VLASOV-POISSON AND MODIFIED KORTEWEG-DE VRIES THEORY AND SIMULATION OF WEAK AND STRONG DOUBLE LAYERS

by

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ABSTRACT

A general graphical method of solving the Vlasov-Poisson system associated with a set of nonlinear eigenvalue conditions is presented.

Analytic evidence for the existence of small amplitude electron and ion acoustic monotonic double layers is presented. These are the nonlinear extensions of the slow electron acoustic wave and the slow ion acoustic wave, respectively: one related to the electron solitary hole, the other related to the ion acoustic solitary hole, both having negative trapping parameters. A modified K-dV equation for a monotonic double layer, showing a relationship among double layer amplitude, its propagation speed and its spatial scale length, is also derived.

We present a general analytic formulation for nonmonotonic double layers and illustrate with some particular solutions. This class of double layers satisfies the time stationary Vlasov-Poisson system while requiring a Sagdeev potential which is a double valued function of the physical potential: it follows that any distribution function having a density representation as any integer or noninteger power series of the physical potential can never satisfy the nonmonotonic double layer boundary conditions. A K-dV like equation is found showing a relationship among the speed of the nonmonotonic double layer, its spatial scale length, and its degree of asymmetry.

Particle simulations of ion acoustic double layers have been successful in short systems ($L = 80\lambda_e$) and with low drift velocities ($v_d = 0.45\, v_{th}$ for the electrons). We present simulation results for systems driven by constant current and by constant applied voltage. By using the analytic formulation, we find that there is a "critical" electron drift velocity (which is considerably smaller than the value reported by previous papers but very close to the value of our simulations) for the existence of ion acoustic double layers. We find that for a given electron drift velocity (exceeding the "critical" drift) there is a corresponding maximum amplitude for the ion acoustic double layer. We show that the net potential jump across the ion acoustic double layer is determined by the temperature difference between the two plasmas. It is also shown that the usual Bohm condition is *not* satisfied for ion acoustic double layers with finite amplitude: the velocity of an ion acoustic double layer decreases (below C_s) as its amplitude increases.

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1. INTRODUCTION

In recent years, there have been considerable research interest in understanding local electrostatic potential formations in plasmas[1-33]. Besides theoretical and experimental interests, there are two practically important applications: one is the recently developed concept of plasma confinement using electrostatic potential structures; the other is that some of these potential structures are considered to be responsible for the acceleration of particles in a variety of plasmas.

There are two frequently used methods for solving a Vlasov-Poisson system describing an electrostatic potential structure. There is first the well known "BGK method", which prescribes both an exact potential structure form $\phi(x)$ and all the distribution functions except one (e.g., one of the trapped particle populations) which it must then solve for self-consistently[32]. It turns out that the BGK method applied e.g., to monotonic double layers, can in fact yield negative (nonphysical) distribution functions[14,15,33]. Therefore, in the second chapter we outline and generalize an alternative "graphical method" or "reduced potential approach" for solving the Vlasov-Poisson system.

There are various configurations of interesting potential structures. Here we describe some of the potential structures of recent interest. A *monotonic* potential double layer is ideally an isolated pair of oppositely charged sheets which results in a narrow region of abrupt potential jump of some amplitude $\Delta\phi = \psi$; well outside of this localized jump, the potential is effectively uniform[1-17]. Even though double layer studies often are restricted to such simple (i.e., monotonic) potential structures in plasmas, the double layer concept is more accurately a generic concern about the rules governing *allowable transitions* between regions of two (or more) different collisionless plasmas. Some recent numerical calculations[14,15] suggested that there may be a low amplitude limit for the monotonic double layer, arguing that the existence of a weak double layer requires a trapped-particle distribution that is nearly a δ function and therefore is subject to strong instabilities.

In the third chapter, we present two different kinds of weak *monotonic* double layer analytic solutions[17], *i.e.* which do have small amplitude. These solutions are the analytic extensions of the electron solitary hole and ion acoustic solitary hole[15-17,29,30], both having negative trapping parameters; these are the nonlinear extensions of the slow electron acoustic wave and the slow ion acoustic wave, respectively.

Often in experiments and in simulations the *observed* double layer exhibits a potential spatial-profile having a potential depression on the low side (or conversely a potential bump on the high side), as shown in Figure 1(a). Such a *non−monotonic* double layer (NDL) is actually a localized region of *three* sheets of alternating charge sign, and thus includes subregions of oppositely directed non-monotonic electric fields[15-16,18-28].

It is increasingly clear that even the straightforward NDL structure can evidence complex nonlinear characteristics, as exhibited many ways in both simulations and experiments. Reports of several recent simulations[16,18-23] indicate that an *ion acoustic* double layer can be formed by reflection of electrons off the negative potential depression; its simulated potential profile has an NDL form as in Figure 1(a). Recent satellite measurements[24] of field aligned potentials in the auroral region, show signatures that are especially consistent with the NDL, having a characteristic potential depression at the low potential side (or a bump on the high potential side). It has been further suggested[24] that a series of such small amplitude non-monotonic double layers might account for a large portion of the total potential drop along auroral field lines, and might also explain the fine structure of auroral kilometric radiation. The recent thermal barrier cell concept for tandem mirror devices is based on the generation of an abrupt potential depressions by means of forced changes in the particle distribution functions[13,25]. Recent experiments with Q-machine plasmas[26] also reported the formation of a potential depression between two plasmas with different electron temperatures; the "non-monotonic" negative potential depression is thought to play a crucial role in the formation of double layers, accounting for both the observed current disruptions (by reflecting the electrons) and also for the high frequency noise excitation seen behind the double layer (caused by a two stream instability involving electrons that pass the negative potential peak[15,28]). A recent triple plasma experiment reported that the formation of an ion acoustic type double layer was observed in the laboratory for the first time[27].

Although there have been many theoretical, numerical and experimental investigations of double layers, recent theoretical work has been devoted to *numerical* evaluations of the Vlasov-Poisson system (or of the fluid system) mainly because of the highly nonlinear properties of double layers[13-16,21,22,28]. In order to explain *non*monotonic double layers, theoretical efforts have attempted to generalize ion hole, ion acoustic soliton or monotonic double layer descriptions[13,15,16,21,22]. It should be noted that to our knowledge there exists only one theory offering a *numerical* solution for a nonmonotonic potential structure obtained from a Vlasov-Poisson system[28]; However, it should be pointed out that the distribution function used in this work was not self consistent with the Vlasov equation.

In the fourth chapter, we present a general non-monotonic double layer formulation and self-consistent *analytic* solutions for *non*monotonic double layers which satisfy a time stationary Vlasov-Poisson system. We further derive a K-dV like equation which describes a *moving* NDL structure related to the ion acoustic wave. Expressions are found relating the NDL two potential amplitudes ψ and ψ_1, the spatial scaling parameter (the NDL structure width), and the NDL speed. In the final chapter, we describe our numerical simulation results of ion acoustic double layers and compare these with our theoretical results for finite amplitude ion acoustic double layers , which were obtained from our theoretical formulation.

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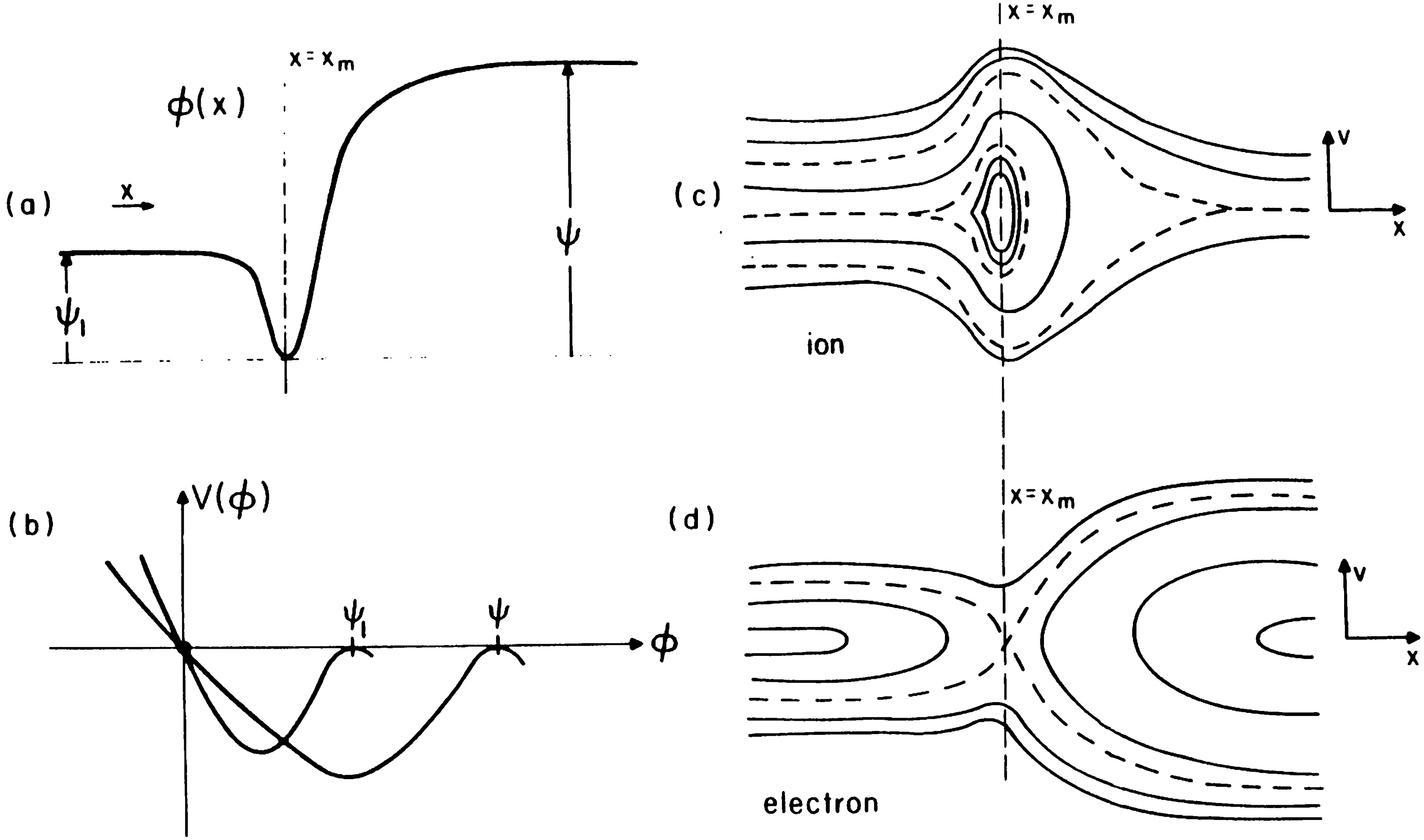

FIG. 1. (a) NDL with potential depression at the low potential side. (b) Sagdeev potential for the above NDL. (c) Ion phase space plot. (d) Electron phase space plot.

2. A General Graphical Method for Solving a Vlasov-Poisson System

To describe propagation of an electrostatic potential structure in a Vlasov-Poisson system. we shift to a frame that has been Galilean-transformed to the wave frame (where the wave is time stationary). The electron and ion Vlasov distribution functions each consist of two components: some particles are energetic enough that they stream freely through the potential structure, while the rest reflect off it. In this frame, we can express the time stationary solution to the Vlasov equation (i.e., the particle distribution functions) as *any* function of the particle constants of motion; usually these are recognized to include (i) the particle total energy and (ii) the sign of the velocity of the *free streaming* (also called untrapped) particles. However, besides these usual constants of motion, it is important to note that a third constant of motion exists for the *reflected* (also called trapped) particles, namely $sgn\ (x - x_m)$ where x_m represents the position of potential minimum (or maximum) for the negatively charged particles (or the positively charged particles). It turns out that this final constant of motion plays an important role in constructing non-monotonic double layers.

There is first the well known "BGK method" for solving the Vlasov-Poisson system. which prescribes both an exact potential structure form $\phi(x)$ and all the distribution functions except one (e.g., one of the trapped particle populations) which it must solve for self—consistently[1]. It turns out that the BGK method applied e.g., to monotonic double layers, can in fact yield negative (nonphysical) distribution functions[2,3].

Therefore we present here an alternative "graphical method" or "reduced potential approach" for solving the Vlasov-Poisson system. Using electron (f_e) and ion distribution functions (f_i) which satisfy the Vlasov equation, the Poisson equation for $\phi(x)$ may be written by introducing a Sagdeev (or reduced) potential $V(\phi)$ as follows:

$$\phi''(x) \equiv \partial^2\phi/\partial x^2 \equiv \phi_{xx}$$
$$= n_e - n_i \equiv \int_{-\infty}^{+\infty} f_e\, dv - \int_{-\infty}^{+\infty} f_i\, dv \tag{1}$$
$$= -\frac{dV(\phi)}{d\phi} \equiv -V'(\phi)$$

Clearly, the electric field amplitude is proportional to the square root of the magnitude of the Sagdeev potential. It should be noted that this approach has already been used successfully to describe some relatively simple potential structures such as ion holes, electron holes, solitons, and monotonic double layers[3-10].

To outline and generalize this reduced-potential approach, we present below a set of simple rules which, with Fig. 1, allows us to construct the corresponding Sagdeev potential $V(\phi)$ for any arbitrary potential form $\phi(x)$. From the six basic graphs of Fig. 1, one can derive a set of solution constraints (or "boundary conditions" or "nonlinear eigenvalue equations" [4] or "nonlinear dispersion relations" [6]). Note that the "reference potential" for $\phi(x)$ is always ψ. Rules (a) through (d) describe eight possible potential configurations as illustrated in Fig.1(a) through Fig.1(d).

(a) This graph represents any physical potential configuration in which the potential changes *curvature* from *positive* value to *negative* value. The corresponding Sagdeev potential should have a local *minimum* with *negative* value; from the plot, the corresponding eigenvalue conditions are seen to be given by $V(\psi) < 0$, $V'(\psi) = 0$ and $V''(\psi) > 0$.

(b) This graph represents any physical potential configuration in which the potential changes *curvature* from *negative* to *positive*. The corresponding Sagdeev potential should have a local *maximum* with *negative* value; from the plot, the corresponding eigenvalue conditons are seen to be given by $V(\psi) < 0$, $V'(\psi) = 0$ and $V''(\psi) < 0$.

(c) This graph represents any physical potential configuration in which the potential approaches asymptotically to some value ψ at infinity with *positive curvature*. The corresponding Sagdeev potential should have local *maximum* with zero value at $\phi = \psi$. from the plot, the corresponding eigenvalue conditions are seen to be given by $V(\psi) = 0$, $V'(\psi) = 0$, and $V''(\psi) < 0$.

(d) This graph represents any physical potential configuration in which the potential approaches asymptotically to some value ψ at infinity with *negative curvature*. The corresponding Sagdeev potential should have local *maximum* with zero value at $\phi = \psi$; from the plot, the corresponding eigenvalue conditions are seen to be given by $V(\psi) = 0$, $V'(\psi) = 0$ and $V''(\psi) < 0$.

(e) This graph represents any physical potential configuration in which the potential $\phi(x)$ has a local *maximum* having *negative curvature* at some position. The corresponding Sagdeev potential should cross ϕ axis with *positive* slope; from the plot, the corresponding eigenvalue equations are seen to be given by $V(\psi) = 0$ and $V'(\psi) > 0$.

(f) This graph represents any physical potential configuration in which the potential $\phi(x)$ has a local *minimum* having *positive curvature* at some position. The corresponding Sagdeev potential should cross ϕ axis with *negative* slope; from the plot, the corresponding eigenvalue equations are seen to be given by $V(\psi) = 0$ and $V'(\psi) < 0$.

The boundary conditions $V(\psi) = 0$ and $V'(\psi) = 0$ in these cases enforce zero electric field and charge neutrality at $\phi = \psi$. Besides these rules, it is important to note that the Sagdeev potential is in general *multiple−valued* function of physical potential when the magnitude of the electric fields for some fixed value of physical potential are multiple-valued; the multiplicity of the Sagdeev potential is equal to the multiplicity of the magnitudes of the electric fields For example in Fig. 2, in an NDL "staircase" there is a double valued section of Sagdeev potential for $0 < \phi \leqslant \psi_1$; and there is a triple valued Sagdeev potential for $\psi_2 < \phi < \psi_3$. Or, in a second example in Fig. 3 which resembles a symmetric thermal barrier potential, the corresponding Sagdeev potential is double valued for $\psi_m \leqslant \phi < \psi$. Or finally, for an asymmetric solitary wave in Fig. 4, the corresponding Sagdeev potential is *double* valued over the entire range $0 < \phi < \psi$.

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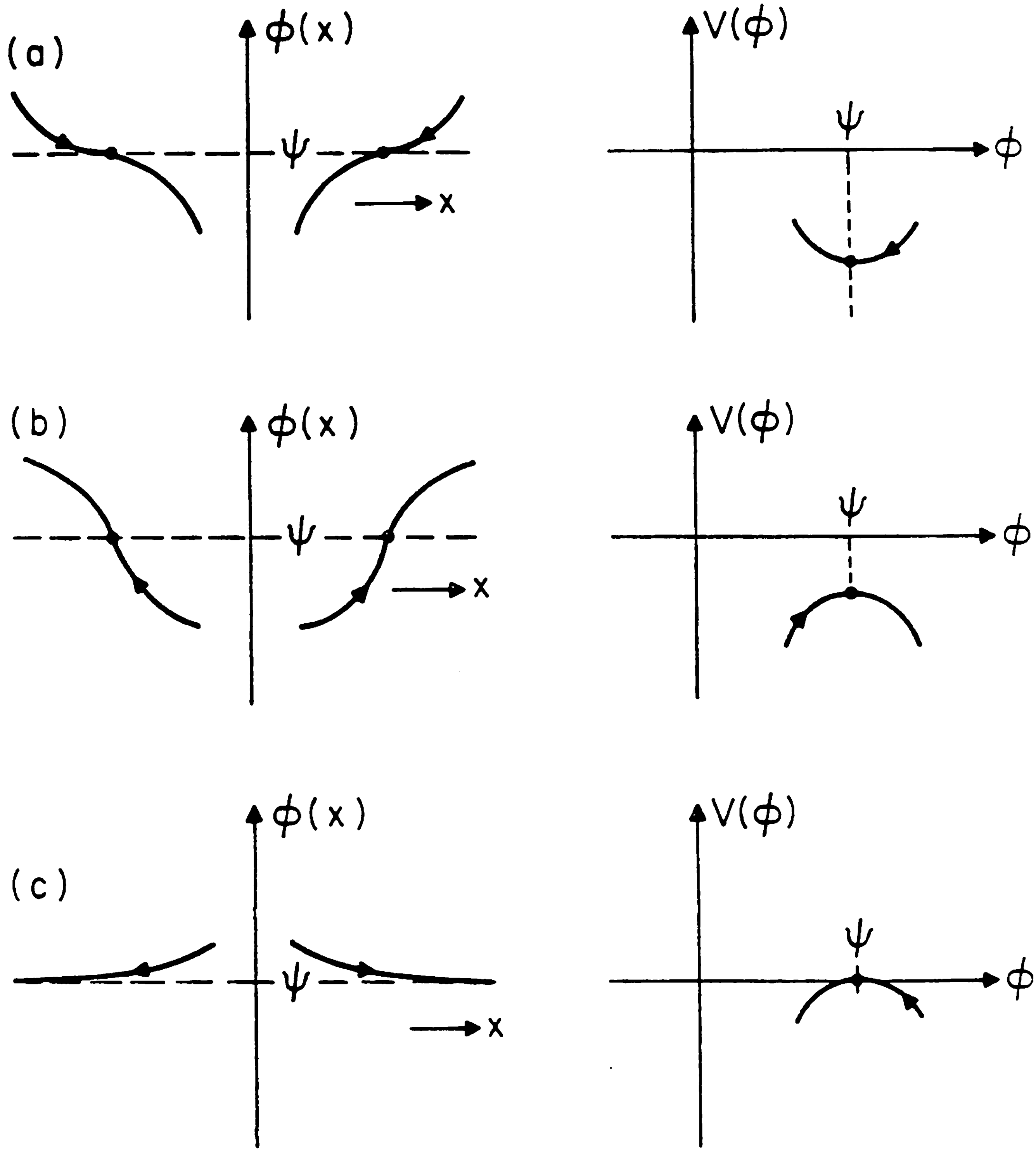

FIG. 1. Potentials $\phi(x)$ and $V(\phi)$.

