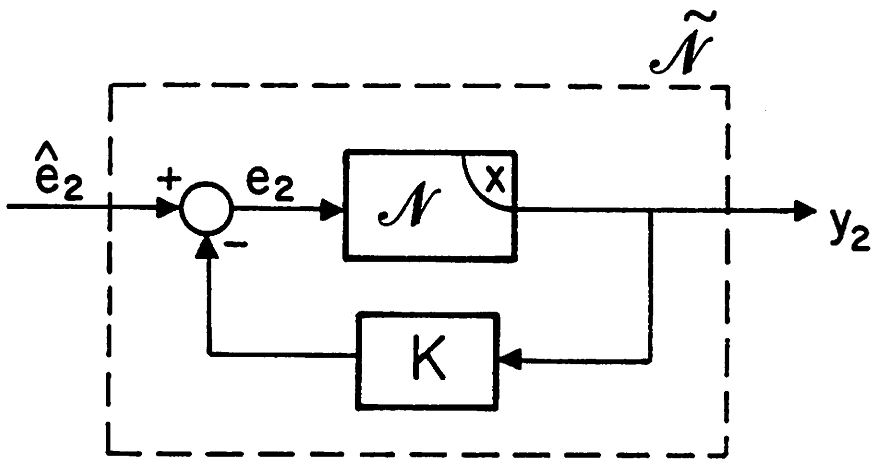

Fig. 4.1

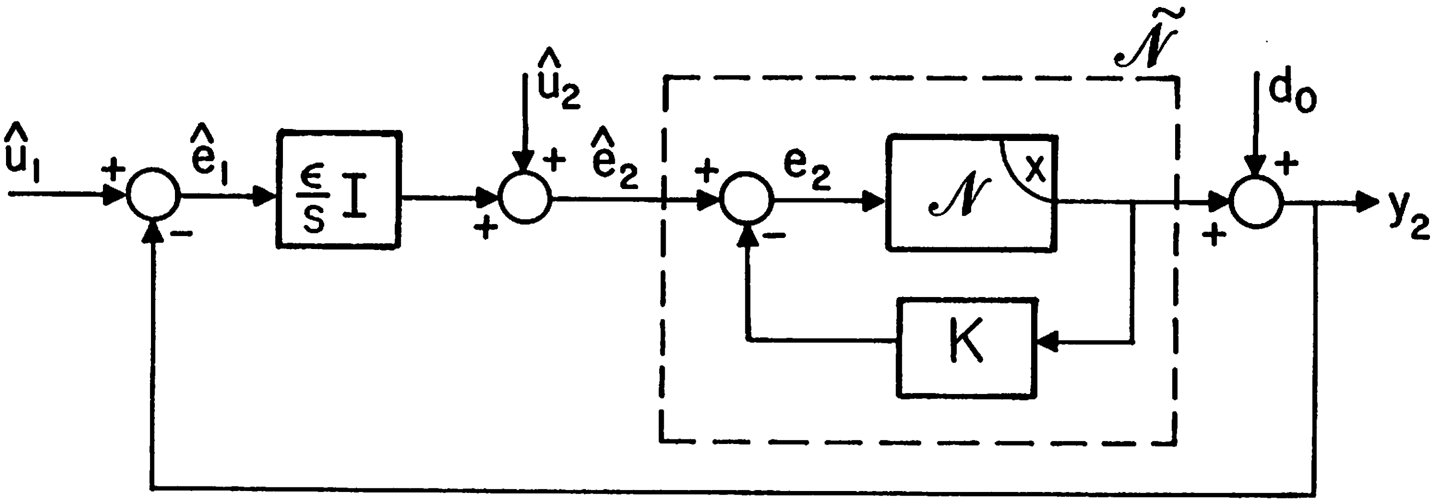

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Fig. 4.2