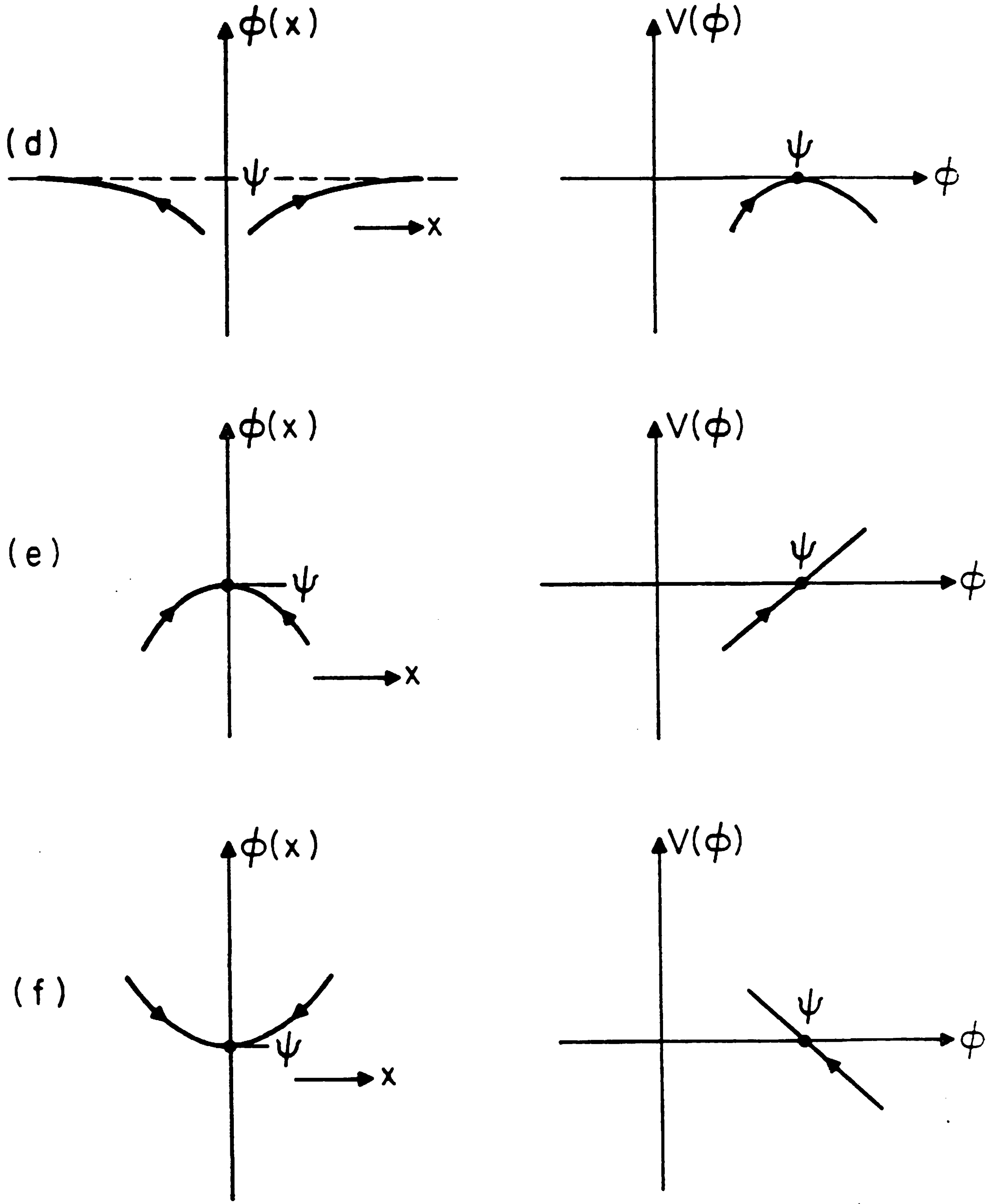

FIG. 1. Potentials $\phi(x)$ and $V(\phi)$.

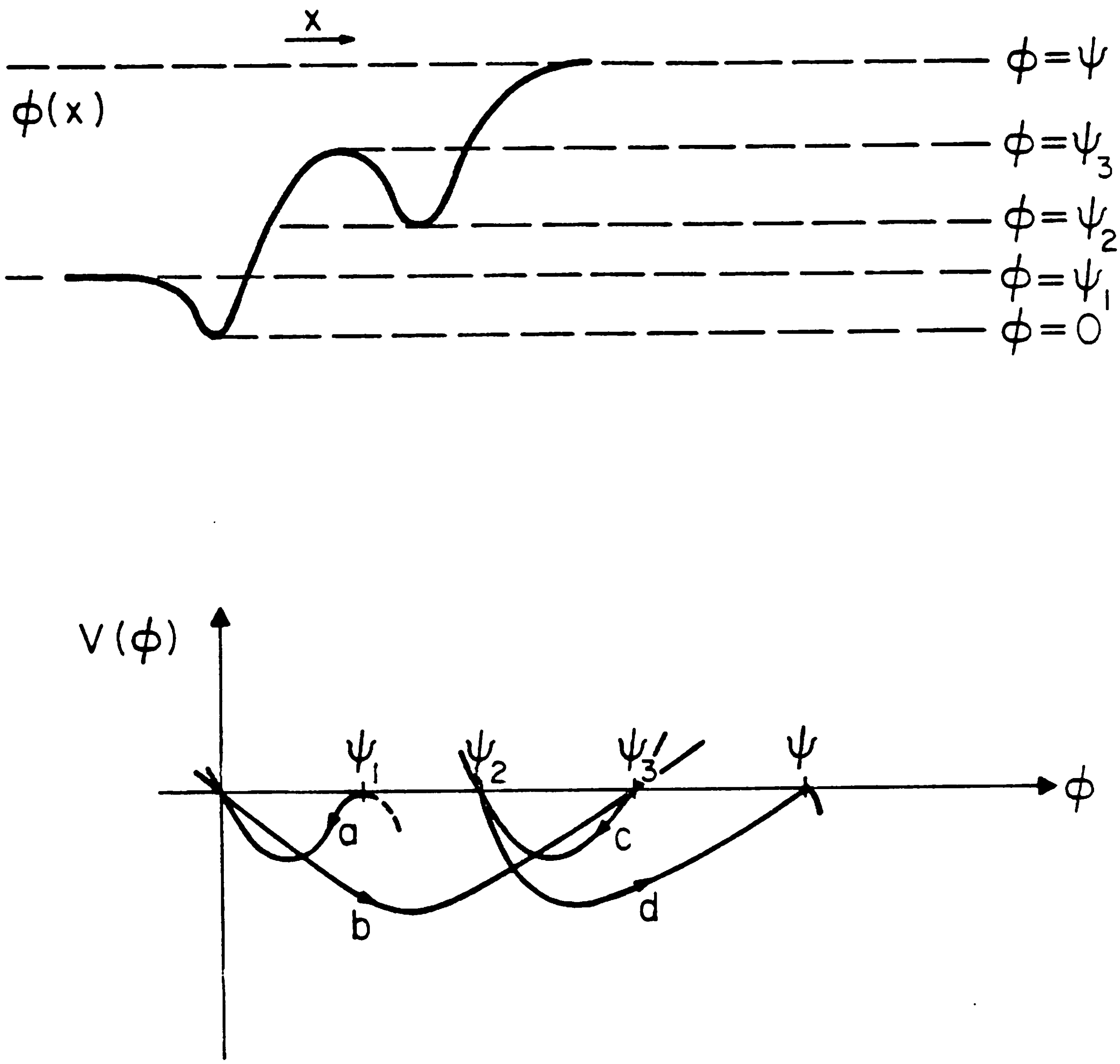

FIG. 2. A nonmonotonic double layer "staircase" $\phi(x)$ and the associated Sagdeev potential $V(\phi)$. Moving along $\phi(x)$ from left to right maps into moving along $V(\phi)$ from ($\phi=\psi_1$, $V=0$) to the origin, then to (ψ_3, 0) etc ... , as shown by direction arrows *a*, *b*, *c*, *d*.

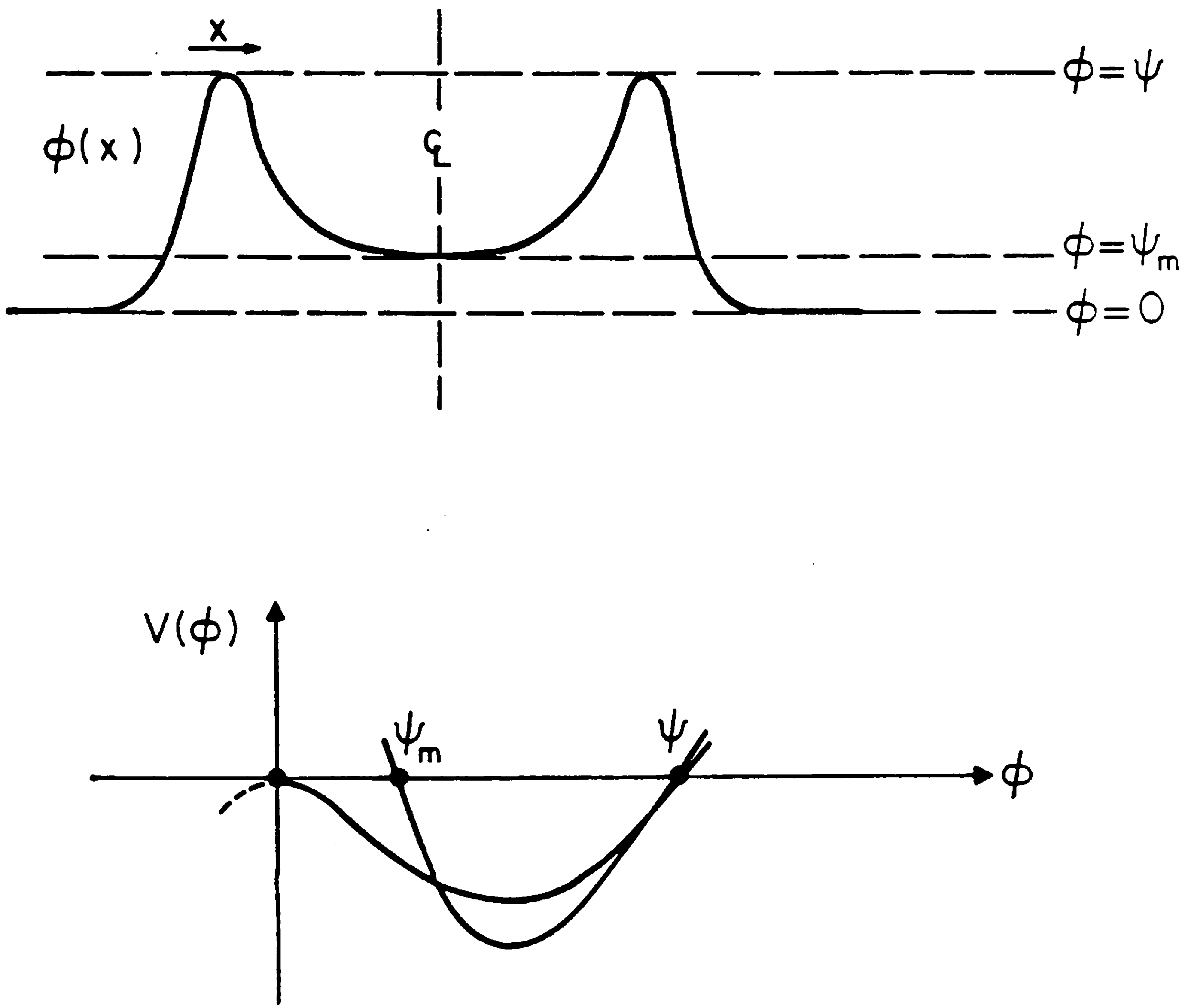

FIG. 3. Schematic symmetric "thermal barrier" $\phi(x)$, and the associated Sagdeev potential $V(\phi)$.

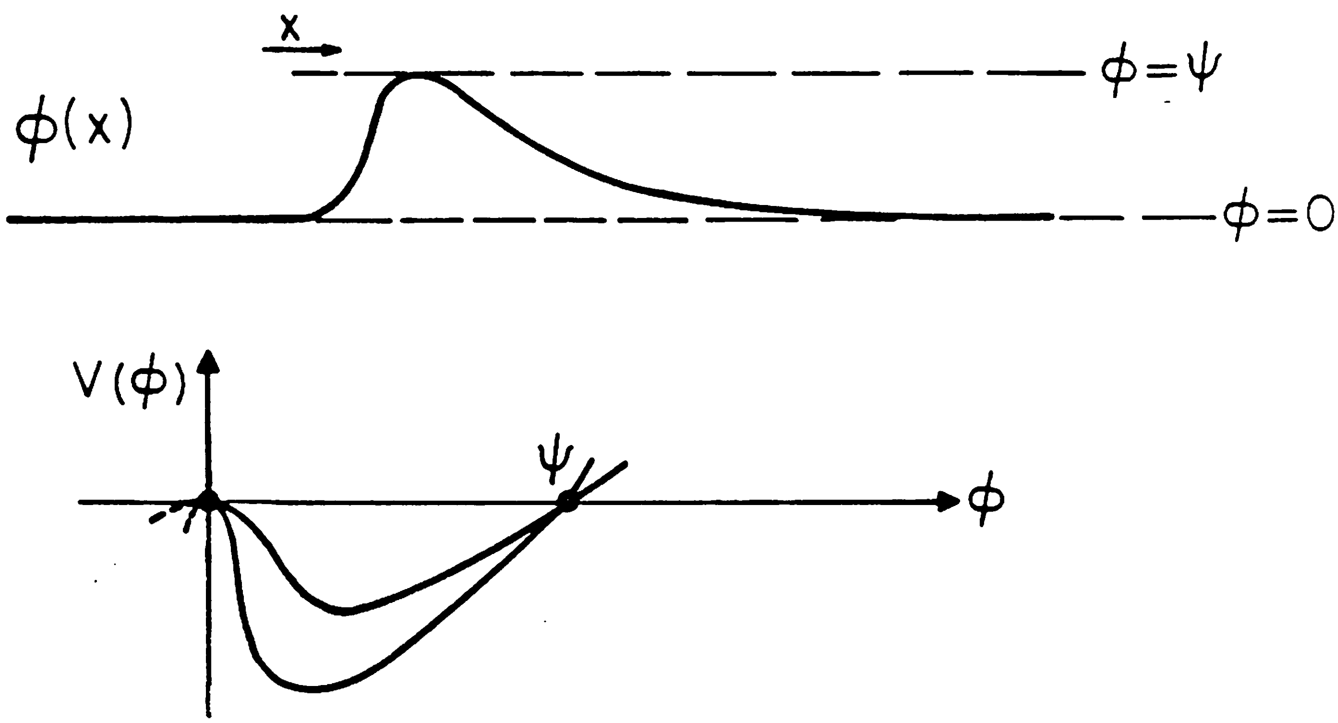

FIG. 4. Asymmetric solitary wave $\phi(x)$ and the associated Sagdeev potential $V(\phi)$.

3. Weak Monotonic Double Layers

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ABSTRACT

Analytic evidence for the existence of small amplitude electron and ion acoustic monotonic double layers is presented. These are the nonlinear extensions of the slow electron acoustic wave and the slow ion acoustic wave, respectively: one related to the electron solitary hole, the other related to the ion acoustic solitary hole, both having negative trapping parameters. A modified K-dV equation for monotonic double layer, showing a relationship among propagation velocity and spatial scale length, is also derived.

1. Introduction

A monotonic double layer is a narrow, isolated region of abrupt potential jump of amplitude ψ , due to a localized dipole-sheet of space charge surrounded by large regions of effectively uniform potential. Although there have been many theoretical and experimental investigations of holes and double layers[1-21], recent theoretical work has been limited to numerical evaluations of the Vlasov-Poisson system (or the fluid equation) mainly because of the highly nonlinear properties of double layers. Recent numerical investigations[10,22] suggested that there may be a low amplitude limit for the monotonic double layer, arguing that the existence of a weak monotonic double layer requires a trapped-particle distribution that is nearly a δ function and therefore is subject to strong instabilities.

In this chapter, we present two different kinds of weak *monotonic* double layer analytic solutions, which do have small amplitude. These monotonic double layer solutions are the analytic extensions to the electron solitary hole and ion acoustic solitary hole which are the nonlinear extensions of the slow electron acoustic wave and the slow ion acoustic wave, respectively[20,21].

To describe propagation of monotonic double layers, we use a Vlasov-Poisson system that has been Galilean transformed to the wave frame (where the wave is time stationary). In this frame, we can express the time stationary solution of the Vlasov equation as any function of the constants of motion: (i) particle total energy and (ii) the sign of the velocity of the untrapped particles[23,24]. Here it is not necessary for us to use third constant of motion, because a monotonic double layer does not require double-valued Sagdeev potential as a function of the physical potential.

2. Weak electron monotonic double layer

In order to describe the monotonic double layer related to the electron solitary hole, we look for a stationary solution in the ion reference frame and take the ion distribution function to be Maxwell-Boltzmann:

$$f_i = \frac{\sqrt{\tau}}{\sqrt{2\pi}} e^{-\frac{\tau}{2}(v^2 - 2\phi)} \qquad (1)$$

We consider the following electron distribution function which is continuous at the separatrix[24]:

$$f_e = (2\pi)^{-1/2} \Big\{ \exp\{-1/2(sgn(v)\, \epsilon^{1/2} - v_d)^2\}\, \Theta(\epsilon) + \exp\{-1/2(v_d^2 + \beta\epsilon)\}\, \Theta(-\epsilon) \Big\} \qquad (2)$$

where $\tau = T_e/T_i$ and $\epsilon = v^2 - 2\phi$ for $0 \leqslant \phi \leqslant \psi$. Here the electron velocity, ion velocity, the wave potential and the spatial coordinates are normalized to the electron thermal velocity $(T_e/m_e)^{1/2}$, ion acoustic velocity $(T_e/m_i)^{1/2}$, the electron temperature T_e/e and the electron Debye length $\lambda_e = (T_e/4\pi n_0 e^2)^{1/2}$, respectively; v_d represents the electrons drift velocity. The electron distribution function at $\phi = 0$ models a drifting Maxwellian. Here Θ represents the Heaviside step function and β called the trapping parameter can be positive and negative depending on the structure of trapped electron phase space. We will show that β should be negative for the existence of small amplitude monotonic double layer.

Again, because the above electron distribution function is expressed entirely in terms of the constants of the motion, it clearly satisfies the time stationary Vlasov equation. Thus Poisson equation may be written by introducing Sagdeev potential($V(\phi)$) as follows:

$$\phi_{xx} = -\frac{dV(\phi)}{d\phi}$$
$$= e^{-\frac{v_d^2}{2}} \left\{ F(\frac{v_d^2}{2},\phi) + T_-(\beta,\phi) \right\} - e^{-\tau\phi} \tag{3}$$

where F and T_- are defined as follows:

$$F(\frac{v_d^2}{2},\phi) = \sqrt{\frac{2}{\pi}}\int_0^\infty dV \frac{V}{\sqrt{V^2+2\phi}} e^{-\frac{V^2}{2}} \cosh(Vv_d) , \tag{4}$$

$$T_-(\beta,\phi) = \frac{2}{\sqrt{\pi|\beta|}} e^{-|\beta|\phi} \int_0^{\sqrt{|\beta|\phi}} dt\, e^{t^2} \text{ with } \beta < 0, \tag{5}$$

Here the Sagdeev potential is given by the following expression:

$$-V(\phi) = e^{-\frac{v_d^2}{2}} \left\{ \bar{F}(\frac{v_d^2}{2},\phi) + \bar{T}_-(\beta,\phi) \right\} + \frac{1}{\tau}(e^{-\tau\phi} - 1) \tag{6}$$

where we have set $V(\phi = 0) = 0$.

$\bar{F}$ and $\bar{T}_-$ are given as follows:

$$\bar{F}(\frac{v_d^2}{2},\phi) = \sqrt{\frac{2}{\pi}}\int_0^\infty dV\, V\{ \sqrt{V^2+2\phi} - V \}\, e^{-\frac{V^2}{2}} \cosh(Vv_d) , \tag{7}$$

$$\bar{T}_-(\beta,\phi) = -\frac{1}{|\beta|} T_-(\beta\phi) + \frac{2}{\sqrt{\pi|\beta|}}\sqrt{\phi} \tag{8}$$

Here F and T_- have the following small amplitude expansion($\psi << 1$):

$$F(E_d,\phi) = e^{E_d}\left[1 - \frac{1}{2}Z'_r(\sqrt{E_d})\phi - \frac{2e^{E_d}}{\sqrt{\pi}}\phi^{1/2} + \frac{4e^{-E_d}}{3\sqrt{\pi}}(2E_d - 1)\phi^{3/2} + \frac{1}{2}G(E_d)\phi^2 + \cdots \right]$$

$$T_-(\beta,\phi) = \frac{2}{\sqrt{\pi}}\phi^{1/2} + \frac{4\beta}{3\sqrt{\pi}}\phi^{3/2} + \ldots. \tag{9}$$

where $G(E_d)$ is a monotonically decreasing function of E_d with $G(0) = 1$ and $G(E_d) = e^{-E_d} + \frac{3}{8E_d^2}$ for $E_d >> 1$. Here $Z'_r(x)$ represents the real part of the derivative of the complex plasma dispersion function(see Fig. 1(a)) and has following properties:

$$-\frac{1}{2}Z'_r(y) = -\frac{(y - y_0)}{y_0} + (y - y_0)^2 + (\textit{ higher order terms }) ,$$

for $|y-y_0| << 1$ and $y_0 = 0.924$

$$-\frac{1}{2}Z'_r(y) = -\frac{1}{2y^2}\left(1+\frac{3}{2y^2}\right) + (\textit{higher order terms}), \text{ for } |y| >> 1 .$$

Monotonic double layer solutions are found by considering the following nonlinear eigenvalue conditions(or nonlinear boundary conditions) associated with our graphic method:

- To impose charge neutrality at $x = \pm\infty$, we require that the *rhs* of Eq.(3) should vanish at the boundaries $\phi = 0$, ψ.
- Existence of the double layer requires that the Sagdeev potential be identically zero at $\phi = 0$, ψ, so that the electric field equals zero outside the double layer.
- An additional condition on the Sagdeev potential {see Fig. 1. (b)-1} is $V(\phi) < 0$ for $0 < \phi < \psi$.

The first condition yields $\frac{dV(0)}{d\phi} = \frac{dV(\psi)}{d\phi} = 0$:

$$-V'(0) = e^{-\frac{v_d^2}{2}}\left\{ F(\frac{v_d^2}{2},0) + T_-(\beta,\phi) \right\} - 1 = 0, \tag{10}$$

$$-V'(\psi) = e^{-\frac{v_d^2}{2}}\left\{ F(\frac{v_d^2}{2},\psi) + T_-(\beta,\psi) \right\} - e^{-\tau\psi} = 0 . \tag{11}$$

The second condition gives rise the following relations:

$$-V(\psi) = \frac{1}{\tau}(e^{-\tau\psi} - 1) + e^{-\frac{v_d^2}{2}}\left\{ \bar{F}(\frac{v_d^2}{2},\psi) + \bar{T}_-(\beta,\psi) \right\} = 0 \tag{12}$$

Solving the above set of nonlinear eigenvalue equations(Eqs.(10)-(12)) together with the third conditions, one can obtain a set of monotonic double layer solutions.

Since we are interested in weak monotonic double layer solution, we will use the following Poisson equation in considering the small amplitude limit of Eq.(3):

$$\phi_{xx} = -\frac{dV(\phi)}{d\phi} \tag{13}$$

$$\phi_{xx} = \{\tau - \frac{1}{2}Z'_r(\sqrt{E_d})\}\,\phi + \frac{4e^{-E_d}}{3\pi^{1/2}}\{2E_d + \beta - 1\}\,\phi^{3/2} \tag{14}$$

$$+ \tfrac{1}{2}(G(E_d) - \tau^2)\,\phi^2 + \{\text{ higher order terms }\}$$

where E_d is the electron drift energy.

Here it should be noted that we have to retain terms in our expansion *at least* up to order ϕ^2 , in order to satisfy the above nonlinear eigenvalue conditions for the double layer; it is sufficient to retain the terms up to only $\phi^{3/2}$ for the case of the solitary hole.

By solving Eqs.(13) and (14) subject to our nonlinear eigenvalue conditions(Eqs.(10)-(12)) together with the third condition, we get the following monotonic double layer solution

$$\phi = \frac{\psi}{4} \, [\, 1 + \tanh \kappa x \,]^2 \tag{15}$$

where

$$\kappa = \pm \tfrac{1}{4}(\tau - \frac{1}{2} Z'_r(\sqrt{E_d}\,)\,)^{1/2} , \tag{16}$$

$$\beta = 1 - 2E_d - \frac{30\sqrt{\pi} e^{E_d} \kappa^2}{\sqrt{\psi}} , \tag{17}$$

$$\frac{1}{\psi} = \frac{1}{48\kappa^2} \, (G(E_d) - \tau^2) \, . \tag{18}$$

Here it should be noted that the first and third coefficients of the *rhs* of Eq.(14) must be positive and that the second coefficient negative: for the first coefficient, this requires $0 < \tau < 0.285$ and $0.924 < \sqrt{E_d}$, in the *long wave length limit* ($\kappa \to 0$): these conditions follow from the fact that $\frac{1}{2} Z'_r$ has an absolute maximum with positive value 0.285 and that $\frac{1}{2} Z'_r$ is positive for $\sqrt{E_d} > 0.924$. In the long wave length and the small amplitude limit, we obtain the following expressions for $\sqrt{E_d}$ and β:

$$\sqrt{E_d} = \sqrt{E_\tau} + \frac{16\sqrt{\psi} e^{-E_\tau}}{15\sqrt{\pi} Z''_r(\sqrt{E_\tau})} [\, 2E_\tau + \beta - 1 \,] , \tag{19}$$

$$\beta = 1 - 2E_\tau - \frac{5\sqrt{\pi}\sqrt{\psi}}{8} \, \{G(E_\tau) - \tau^2\} e^{E_\tau} \, . \tag{20}$$

Here E_τ is determined from the following equation (see Fig. 1. (a)): $\tau - \frac{1}{2} Z'_r(\sqrt{E_\tau}) = 0$.

For some choice of our physical parameters(τ, E_d, β and ψ) we can neglect the third term in our expansion Eq.(14), in that case we would obtain the following *electron solitary hole* solution[20]:

$$\phi = \psi \, sech^4(\kappa x) \, , \tag{21}$$

where

$$\beta = 1 - 2E_d - \frac{15\sqrt{\pi}e^{E_d}\kappa^2}{\sqrt{\psi}} < -0.71 \text{ as } \tau \rightarrow 0 . \tag{22}$$

Thus, the electron *solitary hole* solution makes a transition to a *double layer* solution when we take into account the third term of Eq.(14); this term comes entirely from electron density associated with the first term of Eq.(12), the free (streaming) electrons in f_e, as $\tau \rightarrow 0$. It is important to note that both the electron solitary hole and the electron double layer solutions require that the trapping parameter be negative ($\beta < -0.71$) and are the nonlinear extensions of the *slow* electron acoustic wave[25], whose linear dispersion relation is given by $\omega^2 = 1.71\ k^2 T_e/m_e$ as $\tau \rightarrow 0$ and $\psi \rightarrow 0$: it follows from Eq.(19) and Eq.(20) by noting that $\sqrt{E_\tau} \rightarrow 0.924$ *as* $\tau \rightarrow 0$. Thus we see that there are no possible small amplitude solutions for the case of positive trapping parameter. Negative trapping parameter represents electron phase space hole(or vortex) for the electron hole case and half vortex like phase space structure for the case of monotonic double layer. It should also be noted that our weak electron double layer has high(low) density at the low(high) potential side. From Eq.(20), it follows that the velocity of double layer decreases as its amplitude increases.

3. Weak ion monotonic double layer

Thus far, we have considered only the double layer related to an *electron* solitary hole. We now turn to the problem of a double layer which is related to the *ion acoustic* solitary hole. In order to describe this class of double layer, we assume two temperature Maxwell-Boltzmann electron distributions[26] $\{n_e = (1-f)e^{\phi} + fe^{\gamma\phi}$ with $\gamma > 1$ and $0 < f < 1\}$, which are usually found in the space plasma, and we consider the following ion distribution[24]:

$$f_i = (2\pi)^{-1/2}\left[\exp\{-\tfrac{1}{2}(\,sgn\,(v)\ \epsilon_i^{1/2} + v_0\,)^2\}\ \Theta(\epsilon_i) + \exp\{-\tfrac{1}{2}(\alpha\epsilon_i + v_0^2)\}\ \Theta(-\epsilon_i)\right] \tag{23}$$

where $\epsilon_i = v^2 + 2\tau\phi$ with $-\psi \leqslant \phi \leqslant 0$, $\tau = T_e/T_i$ and ion velocity has been normalized to the ion thermal velocity $(T_i/m_i)^{1/2}$. Here v_0 , E_0 and α represent the ion drift velocity, the ion drift energy and the inverse temperature, respectively. The ion trapping parameter(α) can be positive and negative depending on the structure of phase space. It will turn out that ion trapping parameter(α) should be negative for the existence of a small amplitude monotonic double layer. Thus the Poisson equation may be written by introducing the Sagdeev potential as follows:

$$\phi_{xx} = -\frac{dV(\phi)}{d\phi} \tag{24}$$

$$\phi_{xx} = -e^{-\frac{v_0^2}{2}} \left\{ F(\frac{v_0^2}{2}, -\tau\phi) + T_-(\alpha, -\tau\phi) \right\} + (1-f)e^{\phi} + fe^{\gamma\phi}$$

Here the Sagdeev potential can be written.

$$-V(\phi) = -e^{-\frac{v_0^2}{2}} \left\{ \tilde{F}(\frac{v_0^2}{2}, -\tau\phi) + \tilde{T}_-(\alpha, -\tau\phi) \right\} + (1-f)(e^{\phi} - 1) + \frac{f}{\gamma}(e^{\gamma\phi} - 1)\,. \tag{25}$$

Monotonic double layer solutions are found by considering the following nonlinear eigenvalue conditions(or nonlinear boundary conditions) associated with our graphic method:

- To impose charge neutrality at $x = \pm\infty$, we require that the *rhs* of Eq.(24) should vanish at the boundaries $\phi = 0\,, -\psi$.
- Existence of the double layer requires that the Sagdeev potential be identically zero at $\phi = 0\,, -\psi$, so that the electric field equals zero outside the double layer.
- An additional condition on the Sagdeev potential {see Fig. 1. (b)-2} is $V(\phi) < 0$ for $0 > \phi > -\psi$.

The first condition yields $\frac{dV(0)}{d\phi} = \frac{dV(-\psi)}{d\phi} = 0$:

$$-V'(-\psi) = -e^{-\frac{v_0^2}{2}} \left\{ F(\frac{v_0^2}{2}, \tau\psi) + T_-(\alpha, -\tau\psi) \right\} + (1-f)e^{-\psi} + fe^{-\gamma\psi} = 0\,, \tag{26}$$

$$-V'(0) = -e^{-\frac{v_0^2}{2}} \left\{ F(\frac{v_0^2}{2}, 0) + T_-(\alpha, 0) \right\} + 1 = 0\,. \tag{27}$$

The second condition gives rise the following relation:

$$-V(-\psi) = -e^{-\frac{v_0^2}{2}} \left\{ \tilde{F}(\frac{v_0^2}{2}, \tau\psi) + \tilde{T}_-(\alpha, \tau\psi) \right\} + (1-f)(e^{-\psi} - 1) + \frac{f}{\gamma}(e^{-\gamma\psi} - 1) = 0 \tag{28}$$

Solving the above nonlinear eigenvalue equations(Eqs.(26)-(28)) together with the third conditon, one can obtain a set of monotonic double layer solutions. Since we are interested in a weak monotonic double layer solution, we use the following small amplitude limit of Poisson equation Eq.(24):

$$\phi_{xx} = [\,1-f+f\gamma - \frac{\tau}{2}\,Z'_r(\sqrt{E_0}\,)\,]\,\phi - \frac{4\tau^{3/2}}{3\sqrt{\pi}}e^{-E_0}\,(2E_0+\alpha-1)\,(-\phi)^{3/2} \qquad (29)$$

$$+\,\tau^2\{\,\frac{1-f+f\gamma^2}{2\tau^2} - \frac{1}{2}G(E_0)\,\}\phi^2 \;+\; \{\text{ higher order terms }\}$$

Again solving Eq.(29) and the above nonlinear dispersion relations together with the requirement $V(\phi) < 0$ for $-\psi < \phi < 0$, we obtain the following ion acoustic double layer solution:

$$\phi = -\frac{\psi}{4}\,\{\,1+\tanh \kappa_a x\,\}^2 \qquad (30)$$

where

$$\kappa_a = \pm\,\tfrac{1}{4}[\,1-f+f\gamma - \frac{\tau}{2}Z'_r(\sqrt{E_0}\,)\,]^{1/2}, \qquad (31)$$

$$\alpha = 1 - 2E_0 - \frac{30\sqrt{\pi}e^{E_0}\,\kappa_a{}^2}{\sqrt{\psi}\,\tau^{3/2}}, \qquad (32)$$

$$\frac{1}{\psi} = \frac{\tau^2}{24\kappa_a{}^2}\{\frac{1-f+f\gamma^2}{2\tau^2} - \frac{1}{2}G(E_0)\,\}. \qquad (33)$$

Here it should be noted that all three coefficients of the *rhs* of Eq.(29) should be positive: for the first coefficient, this requires $\tau > 3.51(1-f+f\gamma)$ and $\sqrt{E_0} > 0.924$: these conditions follow from the fact that $\frac{1}{2}Z'_r$ has an absolute maximum with corresponding value $\frac{1}{3.51}$ and that it is positive for the range of $\sqrt{E_0} > 0.924$. By using the asymptotic expansion of plasma dispersion function in Eq.(31) with $\kappa_a \approx 0$ and the positivity requirement of the rhs of Eq.(33), one can show that there are no small amplitude monotonic double layer solutions for the case of $f=0$ and $E_0 >> 1$. In the long wave length limit, we obtain the following expressions for $\sqrt{E_0}$ and α:

$$\sqrt{E_0} = \sqrt{E_l} + \frac{16\sqrt{\psi\tau}e^{-E_l}}{15\sqrt{\pi}Z''_r(\sqrt{E_l})}[\,2E_l+\alpha-1\,], \qquad (34)$$

$$\alpha = 1-2E_l - \frac{5\sqrt{\psi\tau}}{4}e^{E_l}\,\{\,\frac{1-f+f\gamma^2}{2\tau^2} - \frac{1}{2}G(E_l)\,\}. \qquad (35)$$

Here E_l is determined from the following equation (see Fig. 1. (a))

$1-f+f\gamma - \frac{\tau}{2}Z'_r(\sqrt{E_l}) = 0.$

For some choice of our physical parameters we can neglect the third term in the small amplitude expansion. In that case we would instead recover the *ion acoustic solitary hole* solution, which is the nonlinear version of the *slow* ion acoustic wave[21,25] as τ^{-1} , $f \to 0$ and $\psi \to 0$. Our ion acoustic solitary hole becomes an ion acoustic *double layer* solution by adding the third term of the small amplitude expansion of the free streaming ion density (distribution function) as $\tau^{-1} \to 0$. Here we would like to note that a recent experiment reported a transition of ion hole like structure to weak monotonic double layer[27]. It should be noted that both our ion acoustic solitary hole and ion acoustic double layer solution require a negative trapping parameter ($\alpha < -0.71$) and are the nonlinear extension of the *slow* ion acoustic wave[21,25]: these conditions result from Eq.(34) and Eq.(35) by noting that $\sqrt{E_0} \to 0.924$ as $\tau \to \infty$. We would like to emphasize that our ion acoustic double layer solution can exist with the ion drift velocity smaller than the electron thermal velocity[9]. In contrast to our weak electron monotonic double layer, the weak ion monotonic double layer has high(low) density at the high(low) potential side. From Eq.(34), it follows that the velocity of the double layer decreases as its amplitude increases.

4. Modified K-dV equation for weak ion acoustic monotonic double layer

Having obtained the analytic solutions for the time stationary double layers using the Vlasov-Poisson system , we would like to present a derivation of the evolution equation, which describes the one dimensional asymptotic behavior of ion acoustic monotonic double layers of small but finite amplitude. To describe a collisionless plasma of cold ions and warm electrons, we consider the following set of equations for the cold ions:

$$n_t + (nv)_x = 0 \, , \tag{36}$$

$$v_t + vv_x + \phi_x = 0 \, , \tag{37}$$

$$\phi_{xx} = n_e - n \, , \tag{38}$$

where the density, velocity, potential and spatial coordinate are normalized to the unperturbed density n_0, ion acoustic velocity $(T_e/m_i)^{1/2}$, the electron temperature T_e/e and the electron Debye length, respectively. We introduce the Gardner-Morikawa coordinate transformation

$\xi = \delta^{1/2}(x-t)$ and $\tau = \delta^{3/2}t$. Assuming electrons to be in a quasi-equilibrium with the low-frequency ion acoustic wave, we may expand the electron density as before $n_e = 1 + c_1\phi + c_2\delta^{1/2}\phi^{3/2} + c_3\phi^2 + \ldots$. By using reductive perturbation theory, we expand n, v, ϕ in powers of small parameter[28] δ as follows:

$$n = 1 + \delta n^{(1)} + \delta^2 n^{(2)} + \cdots \tag{39}$$
$$v = v_0 + \delta v^{(1)} + \delta^2 v^{(2)} + \cdots$$
$$\phi = \delta\phi^{(1)} + \delta^2\phi^{(2)} + \ldots .$$

Using the Gardner-Morikawa transform, we obtain the following set of equations:

$$-\delta^{1/2}\ \partial_\xi + \delta^{3/2}\,\partial_\tau n + \delta^{1/2}\,\partial_\xi(nv) = 0\ , \tag{40}$$
$$-\delta^{1/2}\ \partial_\xi v + \delta^{3/2}\ \partial_\tau v + \delta^{1/2}\ v\ \partial_\xi v + \delta^{1/2}\ \partial_\xi\phi = 0\ , \tag{41}$$
$$\delta\ \partial_{\xi\xi}\phi = n_e - n\ . \tag{42}$$

From the above set of equation, we obtain the following set of coupled equations by using reductive perturbation expansion:

$$(v_0 - 1)\ \partial_\xi n^{(1)} + \partial_\xi v^{(1)} = 0\ , \tag{43}$$
$$(v_0 - 1)\ \partial_\xi n^{(2)} + \partial_\tau n^{(1)} + \partial_\xi v^{(2)} + \partial_\xi(n^{(1)}v^{(1)}) = 0\ , \tag{44}$$
$$(v_0 - 1)\ \partial_\xi v^{(1)} + \partial_\xi\phi^{(1)} = 0\ , \tag{45}$$
$$(v_0 - 1)\ \partial_\xi v^{(2)} + \partial_\tau v^{(1)} + v^{(1)}\ \partial_\xi v^{(1)} + \partial_\xi\phi^{(2)} = 0\ , \tag{46}$$
$$n^{(1)} = c_1\phi^{(1)}\ , \tag{47}$$
$$\partial_{\xi\xi}\phi^{(1)} = c_1\ \phi^{(2)} - n^{(2)} + c_2(\phi^{(1)})^{3/2} + c_3(\phi^{(1)})^2\ . \tag{48}$$

From Eqs.(43), (45) and (47), it follows $c_1 = 1$ and we set $\bar\lambda\ n^{(1)} = -\ v^{(1)}$ and $\bar\lambda\ v^{(1)} = -\ \phi^{(1)}$ with $\bar\lambda = (1-v_0)^3$.

After a certain amount of algebra, we obtain the following modified K-dV equation from the above set of equations:

$$\partial_\tau\phi^{(1)} - \frac{1}{2}\bar\lambda^3\,\partial_\xi\,\{\ c_2(\phi^{(1)})^{3/2} + \frac{3c_2{}^2\bar\lambda^3}{25M}(\phi^{(1)})^2\ \} + \frac{\bar\lambda}{3}\partial_{\xi\xi\xi}\phi^{(1)} = 0 \tag{49}$$

where $c_2 < 0$, $M\bar\lambda^3 > 0$, $\frac{3}{\psi} = \frac{\bar\lambda^3}{M}\,(c_3 - \frac{3}{2\bar\lambda^3})$, and M represents the velocity of the ion acoustic monotonic double layer in the frame moving with ion acoustic velocity. Here we have used the monotonic double layer boundary conditions so that we can extract some useful physics. The above equation has the following moving ion acoustic double layer solution:

$$\phi(\xi,\tau) = \frac{1}{4}\left(\frac{5M}{\bar{\lambda}^3 c_2}\right)^2 \left[1 + \tanh \pm \frac{\sqrt{M/2\bar{\lambda}^3}}{2} (\xi - M\tau) \right]^2 \tag{50}$$

Here it should be noted that the velocity of double layers can be $M < 0$ or $M > 0$ depending on the drift velocity of cold ions and the electron equation of state: our double layer velocity is $M < 0$ for $v_0 > 1$ and thus the double layer velocity in the lab frame decreases as its amplitude increases, but for $v_0 < 1$ the double layer velocity is $M > 0$ and thus the double layer velocity in the lab frame increases as its amplitude increases.

5. Conclusion

In this chapter, we have obtained the two different monotonic double layer analytic solutions: one related to the electron solitary hole (electron phase space vortex), the other related to the ion acoustic solitary hole (ion phase space vortex), both having negative trapping parameters. We have given the analytic evidence for the existence of the small amplitude ion acoustic and electron monotonic double layers, which are the nonlinear extensions of the slow ion acoustic wave and the slow electron acoustic wave, respectively. Finally, we have derived modified K-dV equation, which describes the moving ion acoustic monotonic double layers having velocity $M < 0$ or $M > 0$ in the frame moving with ion acoustic velocity depending on the drift velocity of cold ions and the electron equation of state.

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Note Added in Proof:

After submmiting this report, we become aware of the *numerical* investigation by Hudson et al.[29] . We would like to note that they also confirmed numerically the existence of small amplitude slow ion acoustic monotonic double layers and holes for the case of negative trapping parameters.

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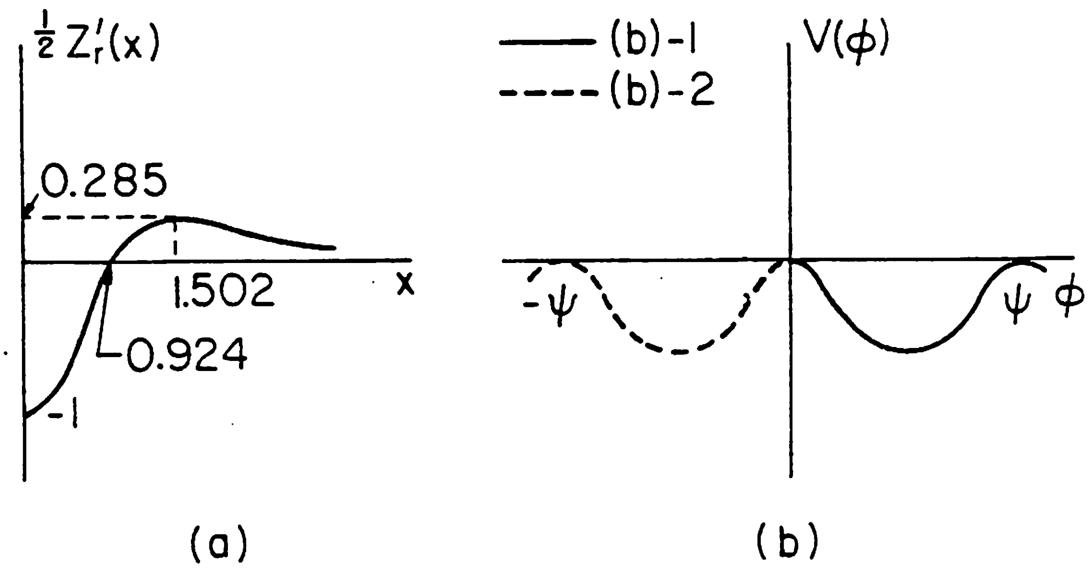

FIG. 1. (a) The real part of derivative of the complex plasma dispersion function: $\frac{1}{2}Z'_r(x)$.
(b) The Sagdeev potential for the monotonic double layer.

4. Theory of Nonmonotonic Double Layers

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ABSTRACT

We present a general analytic formulation for non-monotonic double layers and illustrate with some particular solutions. This class of double layers satisfies the time stationary Vlasov-Poisson system while requiring a Sagdeev potential which is a double valued function of the physical potential: it follows that any distribution function having a density representation as any integer or noninteger power series of potential can never satisfy the non-monotonic double layer boundary conditions. A K-dV like equation is found, showing a relationship among the speed of the non-monotonic double layer, its scale length, and its degree of asymmetry.

1. Introduction

A *monotonic* potential double layer is ideally an isolated pair of oppositely charged sheets which results in a narrow region of abrupt potential jump of some amplitude $\Delta\phi = \psi$; well outside of this localized jump, the potential is effectively uniform[1-17].

In the third chapter, we have presented two different kinds of weak *monotonic* double layer analytic solutions[17], *i.e.* those which do have small amplitude. These solutions are the analytic extensions of the electron solitary hole and ion acoustic solitary hole[15-17,29,30], both having negative trapping parameters; these are the nonlinear extensions of the slow electron acoustic wave and the slow ion acoustic wave, respectively.

Often in experiments and in simulations the *observed* double layer exhibits a potential spatial-profile having a potential depression on the low side (or conversely a potential bump on the high side), as shown in Figure 1(a). Such a *non−monotonic* double layer (NDL) is actually a localized region of *three* sheets of alternating charge sign, and thus includes subregions of oppositely directed non-monotonic electric fields[15-16,18-28].

It is increasingly clear that even the straightforward NDL structure can evidence complex nonlinear characteristics, as exhibited many ways in both simulations and experiments. Reports of several recent simulations[16,18-23] indicate that an *ion acoustic* double layer can be formed by reflection of electrons off the negative potential depression; its simulated potential profile has an NDL form as in Figure 1(a). Recent satellite measurements[24] of field aligned potentials in the auroral region, show signatures that are especially consistent with the NDL, having a characteristic potential depression at the low potential side (or a bump on the high potential side). It has been further suggested[24] that a series of such small amplitude non-monotonic double layers might account for a large portion of the total potential drop along auroral field lines, and might also explain the fine structure of auroral kilometric radiation. The recent thermal barrier cell concept for tandem mirror devices is based on the generation of an abrupt potential depressions by means of forced changes in the particle distribution functions[13,25]. Recent experiments with Q-machine plasmas[26] also reported the formation of a potential depression between two plasmas with different electron temperatures; the "non-monotonic" negative potential depression is thought to play a crucial role in the formation of double layers, accounting for both the observed current disruptions (by reflecting the electrons) and also for the high frequency noise excitation seen behind the double layer (caused by a two stream instability involving electrons that pass the negative potential peak[15,28]). A recent triple plasma experiment reported that the formation of an ion acoustic type double layer was observed in the laboratory for the first time[27].

Although there have been many theoretical, numerical and experimental investigations of double layers, recent theoretical work has been devoted to *numerical* evaluations of the Vlasov-Poisson system (or of the fluid system) mainly because of the highly nonlinear properties of double layers[13-16,21,22,28]. In order to explain *non*monotonic double layers, theoretical efforts have attempted to generalize ion hole, ion acoustic soliton or monotonic double layer descriptions[15,16,21,22]. It should be noted that to our knowledge there exists only one theory offering a *numerical* solution for a nonmonotonic potential structure obtained from a Vlasov-Poisson system[28]. However, it should be pointed out that the distribution function used in this work was not selfconsistent with the Vlasov equation.

In this chapter, we present a general formulation and the first self-consistent *analytic* solution for *non−monotonic* double layers, which satisfy the time stationary Vlasov-Poisson system. We present a derivation of a K-dV like equation describing a *moving* non-monotonic double layer, showing a relationship among the spatial scaling parameter, two amplitudes of non-monotonic double layers and the speed of double layer.

To describe propagation of an electrostatic double layer, we again use a Vlasov-Poisson system that has been Galilean-transformed to the wave frame (where the wave is time stationary). In this frame, we can express the time stationary solution of Vlasov equation as any function of the constants of motion: (i) particle total energy and (ii) the sign of the velocity of the untrapped particles. Besides these usual constants of motion, it is important to note that a third constant of motion exists for the reflected particles: $sgn\ (x - x_m)$ where x_m represents the position of potential minimum (or maximum) for the negatively charged particles (or the positively charged particles). It turns out that this final constant of motion plays an important role in constructing the non-monotonic double layers[23].

2. NDL with potential depression at the low potential side

An important class of NDL jumps from $\phi(x \to -\infty) = \psi_1$ to $\phi(x \to +\infty) = \psi$ and has a potential minimum $\phi(x_m) = 0$ on the low side (see Fig. 1(a)). In order to describe this type of NDL, we next introduce a class of *modified* Schamel distribution functions for the electrons and ions (the regular Schamel[14–17,29–31,33] distribution functions used to describe phase space holes and *monotonic* double layers cannot give rise to NDL). Using all three of the constants of motion mentioned earlier, we construct the following general class of electron and ion distribution functions containing both free-streaming and trapped populations:

$$f_e = \frac{1}{\sqrt{2\pi}} \Bigg\{ e^{-\frac{1}{2}[\mathrm{sgn}\, v \sqrt{\epsilon_e} - v_0]^2}\, \Theta(\epsilon_e) + e^{-\frac{v_0^2}{2}} \left[f_1(1+\mathrm{sgn}\,(x-x_m))\, e^{-\frac{\beta\epsilon_e}{2}} + f_2(1-\mathrm{sgn}\,(x-x_m))\, e^{-\frac{\delta\epsilon_e}{2}} \right] \Theta(-\epsilon_e) \Bigg\} \tag{1}$$

$$f_i = \frac{A}{\sqrt{2\pi}} \Bigg\{ e^{-\frac{1}{2}[\mathrm{sgn}\, u \sqrt{\epsilon_i - 2\tau\psi} - u_0]^2}\, \Theta(\epsilon_i - 2\tau\psi) + e^{-\frac{u_0^2}{2}}\, e^{-\frac{\alpha}{2}(\epsilon_i - 2\tau\psi)}\, \Theta(-\epsilon_i + 2\tau\psi) \Bigg\} \tag{2}$$

where $\epsilon_e = v^2 - 2\phi$, $\epsilon_i = u^2 + 2\tau\phi$, $\tau = T_e/T_i$ and $0 \leqslant \phi(x) \leqslant \psi$.

Here the electron velocity v, the potential ϕ and the spatial coordinate x are respectively normalized to the electron thermal velocity $(T_e/m_e)^{1/2}$, the electron temperature T_e/e and the electron Debye length $\lambda_e = (T_e/4\pi n_0 e^2)^{1/2}$; v_o represents the electron drift velocity. The ion velocity u has been normalized by the ion thermal velocity; u_o is their drift. The density normalization constants A, f_1 and f_2 are positive. The thermal distribution scalings α (for the trapped ions), β and δ (for the trapped electrons) represent effective inverse temperatures. Note that to describe NDL associated, e.g., with an ion phase space hole, the ion trapping parameter (inverse temperature α) may be chosen to be negative; of course, the distribution function itself is everywhere positive and in this sense physically realizable. The symbol Θ is simply the Heaviside step function.

At the NDL's potential minimum ($x = x_m$, $\phi = 0$) the electron distribution models a drifting Maxwellian, Fig. 1(d); the ion distribution function models a drifting Maxwellian at the NDL high side, $\phi = \psi$, Fig. 1(c). Since the reflected particles in either region $x > x_m$ or $x < x_m$, cannot communicate each other, we have introduced two separate temperatures (β and δ) and two normalization constants (f_1 and f_2) for these two separated particle populations. Because we are solving the time stationary Vlasov equation, it follows that the current density (or total particle flux) is uniform in space: electron and ion current density are given by $j_e = -v_0$ and $j_i = Au_0$ throughout the system, respectively.

From the model distribution functions given above, the corresponding densities for electrons and ions can be found by simple velocity-space integrations; these are expressed as functions of physical potential $\phi(x)$ as follows:

$$n_e(\phi) = e^{-\frac{v_0^2}{2}} \left\{ F(\frac{v_0^2}{2},\phi) + f_1(1+sgn(x-x_m))\, T_{\mp}(\beta,\phi) + f_2(1-sgn(x-x_m))\, T_{\mp}(\delta,\phi) \right\} \tag{3}$$

$$n_i(\phi) = A\, e^{-\frac{u_0^2}{2}} \left\{ F(\frac{u_0^2}{2},\tau(\psi-\phi)) + T_{\mp}(\alpha,\tau(\psi-\phi)) \right\} \tag{4}$$

where F, T_- and T_- are defined as follows:

$$F(\frac{v_0^2}{2},\phi) = \sqrt{\frac{2}{\pi}} \int_0^\infty dV \frac{V}{\sqrt{V^2+2\phi}} \cdot e^{-\frac{V^2}{2}} \cosh(Vv_0) \,, \tag{5}$$

$$T_-(\beta,\phi) = \frac{1}{\sqrt{\beta}}\, e^{\beta\phi}\, \mathrm{erf}(\sqrt{\beta\phi}) \qquad \text{with} \quad \beta > 0 \,, \tag{6}$$

$$T_-(\alpha,\phi) = \frac{2}{\sqrt{\pi|\alpha|}}\, e^{-|\alpha|\phi} \int_0^{\sqrt{|\alpha|\phi}} dt\, e^{t^2} \qquad \text{with} \quad \alpha < 0. \tag{7}$$

Thus the Poisson equation expressed in terms of the Sagdeev potential $V(\phi)$ then becomes:

$$\phi''(x) \equiv \phi_{xx} = -\frac{dV(\phi)}{d\phi} \tag{8}$$

$$= e^{-\frac{v_0^2}{2}} \left\{ F(\frac{v_0^2}{2},\phi) + f_1(1+sgn(x-x_m))\, T_{\mp}(\beta,\phi) + f_2(1-sgn(x-x_m))\, T_{\mp}(\delta,\phi) \right\}$$
$$- A\, e^{-\frac{u_0^2}{2}} \left\{ F(\frac{u_0^2}{2},\tau(\psi-\phi)) + T_{\mp}(\alpha,\tau(\psi-\phi)) \right\} \tag{9}$$

From this the Sagdeev potential is obtainable by direct integration, using an integration constant $V(\phi=0) = 0$. This value could have been chosen $\neq 0$; what matters is that we choose a simple constraint, e.g., $\phi = 0$ where $E = 0$.

$$-V(\phi) = e^{-\frac{v_0^2}{2}} \left\{ \bar{F}(\frac{v_0^2}{2},\phi) + f_1(1+sgn(x-x_m))\ \bar{T}_{\mp}(\beta,\phi) \right.$$

$$\left. + f_2(1-sgn(x-x_m))\ \bar{T}_{\mp}(\delta,\phi) \right\}$$

$$- A\ e^{-\frac{u_0^2}{2}} \left\{ \bar{F}(\frac{u_0^2}{2},\tau(\psi-\phi)) + \bar{T}_{\mp}(\alpha,\tau(\psi-\phi)) \right\} \tag{10}$$

In this expression, the functions $\bar{F}$ and $T_{\pm}$ are given as the following integrals:

$$\bar{F}(\frac{v_0^2}{2},\phi) = (\frac{2}{\pi})^{1/2} \int_0^\infty dV\ V\{\sqrt{V^2+2\phi} - V\}\ e^{-\frac{V^2}{2}} \cosh(Vv_0)\ , \tag{11}$$

$$\bar{F}(\frac{u_0^2}{2},\phi) = -(\frac{2}{\pi})^{1/2}\ \frac{1}{\tau} \int_0^\infty dV\ V\{\sqrt{V^2+2\tau(\psi-\phi)}$$

$$- \sqrt{V^2+2\tau\psi}\}\ e^{-\frac{V^2}{2}} \cosh(Vv_0)\ , \tag{11-a}$$

$$\bar{T}_{-}(\beta,\phi) = \pm\frac{1}{|\beta|}T_{\pm}(\beta\phi) \mp \frac{2}{\sqrt{\pi}|\beta|}\sqrt{\phi} \tag{12}$$

$$\bar{T}_{\mp}(\alpha,\tau(\psi-\phi)) = \mp\frac{1}{|\alpha|\tau}\ \{\ T_{\pm}(\alpha,\tau(\psi-\phi)) - T_{\pm}(\alpha,\tau\psi)\ \}$$

$$\pm\ \frac{2}{|\alpha|\tau}\ \{\ \sqrt{|\alpha|\tau(\psi-\phi)} - \sqrt{|\alpha|\tau\psi}\ \}\ . \tag{12-a}$$

For small amplitudes ($\psi << 1$), F and $T_{\pm}$ have the following expansions:

$$F(E_d,\phi) = e^{E_d}\left[1 - \tfrac{1}{2}Z'_r(\sqrt{E_d})\phi - \frac{2e^{E_d}}{\sqrt{\pi}}\phi^{1/2} + \frac{4e^{-E_d}}{3\sqrt{\pi}}(2E_d-1)\phi^{3/2} + \tfrac{1}{2}G(E_d)\phi^2 + \cdots \right]$$

$$T_{\pm}(\beta,\phi) = \frac{2}{\sqrt{\pi}}\phi^{1/2} + \frac{4\beta}{3\sqrt{\pi}}\phi^{3/2} + \ldots. \tag{13}$$

where $G(E_d) = e^{-E_d} + \dfrac{3}{8E_d^2}$ for $E_d >> 1$ is a monotonically decreasing function of E_d with $G(0) = 1$. The function $Z'_r(x)$ represents the real part of the derivative of the complex plasma dispersion function (see Fig.2) and has following expansions:

$$-\tfrac{1}{2}Z'_r(y) = -\frac{(y-y_0)}{y_0} + (y-y_0)^2 + (\text{higher order terms})\ ,$$

for $|y-y_0| << 1$ and $y_0 = 0.924$

$$-\tfrac{1}{2}Z'_r(y) = -\frac{1}{2y^2}\left(1+\frac{3}{2y^2}\right) + (\text{higher order terms}) ,$$

for $|y| >> 1$.

NDL solutions to Eq.(9) are found by considering the following nonlinear eigenvalue conditions associated with our graphic method:

- Outside the NDL region there is charge neutrality and the right hand side of Eq.(9) vanishes, *i.e.*, the physical potential *curvature* is zero: $\phi_{xx} = 0$.
- At the potential minimum, $x = x_m$, the curvature of the physical potential is *positive*. Fig 1(a). Hence $\phi_{xx} > 0$ and therefore $dV(\phi)/d\phi < 0$.
- Outside the NDL region ($x \to \pm\infty$) and also at the potential minimum ($x = x_m$), the *electric field* is zero: $E(x) = -\phi_x$. Hence $V(\phi) = 0$ at values $\phi = \psi_1$, 0, and ψ.
- An additional condition for the existence of NDL requires $V(\phi) < 0$ for $0 < \phi < \psi$ except at $\phi = \psi_1$. Furthermore, the Sagdeev potential (see Fig. 1(b)) must be a double valued function of ϕ for $0 < \phi < \psi_1$.

The required double valuedness of the Sagdeev potential is guaranteed by the use of $sgn(x-x_m)$ in the electron distribution function: the reflected particles on either side of the NDL minimum will in general have different distribution functions, depending on the signature of $(x-x_m)$. In this respect, it should be noted that recent Q-machine experiments have reported a formation of potential depressions between two plasmas with different electron temperatures[26]. From this requirement for the existence of an NDL (that the Sagdeev potential be double valued), it follows that any distribution function resulting in an *analytic* density representation as a function of ϕ can *never* satisfy the necessary nonlinear boundary conditions for an NDL potential form (for example, all of the distribution functions used in references[13-17,29-30,33] to describe phase space holes and double layers).

The first and second boundary conditions mentioned above stipulate that $V'(\psi_1) = 0 = V'(\psi)$ and that $V'(\phi=0) < 0$:

$$-V'(\psi_1) = e^{-\frac{v_0^2}{2}} \left\{ F(\frac{v_0^2}{2},\psi_1) + f_2\, T_{\mp}(\delta,\psi_1) \right\} - A\, e^{-\frac{u_0^2}{2}} \left\{ F(\frac{u_0^2}{2},\tau(\psi-\psi_1)) + T_{\mp}(\alpha,\tau(\psi-\psi_1)) \right\} = 0\,, \tag{14}$$

$$-V'(\psi) = e^{-\frac{v_0^2}{2}} \left\{ F(\frac{v_0^2}{2},\psi) + f_1\, T_{\mp}(\beta,\psi) \right\} - A\, e^{-\frac{u_0^2}{2}} \left\{ F(\frac{u_0^2}{2},0) + T_{\mp}(\alpha,0) \right\} = 0\,. \tag{15}$$

$$-V'(0) = e^{-\frac{v_0^2}{2}} F(\frac{v_0^2}{2},0) - A\, e^{-\frac{u_0^2}{2}} \left\{ F(\frac{u_0^2}{2},\tau\psi) + T_{\mp}(\alpha,\tau\psi) \right\} > 0\,. \tag{16}$$

The third condition (above) gives rise the following relations:

$$V(\psi_1) = Ae^{-\frac{u_0^2}{2}} \left\{ \tilde{F}(\frac{u_0^2}{2},\tau(\psi-\psi_1)) + \tilde{T}_{\mp}(\alpha,\tau(\psi-\psi_1)) \right\} - e^{-\frac{v_0^2}{2}} \left\{ \tilde{F}(\frac{v_0^2}{2},\psi_1) + 2f_2\tilde{T}_{\mp}(\delta,\,\psi_1) \right\} = 0\,, \tag{17}$$

$$V(\psi) = Ae^{-\frac{u_0^2}{2}} \left\{ \tilde{F}(\frac{u_0^2}{2},0) + \tilde{T}_{\mp}(\alpha,0) \right\} - e^{-\frac{v_0^2}{2}} \left\{ \tilde{F}(\frac{v_0^2}{2},\psi) + 2f_1\tilde{T}_{\mp}(\beta,\psi) \right\} = 0 \tag{18}$$

Solving the above set of nonlinear eigenvalue equations (Eqs.(14)-(18)) together with the fourth condition, one can obtain a set of NDL solutions.

In general, the above nonlinear dispersion relation(or nonlinear boundary conditions) can only be solved by numerical means. However, there is one case where an analytic solution is possible. In the ion reference frame, assume a simple Maxwell-Boltzmann ion distribution (i.e., set $\alpha = 1$ and $u_0 = 0$ in Eq.(2)), $f_i = \frac{\tilde{n}_i\sqrt{\tau}}{\sqrt{2\pi}} e^{-\frac{\tau}{2}(v^2 - 2\phi)}$ where $\tilde{n}_i$ is some normalization constant. The electron distribution function is given by Eq.(1) (i.e., $\beta \neq 0 \neq \delta$). With these distribution functions f_i and f_e, the Poisson equation (Eq.(9)) may be written in the small amplitude limit (*i.e.*, expansion in $\phi^{1/2}$ up to terms of $O(\phi^2)$) as follows:

$$\phi_{xx} = -\frac{dV(\phi)}{d\phi} \tag{19}$$

$$\phi_{xx} = \{1-\tilde{n}_i\} + \frac{2e^{-E_d}}{\pi^{1/2}}\{f_1+f_2-1+(f_1-f_2)\, sgn\,(x-x_m)\,\}\phi^{1/2}$$

$$+ \{\tilde{n}_i\tau - \tfrac{1}{2}Z'_r(\sqrt{E_d})\,\}\phi$$

$$+ \frac{4e^{-E_d}}{3\pi^{1/2}}\,\{2E_d + f_1(\,1+ sgn\,(x-x_m)\,)\,\beta + f_2(\,1- sgn\,(x-x_m)\,)\,\delta - 1\,\}\phi^{3/2}$$

$$+ \tfrac{1}{2}\,\{\,G(E_d) - \tilde{n}_i\,\tau^2\}\,\phi^2 + \{\text{ higher order terms }\} \tag{20}$$

where E_d is the electron drift energy and we have used our small amplitude expansion(Eq.(13)).

Eqs.(19) and (20) subject to NDL nonlinear boundary conditions (i.e., Eqs.(14)-(18) together with the fourth condition), can be solved to obtain the following simple NDL form:

$$\phi = \tilde{\psi}\{\,a_s + \tanh \pm|\kappa_r|\,x\,\}^2 \tag{21}$$

where for convenience only $\psi > \psi_1 > 0$ is considered and we have defined

$$\tilde{\psi} = \frac{(\sqrt{\psi_1}+\sqrt{\psi})^2}{4}\,,\quad a_s = \frac{\sqrt{\psi}-\sqrt{\psi_1}}{\sqrt{\psi_1}+\sqrt{\psi}}\,,\quad \kappa_r{}^2 = \frac{\nu\tilde{\psi}}{6} > 0\,. \tag{22}$$

Here $\tilde{\psi}$, a_s, κ_r and ν are related to our system parameters as follows:

$$1-\tilde{n}_i = \frac{1}{3}\psi_1\psi\,\nu\,, \tag{23}$$

$$\frac{2}{\sqrt{\pi}}\,(f_1 - f_2)\,e^{-E_d} = \pm\,\sqrt{\psi_1}\sqrt{\psi}(\sqrt{\psi_1}-\sqrt{\psi})\,\nu\,, \tag{24}$$

$$\{\,\tilde{n}_i\tau - \frac{1}{2}Z'_r(\sqrt{E_d})\,\} = \frac{2}{3}(\psi_1 + \psi - 4\sqrt{\psi_1}\sqrt{\psi})\,\nu\,, \tag{25}$$

$$\frac{4e^{-E_d}}{3\sqrt{\pi}}\,(f_1\beta - f_2\delta) = \mp\,\frac{1}{6}(\sqrt{\psi_1}-\sqrt{\psi})\,\nu\,, \tag{26}$$

$$\nu = \frac{1}{2}\{\,G(E_d) - \tilde{n}_i\,\tau^2\,\} \tag{27}$$

We may consider a_s to be an "asymmetry parameter". For example, if we were to set $\psi_1 = 0$ (*i.e.*, $a_s = 1$) in Eq.(22), then we would obtain our previous [17] monotonic double layer (MDL) solution; the MDL is the nonlinear version of the slow electron acoustic wave in the limits $\tau \to 0$, $\psi \to 0$. Similarly, the condition $\psi_1 = \psi$ ($a_s = 0$) corresponds to the solitary ion hole solution. It should be noted that for the above monotonic double layer and hole solutions, the Sagdeev potential is no longer double-valued. The more general condition $|a_s|<1$ gives rise to the *double*-valued Sagdeev potential, and the above solution Eq.(21) describes a *non-monotonic* double layer, with a potential depression at the low potential side. From Eq.(23), we see that the existence of the non-monotonic double layer, with a potential depression at the low potential side, requires $\bar{n}_i < 1$ in order to have a positive curvature at $\phi = 0$. For $a_s^2 > 1/3$ and in the long wave length limit, the coefficient of ϕ (Eq.(25)) should be positive: this requires $0 < \tau < 0.285$ and $0.924 < \sqrt{E_d}$ and these conditions follow from the fact that $\frac{1}{2}Z'_r$ has an absolute maximum with positive value 0.285 and that it is positive for $\sqrt{E_d} > 0.924$. Note especially that even though this NDL has a potential structure that is similar to recently reported *ion acoustic* double layer simulations, its origin is *not* related to any ion acoustic wave (we give an ion acoustic double layer description in a later chapter). It should be noted that this NDL has high (low) density at low (high) potential side.

3. NDL with potential hump at the high potential side

In order to describe this type of NDL (see FIG. 3. (a) and (b)), we introduce the following *modified* Schamel distribution functions for the electrons and ions: the roles of electrons and ions are interchanged compared with those of the earlier NDL:

$$f_i = \frac{1}{\sqrt{2\pi}}\Big\{ e^{-\frac{1}{2}[\mathrm{sgn}\, u\sqrt{\epsilon_i} - u_0]^2}\Theta(\epsilon_i) + e^{-\frac{u_0^2}{2}}\big[f_1(1+\mathrm{sgn}(x-x_m))\, e^{-\frac{\alpha\epsilon_i}{2}} + f_2(1-\mathrm{sgn}(x-x_m))\, e^{-\frac{\gamma\epsilon_i}{2}}\big]\Theta(-\epsilon_i)\Big\} \tag{28}$$

$$f_e = \frac{A}{\sqrt{2\pi}}\Big\{ e^{-\frac{1}{2}[\mathrm{sgn}\, v\,(\epsilon_e)^{1/2} - v_0]^2}\Theta(\epsilon_e) + e^{-\frac{v_0^2}{2}}\, e^{-\frac{\beta}{2}\epsilon_e}\,\Theta(-\epsilon_e)\Big\} \tag{29}$$

where $\epsilon_i = u^2 + 2\tau\phi$, $\epsilon_e = v^2-2(\phi+\psi)$, $\tau = \dfrac{T_e}{T_i}$ and $0 \geqslant \phi \geqslant -\psi$.

The electron velocity, the wave potential and the spatial coordinates are normalized again to the electron thermal velocity $(T_e/m_e)^{1/2}$, the electron temperature T_e/e and the electron

Debye length $\lambda_e = (T_e/4\pi n_0 e^2)^{1/2}$, respectively; v_o represents the electron drift velocity. The ion velocity has been normalized by the ion thermal velocity. α, β and γ represent effective inverse temperatures. The electron distribution represents the drifting Maxwellian at $\phi = -\psi$ and the ion distribution function represents the drifting Maxwellian at $\phi = 0$. Since reflected particles in either $x > x_m$ or $x < x_m$ can not communicate each other, we have also introduced two different temperatures for the ions. Electron and ion current densities are given by $j_e = -Av_0$ and $j_i = u_0$ throughout the system, respectively.

The corresponding densities for electrons and ions are given as follows:

$$n_i(\phi) = e^{-\frac{u_0^2}{2}} \left\{ F(\frac{u_0^2}{2}, -\tau\phi) \right.$$
$$\left. + f_1(1+sgn(x-x_m))\ T_=(\alpha,-\tau\phi) + f_2(1-sgn(x-x_m))\ T_=(\gamma,-\tau\phi) \right\} \tag{30}$$

$$n_e(\phi) = A\ e^{-\frac{v_0^2}{2}} \left\{ F(\frac{v_0^2}{2}, \phi+\psi) + T_=(\beta,\phi+\psi) \right\} \tag{31}$$

where F, T_- and T_- are defined as before(Eqs. (6)-(8)). Thus the Poisson equation may be written by using the Sagdeev potential as follows:

$$\phi_{xx} = -\frac{dV(\phi)}{d\phi} \tag{32}$$

$$\phi_{xx} = -e^{-\frac{u_0^2}{2}} \left\{ F(\frac{u_0^2}{2}, -\tau\phi) + f_1(1+sgn(x-x_m))\ T_=(\alpha,-\tau\phi) + f_2(1-sgn(x-x_m))\ T_=(\gamma,-\tau\phi) \right\}$$
$$+ A\ e^{(-\frac{v_0^2}{2})} \left\{ F(\frac{v_0^2}{2}, \phi+\psi) + T_=(\beta,\phi+\psi) \right\} \tag{33}$$

The Sagdeev potential can be written as follows:

$$-V(\phi) = -e^{-\frac{u_0^2}{2}} \left\{ \tilde{F}(\frac{u_0^2}{2}, -\tau\phi) + f_1(1+sgn(x-x_m))\ \tilde{T}_=(\alpha,-\tau\phi) + f_2(1-sgn(x-x_m))\ \tilde{T}_=(\gamma,-\tau\phi) \right\}$$
$$+ A\ e^{-\frac{v_0^2}{2}} \left\{ F(\frac{v_0^2}{2}, \phi+\psi) + T_=(\beta,\phi+\psi) \right\} \tag{34}$$

NDL solutions with a potential hump at the high potential side are found by considering the following nonlinear eigenvalue(nonlinear boundary conditions) associated with our graphic method:

- Charge neutrality at $\phi = -\psi_1$ and $-\psi$ with $\psi > \psi_1 > 0$ requires the right hand side of the Eq.(33) should vanish at those values of ϕ. Negative curvature at $\phi = 0$ requires $\frac{dV(\phi)}{d\phi} > 0$ at $\phi = 0$.
- Existence of the NDL requires that the Sagdeev potential be identically zero at $\phi = 0, -\psi_1$ and $-\psi$, so that electric field is equal to zero at those values of ϕ
- An additional condition for the existence of NDL requires $V(\phi) < 0$ for $0 > \phi > -\psi$ except at $\phi = -\psi_1$. Furthermore, the Sagdeev potential(see Fig 3. (c)) should be a double valued function of ϕ for $0 > \phi > -\psi_1$.

It should also be noted that the double valuedness of the Sagdeev potential is guaranteed by the use of $sgn(x-x_m)$ for the reflected ions: reflected ions should have different distribution functions depending on $sgn(x-x_m)$.

The first condition yields $\frac{dV(-\psi_1)}{d\phi} = \frac{dV(-\psi)}{d\phi} = 0$:

$$-V'(-\psi_1) = -e^{-\frac{u_0^2}{2}}\left\{F(\frac{u_0^2}{2},\tau\psi_1) + f_1\, T_=(\alpha,\tau\psi_1)\right\}$$

$$+ A\, e^{-\frac{v_0^2}{2}}\left\{F(\frac{v_0^2}{2},-\psi_1+\psi) + T_=(\beta,-\psi_1+\psi)\right\} = 0\,. \qquad (35)$$

$$-V'(-\psi) = -e^{-\frac{u_0^2}{2}}\left\{F(\frac{u_0^2}{2},\tau\psi) + f_2\, T_=(\gamma,\tau\psi)\right\}$$

$$+ A\, e^{-\frac{v_0^2}{2}}\left\{F(\frac{v_0^2}{2},0) + T_=(\beta,0)\right\} = 0\,. \qquad (36)$$

$$-V'(0) = -e^{-\frac{u_0^2}{2}}F(\frac{u_0^2}{2},0) + A\, e^{-\frac{v_0^2}{2}}\left\{F(\frac{v_0^2}{2},\psi) + T_=(\beta,\psi)\right\} > 0\,. \qquad (37)$$

The second condition gives rise the following relations:

$$V(-\psi_1) = -Ae^{-\frac{v_0^2}{2}}\left\{\bar{F}(\frac{v_0^2}{2},\psi-\psi_1)) + \bar{T}_=(\beta,\psi-\psi_1)\right\} \qquad (38)$$

$$+ e^{-\frac{u_0^2}{2}}\left\{\bar{F}(\frac{u_0^2}{2},\tau\psi_1) + 2f_1\bar{T}_=(\alpha,\tau\psi_1)\right\} = 0\,.$$

$$V(-\psi) = -Ae^{-\frac{v_0^2}{2}}\left\{\bar{F}(\frac{v_0^2}{2},0) + \bar{T}_=(\beta,0)\right\} \qquad (39)$$

$$+ e^{-\frac{u_0^2}{2}} \left\{ \tilde{F}(\frac{u_0^2}{2}, \tau\psi) + 2 f_2 \tilde{T}_3(\gamma, \tau\psi) \right\} = 0$$

Solving the above nonlinear eigenvalue equations(Eqs.(35)-(39)) together with the third conditon, one can obtain a set of NDL solutions with a potential hump at the high potential side.

Again, the above nonlinear dispersion relation(or boundary conditions) can only be solved by nummerical means in general. As before there is one special case, where analytic solution is possible. In the electron reference frame, we assume a Maxwell-Boltzmann electron distribution(we set $\beta = 1$ and $v_0 = 0$ in Eq.(29)), $f_e = \frac{\tilde{n}_e}{\sqrt{2\pi}} e^{-\frac{1}{2\tau}(v^2 - 2\phi)}$ where $\tilde{n}_e$ is some normalization constant, and we consider the ion distribution function given by Eq.(28). With the distribution functions f_i and f_e, the Poisson equation may be written in the small amplitude limit by using a Sagdeev potential $V(\phi)$(Eq.(32)-(33)):

$$\phi_{xx} = -\frac{dV(\phi)}{d\phi} \tag{40}$$

$$\phi_{xx} = \{-1+\tilde{n}_e\} + \frac{2\sqrt{\tau}e^{-E_0}}{\pi^{1/2}}\{f_1+f_2-1 + (f_1-f_2)\, sgn(x-x_m)\}\, (-\phi)^{1/2}$$

$$+ \{\tilde{n}_e - \tfrac{1}{2}\tau Z'_r(\sqrt{E_0})\}\phi$$

$$- \frac{4\tau^{3/2}e^{-E_0}}{3\pi^{1/2}}\{2E_0 + f_1(1+ sgn(x-x_m))\,\alpha + f_2(1- sgn(x-x_m))\,\gamma - 1\}(-\phi)^{3/2}$$

$$+ \frac{\tau^2}{2}\{\frac{\tilde{n}_e}{\tau^2} - G(E_0)\}\phi^2 + \{\text{ higher order terms }\} \tag{41}$$

where E_0 is the ion drift energy and $Z'_r(x)$ again represents the real part of the derivative of the complex plasma dispersion function(see Fig. 2.). In this case, the double valued Sagdeev potential is guaranted by the use of the constant of motion for reflected ions. By solving Eqs.(40) and (41) subject to the NDL boundary conditions(Eqs.(35)-(39) together with the third condition), we get the following double layer solution:

$$\phi = -\tilde{\psi}\{ a_s + \tanh \pm |\kappa_e|\, x \}^2 \tag{42}$$

where for convenience we consider only $\psi > \psi_1 > 0$ and we have defined

$$\tilde{\psi} = \frac{(\sqrt{\psi_1} + \sqrt{\psi})^2}{4}, \quad a_s = \frac{\sqrt{\psi} - \sqrt{\psi_1}}{\sqrt{\psi_1} + \sqrt{\psi}}, \quad \kappa_e^2 = -\frac{\mu\tilde{\psi}}{6} > 0 . \tag{43}$$

$\bar{\psi}$, a_s, κ_e and μ are related to our system parameters as follows:

$$-1+\tilde{n}_e = \frac{1}{3}\psi_1\psi\,\mu\,, \tag{44}$$

$$\frac{2\tau}{\sqrt{\pi}}\,(f_1 - f_2)\,e^{-E_0} = \pm\,\sqrt{\psi_1}\sqrt{\psi}(\sqrt{\psi_1} - \sqrt{\psi})\,\mu\,, \tag{45}$$

$$\{\,\tilde{n}_e - \frac{\tau}{2}Z'_r(\sqrt{E_0})\,\} = \frac{-2}{3}(\psi_1 + \psi - 4\sqrt{\psi_1}\sqrt{\psi})\,\mu\,, \tag{46}$$

$$-\frac{4\tau^{3/2}e^{-E_0}}{3\sqrt{\pi}}\,(f_1\,\alpha - f_2\,\gamma) = \mp\,\frac{1}{6}(\sqrt{\psi_1} - \sqrt{\psi})\,\mu\,, \tag{47}$$

$$\mu = 1/2\{\,\tilde{n}_e - \tau^2 G(E_0)\,\}\,. \tag{48}$$

Clearly a_s may again be thought of as an NDL asymmetry parameter. But now if we consider $\psi_1 = 0$ ($a_s = 1$) in Eq.(42), then our previous *ion acoustic* monotonic double layer solution is recovered, and is the nonlinear version of the slow ion acoustic wave. Similarly the condition $\psi_1 = \psi$ ($a_s = 0$) would return the *electron* solitary hole solution. It should also be noted that the above two solutions have single valued Sagdeev potentials. The more general condition $|a_s|<1$ gives rise to a double-valued Sagdeev potential, and the above solution describes an *non−monotonic* double layer, with a potential hump at the high potential side. From Eq.(44), we note that the existence of the non-monotonic double layer with a potential hump at the high potential side, requires $\tilde{n}_e < 1$ in order to have a negative curvature at $\phi = 0$. For $a_s^2 > 1/3$, the long wave length limit, the coefficient of ϕ should be positive: this requires $\tau > 3.51$ and $\sqrt{E_0} > 0.924$ and these conditions follow from the fact that $\frac{1}{2}Z'_r$ has an absolute maximum with corresponding value $1/3.51$ and that it is positive for the range of $\sqrt{E_0} > 0.924$. Unlike the NDL in Section 2, this NDL has high (low) density at high (low) potential side.

4. MK-dV like solution, moving NDL

In this section we derive a K-dV like equation, which applies to the one dimensional non-monotonic double layers of small but finite amplitude (a weak ion acoustic double layer) having a potential depression at the low potential side.

To describe a collisionless plasma of cold ions and warm electrons, we consider the following set of equations for the cold ions:

$$n_t + (nv)_x = 0\ , \tag{49}$$

$$v_t + vv_x + \phi_x = 0\ , \tag{50}$$

$$\phi_{xx} = n_e - n\ , \tag{51}$$

where the density, velocity, potential and spatial coordinate are normalized to the unperturbed density n_0, ion acoustic velocity $(T_e/m_i)^{1/2}$, the electron temperature T_e/e and the electron Debye length, respectively. We introduce the Gardner-Morikawa coordinate transformation $\xi = \delta^{1/2}(x-t)$ and $\tau = \delta^{3/2}t$. Assuming the electrons to be in a quasi-equilibrium with the low-frequency ion acoustic wave, we may expand the electron density as

$$n_e = 1+\delta^2\tilde{n}_e+\delta^{3/2}c_0\, sgn(\xi-\xi_m)\,\phi^{1/2}+c_1\phi+\delta^{1/2}c_2\, sgn(\xi-\xi_m)\phi^{3/2}+c_3\phi^2+\ .....$$

By using reductive perturbation theory, we expand n, v, ϕ in powers of small parameter[34] δ as follows:

$$n = 1+\delta n^{(1)}+\delta^2 n^{(2)}+\cdots \tag{52}$$

$$v = v_0+\delta v^{(1)}+\delta^2 v^{(2)}+\cdots$$

$$\phi = \delta\phi^{(1)}+\delta^2\phi^{(2)}+\ ....$$

From the above prescription, we obtain the following set of coupled equations:

$$(v_0-1)\,\partial_\xi n^{(1)}+\partial_\xi v^{(1)} = 0\ , \tag{53}$$

$$(v_0-1)\,\partial_\xi n^{(2)}+\partial_\tau n^{(1)}+\partial_\xi v^{(2)}+\partial_\xi(n^{(1)}v^{(1)}) = 0\ , \tag{54}$$

$$(v_0-1)\,\partial_\xi v^{(1)}+\partial_\xi\phi^{(1)} = 0\ , \tag{55}$$

$$(v_0-1)\,\partial_\xi v^{(2)}+\partial_\tau v^{(1)}+v^{(1)}\,\partial_\xi v^{(1)}+\partial_\xi\phi^{(2)} = 0\ , \tag{56}$$

$$n^{(1)} = c_1\phi^{(1)}\ , \tag{57}$$

$$\partial_{\xi\xi}\phi^{(1)} = \tilde{n}_e+c_1\,\phi^{(2)}-n^{(2)}+c_0(\phi^{(1)})^{1/2}+c_2(\phi^{(1)})^{3/2}+c_3(\phi^{(1)})^2\ . \tag{58}$$

From Eqs.(53), (55) and (57), it follows $c_1 = \dfrac{1}{(1-v_0)^2}$ and we set $\phi^{(1)} = (1-v_0)^2\, n^{(1)}$ and $v^{(1)} = \dfrac{1}{1-v_0}\phi^{(1)}+v_0^{(1)}$ where $v_0^{(1)}$ is some constant. We obtain the following K-dV like equation from the above set of equations, after a certain amount of algebra

$$0 = \frac{1}{\bar{\psi}\kappa^2}\partial_\tau\phi+\frac{1}{2\bar{\psi}\kappa^2}\partial_{\xi\xi\xi}\phi \tag{59}$$

$$-\,\partial_\xi\left[\,6a_s(1-a_s^2)\, sgn(\xi-\xi_m)\left(\frac{\phi}{\bar{\psi}}\right)^{1/2}\right.$$

$$\left.+\left\{4(3a_s^2-1)-\frac{M}{\kappa^2}\right\}\frac{\phi}{\bar{\psi}}-10a_s\, sgn(\xi-\xi_m)\left(\frac{\phi}{\bar{\psi}}\right)^{3/2}+3\left(\frac{\phi}{\bar{\psi}}\right)^2\ \right]$$

Here we have defined

$$\bar{\psi} = \frac{(\sqrt{\psi_1} + \sqrt{\psi})^2}{4}, \qquad a_s = \frac{\sqrt{\psi} - \sqrt{\psi_1}}{\sqrt{\psi_1} + \sqrt{\psi}} > 0, \qquad \kappa^2 = \frac{\eta\bar{\psi}}{6} > 0. \tag{60}$$

Here $\bar{\psi}$, a_s and κ are related to our system parameters as follows:

$$\bar{n}_e - \frac{1}{2}\left(\frac{v_0^{(1)}}{\lambda}\right)^2 = 2\bar{\psi}\kappa^2(1-a_s^2)^2, \qquad c_0 = 12\sqrt{\bar{\psi}}\kappa^2 a_s(1-a_s^2),$$
$$\frac{v_0^{(1)}}{\lambda^3} = M - 4\kappa^2(3a_s^2 - 1), \qquad c_2 = -\frac{20}{\sqrt{\bar{\psi}}}\kappa^2 a_s, \qquad \eta = c_3 - \frac{3}{2\lambda^4} \tag{61}$$

where $\lambda = 1 - v_0$ and M represents the velocity of the ion acoustic double layer in the frame moving with ion acoustic velocity. It should be noted that we have used our nonlinear boundary conditions for a *moving* non-monotonic double layer so that we can extract some useful physics. The corresponding moving ion acoustic double layer solution of Eq.(59), with a potential depression at the low potential side, is given by

$$\phi(\xi,\tau) = \bar{\psi}\{\, a_s + \tanh \kappa(\, \xi - M\,\tau\,)\,\}^2 \tag{62}$$

where ξ_m is given by the equation $\phi\ (\,\xi_m - M\ \tau\ ,\ 0\ ) = 0$. Here $a_s = 1$ and $a_s = 0$ represent monotonic double layer and solitary structure, respectively.

5. Conclusion

Using our graphic method, we have given a general formulation of NDL and obtained two new non-monotonic double layer *analytic* solutions: one has a potential depression at the low potential side; the other has a potential hump at the high potential side. From the double-valued properties of the Sagdeev potential (required for the existence of a non-monotonic double layer), it follows that any distribution function having a density representation as any integer or non-integer power series of potential can *never* satisfy the non-monotonic double layer boundary conditions. This shows the importance of using the *third* constant of motion for *reflected* particles, in order to provide a *double*-valued Sagdeev potential for the *non*-monotonic double layers. We have also given a derivation of the K-dV like equation, which describes the non-monotonic *moving* double layer with a potential depression at the low potential side. Thus we have found that there is relation among physical parameters.

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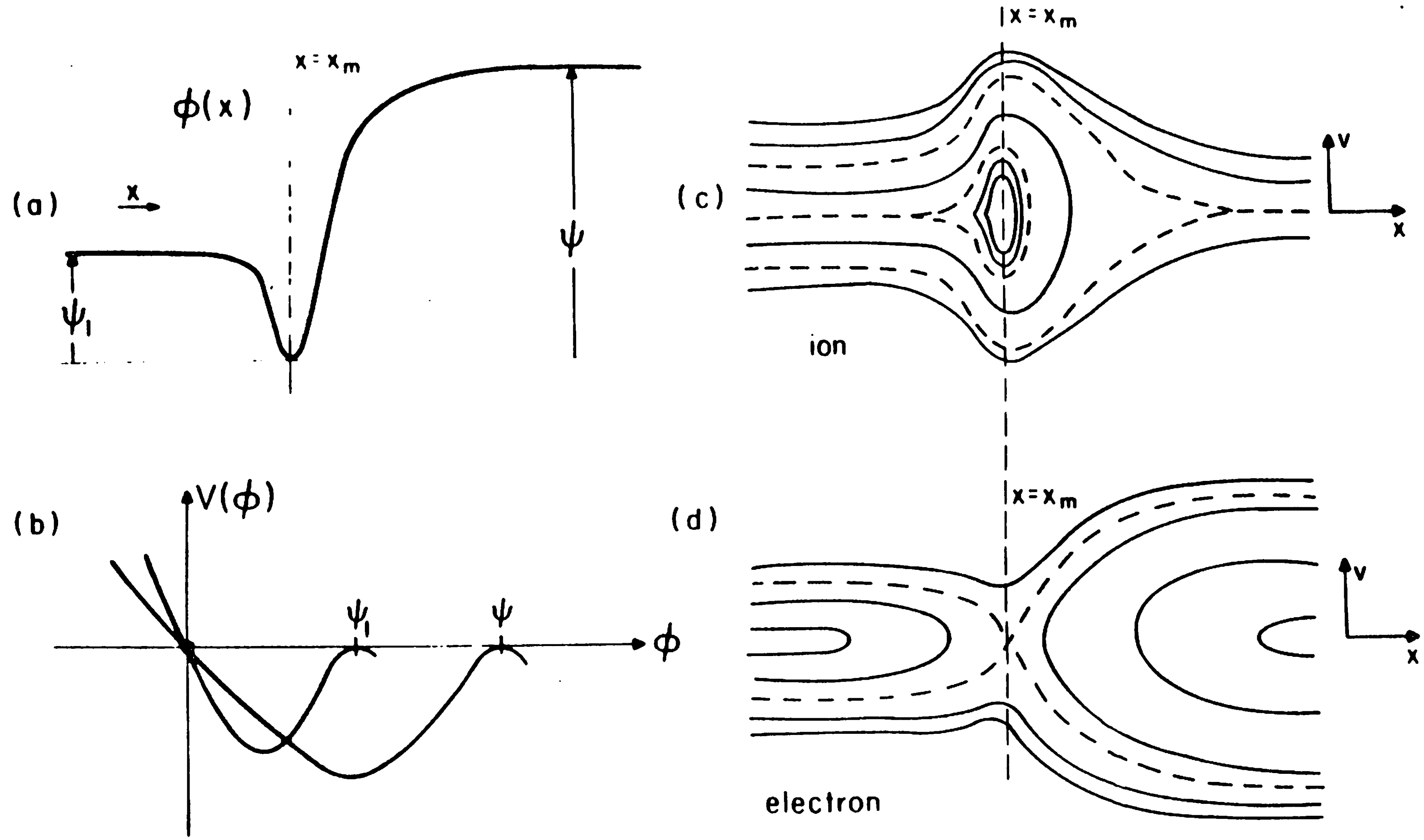

FIG. 1. (a) NDL with potential depression at the low potential side. (b) Sagdeev potential for the above NDL. (c) Ion phase space plot. (d) Electron phase space plot.

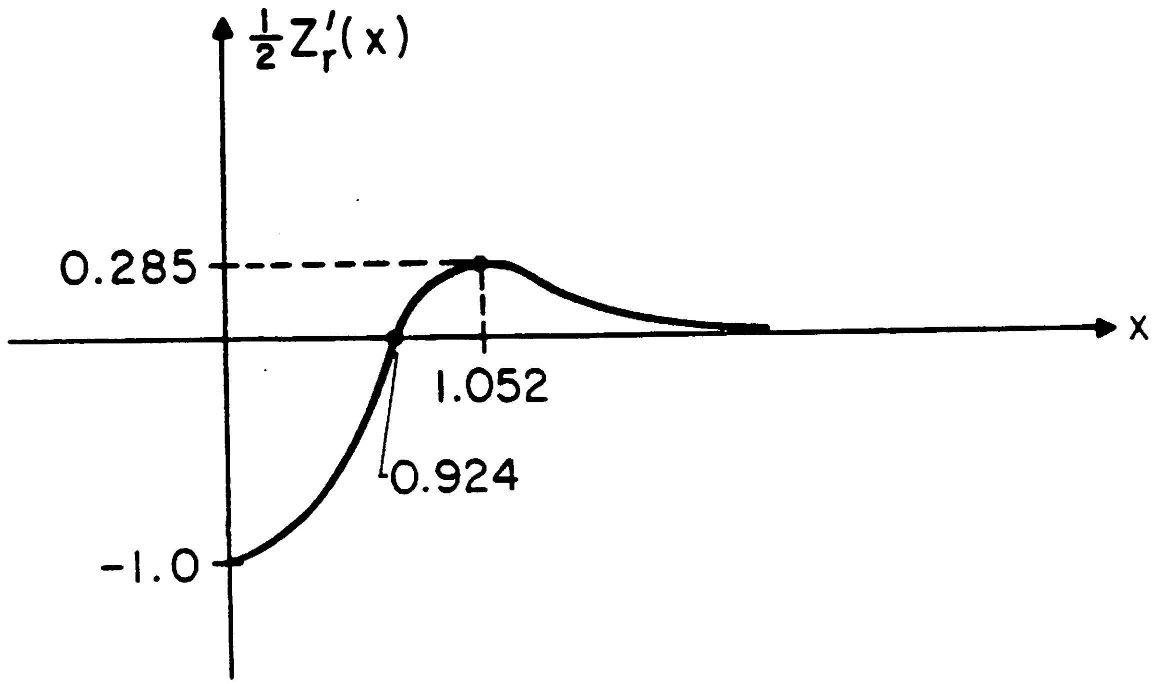

FIG. 2. The real part of derivative of the complex plasma dispersion function: $\frac{1}{2}Z'_r(x)$.

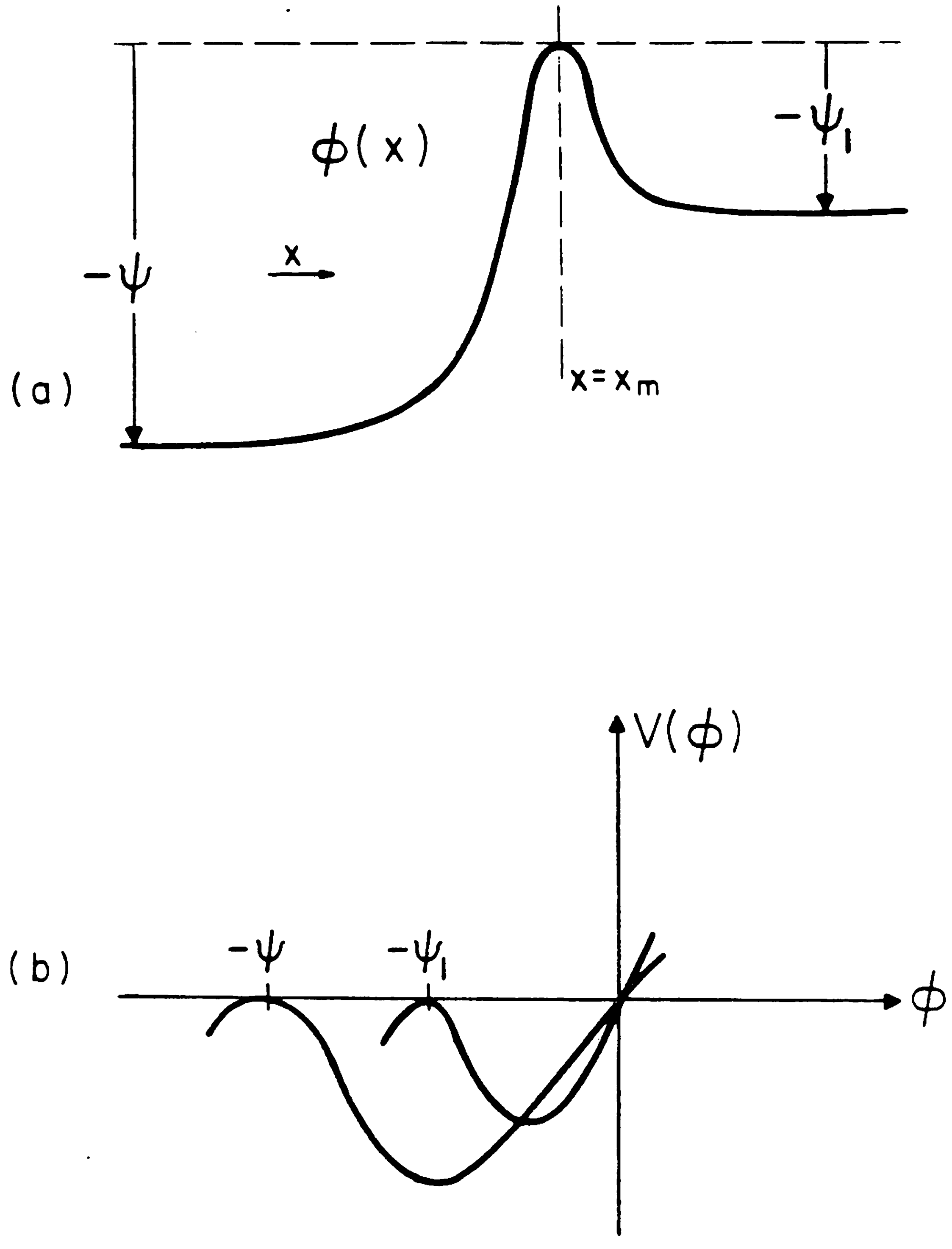

FIG. 3. (a) NDL with potential hump at the high potential side. (b) Sagdeev potential for the above NDL.

5. Theory and Simulation of ion acoustic double layers

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ABSTRACT

Particle simulations of ion acoustic double layers are successful in *short* systems ($L = 80\lambda_D$) and with *low* drift velocities ($v_d = 0.45\, v_{th}$) for the electrons. We present simulation results for systems driven by constant current and by constant applied voltage. By using an analytic formulation, we find that there is a "critical" electron drift velocity (which is considerably smaller than the value reported in previous papers but very close to the value of our simulations) for the existence of ion acoustic double layers. We find that for a given electron drift velocity (exceeding the "critical" drift) there is a corresponding maximum amplitude for the ion acoustic double layer. We show that the net potential jump across the ion acoustic double layers is determined by the temperature difference between the two plasmas. It is also shown that usual Bohm condition is *not* satisfied for ion acoustic double layers with finite amplitude the velocity of the ion acoustic double layer decreases (below C_s) as its amplitude increases.

1. Introduction

In several recent simulation papers[1-6], it has been reported that a weak ion acoustic double layer of $\delta\phi \approx T_e$ can be formed via electron reflection off the potential depression taking

$T_e >> T_i$, it was found that for these weak ion-acoustic double layer to form, there was a threshold drift velocity $v_d \geqslant 0.6\, v_{th}$, and a necessary, relatively long simulation system ($L \geqslant 512\lambda_e$). Here v_{th} represents the thermal velocity of electrons and λ_e is the electron Debye length.

Recent satellite measurements[7] of field aligned potential in the auroral region, show signatures that are especially consistent with the non-monotonic double layer, having the characteristic potential depression at the low potential side (or bump on the high potential side). It has been suggested[7] that a series of such small amplitude double layers might account for a large portion of the total potential drop along auroral field lines, and might also explain the fine structure of auroral kilometric radiation. Recent experiments with Q-machine plasmas[8] also reported the formation of a potential depression between two plasmas with different electron temperatures; the "non-monotonic" negative potential depression is thought to play a crucial role in the formation of double layers, accounting for both the observed current disruptions (by reflecting the electrons) and also for the high frequency noise excitation seen behind the double layer (caused by a two stream instability involving electrons that pass the negative potential peak[9-10]). In a recent experiment[11], it has been reported that ion acoustic type double layer has been observed for the first time in a triple plasma machine and this ion acoustic double layer has a subsonic propagation velocity.

Although there have been many attempts at understanding double layers, recent theoretical work has been devoted to *numerical* evaluations of the Vlasov-Poisson system (or of the fluid system) mainly because of the highly nonlinear properties of double layers[4-6,10-12,15]. To our knowledge, there exists only one theory offering even a numerical solution for a *non-monotonic* potential structure obtained from a Vlasov-Poisson system[7]. Theoretical efforts have attempted to generalize ion hole, ion acoustic soliton, or monotonic double layer descriptions in order to explain non-monotonic double layers[4,5,6,10].

In a previous chapter, we gave a simple graphic method of solving the Vlasov-Poisson system associated with nonlinear eigenvalue conditions for arbitrary potential structures, presented a general analytic formulation for non-monotonic double layers, and illustrated with some particular solutions[16]. In this chapter, we present a theory of ion acoustic holes and ion acoustic double layers and compare this with our simulation results.

2. Simulation of Ion Acoustic Double Layer

First we shall describe some of the results of simulations with no applied dc potential ("current driven"), and then describe briefly the results of simulations that have an applied dc potential ("voltage driven"). All simulations are done with a 1d axially-bounded electrostatic PIC code. In all of our simulations, we have used the same mass ratio ($m_i/m_e = 100$), the time step is $\omega_e\, \delta t = 0.2$. Initially, the simulation plasma density is uniform in space. The ion and electron distribution functions are both Maxwellian; the ions are cold, $T_i << T_e$, and the electrons are drifting relative to them with drift velocity v_d. This relative drift between the electrons and ions constitutes a current and can result in instability depending on v_d.

(a) Simulations of current driven systems

In our simulations of constant current driven system, we have found that, weak ion acoustic double layers can be formed even in a very short system ($80\lambda_e$), and even when the electron drift velocity is small compared to previous simulations ($v_d = 0.45\, v_{te} \approx 4.5\, C_s$); the double layer formation mechanism is still based essentially on amplification of a small negative potential dip, due to reflection of electrons.

Using a temperature ratio $\tau = T_e/T_i = 20$ and plasma parameter $n\lambda_e = 100$, the system plasma is loaded uniformly in space; there is thus no electric field initially. From Figures 1 and 2, note that by time $\omega_e\, t = 480$ a small negative potential dip has developed that is associated with an ion phase space distortion as well as with an ion density dip. From subsequent "snapshots" of this potential dip, it is seen to be moving with nearly ion acoustic velocity (C_s). Therefore it can start to trap those ions which are resonant with the structure (*i.e.*, ions in the *positive* velocity tail of the distribution), and an ion hole starts to form there. At the left side of the growing potential dip, electrons with velocity slightly greater than the ion acoustic velocity contribute to the structure's growth; their velocity distribution (in the moving frame of potential dip) has positive slope, so that they give up their energy to this potential.

As the negative potential structure grows and decelerates due to this electron reflection, the dip is able both to trap more ions and, at the same time, to reflect more electrons. "Positive" momentum transfer due to electron reflection leads to "deceleration" of the ion hole, and the potential thus appears to have a negative effective mass as well as negative effective charge. This deceleration and growth of the potential dip can lead to increased ion trapping, because the structure sees more densely populated ion distribution as it decelerates. This electron reflection causes the asymmetry of potential due to more electron density buildup at the left hand side of potential dip than that of right hand side. At time $\omega_e t = 880$, an ion acoustic double layer is well developed, with a negative net potential jump $|\delta\phi| = 0.3\ T_e$ over a distance of about 10λ.

One reason for initiation with the low drift velocity of our simulation is the condition of *constant* current injection as opposed to the previous simulations using decaying current injection. Of cource, there is a similarity between our system (with constant current condition) and previous long periodic simulations with decaying current conditions; because the system length is *long* in the periodic simulation and because of periodicity (a particle leaving at one boundary is replaced at the opposite boundary by an incoming particle with the same initial velocity), early in time there is *nearly* constant current coming in and going out. This nearly constant current in a *long* periodic simulation acts as a source of energy which leads to the formation of weak ion acoustic double layers by the reflection of electron current. However, our bounded simulation system, with constant current injection has more energy available than that of a long periodic simulation system, resulting in a lower threshold drift velocity than that of a periodic system.

In all of our simulations, the negative potential dip always moves in the same direction (*i.e.*, that of the electron drift velocity). This is because right going ion acoustic waves are preferentially amplified by the inverse electron Landau damping but left going ion acoustic waves are severely attenuated by the Landau damping. The early growth and deceleration of the potential dip can be understood qualitatively as a simple momentum exchange between the structure and the particles. The rate of change of ion acoustic double layer momentum should be approximately equal to the negative of the rate of change of particle momentum; this can be expressed by considering the reflection of electrons and ions by the ion acoustic double layer as follows(see Fig. 3(b)):

$$-\frac{dP_{iadl}}{dt} = -2m_e \int_0^{(p_{e1}-p_e)^{1/2}} dv\, v^2 f_e(M+v) + 2m_e \int_0^{(p_{e1})^{1/2}} dv\, v^2 f_e(M-v)$$
$$- 2m_i \int_0^{p_i^{1/2}} du\, u^2 f_i(M+u)\,. \qquad (1)$$

where $p_e = \frac{e}{m_e}\psi$, $p_{e1} = \frac{e}{m_e}\psi_1$, $p_i = \frac{e}{m_i}\psi$ and M represents the velocity of ion acoustic double layer. In the small amplitude limit(ψ_1 , $\psi << 1$), the above equation can be written as follows:

$$-\frac{dP_{iadl}}{dt} = -2m_e f_e(M)\{\, [\frac{e}{m_e}(\psi_1-\psi)]^{3/2} - [\frac{e}{m_e}\psi_1]^{3/2}\,\}$$
$$-2m_e f'_e(M)\{\, [\frac{e}{m_e}(\psi_1-\psi)]^2 + [\frac{e}{m_e}\psi_1]^2\,\}$$
$$-2m_i f_i(M)[\frac{e}{m_i}\psi]^{3/2}$$
$$-2m_i f'_i(M)[\frac{e}{m_i}\psi]^{3/2}$$

The first term in Eq.(1) represents the positive momentum transfer(i.e. electron momentum loss) due to electron reflection from the left hand side of ion acoustic double layer The second term in Eq.(1) represents the negative momentum transfer due to electron reflection from the right hand side of ion acoustic double layer. The third term in Eq.(1) represents the positive momentum transfer due to ion reflection from the left hand side of ion acoustic double layer.

Initially there is no appreciable asymmetry in the potential structure (*i.e.*, $\psi \simeq 0$) and $dP_{iadl}/dt > 0$ since electron drift is in the right direction and thus $\frac{df_e(M)}{dv} > 0$. Due to this *positive* momentum transfer(i.e. electron momentum loss) by electron reflection, the ion acoustic double layer having negative effective mass decelerates; the negative effective mass results from the ion density dip associated with the ion acoustic double layer. As potential asymmetry develops by electron reflection(i.e. $\psi > 0$), ion reflection starts come into play and the second term starts to compete with the first term. For the cold ion case, the third term can be considered negligible, and ion acoustic double layer can receive net positive or negative momentum transfer and thus decelerate or accelerate, depending on the relative magnitude of ψ_l and ψ_r the velocity of the ion acoustic double layer and the electron distribution function at both sides of the ion acoustic double layer.

(b) Simulation of voltage driven systems

For the constant voltage driven system with current injection ($\delta\phi = 0.5$), we have found results that are similar to the above current driven simulation. An important new feature however, is that the ion acoustic double layer now *always* appears first near the left side of wall and develops more quickly than in the earlier current driven simulation (Fig. 4 and 5). In addition, our constant voltage driven simulation requires a lower electron drift velocity ($0.2v_{te}$) for the formation of ion acoustic double layers than was necessary in the constant current driven simulations.

In our constant voltage driven system with current injection at the boundary, an applied potential across the system acts to increase the *effective* drift velocity of electrons. Since ions injected at left boundary see a potential barrier due to the applied potential, they will be reflected, thereby forming an ion phase space density dip, which contributes to the formation of a negative potential dip near the left side of the wall(Fig. 4(a)). Therefore, we expect to find formation of ion acoustic double layers at a lower electron drift velocity and near the left side of the wall.

3. Theory of Ion Acoustic Double Layers

Having described our simulation results, we are in a position to explain theoretically some of their features. Using our graphic method, we shall show that there is a "critical" velocity for the existence of ion acoustic double layers, which is smaller than the value reported in previous papers, and that there are maximum amplitudes of the potential dip corresponding to the drift velocities exceeding critical velocity. We also find that the net potential jump across the ion acoustic double layers is determined by the temperature difference between two plasma regions.

From the simulations we see that the electron kinetics are important and our theory accordingly must start with a reasonable yet tractable kinetic electron model which uses all three constants of motion. To this end, we introduce the following "modified Schamel" type of electron distribution function, having three electron components:

$$f_e = \frac{1}{\sqrt{2\pi}} \left\{ e^{-\frac{1}{2}(sgn\, v \sqrt{\epsilon} - v_d)^2} \Theta(\epsilon - 2\psi_1) + e^{-\frac{v_0^2}{2}} \left[f_\beta \left[1 - sgn\,(x - x_m)\right] e^{-\frac{\beta\epsilon}{2}} + f_\delta \left[1 + sgn\,(x - x_m)\right] e^{-\frac{\delta\epsilon}{2}} \right] \Theta(-\epsilon + 2\psi_1) \right\} \qquad (2)$$

where $\epsilon = v^2 - 2\phi$. The first component is the "free" (or untrapped) group of electrons. They make up the bulk of the electrons, and are modeled here as a drifting Maxwellian function at $\phi = 0$; as given, their temperature has been normalized (*i.e.*, to T_e/e). Reflected (or "trapped") particles populate the regions either $x > x_m$ or $x < x_m$ on each side of the double layer, and cannot communicate with each other; therefore, we have introduced two separate temparatures (β and δ) and two normalization constants (f_β and f_δ) for them. Here, electron velocity (v), the potential (ϕ), and the distance (x) are normalized to the electron thermal velocity $(T_e/m_e)^{1/2}$, the "free" electron temperature T_e/e, and the electron Debye length $\lambda_e = (T_e/4\pi n_0 e^2)^{1/2}$, respectively. Θ represents simply the Heaviside step function.

Since we are interested in describing ion acoustic double layers, we shall use a fluid formalism to describe the essentially cold ions:

$$(n_i)_t + (n_i u)_x = 0\,, \qquad u_t + u\, u_x + \phi_x = 0 \qquad (3)$$

Considering a time stationary situation, the above fluid ion model yields $n_i u = n_0 u_0$ and $u^2/2 + \phi = E_0$ where n_0, u_0 and E_0 are constants.

From these two species models, the corresponding densities for electrons and ions in the ion acoustic double layer frame are given as follows:

$$n_e(\phi) = e^{-\frac{v_d^2}{2}} \left\{ N_f(\frac{v_d^2}{2},\phi) + f_\beta\left[1-sgn\,(x-x_m)\right] R\,(\beta,\phi) + f_\delta\left[1+sgn\,(x-x_m)\right] R\,(\delta,\phi) \right\} \quad (4)$$

$$n_i(\phi) = -\frac{n_0 u_0}{\sqrt{2(E_0-\phi)}} \quad (5)$$

where N_f and R are defined as follows:

$$N_f(\frac{v_d^2}{2},\phi) = (\frac{2}{\pi})^{\frac{1}{2}} e^{\phi} \int_{y_l}^{\frac{\pi}{2}} dy\ (1+\tan^2 y)\ e^{-\frac{\tan^2 y}{2}} \cosh\,(v_d\sqrt{\tan^2 y - 2\phi})\ , \quad (6)$$

$$R\,(\beta,\phi) = \frac{1}{\sqrt{\beta}} e^{\beta\phi}\, erf\,(\sqrt{\beta(\phi+\psi_1)}\,) \quad (7)$$

$$y_l = \tan^{-1}\sqrt{2(\psi_1+\phi)} \quad \text{and} \quad \beta > 0$$

Thus Poisson's equation may be written by introducing a Sagdeev potential $V(\phi)$ as follows

$$\phi_{xx} = -\frac{dV(\phi)}{d\phi} = e^{-\frac{v_d^2}{2}} \left\{ N_f(\frac{v_d^2}{2},\phi) + f_\beta\left[1-sgn\,(x-x_m)\right] R\,(\beta,\phi) + f_\delta\left[1+sgn\,(x-x_m)\right] R\,(\delta,\phi) \right\} + \frac{n_0 u_0}{\sqrt{2(E_0-\phi)}} \quad (8)$$

The Sagdeev potential is given by the obvious first integral of Poisson's equation with appropriately chosen boundary conditions:

$$-V(\phi) = e^{-\frac{v_d^2}{2}} \left\{ \tilde{N}_f(\frac{v_d^2}{2},\phi) + f_\beta(1-sgn\,(x-x_m))\ R\,(\beta,\phi) + f_\delta(1+sgn\,(x-x_m))\ R\,(\delta,\phi) \right\} + n_0 v_d\left[\sqrt{2(E_0-\phi)} - \sqrt{2(E_0+\psi_1)}\right] \quad (9)$$

where we have set $V(\phi = -\psi_1) = 0$. Here $\tilde{N}_f$ and $\tilde{R}$ are given as follows:

$$\tilde{N}_f\left(\frac{v_d^2}{2},\phi\right) = \sqrt{\frac{2}{\pi}} \int_{\tan^{-1}\sqrt{2\psi_1}}^{\frac{\pi}{2}} dy \tan y \,(1+\tan^2 y\,)\{\sqrt{\tan^2 y + 2\phi} - \sqrt{\tan^2 - 2\psi_1}\,\}$$

$$e^{-\frac{\tan^2 y}{2}} \cosh(v_d \tan y)\,, \tag{10}$$

$$\tilde{R}(\beta,\phi) = \frac{1}{\beta} R(\beta\phi) - \frac{2}{\sqrt{\pi}\beta}\sqrt{\phi + \psi_1}\, e^{-\beta\psi_1} \tag{11}$$

A prototypical ion acoustic double layer potential structure is shown coming from the right at some reference potential $\phi = 0$, dipping to some negative value $\phi = -\psi_1$, then rising to a final negative value $\phi = -\psi$ on the left(Fig. 3(b)). An ion acoustic double layer solutions to these equations can be found by applying the following nonlinear eigenvalue conditions[14] (also called "nonlinear boundary conditions"), associated with the graphic method:

(1) Charge neutrality outside the double layer region requires that the right side of the Poisson equation vanish at the right and left extremes of the potential structure *i.e.*, at $\phi = 0$ and $\phi = -\psi$.

(2) The electric field is, of course, zero where $\phi(x)$ is flat *i.e.*, outside the locality of the potential structure. Thus the existence of ion acoustic double layer solutions requires that the corresponding Sagdeev potential be identically zero at the three places this happens. *i.e.*, at $\phi = 0, -\psi_1$ and $-\psi$.

(3) Positive curvature at $\phi = -\psi_1$ requires that $dV(\phi)/d\phi < 0$ at $\phi = -\psi_1$.

(4) An additional condition for the existence of ion acoustic double layers requires that $V(\phi)$ be negative for $0 > \phi > -\psi_1$ except at $\phi = -\psi$. Additionally, the Sagdeev potential should be a ***double valued*** function of ϕ for $-\psi > \phi > -\psi_1$ (see Fig. 3(a)). It is important to note that double valuedness of the Sagdeev potential is guaranteed by the use of $sgn(x-x_m)$ in our kinetic model for electrons (2): reflected electrons will have different distribution functions depending on the sign of $(x-x_m)$. In this respect, it should be noted that a recent Q-machine experiment reported formation of potential depressions between two plasmas with different electron temperatures[8].

By way of illustration, we set $\delta = 1$ and $f_\delta = 1/2$: these choices simply match one of the "trapped" populations, the "δ" group, to the "free"electrons, both in temperature and distribution amplitude resulting simple Maxwell-Boltzmann distribution function for $v_d = 0$.

The first and third conditions above require that $\frac{dV(0)}{d\phi} = \frac{dV(-\psi)}{d\phi} = 0$ and $\frac{dV(-\psi_1)}{d\phi} < 0$:

$$V'(0) = -e^{-\frac{v_d^2}{2}} \left\{ N_f(\frac{v_d^2}{2},0) + R(1,0) \right\} - \frac{n_0 u_0}{\sqrt{2E_0}} = 0\,, \tag{12}$$

$$V'(-\psi) = -e^{-\frac{v_d^2}{2}} \left\{ N_f(\frac{v_d^2}{2},-\psi) + f_\beta R(\beta,-\psi) \right\} - \frac{n_0 u_0}{\sqrt{2(E_0+\psi)}} = 0\,, \tag{13}$$

$$V'(-\psi_1) = -\frac{n_0 u_0}{\sqrt{2(E_0+\psi_1)}} - e^{-\frac{v_d^2}{2}} N_f(\frac{v_d^2}{2}, -\psi_1) < 0 \tag{14}$$

The second condition above gives rise the following relations:

$$V(0) = n_0 u_0 \left[\sqrt{2E_0} - \sqrt{2(E_0+\psi_1)}\right] - e^{-\frac{v_d^2}{2}} \left[\bar{N}_f(\frac{v_d^2}{2},0) + \bar{R}(1,0)\right] = 0\,, \tag{15}$$

$$V(-\psi) = n_0 u_0 \left[\sqrt{2(E_0+\psi)} - \sqrt{2(E_0+\psi_1)}\right] - e^{-\frac{1}{2}v_d^2} \left[\bar{N}_f(\frac{v_d^2}{2},-\psi) + 2f_\beta \bar{R}(\beta,-\psi)\right] = 0\,. \tag{16}$$

Solving the above set of nonlinear eigenvalue equations(Eqs.(12)-(16)) together with the third conditions, one can obtain a set of ion acoustic double layer solutions. In order to solve the above set of equations, it is convenient to rewrite them as follows (here we have set $n_0 = 1$):

$$u_0 = -ne0\sqrt{2E_0} \tag{17}$$

$$\sqrt{2(E_0+\psi)} = -\frac{u_0}{ne1} \tag{18}$$

$$u_0 = -\frac{nb0}{\sqrt{2}}(\psi_1 - \frac{nb0}{ne0})^{-\frac{1}{2}} \tag{19}$$

$$V'(-\psi_1) = -u_0\left[2\psi_1 + (\frac{u_0}{ne0})^2\right]^{-\frac{1}{2}} - nem < 0 \tag{20}$$

$$V(-\psi) = -\frac{u_0^2}{ne1} - u_0\left[2\psi_1 + (\frac{u_0}{ne0})^2\right] - nb1 = 0\,. \tag{21}$$

Here we used Eq.(17) to obtain Eqs.(19)-(20), and Eq.(21) follows from Eq.(15) by using Eqs.(15), (17) and (18). The definitions of $ne0$, $ne1$, $nb0$ and $nb1$ are as follows

$$ne0 = e^{-\frac{v_d^2}{2}} \left\{ N_f(\frac{v_d^2}{2},0) + R(1,0) \right\} \tag{22}$$

$$ne1 = e^{-\frac{v_d^2}{2}} \left\{ N_f(\frac{v_d^2}{2},-\psi) + f_\beta R(\beta,-\psi) \right\} \tag{23}$$

$$nem = e^{-\frac{v_d^2}{2}} N_f(\frac{v_d^2}{2}, -\psi_1) \tag{24}$$

$$nb0 = e^{-\frac{v_d^2}{2}} \left\{ \bar{N}_f(\frac{v_d^2}{2},0) + \bar{R}(1,0) \right\}, \tag{25}$$

$$nb1 = e^{-\frac{1}{2}v_d^2}\left\{ \tilde{N}_j(\frac{v_d^2}{2},-\psi) + 2f_\beta \tilde{R}(\beta,-\psi) \right\} \tag{26}$$

First of all, we show analytically that there are no possible ion acoustic double layer solutions without electron drift ($v_d = 0$).

Setting $v_d = 0$ in Eq.(17), we get $u_0 = -\sqrt{2E_0}$. Here $n_0 = 1$ coresponds to the ion density at $\phi = 0$ and $u = u_0$. In this case, Eq.(19) and Eq.(20) yield following relations

$$u_0 = -\frac{1 - e^{-\psi_1}}{\sqrt{2(\psi_1 - 1 + e^{-\psi_1})}} \tag{27}$$

$$u_0^2 < \frac{2\psi_1 e^{-2\psi_1}}{1 - e^{-2\psi_1}} \tag{28}$$

Let us define u_m so that it satisfies Eq.(28) with equality sign instead of inequality sign. Looking first at the large amplitude limit ($\psi_1 >> 1$), we find that $u_0 - u_m \approx -\frac{1}{\sqrt{2\psi_1}} + \sqrt{2\psi_1 e^{-2\psi_1}} < 0$; thus there are no large amplitude ion acoustic double layer solutions. In the small amplitude limit ($\psi_1 << 1$), we again find that $u_0 - u_m \approx -\psi_1/6 < 0$; this again means that there are no small amplitude solutions possible

This calculation also implies that there exist positive polarity soliton solutions with no electron drift. We have examined Eqs.(27) and (28) numerically and found no possible ion acoustic double layer solutions for all amplitudes, so long as we hold $v_d = 0$. This already suggests that there may be a critical drift velocity for the existence of ion acoustic double layers as was found experimentally in our simulations.

Numerically solving our nonlinear eigenvalue equation, we have been able to find that, in fact, there is a "critical" velocity for ion acoustic double layer solutions to exist. Arguing that the smallest *visible* potential dip in a simulation would be of order $\delta\phi \geqslant 0.1$, the corresponding "critical" velocity for this amplitude solution to exist is found to be $v_d \approx 0.4 v_{th}$; calculation shows that electron drift velocity should be greater than $0.3v_{th}$ to have $\delta\phi \geqslant 0.02$. This value is considerablly smaller than has been reported in other papers, and much closer to our simulation result (*i.e.*, that $v_d \geqslant 0.45v_{th}$ was necessary before the current driven double layer would form).

Examining solutions to our nonlinear eigenvalue system, we have found that there are maximum amplitude limits for the negative potential dip $(-\psi_1)$; these depend on the electron drift velocity exceeding the critical value (Fig. 6.).

We have also calculated ion drift velocities in the frame of ion acoustic double layers and found that the usual Bohm condition is *not* met in the case of ion acoustic double layers In fact, the ion drift velocity decreases (below C_s) as both amplitudes of negative dip $(-\psi_1)$and net potential drop $(+\psi)$ increase (see Fig. 7.). With regard to the common identification made between the ion acoustic soliton and our double layer solutions, we point out that the velocity of the usual (rarefactive, having negative polarity) ion acoustic soliton *increases* with increasing amplitude; this character is in direct conflict with our earlier observation that the ion acoustic double layer slows down as it grows. It is important to note that net amplitude of an ion acoustic double layer correlates directly with the temperature difference between the two plasmas the net potential drop (ψ)increases with increased temperature of reflected electrons on the high potential side(see Fig. 3(b)). In fact, a recent Q-machine experiment reported that formation of a negative potential depression has been observed in a system with two different plasma sourses and that potential at the high potential side increases as the electron temperature at high potential side is increased by heating.

4. Conclusion

Using our general formulation, we have shown *analytically* that there are no possible ion acoustic solutions *without* electron drift. Numerically solving our nonlinear eigenvalue equation we found that, in fact, there is a critical (minimum) velocity for the existence of an ion acoustic double layers. Theoretically calculated "critical" electron drift velocity for the existence of ion acoustic double layer is found to be $0.4v_{th}$; our simulated ion acoustic double layer was found at $0.45v_{th}$.

We have also found that there are limits on the maximum amplitude of the negative potential dip, and these depend on the electron drift velocity (which must exceed the critical value). We have also calculated ion drift velocities in the frame of ion acoustic double layers and found that the usual Bohm criterion is *not* valid in the case of ion acoustic double layers. In fact, the ion drift velocity decreases as both the amplitude of the negative dip and the net potential drop of the ion acoustic double layers increase. It should be noted that the velocity of the usual rarefactive ion acoustic soliton with negative polarity increases with increasing amplitude, as opposed to the ion acoustic double layer which slows down. It is important to note that the net amplitude of the ion acoustic double layer is determined by temperature difference between two plasmas: the net potential drop increases with the temperature of reflected electrons at high potential side.

Finally, we have found the following new results from our simulation in a very short system ($L = 80\lambda_e$), we have found the formation of weak non-monotonic double layers (NDL) with drift velocity ($0.45v_{th}$); this is significantly shorter ($\approx 512\lambda_e$) and slower ($v_c \approx 0.6v_{th}$) than that of previous simulations. We have also given some physical explanations for the low threshold drift velocity for the formation of NDL, and for the formation of the NDL near the left wall in the *constant voltage* driven system with current injection.

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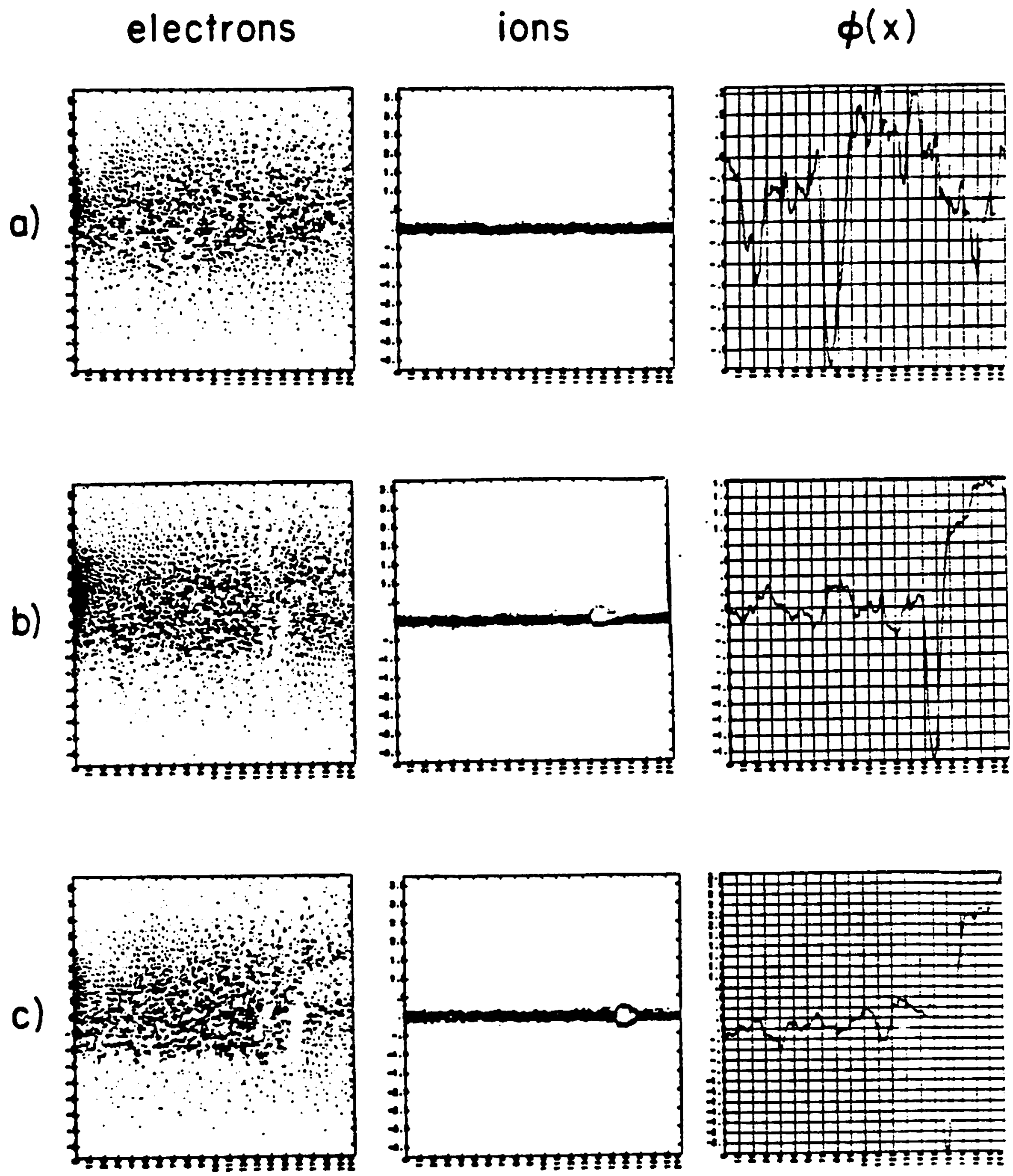

Fig 1. (a) Simulation phase space for electrons, and ions and the potential profile at time $\omega_r t = 480$ for current driven system Here, (b) and (c) is given at time $\omega_r t = 880$ and $\omega_r t = 1080$ respectively

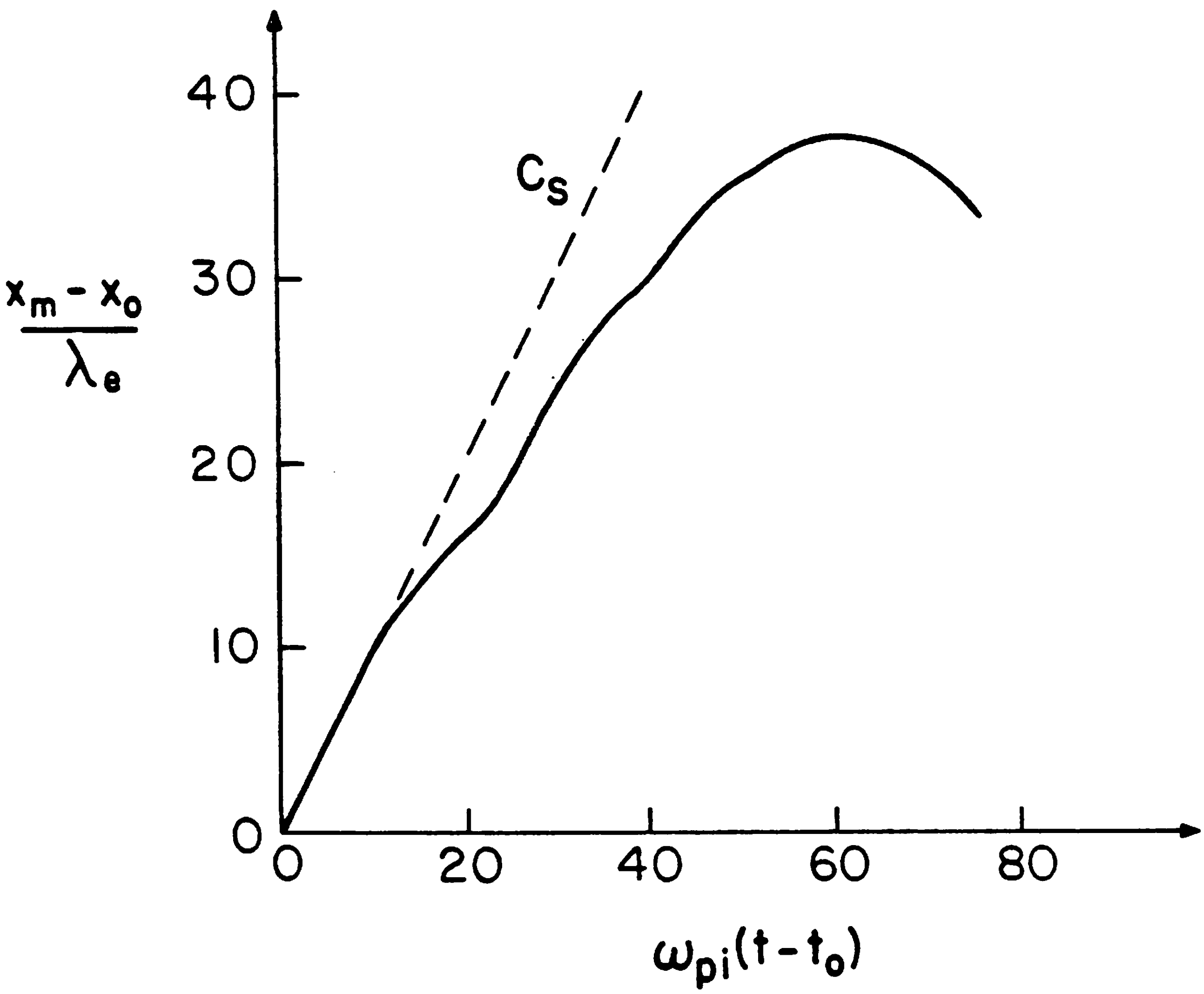

Fig. 2. Observed position of potential minimum v.s. time for the current driven simulation Here, x_0 is the position of potential minimum at time t_0 and $\omega_r = 0.1\omega_c$.

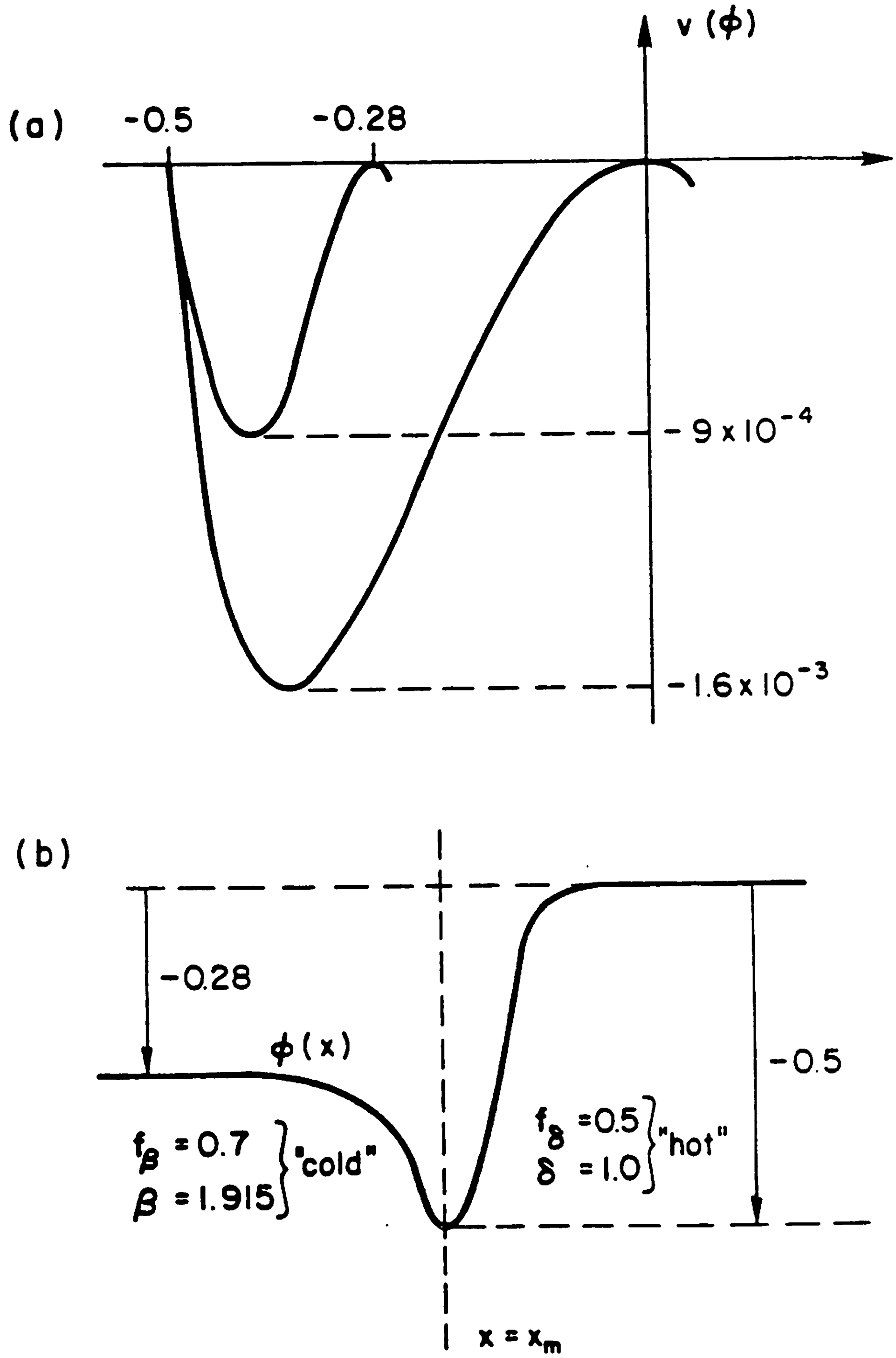

Fig 3 (a) Calculated Sagdeev potential for ion acoustic double layer (b) Corresponding ion acoustic double layer, using left side potential as a zero reference, then $\psi = 0.5$ and $\psi = 0.28$ (see text) with $v_d = 0.9$

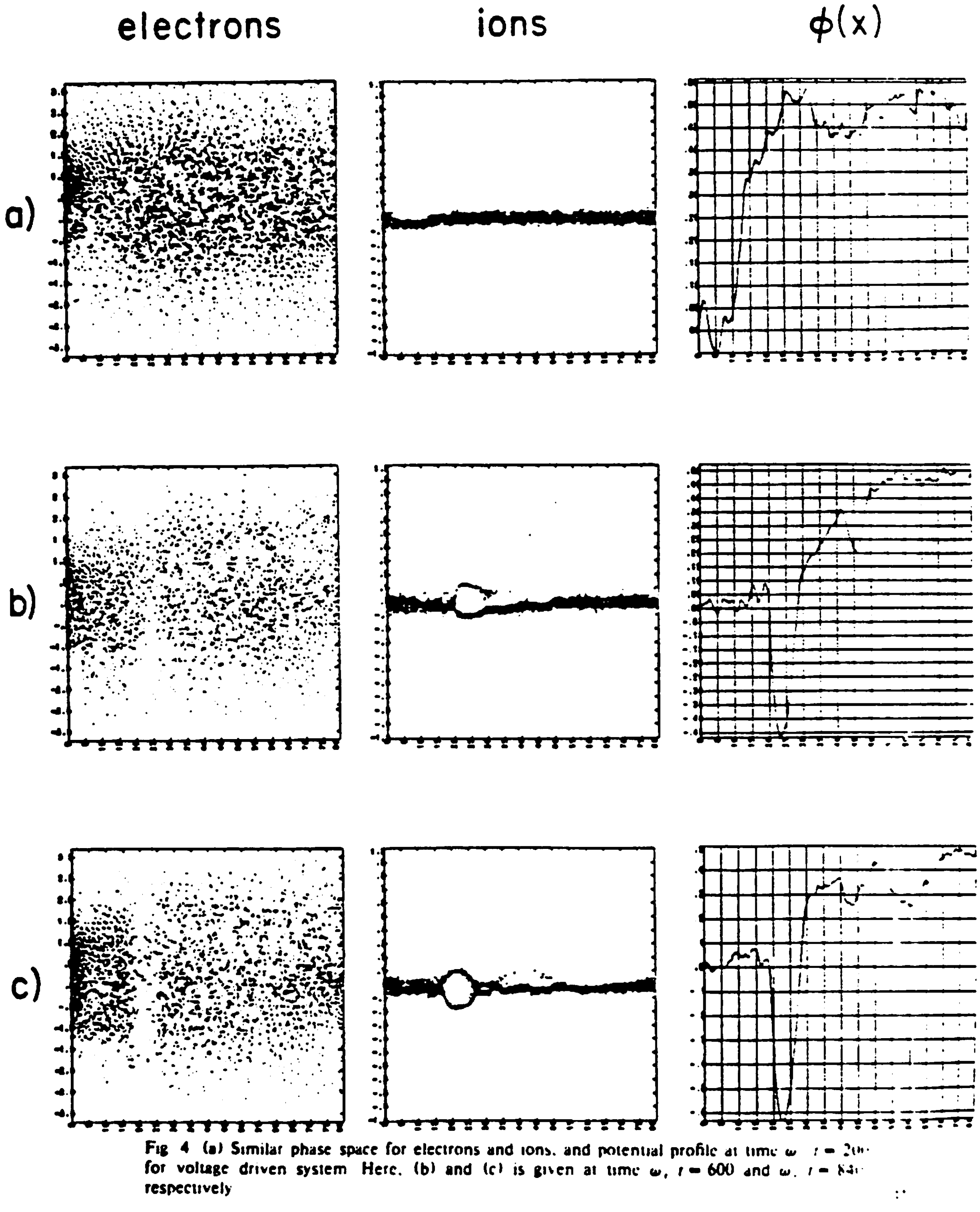

Fig 4 (a) Similar phase space for electrons and ions, and potential profile at time $\omega_i t = 2$[illegible] for voltage driven system Here, (b) and (c) is given at time $\omega_i t = 600$ and $\omega_i t = 84$[illegible] respectively

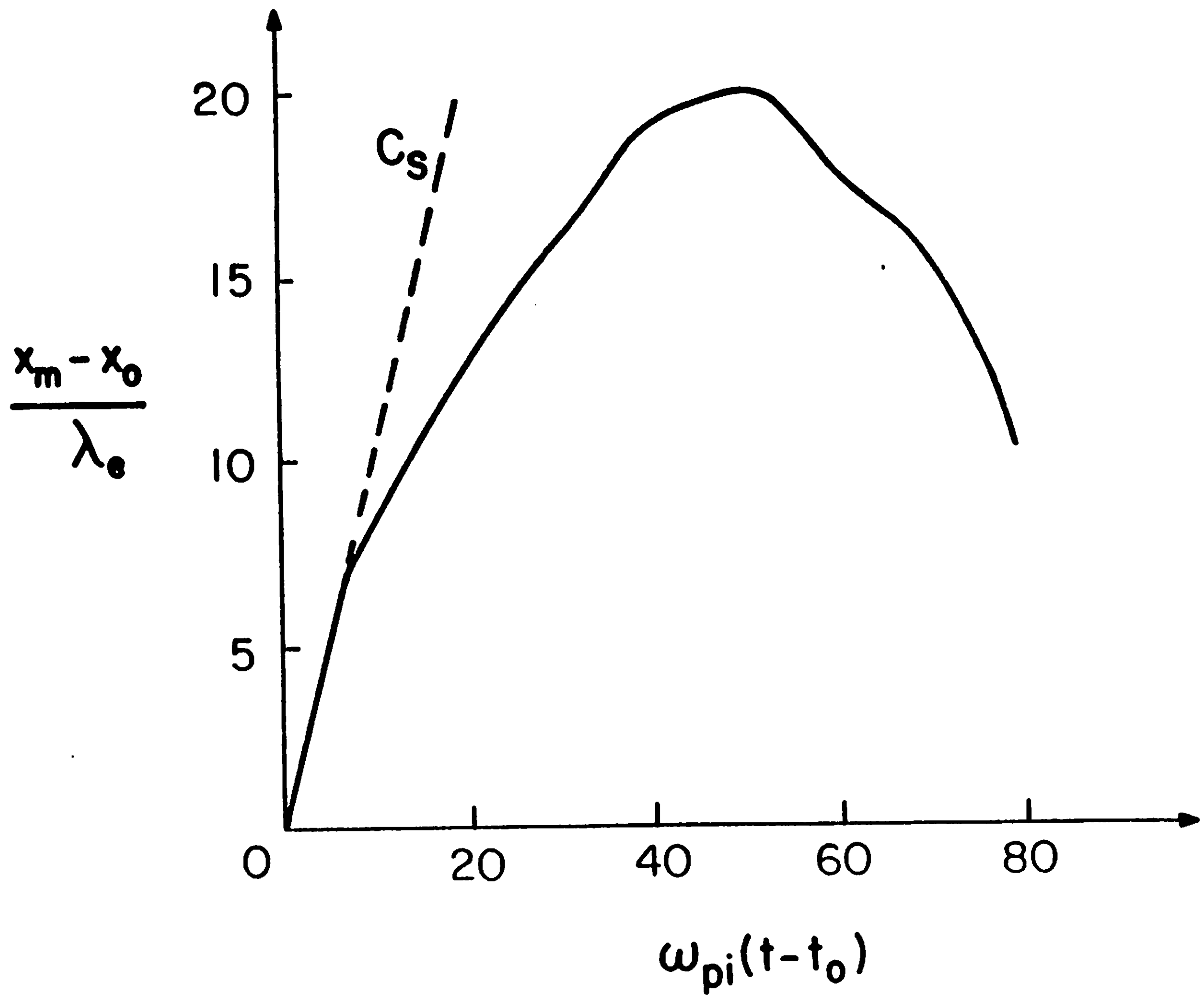

Fig. 5. Observed position of potential minimum (x_m) v.s. time for the voltage driven simulation

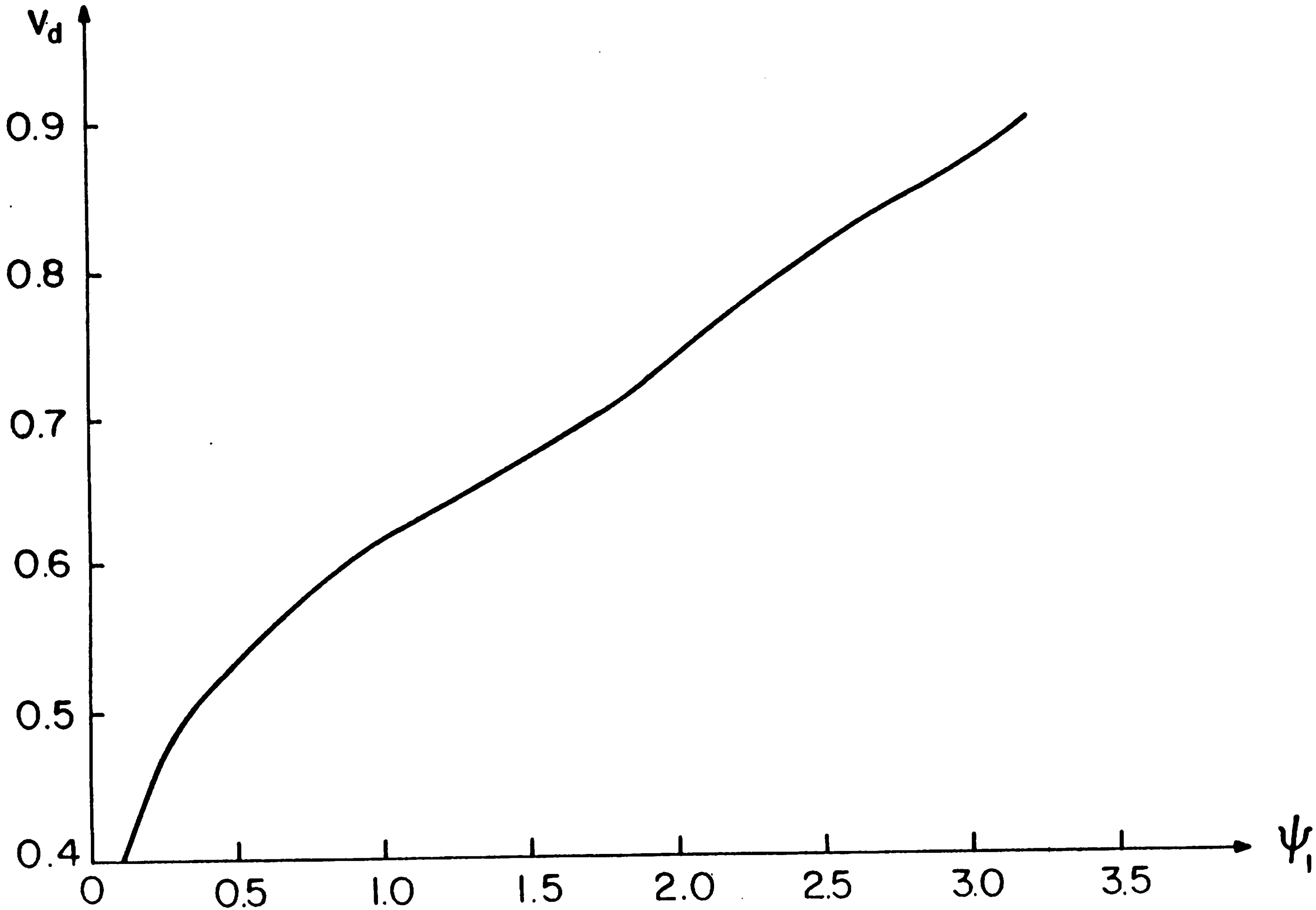

Fig. 6. Calculated electron drift velocity (v_d) and corresponding maximum ion acoustic double layer amplitude (ψ_1)

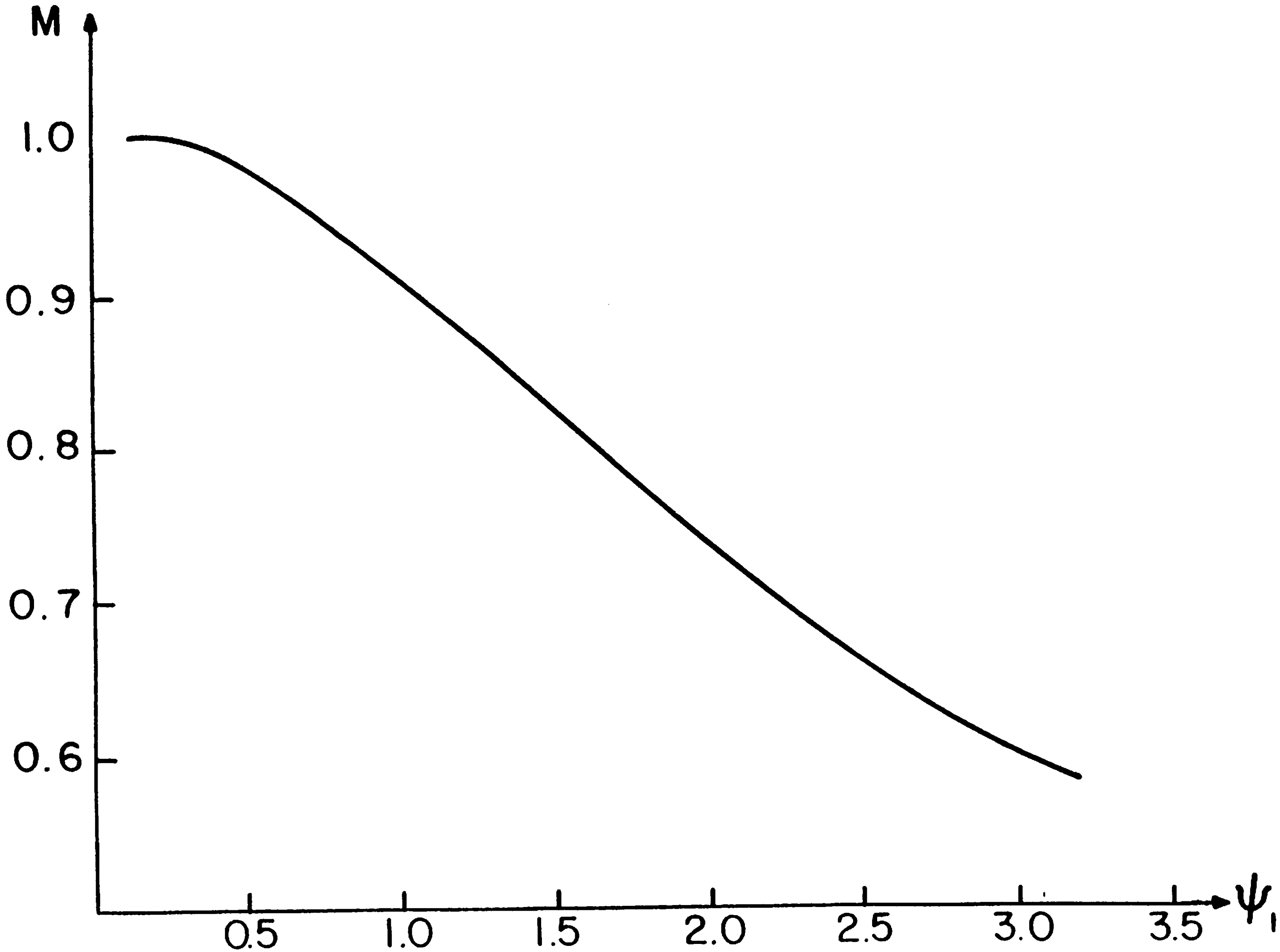

Fig. 7. Calculated ion acoustic double layer velocity(M) v.s. maximum amplitude(ψ_1).